

# Coding Non-Orientable Laminations

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**Abstract.** Surface laminations are classic closed sets of disjoint curves in surfaces. We give here a full description of how to obtain codings of such laminations when they are non-orientable by using lamination languages, i.e. specific linear complexity languages of two-way infinite words. We also compare lamination languages with symbolic laminations, i.e. the coding counterparts of algebraic laminations.

## 1 Introduction

A *surface lamination* is essentially a closed set of pairwise disjoint closed or two-way infinite curves (leaves) rolling around a surface, a notion related to *foliations* of surfaces [18, 6]. The curves of a lamination can always be continuously deformed onto paths of labeled embedded graphs – closely related to classic *train-tracks* [18, 16] –, that we call *train-track like (ttl) graphs*. This deformation of curves onto a graph is called *carrying*, and the set of labels of the paths the curves of a lamination are deformed onto is called a *lamination language* [14]. Lamination languages not only reflect some of the geometric behavior of the laminations they represent, they are also a family of languages with specific combinatorial properties. They are shifts, characterized by their sequences of embedded *Rauzy graphs*, and by their sets of *bispecial factors* [14] (as an extension of results about *interval exchange transformations* obtained in [10, 1]). Lamination languages always have *linear factor complexity*, whose possible functional forms have been described in [15]. The fact that the non-orientable case – that is, when laminations and their carrier train-tracks are non-orientable –, reduces to the orientable case [18] was traditionally taken for granted. However, recent advances in the non-orientable case through a detailed symbolic study of *linear involutions* [2], together with the general framework in [8] readily including the non-orientable case and using a representation called *symbolic laminations*, show that this reduction to the orientable case is not as straightforward as it seems. The purpose of this paper is to describe how this reduction goes, and in particular we prove:

**Proposition A.** *Every ttl graph  $\Gamma$  admits a canonically oriented ttl graph  $\tilde{\Gamma}$  as an orientation covering space, and any lamination carried by  $\Gamma$ , orientable or not, can be coded by  $\tilde{\Gamma}$ , yielding a lamination language.*

Lamination languages are mostly included into the set of symbolic laminations, which is a much larger set of shifts. In fact, symbolic laminations are in

bijection with *algebraic laminations*, a notion defined for free groups of finite type, and among the algebraic laminations are those which describe geodesic laminations on compact surfaces, called *algebraic surface laminations* [8]. With this respect, the second main result we prove here is the following, making a link possible between the results of [8] and [14]:

**Proposition B.** *The set of symbolic laminations  $\Lambda_{surf}$  corresponding to algebraic surface laminations is equal to the set  $\mathbf{L}_d$  of lamination languages obtained through Proposition A.*

## 2 Context and Basic Definitions

We first give basic definitions (details can be found in [14]). Let  $\Gamma = (V, E)$  be a graph with  $V$  as a set of vertices and  $E$  as a set of edges.  $\Gamma$  is **connected** if any two of its vertices are linked by a path. It is **directed** (or **oriented**) if an orientation is given to each of its edges, thus every edge has an initial vertex and a terminal vertex. A directed graph is **connected** if any two of its vertices are linked by a **(directed) admissible path**, i.e. a sequence of consecutive edges with the same orientation. Now, a classic notion in lamination theory is *train-tracks* [18, 16], usually defined as *1D branched differentiable submanifolds* of surfaces, but which can also be defined as a kind of graphs embedded in surfaces:

**Definition 1** *A graph  $\Gamma = (V, E)$  is said to be **train track-like** (**ttl** for short) if it is embedded in a surface  $\Sigma$ , and for each vertex  $v \in V$  of degree  $> 1$ , the set of incident edges at  $v$  has been non-trivially partitioned into two sets  $E_{1,v}, E_{2,v}$ , both formed of incident edges which are next to each other around  $v$ .*

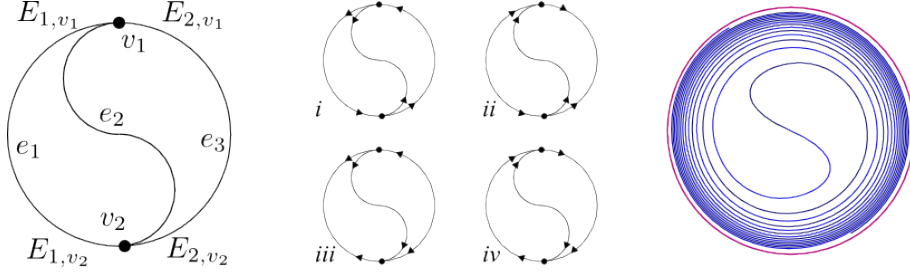
A ttl graph can be thought of as a “semi-directed” graph: at each vertex  $v$ , the sets  $E_{1,v}, E_{2,v}$  induce two possible local orientations, where the edges in  $E_{1,v}$  are considered incoming at  $v$  while the ones of  $E_{2,v}$  outgoing from  $v$ , or the other way around. Thus a **(ttl) admissible path** (or *trainpath*) in a ttl graph  $\Gamma$  is a sequence of consecutive edges such that at each vertex  $v$  crossed, the path enters  $v$  by an edge in  $E_{1,v}$  (resp.  $E_{2,v}$ ) and leaves it by an edge in  $E_{2,v}$  (resp.  $E_{1,v}$ ). This semi-directedness reflects the idea of “railroad switches” in a train-track.

**Definition 2** *A curve  $\gamma$  in a surface  $\Sigma$  is said to be **carried** by a ttl graph  $\Gamma$  in  $\Sigma$  if there is a continuous deformation – a homotopy – of  $\gamma$  onto a ttl admissible path of  $\Gamma$ .*

A ttl graph  $\Gamma$  is **orientable** if at each vertex  $v$  of  $\Gamma$ , there is a choice of one of the local orientations among the two induced by  $E_{1,v}$  and  $E_{2,v}$ , so that these orientations consistently match on each edge in  $\Gamma$ . A ttl graph  $\Gamma$  is **directed** (or **oriented**) when such orientation choices for  $\Gamma$  have been made, and  $\Gamma$  becomes then a plain embedded directed graph.

**Lemma 1** (Two orientations). *Let  $\Gamma$  be a connected and orientable ttl graph. Then  $\Gamma$  has only two possible orientations, opposite of each other on each edge.*

**Example 1** (A non-orientable graph). *One of the simplest non-orientable ttl graph is the “yin-yang graph” which is made of two vertices  $v_1, v_2$ , three edges  $e_1, e_2, e_3$  linking them, and such that  $E_{1,v_1} = \{e_1, e_2\}$ ,  $E_{2,v_1} = \{e_3\}$ ,  $E_{1,v_2} = \{e_1\}$ ,  $E_{2,v_2} = \{e_2, e_3\}$ . This graph is shown on the left of the next figure. In the middle are represented the four possible sets of local orientations derived from the  $E_{\dots}$  sets; each of them induces some non-consistently oriented edge, hence the non-orientability of the yin-yang graph:*



A **geodesic lamination**  $\mathcal{L}$  on a surface  $\Sigma$  (with a Poincaré metric) is a non-empty closed subset of  $\Sigma$  forming a union of simple and pairwise disjoint geodesics [18]. Another definition, equivalent up to continuous deformation, goes essentially as follows [14]: A **(topological) lamination**  $\mathcal{L}$  in  $\Sigma$  is a set of simple closed or two-way infinite curves in  $\Sigma$ , all pairwise disjoint and non-homotopic, such that there exists an embedded ttl graph  $\Gamma$  which carries  $\mathcal{L}$  in a maximal way with respect to inclusion (no other curve carried by  $\Gamma$  can be added to  $\mathcal{L}$  while preserving all its curve set properties). The right part of the above figure shows an example of a lamination, carried by the yin-yang graph, and made of two curves: one looks like a Fermat’s spiral rescaled so as to accumulate at the other one, a surrounding circle. Orientability for a lamination, can be defined via the notion of carrying [4]: A lamination  $\mathcal{L}$  is **orientable** if all its curves can be carried by a directed ttl graph  $\Gamma$ , using only directed admissible paths of  $\Gamma$ . A lamination is **oriented** when it is carried by a given directed graph  $\Gamma$ .

An embedded directed graph is **coherent** if at each vertex  $v$  of  $\Gamma$  all the incoming edges are next to each other around  $v$  (hence the outgoing edges are too). Following Definition 1 a directed ttl graph  $\Gamma$  is always coherent. Conversely, an embedded directed graph  $\Gamma$  can become a ttl graph only when it is coherent so that the sets  $E_{1,v}, E_{2,v}$  at each vertex  $v$  can be accordingly defined. Nevertheless, when the focus is on curve carrying by directed graphs, non-coherent carrier graphs can be considered too. These graphs have directed admissible paths as any other directed graphs, and Definition 2 applies to them as well. Thus we can drop the constraint in Definition 1 that  $E_{1,v}, E_{2,v}$  need to be formed by consecutive incident edges around  $v$ . A non-necessarily coherent directed ttl graph is an embedded directed graph, and conversely. In fact, a non-coherent graph  $\Gamma$  is always the result of *edge contractions* applied to a coherent graph while preserving its carried laminations (see Sections 3 and 5.4 in [14]). Non-coherent ttl graphs can indeed be seen as contracted train tracks.

Let  $A$  be a finite alphabet, and let  $A^{\mathbb{Z}}$  denote the set of the two-way infinite words over  $A$ . Any subset of  $A^{\mathbb{Z}}$  is a **language** of two-way infinite words.  $A^{\mathbb{Z}}$  can be endowed with the topology coming from the *Cantor metric*, and the **shift map**  $\sigma$  is a continuous map on  $A^{\mathbb{Z}}$  which sends  $\dots a_{-1}a_0a_1\dots$  to  $\dots a'_{-1}a'_0a'_1\dots$  where  $a'_i = a_{i+1}$  for  $i \in \mathbb{Z}$ . A **shift** is a closed  $\sigma$ -invariant language in  $A^{\mathbb{Z}}$ . Now, an embedded directed graph  $\Gamma$  is here said to be **labeled** by  $A$  if its edges are bijectively labeled by  $A$ , and the **label** of a directed admissible path of  $\Gamma$  is the word obtained by concatenating the labels of its edges. If  $\gamma$  is a curve carried by  $\Gamma$ , and if it is carried by a unique path (up to indexing), its **coding** is the label of this path. In this case, we also say that  $\gamma$  is **coded** by this label, or **coded** by  $\Gamma$ . The coding of a carried closed curve  $\gamma$  is the two-way infinite periodic word  ${}^\omega u^\omega$ , where  $u$  is the label of the closed directed path in  $\Gamma$  carrying  $\gamma$ .

**Definition 3** *A lamination language is the  $\sigma$ -closure in  $A^{\mathbb{Z}}$  of the codings of all the curves of a lamination  $\mathcal{L}$  coded by a directed graph  $\Gamma$  labeled by  $A$ .*

Embedded directed graphs not only occur in the definition of lamination languages, they are also useful when analyzing the combinatorial properties of these languages [14]. But then, the assumption that laminations should be orientable seems necessary. The next sections show that the non-orientable case reduces to the orientable one, thus making possible to stick to directed graphs.

### 3 Orientation Coverings for Carrier Graphs

First of all, to have a more combinatorial description of laminations together with carrier graphs less dependent on the embedding surfaces, the idea in [14] was to consider ttl graphs including their embedding information, that is, to consider *ribbon graphs* (close to *combinatorial maps* [12]). This technical move is possible since the coding of lamination curves only depends on the way curves are carried by a graph, not on the *genus* or the *punctures* of their embedding surfaces.

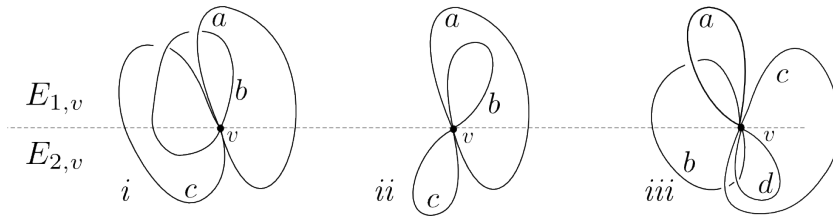
**Definition 4** *A ribbon graph (or fat graph)  $\Gamma$  is a quintuple  $(V, H, h, i, \xi)$  where:  $V$  is a set of vertices;  $H$  is a set of half-edges;  $h : H \rightarrow H$  is an involution without fixed points, which exchanges the pairs of half-edges, thus inducing a set  $E$  of (full) edges (giving  $\Gamma$  an associated usual graph structure);  $i : H \rightarrow V$  is an incidence map, which indicates the vertex of  $\Gamma$  each half-edge is incident with;  $\xi$  is a permutation on  $H$  defined as the product of the cyclic orderings defined on each subset  $i^{-1}(v)$ , with  $v \in V$ .*

An embedding of a graph in a surface naturally endows it with a unique ribbon structure. Also, similarly to a graph, a ribbon graph  $\Gamma$  is **connected** if its associated graph is connected; it is **orientable** if for each half-edge in  $H$ , there is a choice of orientation so that these orientations consistently match on each edge in  $E$ ; it is **directed** (or **oriented**) when such orientation choices for  $\Gamma$  have been made; it is **coherent**, if  $\xi$  implies that at each vertex in  $V$ , all the incoming edges are next to each other around. A ttl structure is assigned to a ribbon graph by partitioning the half-edges of  $H$  incident at each  $v \in V$  into sets  $E_{1,v}$  and  $E_{2,v}$ .

**Example 2** (A non-orientable coherent ttl ribbon graph). *Reconsidering the yin-yang graph of Example 1 with its embedding in the plane, and denoting by  $e_{j,s}, e_{j,t}$  the two half-edges of the edge  $e_j$ , we get the following ribbon graph:  $V = \{v_1, v_2\}$ ,  $H = \{e_{1,s}, e_{1,t}, e_{2,s}, e_{2,t}, e_{3,s}, e_{3,t}\}$ ,  $h(e_{j,s}) = e_{j,t}$  and  $h(e_{j,t}) = e_{j,s}$  for  $j = 1, \dots, 3$ ,  $i(e_{j,s}) = v_1$  and  $i(e_{j,t}) = v_2$  for  $j = 1, \dots, 3$ , and  $\xi = (e_{1,s} e_{2,s} e_{3,s})(e_{1,t} e_{3,t} e_{2,t})$ . We can also define the same ttl structure as in Example 1:  $E_{1,v_1} = \{e_{1,s}, e_{2,s}\}$ ,  $E_{2,v_1} = \{e_{3,s}\}$ ,  $E_{1,v_2} = \{e_{1,t}\}$ ,  $E_{2,v_2} = \{e_{2,t}, e_{3,t}\}$ .*

Let us give other instances of embedded graphs describable as ribbon graphs. First of all, a ttl graph  $\Gamma$  is **recurrent** if for every edge  $e$  in  $\Gamma$  there exists a closed admissible path which includes  $e$ . Then  $\Gamma$  is recurrent iff its edges can be weighted by a positive map  $\mu : E \rightarrow \mathbb{R}_+^*$  such that at every vertex  $v$  of  $\Gamma$  the branch equation  $\sum_{e \in E_{1,v}} \mu(e) = \sum_{e \in E_{2,v}} \mu(e)$  holds [16]. A ttl graph  $\Gamma$  endowed with such a map  $\mu$  is said to be **weighted**. A **bouquet of circles** is a graph having only one vertex  $v$ . Now, **interval exchange transformations** [7] (**iets** for short) are orientation-preserving and piecewise isometric maps of bounded intervals, and they are characterized by the fact their dynamics correspond to laminations carried by orientable (thus trivially recurrent), coherent and weighted ttl ribbon bouquets of circles [4]. When these laminations are coded by their bouquets, they define a subset of lamination languages that we call **iet languages**, corresponding to the usual way of coding iets, and characterized in [10, 1] (note that the languages coding iets over two subintervals are the *Sturmian languages*). **Non-classical iets** can be defined similarly to iets, as they are also characterized by the fact their dynamics correspond to laminations carried by coherent and weighted (hence recurrent) ttl ribbon bouquets of circles, but non-orientable ones [11]. Note that for a ttl bouquet, non-orientability is equivalent to the presence of at least one **reversing loop**, that is, an edge whose both half-edges lie either in  $E_{1,v}$  or in  $E_{2,v}$ . A non-orientable ttl bouquet is recurrent iff there are reversing loops of both kinds, with half-edges in  $E_{1,v}$  and  $E_{2,v}$ . Iets and non-classical iets dynamics are thus represented by weighted ttl ribbon bouquets of circles, and they characterize what is called **linear involutions** [9, 2].

**Example 3** (Some ttl ribbon bouquets of circles). *In the following figure, three recurrent ttl ribbon bouquets of circles are shown; their embeddings induce a ribbon structure at their unique vertex  $v$ , and their ttl structures are indicated by the fact that every half-edge starting above the dotted line belongs to  $E_{1,v}$ , while every one below belongs to  $E_{2,v}$ . The bouquet (i) corresponds to an iet over three intervals; (ii) and (iii) corresponds to non-classical iets (linear involutions), respectively given in [11, Section 2.4], and in [2, Section 2.2]:*



Ribbon graphs allow us to make abstraction of the embedding surfaces. We can show then directly how to associate an oriented ttl ribbon graph with any non-orientable one, reflecting the two possible edge orientations in their admissible paths, and using a construction based on the classic method [17, 13]:

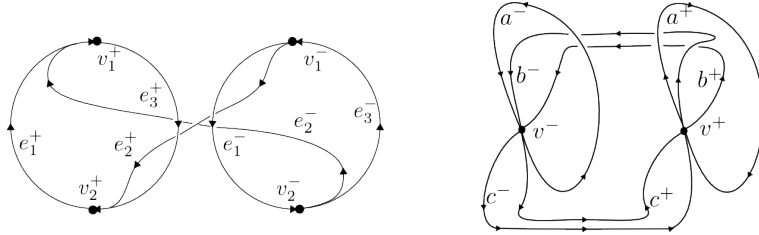
**Definition 5** Let  $\Gamma = (V, H, h, i, \xi)$  be a ttl ribbon graph. Let  $(V^+, H^+, i^+, \xi^+)$  and  $(V^-, H^-, i^-, \xi^-)$  be two copies of  $(V, H, i, \xi)$ , that is, copies of  $\Gamma$  without half-edges linking information. For every  $v \in V$ , let  $v^+ \in V^+$ ,  $v^- \in V^-$  denote its corresponding copies, and define the local orientation at  $v^+ \in V^+$  and  $v^- \in V^-$  so that the half-edges in  $E_{1,v^+}$  are outgoing (and those in  $E_{2,v^+}$  incoming), and the half-edges in  $E_{1,v^-}$  are incoming (and those in  $E_{2,v^-}$  outgoing).

Then the **orientation covering** of  $\Gamma$  is a ribbon graph  $\tilde{\Gamma} = (V^+ \sqcup V^-, H^+ \sqcup H^-, \tilde{h}, i^+ \sqcup i^-, \xi^+ \sqcup \xi^-)$ , where  $\tilde{h}$  is an involution without fixed point defined as follows. For each  $e \in H$ , let  $e^+ \in H^+$ ,  $e^- \in H^-$  denote its copies, and:

- If the orientation of  $e^+$  is consistent with that of  $h(e)^+$ ,  $\tilde{h}(e^+) = h(e)^+$  (defining an oriented edge linking vertices in  $V^+$ ); and similarly for its copy,  $\tilde{h}(e^-) = h(e)^-$  (linking vertices in  $V^-$ ).
- If the orientation of  $e^+$  is consistent with that of  $h(e)^-$ ,  $\tilde{h}(e^+) = h(e)^-$  (defining an oriented edge linking a vertex in  $V^+$  to a vertex in  $V^-$ ); similarly for its copy,  $\tilde{h}(e^-) = h(e)^+$  (linking a vertex in  $V^-$  to a vertex in  $V^+$ ).

The ribbon graph  $\tilde{\Gamma}$  comes with a **canonical orientation**, since Cases (a) and (b) above fix an orientation for each edge. With respect to the classic theory [17, 13],  $\tilde{\Gamma}$  is just the two-sheeted (two-fold) **orientation covering space** of  $\Gamma$  coming with a **covering map**  $\psi : \tilde{\Gamma} \rightarrow \Gamma$  defined by sending vertices and half-edges of  $\tilde{\Gamma}$  to the corresponding ones of  $\Gamma$  in the obvious way. Also, since  $\tilde{\Gamma}$  is a two-sheeted covering, it has a covering involution  $s$ , which is the map exchanging  $V^+$  and  $V^-$  together with their incident half-edges.

**Example 4** (Some orientation covering graphs). Considering again the yin-yang graph of Example 1, and the non-orientable bouquet of circles (ii) in the figure p. 5, here are their respective orientation coverings following Definition 5:



The next lemma is the analogous to a classic result in manifold theory [13]:

**Lemma 2** (Connectedness and Orientation). Let  $\Gamma$  be a connected ttl ribbon graph. Then  $\tilde{\Gamma}$  is connected iff  $\Gamma$  is non-orientable.

**Lemma 3** (Uniqueness of  $\tilde{\Gamma}$ ). Let  $\Gamma$  be a non-orientable ttl ribbon graph. Then the possible orientations of  $\tilde{\Gamma}$  yield isomorphic directed graphs.

## 4 Orientation Coverings for Laminations

In order to make a ttl ribbon graph  $\Gamma$  as an effective carrier graph for laminations, there are ways of building surfaces which embed  $\Gamma$  and laminations carried by  $\Gamma$  in a unique way. A **ribbon graph surface**  $\Sigma(\Gamma)$ , unique up to isometry, can be obtained as follows [14]: each vertex of  $\Gamma$  with degree  $\leq 2$  and each half-edge in  $H$  is replaced by an Euclidean square having side length 1, and each vertex in  $V$  with degree  $d > 2$  is replaced by an Euclidean regular polygon with  $d$  sides of length 1; these polygons are then glued together by Euclidean positive isometries according to the patterns given by  $h, i$  and  $\xi$ . Laminations can then be defined on  $\Sigma(\Gamma)$  so as to be carried by  $\Gamma$ . To obtain surfaces without boundary embedding  $\Sigma(\Gamma)$ , the boundary components of  $\Sigma(\Gamma)$  can be capped off, e.g. with disks, but as we already said, we mostly focus here at how a lamination is carried by  $\Gamma$ , not at the specifics of their embedding surfaces. Also, any lamination  $\mathcal{L}$  on a surface  $\Sigma$  carried by an embedded graph  $\Gamma$  can always be moved on  $\Sigma$  so as to lie in a regular neighborhood  $N(\Gamma)$  of  $\Gamma$  in  $\Sigma$ . The embedding of  $\Gamma$  in  $\Sigma$  endows it then with a ribbon structure, and  $\Sigma(\Gamma)$  is homeomorphic to  $N(\Gamma)$  by uniqueness of regular neighborhoods. Thus, to be able to speak about laminations, there is no loss of generality using only surfaces of the form  $\Sigma(\Gamma)$ .

Now, considering the orientation covering graph  $\tilde{\Gamma}$  of  $\Gamma$ , we get  $\Sigma(\tilde{\Gamma})$ , where  $\psi$  extends to a covering map to  $\Sigma(\Gamma)$ , whereto corresponding laminations can be then defined consistently:

**Definition 6** *Let  $\mathcal{L}$  be a lamination carried by  $\Gamma$  in  $\Sigma(\Gamma)$ . Let  $\Gamma^+$  and  $\Gamma^-$  be two copies of  $\Gamma$ , and let  $\mathcal{L}^+$  and  $\mathcal{L}^-$  be two copies of  $\mathcal{L}$  embedded in  $\Sigma(\Gamma^+)$  and  $\Sigma(\Gamma^-)$ , respectively. Then, the **orientation covering lamination**  $\tilde{\mathcal{L}}$  of  $\mathcal{L}$  in  $\Sigma(\tilde{\Gamma})$  carried by  $\tilde{\Gamma}$  is built after Definition 5: the surface  $\Sigma(\tilde{\Gamma})$  is obtained by gluing together the polygons making  $\Sigma(\Gamma^+)$  and  $\Sigma(\Gamma^-)$ , using  $h, i^+ \sqcup i^-$ , and  $\xi^+ \sqcup \xi^-$  of  $\tilde{\Gamma}$ . These polygons contain pieces of curves of  $\mathcal{L}^+$  and  $\mathcal{L}^-$ , fitting together on  $\Sigma(\Gamma^+)$  by construction. The result is  $\tilde{\mathcal{L}}$  and, by extension  $\psi(\tilde{\mathcal{L}}) = \mathcal{L}$ .*

A technical point is that if we put some *Poincaré metric* on  $\Sigma(\Gamma)$  (with geodesic boundary), then  $\Sigma(\tilde{\Gamma})$  inherits one too in such a way that  $\psi$  becomes a local isometry, and in this case the covering involution  $s$  becomes an isometry.

The main properties of carrier graphs and laminations are preserved by their orientation coverings, showing that the tools described in [14] remain effective on them. First, a carrying of a lamination  $\mathcal{L}$  by  $\Gamma$  is said to be **full** if every edge of  $\Gamma$  is used to carry  $\mathcal{L}$ :

**Lemma 4** (Fullness of the carrying). *Let  $\mathcal{L}$  be a lamination fully carried by  $\Gamma$ . Then  $\tilde{\mathcal{L}}$  is fully carried by  $\tilde{\Gamma}$ .*

**Lemma 5** (Coherence) *Let  $\Gamma$  be a coherent ttl ribbon graph. Then  $\tilde{\Gamma}$  is coherent.*

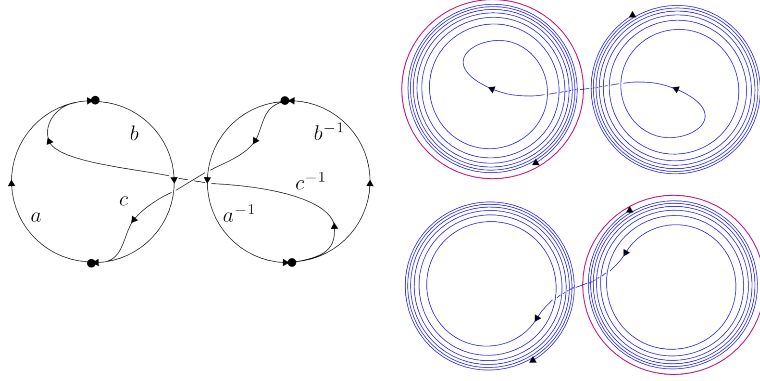
In reference to the topological definition of laminations, a carrying of a lamination  $\mathcal{L}$  by  $\Gamma$  is said to be **maximal** if no other curve carried by  $\Gamma$  can be added to  $\mathcal{L}$  while at the same time preserving its nature of lamination (the definition of carrying does not require maximality, only the definition of laminations does):

**Lemma 6** (Maximality). *Let  $\mathcal{L}$  be a lamination maximally carried by a coherent graph  $\Gamma$ . Then  $\tilde{\mathcal{L}}$  is maximally carried by  $\tilde{\Gamma}$ .*

A lamination  $\mathcal{L}$  is said to be **minimal** if it does not contain any lamination as a proper non-empty subset:

**Lemma 7** (Minimality). *Let  $\mathcal{L}$  be a minimal lamination carried by a graph  $\Gamma$ . Then  $\tilde{\mathcal{L}}$  is minimal iff  $\mathcal{L}$  is non-orientable.*

**Example 5** (An orientation covering non-minimal lamination). *Considering the yin-yang graph  $\Gamma$  and the lamination  $\mathcal{L}$  of Example 1, together with the orientation covering graph  $\tilde{\Gamma}$  shown in Example 4, the corresponding orientation covering lamination  $\tilde{\mathcal{L}}$  can be seen to be made of four curves. We show them on the right of the next figure in two pairs, each pair containing one curve with its two spiralling ends and one surrounding limit cycle:*



Using the above labeling of  $\tilde{\Gamma}$ , the lamination language coding  $\tilde{\mathcal{L}}$  is the  $\sigma$ -closure of  $\{\omega(b^{-1}a^{-1})c^{-1}(ba)^\omega, \omega(a^{-1}b^{-1})c(ab)^\omega, \omega(ab)^\omega, \omega(a^{-1}b^{-1})^\omega\}$ .

## 5 Coding Laminations

Given a lamination  $\mathcal{L}$  carried by a ttl ribbon graph  $\Gamma$ , we are now ready for:

*Proof (of Proposition A).* First, we can consider their respective orientation coverings  $\tilde{\Gamma}$  and  $\tilde{\mathcal{L}}$  (cf. Definitions 5 and 6). As a consequence of Definition 5,  $\tilde{\Gamma}$  is directed, and by Lemma 3, it is uniquely determined. Now, since  $\tilde{\mathcal{L}}$  is carried by  $\tilde{\Gamma}$ , it is oriented too, thus it can be coded by  $\tilde{\Gamma}$  as a lamination language following Definition 3.  $\square$

As a consequence, here are all the possible ways of coding  $\mathcal{L}$  by  $\Gamma$  into a lamination language considering its orientability status:

- i. If  $\mathcal{L}$  is oriented,  $\Gamma$  is too, and the coding process is just based on taking the labels of the admissible paths in  $\Gamma$  carrying the curves of  $\mathcal{L}$ .



- ii. If  $\mathcal{L}$  is orientable but does not come with an orientation:
  - a. Either we fix one of the two orientations (see Lemma 1) and  $\mathcal{L}$  is coded through this orientation (as in Case (i)),
  - b. Or we consider both orientations, that is,  $\mathcal{L}$  is coded through the coding of  $\tilde{\mathcal{L}}$  by  $\tilde{\Gamma}$  (see Proposition A), and we are in Case (i) again. Recall that  $\tilde{\Gamma}$  is then made of two disjoint copies of  $\Gamma$  (see Lemma 2), respectively having one of the two possible orientations of  $\Gamma$  (see Lemma 3).
- iii. If  $\mathcal{L}$  is non-orientable,  $\mathcal{L}$  is coded through the coding of  $\tilde{\mathcal{L}}$  by  $\tilde{\Gamma}$  (see Proposition A), and we are in Case (i) again. Here,  $\tilde{\Gamma}$  is a connected graph.

As an example of a combinatorial property, the (factor) **complexity** [5] of a language  $L$  is the map  $p_L : \mathbb{N}^* \rightarrow \mathbb{N}^*$ , where  $p_L(n)$  is the number of distinct factors (*subblocks*) in the words of  $L$ , and as a consequence of [14, 4.1.1]:

**Remark 1** (Factor complexity of non-orientable laminations). *Let  $\mathcal{L}$  be a non-orientable lamination, maximally carried by a ttl graph  $\Gamma$  with a set  $V$  of vertices and a set  $E$  of edges. Let  $L$  be the lamination language obtained by coding  $\mathcal{L}$  by  $\tilde{\Gamma}$ . Then we have:  $p_L(n) = 2(|E| - |V|)n + 2|V|$ ,  $\forall n > 0$ .*

For instance, the complexity of the lamination language  $L$  of Example 5 is  $p_L(n) = 2n + 4$ . An additional remark is that for a lamination language  $L$  coming from some linear involution *without connections*, i.e. without simplifications so that maximality of the corresponding laminations holds, we have  $p_L(n) = 2(|E| - 1)n + 2$ ,  $\forall n > 0$  (in accordance with [2]).

Note that the context of lamination languages is mostly the one of free monoids for their sets of factors. Thus the way the carrier graphs are labeled is of no importance (in particular from a word combinatorics point-of-view) as long as we use as many distinct letters as there are edges. Nevertheless, considering the orientation covering graphs allows one to extend the context to free groups, as these graphs include edges with both orientations. In order to indicate these orientations in the coding, there is a way of labeling an orientation covering graph  $\tilde{\Gamma}$ , so that the edge directions taken by an admissible path  $\gamma$  in  $\Gamma$  as a ttl graph are reflected in  $\psi^{-1}(\gamma)$ :

**Definition 7** *Let  $\Gamma$  be a non-orientable graph, and  $\tilde{\Gamma}$  its orientation covering. Let  $A = \{a_1, \dots, a_n\}$  and  $A^{-1} = \{a_1^{-1}, \dots, a_n^{-1}\}$ , for which  $s$  is defined as  $s(a_i) = a_i^{-1}$  and  $s(a_i^{-1}) = a_i$ , for every  $i = 1 \dots n$ , and where  $n$  is the number of edges of  $\Gamma$ . Then, following Definition 5, a **natural labeling** of  $\tilde{\Gamma}$  is given as:*

- a. *For every edge  $e$  built by Case (a) of Definition 5 of  $\tilde{\Gamma}$ , we assign a distinct letter  $a \in A$ , and  $a^{-1}$  to  $s(e)$  (its corresponding copy);*
- b. *For every edge  $e$  built by Case (b) of Definition 5 and its copy  $s(e)$ , we assign letters  $a \in A$  and  $a^{-1}$ , with an arbitrary choice (since they correspond to edges linking vertices in  $V^+$  and  $V^-$ ).*

For instance, the orientation covering of the yin-yang graph shown in Example 5 is labeled according to a natural labeling.

## 6 Lamination Languages and Symbolic Laminations

Let  $w = \dots w_{-2}w_{-1}w_0w_1w_2\dots$  be a two-way infinite word over an alphabet  $A \sqcup A^{-1}$  equipped with the involution  $s$  exchanging  $a$  and  $a^{-1}$  (see Definition 7), then the **symmetric** word of  $w$  is  $\nu(w) = \dots s(w_2)s(w_1)s(w_0)s(w_{-1})s(w_{-2})\dots$ . By extension, the symmetric language  $\nu(L)$  of a language  $L$  over  $A \sqcup A^{-1}$  is the language made of the symmetric words of all the words in  $L$ . The word  $w$  is said to be **reduced** if  $w_i \neq s(w_{i+1})$  for all  $i$  (neither  $a_i a_i^{-1}$  nor  $a_i^{-1} a_i$  occurs).

**Lemma 8** *Let  $\mathcal{L}$  be a lamination carried by a ttl ribbon graph  $\Gamma$ . Let  $\tilde{\Gamma}$  be labeled with a natural labeling. Let  $L$  be the lamination language of  $\mathcal{L}$  coded by  $\tilde{\Gamma}$ . Then  $L = \nu(L)$ , and all the words in  $L$  are reduced.*

Now, there exists another symbolic way of on dealing with laminations, which is defined in a very general setting in [8] by:

**Definition 8** *A symbolic lamination is a symmetric-invariant shift over an alphabet  $A \sqcup A^{-1}$  made of reduced words.*

Let us denote the set of all symbolic laminations by  $\Lambda$ , the set of lamination languages by  $\mathbf{L}$ , and the subset of all lamination languages obtained by coding laminations using the orientation coverings of their carrier graphs (cf. Cases (ii)(b) and (iii) in p. 9) by  $\mathbf{L}_d \subsetneq \mathbf{L}$ . According to Lemma 8, all the words of these sets of languages coded over alphabets of the form  $A \sqcup A^{-1}$  are reduced, thus these sets are comparable, combinatorially fitting in the context of free monoids:

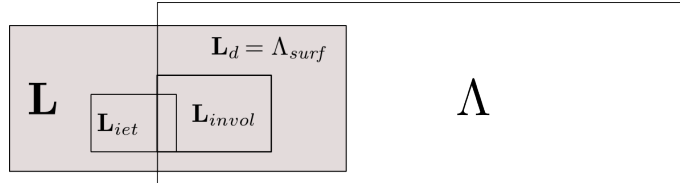
**Lemma 9**  $\mathbf{L} \not\subset \Lambda$  and  $\mathbf{L}_d \subsetneq \Lambda$ .

An important case illustrating the preceding result is given by the set of all the iet languages (cf. p. 5), that we denote by  $\mathbf{L}_{iet}$ :

**Lemma 10**  $\mathbf{L}_{iet} \subsetneq \mathbf{L}$ ,  $\mathbf{L}_{iet} \not\subset \Lambda$ , and  $\mathbf{L}_{iet} \cap \Lambda \neq \emptyset$ .

In [8], a symbolic lamination in  $\Lambda$  is said to be *orientable* if it can be written as a disjoint union  $L \sqcup \nu(L)$ , where  $L$  and  $\nu(L)$  are both closed shifts, and  $\Lambda$  is called *positive* if either  $L$  or  $\nu(L)$  uses letters from  $A$  only (the other one from  $A^{-1}$  only). This situation corresponds to the case when orientable laminations are coded with lamination languages using their orientation covering graphs (cf. Case (ii)(b) in p. 9), made then of two copies of the carrier graphs. This situation is also the one of orientable linear involutions, which can be seen as a pair of mutually inverse iets [2]. Now, similarly to iets, let us denote by  $\mathbf{L}_{invol}$  all the codings of the dynamics of the linear involutions by the orientation coverings of their carrying ttl bouquets of circles (that is, the codings studied in [2]):

**Lemma 11**  $\mathbf{L}_{invol} \subsetneq \mathbf{L}_d$ , and  $(\mathbf{L}_{invol} \cap \mathbf{L}_{iet}) = (\mathbf{L}_{iet} \cap \Lambda)$ .



In order to explain some of the differences in the above language sets, note first that the choices underlying the notion of lamination language were made so that these languages code laminations according to the way their curves roll up on a surface, and to their carrier graphs, thus yielding specific combinatorial properties [14]. As for orientability, the concreteness of lamination languages makes them to include the codings of orientable laminations without their symmetric counterpart, using their directed carrier graphs, and not necessarily their orientation coverings. Instead, symbolic laminations always preserve the link with group theory, so that in particular free groups can be exploited together with the methods used to study their *outer automorphisms* [8, Remark 4.5].

Also, in order to have a more precise view at the inclusion  $\mathbf{L}_d \subsetneq \Lambda$  (cf. Lemma 9), i.e. how the lamination languages coded by the orientation covering graphs are included into the set of symbolic laminations, we now introduce some more notions from [8]. Let  $F_n$  denote the free group on a set  $A$  of  $n$  generators, where  $A^{-1}$  denote the set of  $A$ 's inverses, and whose elements can be seen to be the reduced finite words over  $A \sqcup A^{-1}$ . The group  $F_n$  can be represented by a labeled tree  $\mathcal{T}_n$  (a *Cayley graph* of  $F_n$ ), defined by associating one vertex with each reduced word in  $F_n$ , and one edge, with label  $x \in A \sqcup A^{-1}$ , from  $v \in F_n$  to  $v' \in F_n$  if  $v' = vx$  without reduction. The vertex associated with the empty word is taken as the origin of  $\mathcal{T}_n$ . It is then possible to consider the limit language  $\partial^2 F_n$  made of all the possible pairs  $(w, w')$ ,  $w \neq w'$ , where  $w, w'$  are right-infinite labels of admissible paths in  $\mathcal{T}_n$  starting at the origin. The **flip involution** is the map on  $\partial^2 F_n$  which sends  $(w, w')$  to  $(w', w)$ . Also,  $F_n$  acts on  $\partial^2 F_n$  as  $v \cdot (w, w') = (vw, vw')$ ,  $v \in F_n$ , where  $vw, vw'$  are reduced if necessary. These pairs  $(w, w')$  give rise to two-way infinite words  $w'^{-1}w$ , where all the reductions (if any) have been applied, and the action of  $F_n$  preserves the set of these two-way infinite words. Geometrically, it is then known that the words  $w'^{-1}w$  correspond to geodesics in some surfaces (the geodesics are determined by two limit points on the boundary of what is called the *universal covering*, where  $\mathcal{T}_n$  embeds, of each of these surfaces), leading to the following definition:

**Definition 9** [8]. *An algebraic lamination is a non-empty, closed set in  $\partial^2 F_n$ , invariant by the flip and by  $F_n$ 's action.*

The set of algebraic laminations is denoted by  $\Lambda^2$ . By construction there is a bijection  $\rho$  between  $\Lambda^2$  and  $\Lambda$  [8, Proposition 4.4] (the flip corresponds to the symmetry property, and the  $F_n$ 's action to the shift-closure, see Definition 8).

Now, accordingly, an algebraic lamination determines a closed set of geodesics on a surface, but this set is generally not a geodesic lamination since these geodesics intersect on the surface (it is even very rare that they do not intersect [3]). As a consequence, symbolic laminations form a much larger set of languages than lamination languages, since lamination languages only code geodesic laminations. When an algebraic lamination actually determines a geodesic lamination, it is called an **algebraic surface lamination** [8]. Their set is denoted by  $\Lambda_{surf}^2 \subsetneq \Lambda^2$ , and let  $\Lambda_{surf} = \rho(\Lambda_{surf}^2)$  denote their corresponding symbolic laminations. We are now in position to prove Proposition B, i.e.  $\Lambda_{surf} = \mathbf{L}_d$  (also completing the figure of the set inclusions shown in p. 10):

*Proof (of Proposition B, sketch).* That  $\Lambda_{surf} \subseteq \mathbf{L}_d$  comes from the fact that any algebraic surface lamination is related to a lamination carried by a bouquet of  $n$  circles corresponding to the quotient of  $\mathcal{T}_n$  by the action of  $F_n$ . The converse  $\Lambda_{surf} \supseteq \mathbf{L}_d$  is essentially obtained by the fact that any ttl carrier graph can be continuously transformed into a ttl bouquet of  $n$  circles by identifying all its vertices onto one single vertex.  $\square$

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