Substitutions from Rauzy Induction

(Extended Abstract)

Luis-Miguel Lopez[1] **and Philippe Narbel**[2]

(1) Takasaki Art Center College, 2229 Iwasaki, Yoshii-machi, Gunma-ken, 370-2131 Japan.

(2) LABRI, Université Bordeaux I. 351, Cours de la Libération 33405 Talence, France

e-mails: lopez@is.titech.ac.jp, narbel@labri.u-bordeaux.fr

Published in "Developments in Language Theory, Foundations, Applications, and Perspectives, DLT'99", G. Rozenberg & W. Thomas eds., pp. 200-210, World Scientific, 2000.

1 Introduction

Interval exchange transformations are piecewise linear maps $[0,1) \to [0,1)$. When iterated, they have been extensively studied as generic examples of dynamical systems (see e.g. [9, 23, 24, 15, 13]). One of its direct quite well-known relationship to formal language theory is that Sturmian words [2, 3] can be obtained as symbolic codings of the orbits of such dynamical systems when only 2 intervals are involved. Moreover, these can be generated essentially by applying compositions of two substitutions controlled by the classical continued fraction algorithm (see e.g. [14, 25, 8, 3]).

We here adress the question of how to describe all the symbolic orbits of any interval exchange transformation over n intervals $n \geq 2$ by composing a finite set of substitutions. In this respect, it is a full generalization of the main results in [11, 12] where the investigated interval exchange transformations were restricted to those of rotation class. The theorem we present here has no such constraint:

Theorem: *Consider any irreducible and irrational interval exchange transformation over n intervals, $n > 2$ (respect. $n = 2$). The set of its symbolic orbits can be constructively described from compositions of an explicit set of $n + 1$ substitutions (respect. 2 substitutions) over an alphabet of n letters.*

This full generalization has been made possible by setting a tighter relationship between the geometric framework used in [12] and *Rauzy induction* used as a nowadays classical tool to investigate ergodic properties of interval exchange transformations (see mainly [20, 24, 26, 27]). The main difference with [12] is that, focussing on generating the symbolic orbits, one can give up most of the references to the classical theory of surfaces and foliations.

This presentation is organized as follows: We first recall the main definitions for the interval exchange transformations. Next, a second part is devoted to introduce Rauzy induction through a geometric interpretation based on the one used in [12]. This allows us to have an effective grasp on Rauzy induction, which leads, in a third part, to describe its associated substitutions, and to the above theorem. Since this result is effective, a full example is finally presented.

2 Interval Exchange Transformations

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n$, $n \geq 2$, such that $\sum_j \lambda_j = 1$, called a **length vector**, and let π be a permutation of $\{1, \cdots, n\}$. An **interval exchange transformation** (see e.g. [9, 24, 15, 13]) is a function $T_{\lambda,\pi} : [0,1) \to [0,1)$ whose domain is decomposed according to $b_0 = 0$ and $b_i = \sum_{j=1}^{i} \lambda_j$, for $i = 1, \ldots, n$, i.e. as $\bigsqcup_{i=1}^{n} I_i$ where $I_i = [b_{i-1}, b_i)$, and whose range is decomposed according to the length vector $(\lambda_{\pi^{-1}(1)}, \ldots, \lambda_{\pi^{-1}(n)})$ with $b_0^\pi = 0$ and $b_i^\pi = \sum_{j=1}^{i} \lambda_{\pi^{-1}(j)}$, i.e. as $\bigsqcup_{i=1}^{n} J_i$ where $J_i = [b_{i-1}^\pi, b_i^\pi)$. The complete expression of $T_{\lambda,\pi}$ is then given as $T_{\lambda,\pi}(x) = x - b_{i-1} + b_{\pi(i)-1}^\pi$ for all $x \in I_i$, $i = 1, \ldots, n$. An interval exchange transformation is said to be **irreducible** when its permutation π does not fix (setwise) any strict subset $\{1, ..., k\} \subset \{1, ..., n\}$. An interval exchange transformation is said of **rotation class** iff it has up to two discontinuities (see [17, 12]). Let the **positive orbit** (respect. **orbit**) of a point $x \in [0,1)$ be $O^+(x) = \{T_{\lambda,\pi}^i(x),\ i \in \mathbb{N}\}$ (respect. $O(x) = \{T_{\lambda,\pi}^i(x),\ i \in \mathbb{Z}\}$), and let $\mathcal{I}$ be $[0,1) \setminus \bigcup_{i=1}^{n-1} O(b_i)$. Then the pair $(\mathcal{I}, T_{\lambda,\pi})$ is a **dynamical system**, i.e. a pair (X, T) such that X is a metric space and $T : X \to X$ is continuous. Such a system is said to be **minimal** iff for $Y \subset X$, $Closure(T(Y)) = Y$ implies $Y = X$ or $Y = \emptyset$. The system $(\mathcal{I}, T_{\lambda,\pi})$ is minimal iff for each $x \in [0,1)$, the orbit $O(x)$ is dense in $[0,1)$ [9]. Moreover, if $T_{\lambda,\pi}$ **irrational**, i.e. irreducible and the only rational relations between the λ_i's are multiples of $\lambda_1 + ... + \lambda_n = 1$, then $T_{\lambda,\pi}$ is minimal [9]. A **topological conjugacy** between two dynamical systems (X_1, T_1) and (X_2, T_2) is a homeomorphism $\phi : X_1 \to X_2$ such that $\phi T_1 = T_2 \phi$. It is known that, up to a compactification of $\mathcal{I}$ (see e.g. [9]), $(\mathcal{I}, T_{\lambda,\pi})$ has such a conjugacy towards a language of two-way infinite words over an alphabet $A = \{x_1, x_2, \ldots, x_n\}$: Let cod be defined as $cod(y) = x_j$ if $y \in I_j$, and be extended by setting $cod(O(y)) = \ldots cod(T_{\lambda,\pi}^k(y)) cod(T_{\lambda,\pi}^{k+1}(y)) \ldots = \ldots x_{i_k} x_{i_{k+1}} \ldots$: the conjugacy is defined by $\phi(y) = cod(O(y))$. We call $\phi(\mathcal{I})$ the **symbolic orbits** of the interval exchange transformation $T_{\lambda,\pi}$. Note that if minimality holds then in each word in $\phi(\mathcal{I})$, every subword of every word in $\phi(\mathcal{I})$ occurs, and every of its subword occurs with bounded gaps (see e.g. [18]). So each distinct orbit contains most of the information about the whole system, and can be studied locally.

3 Rauzy Induction and its Geometric Interpretation

We recall here the construction used in [12] based on the one of Kerckoff [10] which allows one to consider Rauzy induction from a geometrical viewpoint. The idea is the following: we first replace the orbits of the system $(\mathcal{I}, T_{\lambda,\pi})$ by continuous trajectories winding around a particular surface. Second, we introduce simple moves of this surface leading to other surfaces of the same kind, and which can be interpreted as interval exchange transformations as well. This amounts to have operators on the space of all the interval exchange transformations. The inverse of these operators gives an instance of Rauzy induction, which in turn gives an algorithm for decomposing any interval exchange transformation.

The system $(\mathcal{I}, T_{\lambda,\pi})$ can be represented as a surface by a classical operation called a **suspension** (see e.g. [1, 5]): A **foliated box** is a product of two intervals $I \times I'$ for which individual **leaves** are of the form $I \times \{x\}$, with $x \in I'$. Let the length

vector of $T_{\lambda,\pi}$ be $\lambda = (\lambda_1, ...\lambda_n)$. Put $\mathcal{R} = [0,1] \times [0,1)$ and $\mathcal{R}_i = [0,1] \times [0,\lambda_i)$ be n foliated boxes and apply the following identification rules for each $i = 1, \ldots, n$:

$$\{1\} \times [0, \lambda_i) \text{ with } \{0\} \times [1 - b_i, 1 - b_{i-1}),$$
$$\{0\} \times [0, \lambda_i) \text{ with } \{1\} \times [1 - b^{\pi}_{\pi(i)}, 1 - b^{\pi}_{\pi(i)-1}).$$

This gives a foliated surface $\mathcal{M}_{\lambda,\pi}$ henceforth called the **stripped surface** of $T_{\lambda,\pi}$. Since individual leaves of the boxes fit together through the identification rules, we get full leaves running on $\mathcal{M}_{\lambda,\pi}$. Examples of parts of stripped surfaces are pictured in Figure 3.1. Each leaf corresponds to one orbit of $T_{\lambda,\pi}$. To get again the discrete initial orbits, consider the transverse arc corresponding to the left side of the central $\mathcal{R}$ box, i.e. $\{0\} \times [0,1)$ and the corresponding first-return map. Basic moves can be applied to such stripped surfaces. They consist in glueing boxes $\mathcal{R}_i$ along each other. The construction here described rely on two of them:

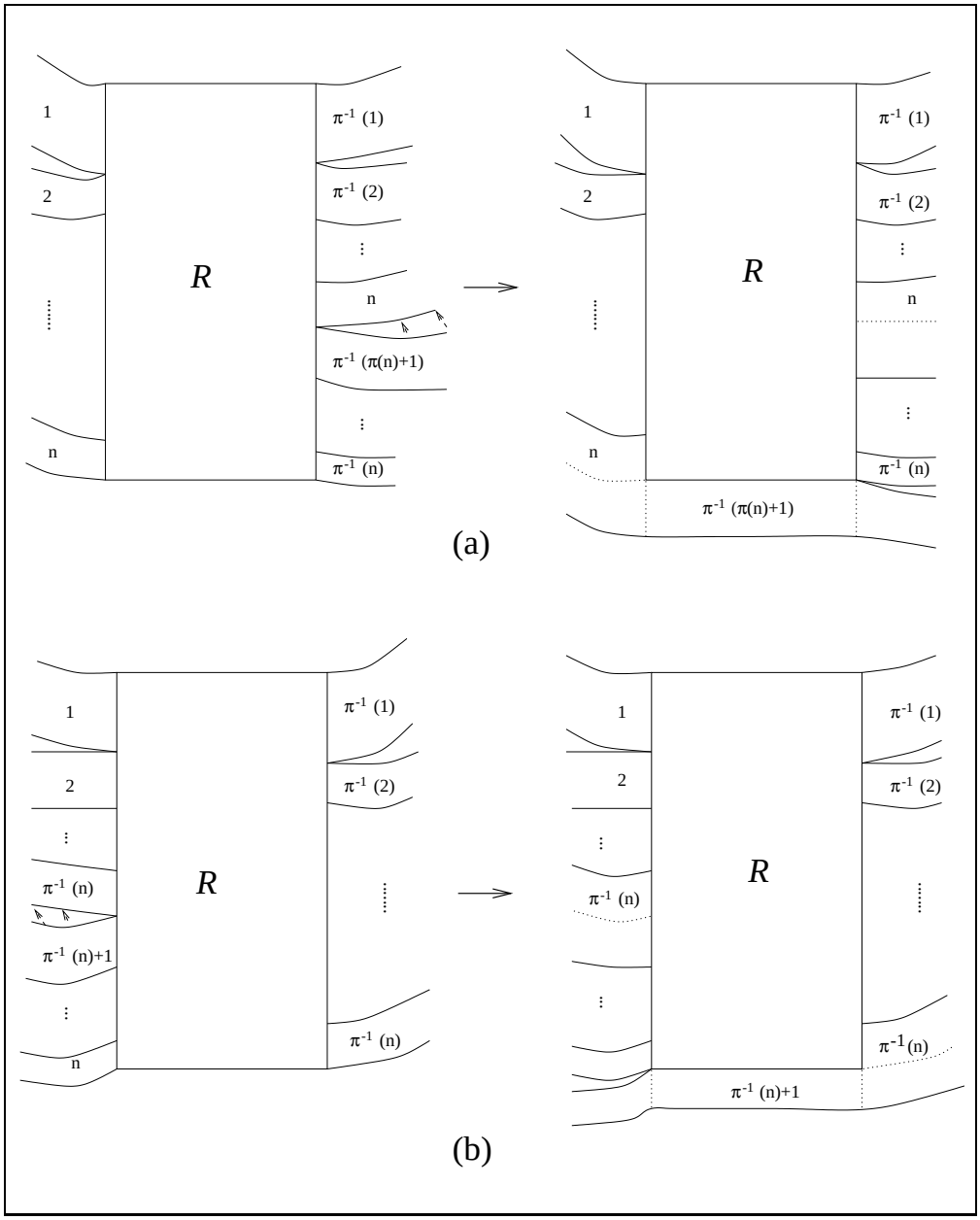

Fig. 3.1

- A **move under** $\mathcal{R}_n$ (n is the number of intervals of $T_{\lambda,\pi}$) consists in glueing the box indexed by $\pi^{-1}(\pi(n)+1)$ under the one indexed n, the glueing being undergone in the direction opposite to the leaves orientation until the lower part of $\mathcal{R}$ is reached. The glueing is continued under $\mathcal{R}$, and stopped as soon as the lower corner of the box indexed by $\pi^{-1}(n)$ is reached (see Figure 3.1(a)).

- A **move under** $\mathcal{R}_{\pi^{-1}(n)}$ consists in glueing the box indexed by $\pi^{-1}(n)+1$ under the one indexed by $\pi^{-1}(n)$, the glueing being undergone in the direction of the leaves orientation until the lower part of $\mathcal{R}$ is reached. The glueing is continued under $\mathcal{R}$, and stopped as soon as the lower corner of the box indexed by n is reached (see Figure 3.1(b)).

Since π is irreducible, $\pi(n) \neq n$, and these moves can always be applied. These can also be straightforwardly interpreted first as maps over the set of irreducible interval exchange transformations, and second as maps defined on the set of all the leaves of $\mathcal{M}_{\lambda,\pi}$. Denote by τ_k the partial circular permutation, $\tau_k(i) = i$ if $i \leq k$, $\tau_k(i) = i+1$ if $k+1 \leq i < n$, and $\tau_k(n) = k+1$.

Lemma **3.1** *Let $T_{\lambda,\pi}$ be an irreducible interval exchange transformation:*

- *The effect of a move under $\mathcal{R}_n$ is to send:*

$$\begin{array}{l}\lambda = (\lambda_1, \ldots, \lambda_n) \text{ to} \\ (\lambda_1, \ldots, \lambda_n + \lambda_{\pi^{-1}(\pi(n)+1)}),\end{array}$$

  and π to $\tau_k^{-1} \circ \pi$, with $k = \pi(n)$,

- *The effect of a move under $\mathcal{R}_{\pi^{-1}(n)}$ is to send:*

$$\begin{array}{l}(\lambda_1, \ldots, \lambda_{\pi^{-1}(n)}, \lambda_{\pi^{-1}(n)+1}, \ldots, \lambda_n) \text{ to} \\ (\lambda_1, \ldots, \lambda_{\pi^{-1}(n)} + \lambda_{\pi^{-1}(n)+1}, \lambda_{\pi^{-1}(n)+2}, \ldots, \lambda_n, \lambda_{\pi^{-1}(n)+1}),\end{array}$$

  and π to $\pi \circ \tau_k$, with $k = \pi^{-1}(n)$.

Inverting the application of each above glueing move, we get **cutting moves**: beginning with $T_{\lambda^{(j)},\pi^{(j)}}$ and trying to find a $T_{\lambda^{(j+1)},\pi^{(j+1)}}$ such that the former is obtained from the latter by one of the glueing moves. According to Lemma 3.1, this is equivalent to solve the following equation systems (the upper index of $\pi^{(j)}$ has been omitted in order to save notation):
- If the move is under $\mathcal{R}_n$:

$$\begin{array}{rl}\lambda_1^{(j)} & = \lambda_1^{(j+1)}, \\ \ldots & \\ \lambda_{n-1}^{(j)} & = \lambda_{n-1}^{(j+1)}, \\ \lambda_n^{(j)} & = \lambda_n^{(j+1)} + \lambda_{\pi^{-1}(\pi(n)+1)}^{(j+1)}.\end{array} \tag{1}$$

- If the move is under $\mathcal{R}_{\pi^{-1}(n)}$:

$$\begin{array}{rl} \lambda_1^{(j)} & = \lambda_1^{(j+1)}, \\ \cdots & \\ \lambda_{\pi^{-1}(n)-1}^{(j)} & = \lambda_{\pi^{-1}(n)-1}^{(j+1)}, \\ \lambda_{\pi^{-1}(n)}^{(j)} & = \lambda_{\pi^{-1}(n)}^{(j+1)} + \lambda_{\pi^{-1}(n)+1}^{(j+1)}, \\ \lambda_{\pi^{-1}(n)+1}^{(j)} & = \lambda_{\pi^{-1}(n)+2}^{(j+1)}, \\ \cdots, & \\ \lambda_{n-1}^{(j)} & = \lambda_n^{(j+1)}, \\ \lambda_n^{(j)} & = \lambda_{\pi^{-1}(n)+1}^{(j+1)}. \end{array} \tag{2}$$

Application of these cutting moves also leads to a transformation of the permutations $\pi^{(j)}$ which is the way around of those given in Lemma 3.1, i.e. a cutting move under $\mathcal{R}_n$ sends $\pi^{(j)}$ to $\tau_k \circ \pi^{(j)}$, with $k = \pi^{(j+1)}(n) = \pi^{(j)}(n)$. A cutting move under $\mathcal{R}_{\pi^{(j+1)^{-1}}(n)}$ sends $\pi^{(j)}$ to $\pi^{(j)} \circ \tau_k^{-1}$, with $k = \pi^{(j+1)^{-1}}(n) = \pi^{(j)^{-1}}(n)$.

Remark **3.2** *The above cutting moves are exactly the two main maps acting on the space of all the irreducible interval exchange transformations which define the so-called* **Rauzy induction** *(also used in e.g. [24, 26]).*

In fact, one can check that following the original notations ([20] p.322), $\tau_{\pi(n)} \circ \pi = a(\pi)$ and $\pi \circ \tau_{\pi^{-1}(n)}^{-1} = b(\pi)$. We recall now the following (see e.g. [12]):

Remark **3.3** *Let $T_{\lambda,\pi}$ be irrational. Then Rauzy induction can be iterated to infinity on $T_{\lambda,\pi}$.*

Remark **3.4** *At each step of Rauzy induction, the choice of the cutting move to be applied is deterministic.*

Thus, accordingly to its initial purpose, Rauzy induction can be summed up as an algorithm to decompose interval exchange transformations:

Algorithm **3.5** **Input:** *$T_{\lambda,\pi}$ with $\lambda \in \mathbb{R}_+^n$ irrational, $n > 1$ and π irreducible.* **Output:** *A sequence $\{t_j\}_{j=1,\ldots}$ of cutting moves.*

1. $j \leftarrow 0$, $\lambda^{(0)} \leftarrow \lambda$, $\pi^{(0)} \leftarrow \pi$.
2. - `if` $\lambda_n^{(j)} > \lambda_{\pi^{(j)^{-1}}(n)}^{(j)}$:
     (a) `solve Eq.Sys.` 1 `to get` $\lambda^{(j+1)}$,
     (b) $\pi^{(j+1)} \leftarrow \tau_{\pi^{(j)}(n)} \circ \pi^{(j)}$.
     (c) $t_{j+1} \leftarrow$ `a move under` $\mathcal{R}_n$.
   - `if` $\lambda_n^{(j)} < \lambda_{\pi^{(j)^{-1}}(n)}^{(j)}$:
     (a) `solve Eq.Sys.` 2 `to get` $\lambda^{(j+1)}$,
     (b) $\pi^{(j+1)} \leftarrow \pi^{(j)} \circ \tau_{\pi^{(j)^{-1}}(n)}^{-1}$.
     (c) $t_{j+1} \leftarrow$ `a move under` $\mathcal{R}_{\pi^{(j)^{-1}}(n)}$.
3. $j \leftarrow j + 1$, `goto` (2).

4 Substitutions

We are now in the same situation as for the interval exchange transformations of rotation class in [12]: all the above can be translated into a symbolic framework. Given an alphabet A, a **substitution** θ over A is a transformation which sends any letter x of A to some word over A, and which is extended to any word $w = ...w_i w_{i+1}...$ with $w_i \in A$ by sending it to $...\theta(w_i)\theta(w_{i+1})...$[1] Now, recall that the leaves of the stripped surface $\mathcal{M}_{\lambda,\pi}$ associated to the orbits of $T_{\lambda,\pi}$ can be represented into words over $\{x_1, x_2, \ldots, x_n\}$ by using a first-return map, which amounts to mark the boxes as the leaves visit them. This corresponds to the topological conjugacy ϕ introduced in Section 2, which maps each point of $\mathcal{I} \subset [0,1)$ to its symbolic orbit. The advantage of the geometric viewpoint we got from the last section is that substitutions can be readily read off from the glueing moves:

Proposition **4.1** *Let t be one of the glueing move of Lemma 3.1. Then there exists a unique substitution θ_t such that $\phi \circ t = \theta_t \circ \phi$.*

Remark **4.2** *The substitutions obtained above are the same as for interval exchange transformations of rotation class [12].*

Indeed, put $\alpha = \pi^{-1}(\pi(n)+1) - 1$ for $\theta_{n,\alpha,0}$, and $\alpha = \pi^{-1}(n)$ for $\theta_{n,\alpha,1}$:

$$\theta_{n,\alpha,0}: \begin{array}{ll} x_1 & \mapsto x_1 \\ & \ldots \\ x_\alpha & \mapsto x_\alpha \\ x_{\alpha+1} & \mapsto x_{\alpha+1}x_n \\ x_{\alpha+2} & \mapsto x_{\alpha+2} \\ & \ldots \\ x_n & \mapsto x_n \end{array} \qquad \theta_{n,\alpha,1}: \begin{array}{ll} x_1 & \mapsto x_1 \\ & \ldots \\ x_\alpha & \mapsto x_\alpha \\ x_{\alpha+1} & \mapsto x_\alpha x_n \\ x_{\alpha+2} & \mapsto x_{\alpha+1} \\ & \ldots \\ x_n & \mapsto x_{n-1} \end{array}$$

Moreover, taking into account irreducibility of the considered interval exchange transformations, this gives a set of $2(n-1)$ substitutions since $\theta_{n,\alpha,0}$ is defined for $\alpha \in \{0, .., n-2\}$, and $\theta_{n,\alpha,1}$ for $\alpha \in \{1, .., n-1\}$. We proved in [12] that this set can be generated from a set of $(n+1)$ substitutions when $n > 2$: Let ρ be the substitution $x_2 \mapsto x_1 x_2$ (the other letters remain unchanged), and τ_i be the substitutions induced by the $n-1$ transpositions $(1\ i)$, for $i = 2, \ldots n$, i.e. $x_i \mapsto x_1$ and $x_1 \mapsto x_i$ (the other letters remain unchanged). Then the substitutions $\theta_{n,\alpha,\epsilon}$, with $\epsilon = 0, 1$, can be obtained by a finite composition of ρ, τ_i and $\theta_{n,0,0}$. Therefore, the increased power of these substitutions compared to [12] is only due by new possibilities for composing them:

Theorem **4.3** *Let $T_{\lambda,\pi}$ be irreducible, irrational over n intervals, $n > 2$ (respect. $n = 2$). Every subword of its symbolic orbits can be obtained to any prescribed length by applying a composition of $n+1$ substitutions (respect. 2 substitutions) over $\{x_1, ..., x_n\}$.*

[1] This is a definition of substitutions as used in dynamical system theory (see e.g. [18]). In fact, substitutions in their full extent also support non-determinism and images are not necessarily on the same alphabet, so that they are not necessarily iterable (see e.g. [21]).

We next give the algorithm coming from the proof of the above theorem as in [12]. Recall first that **recurrence** of the generated finite subwords is due to minimality of $T_{\lambda,\pi}$. It means that each subword occurs in all the infinite symbolic orbits with bounded gaps (see e.g. [18]):

Algorithm **4.4** **Input:** *$T_{\lambda,\pi}$ with $\lambda \in \mathbb{R}_+^n$ irrational, $n > 1$, π irreducible, and some $N > 0$.* **Output:** *A word over $\{x_1, ..., x_n\}$ which is a recurrent subword of any of the symbolic orbits of $T_{\lambda,\pi}$*

1. Apply Algorithm 3.5 to $T_{\lambda,\pi}$ and get the composition $t_1 \circ ... \circ t_N$ made of the first N cutting moves t_i.
2. Transform each cutting move t_i into a substitution θ_{t_i} according to Proposition 4.1 to get a composition $\Theta = \theta_{t_1} \circ ... \circ \theta_{t_N}$.
3. Output $\Theta(x_1)$.

Corollary **4.5** *Let $T_{\lambda,\pi}$ be any irrational irreducible interval exchange transformation. Then the sequence $\{\theta_1 \circ .... \circ \theta_N(x_1)\}_{N \in \mathbb{N}}$ obtained by the above algorithm converges to a one-way infinite word which is a positive orbit of $T_{\lambda,\pi}$.*

The complexity of an infinite word w is defined as a map $\mathbb{N}^* \to \mathbb{N}$ which gives the number of subwords of length m that occur in w. It is known that the one-way infinite words generated by the above algorithm have complexity $(n-1)m+1$ where n is the number of intervals[9, 12]. Now, according to the recurrence property, there exists a $k > 0$ such that every subword with a prescribed length belongs to $w = \theta_{i_1} \circ ... \circ \theta_{i_N}(x_1)$. Thus to generate the set of subwords of a prescribed length m, it suffices to input a sufficiently large N in the above algorithm.

These algorithms have been implemented and here is the result for an interval exchange transformation $T_{\lambda,\pi}$ over 4 intervals where $\lambda = (1, \sqrt{2}, (\sqrt[3]{2})^2, (\sqrt[5]{2})^3)$ and π is $1 \to 4,\ 2 \to 1,\ 3 \to 3,\ 4 \to 2$ (which is neither an interval exchange transformation of rotation class as in [12], nor of the type investigated in [20]). By applying the induction, one may check that the decomposition of $T_{\lambda,\pi}$ goes as follows for the ten first iterations (`pi` indicates the state of the permutation, `lambda` indicates the state of the length vector λ, and at the end of the line, the equation system which must be solved is given):

```
pi:(1->4, 2->1, 3->3, 4->2), lambda: (1 1.414213562 1.587401052 1.515716567)           => Eq.Sys 1
pi:(1->3, 2->1, 3->4, 4->2), lambda: (1 1.414213562 1.587401052 0.5157165669)          => Eq.Sys 2
pi:(1->3, 2->1, 3->4, 4->2), lambda: (1 1.414213562 1.071684485 0.5157165669)          => Eq.Sys 2
pi:(1->3, 2->1, 3->4, 4->2), lambda: (1 1.414213562 0.555967918 0.5157165669)          => Eq.Sys 2
pi:(1->3, 2->1, 3->4, 4->2), lambda: (1 1.414213562 0.040251351 0.5157165669)          => Eq.Sys 1
pi:(1->4, 2->1, 3->3, 4->2), lambda: (1 1.414213562 0.040251351 0.475465215)           => Eq.Sys 2
pi:(1->4, 2->2, 3->1, 4->3), lambda: (0.524534784 0.475465215 1.414213562 0.040251351) => Eq.Sys 2
pi:(1->4, 2->3, 3->2, 4->1), lambda: (0.484283433 0.040251351 0.475465215 1.414213562) => Eq.Sys 1
pi:(1->2, 2->4, 3->3, 4->1), lambda: (0.484283433 0.040251351 0.475465215 0.929930128) => Eq.Sys 1
pi:(1->3, 2->2, 3->4, 4->1), lambda: (0.484283433 0.040251351 0.475465215 0.889678777) => Eq.Sys 1
```

According to step (1) of Algorithm 4.4, the applications of the respective equation systems yields the composition of the first ten cutting moves. Next, according

to Proposition 4.1, we obtain 6 different substitutions of type $\theta_{4,\alpha,\epsilon}$. We then get a composition Θ of these which is applied to the letter x_1 according to step (2) and (3) of Algorithm 4.4. This generates a specific word which is a subword of the symbolic orbits of $T_{\lambda,\pi}$. Here, i.e. for $N = 10$, the obtained subword is $x_1x_4x_2$. Because of minimality of $T_{\lambda,\pi}$, the bigger is N, the longer the word: for $N = 12$, we get the subword $x_1x_4x_2x_1x_4x_3x_3x_3x_4x_2$; for $N = 50$, we get a subword of length 524 and for $N = 80$, it is of length 12449.

5 Discussion

- Applying Algorithm 3.5 is known to define a multidimensional continued fraction in its additive form (see e.g. [19, 26]). In fact, Algorithm 3.5 gives the traditional continued fraction algorithm in case $n = 2$. In [12], it has been shown that this algorithm fits into one of the existing frameworks for generalizing continued fractions defined in [22, 4] and convergence results have been obtained by using unique ergodicity of the interval exchange transformations of rotation class. However, this property is known to be false in the general case discussed here [7, 15, 24] and convergence is still to be set. Also, characterization of periodic expansions of this continued fraction algorithm could lead to a generalization of the results corresponding to the 2 intervals case [6, 2], i.e. to a description of the orbits which can be obtained by iterating a single substitution.

- Sturmian words [16] (see the surveys [2, 3]) are words over two letters with many characterizations and properties. They are known to correspond to symbolic orbits of interval exchange transformations over 2 intervals. Having linear complexity and being described by compositions of substitutions controlled by continued fractions expansions are two important properties of these words. We have seen that these two properties still hold for interval exchange transformations over n intervals. Other relationships can be found in [12].

- The approach of followed here compared to [12] is different in that we did not make use of any surface embeddings of the stripped surfaces $\mathcal{M}_{\lambda,\pi}$. The interval exchange transformations of rotation class have indeed the property that they can all be embedded into torii of genus 1 with a certain number of punctures. Among others, this allowed to consider Rauzy induction in a very geometric manner, i.e. as a transformation on the set of singular foliations on such surfaces. In theory, this would still be possible in the general case discussed here, since Rauzy induction leaves stable the surface on which the successive states of the interval exchange transformation can be embedded. Indeed, apply the Euler-Poincaré characteristic formula: $2-(n+1)+C = 2-2g$ where n is the number of intervals, C is equal to the number of boundary components of $\mathcal{M}_{\lambda,\pi}$ and g is the genus of the surface. Both glueing moves can be seen to leave invariant the number of boundary components C. But, the main difference is that one would have to deal with torii of genus > 1.

- In [12], we described two distinct systems of cutting moves leading to Rauzy induction. Here, we discussed the one for which boxes $\mathcal{R}_i$ of $\mathcal{M}_{\lambda,\pi}$ are always glued under the lower side of $\mathcal{R}$ (see Figure 3.1). The other natural system of moves is given symetrically by having the glueings take place along the upper side of $\mathcal{R}$. For this latter system, the resulting set of substitutions is similar, but different from the one obtained in the former (see [12]). We still do not know whether these are the only ways of defining such an induction and therefore such systems of substitutions. We also do not know whether the number of basic substitutions can be less than $n+1$ in case $n > 2$.

References

[1] P. Arnoux, *Echanges d'intervalles et flots sur les surfaces*, Mono. Enseign. Math. **29** (1981), 5–39.

[2] J. Berstel, *Recent results on Sturmian words*, Developpements in Language Theory 95 (1995), 1–12, World Scientific.

[3] J. Berstel and P. Séébold, *Sturmian words*, Combinatorics on words. 2nd ed., Encyclopedia of Mathematics and Its Applications, no. 17, Cambridge University Press, 1997, Lothaire Editor, pp. 30–93.

[4] A. J. Brentjes, *Multidimensional continued fraction algorithms*, Mathematical Centre Tracts, 145., Mathematisch Centrum, Amsterdam, 1981.

[5] C. Camacho and A. Lins Neto, *Geometric theory of foliations*, Birkäuser, 1985.

[6] D. Crisp, W. Moran, A. Pollington, and P. Shiue, *Substitution invariant cutting sequences*, Journal de Théorie des Nombres de Bordeaux **5** (1993), no. 1, 123–138.

[7] Keynes H. and D. Newton, *A minimal non-uniquely ergodic interval exchange transformation*, Math. Zeit. **148** (1976), 101–105.

[8] S. Ito and S. Yasutomi, *On continued fractions, substitutions and characteristic sequences*, Japan. J. Math. **16** (1990), no. 2, 287–306.

[9] M. Keane, *Interval exchange transformations*, Math. Zeit. **141** (1975), 25–31.

[10] S.P. Kerckhoff, *Simplicial systems for interval exchange maps and measured foliations*, Ergodic Th. and Dynamical Sys. **5** (1985), 257–271.

[11] L.-M. Lopez and Ph. Narbel, *An algorithm to symbolically describe flows on surfaces*, Tenth Annual ACM-SIAM Symposium on Discrete Algorithms (Baltimore, Maryland, USA), 1999, pp.941-942.

[12] ———, *Substitutions and interval exchange transformations of rotation class*, Theoretical Computer Science **1-2** (2001), no. 255, 323–344.

[13] R. Mañé, *Ergodic theory and differentiable dynamics*, Springer-Verlag, 1983.

[14] A.A. Markov, *Sur une question de Jean Bernoulli*, Math. Ann. **19** (1882), 27–36.

[15] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. **115** (1982), 169–200.

[16] M. Morse and G.A. Hedlund, *Symbolic dynamics II. Sturmian trajectories*, American Journal of Mathematics **62** (1940), 1–42.

[17] A. Nogueira and D. Rudolph, *Topological weak-mixing of interval exchange maps*, Ergod.Th. and Dynam. Sys **17** (1997), 1183–1209.

[18] M. Quéffelec, *Substitution dynamical systems. spectral analysis*, Lecture Notes in Mathematics, no. 1294, Springer-Verlag, 1987.

[19] Rauzy, G., *Une généralisation du développement en fraction continue.*, Semin. Delange-Pisot-Poitou, 18e Année **1** (1977), 1501–1515.

[20] ———, *Échanges d'intervalles et transformations induites*, Acta Arith. **34** (1979), 315–328.

[21] G. Rozenberg and A. Salomaa, *Handbook of formal languages (vol. 1) word languages grammar*, Springer Verlag, Berlin, 1997.

[22] G. Szekeres, *Multidimensional continued fractions*, Ann. Univ. Sci. Budapest. Rolando Eoetvoes, Sect. Math. **13** (1970), 113–140.

[23] W. Veech, *Interval exchange transformations*, J. Anal. Math. **33** (1978), 222–272.

[24] ———, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. **115** (1982), 201–242.

[25] B.A. Venkov, *Elementary number theory*, Wolters-Noordhoff, Groningen, 1970.

[26] A. Zorich, *Finite Gauss measure on the space of interval exchange transformations and Lyapunov exponents*, Ann. Inst. Fourier **46** (1996), no. 2, 325–370.

[27] ———, *Deviation for interval exchange transformations*, Ergod. Th. and Dynam. Sys **17** (1997), 1477–1499.