

SUPPLEMENTARY EXAMPLES FOR
“INFINITE INTERVAL EXCHANGE TRANSFORMATIONS
FROM SHIFTS” ([LN15])

Luis-Miguel Lopez Philippe Narbel

Version 1.0, April, 2016

Contents

A	The Shifted Intervals Preservation (Lemma 3.2.4)	2
B	Constructions of $PART_L$ (Lemma 3.2.5)	3
C	Recodings (towards Proposition 3.3.6)	5
D	IET Graph Sequences $\{T_n\}_{n>1}$ (Proposition 5.1.3)	6
E	The Thue-Morse IET Graph $T_{L_{tm}}$	7
F	Other IET Graphs	8
	F.1 The Chacon Shift	8
	F.2 S-Adic Shifts	9
G	Speculative Constructions of IET Graphs	11
	G.1 Toeplitz Shifts	11
	G.2 The Kolakoski Shift	13

NB: The numberings of lemmas, propositions, etc. correspond to the ones in [LN15].

A The Shifted Intervals Preservation (Lemma 3.2.4)

A technical property for the construction of the conjugate IET T_L is given by Lemma 3.2.4, that is: for a shift L with a prolongable set $Fact_L$, then for every factor $u \in Fact_L$ which is not left special, we have for every $w, w' \in Cyl_L(au)$, with $a \in A$:

$$\sigma([w, w']) = [\sigma(w), \sigma(w')]. \quad (2)$$

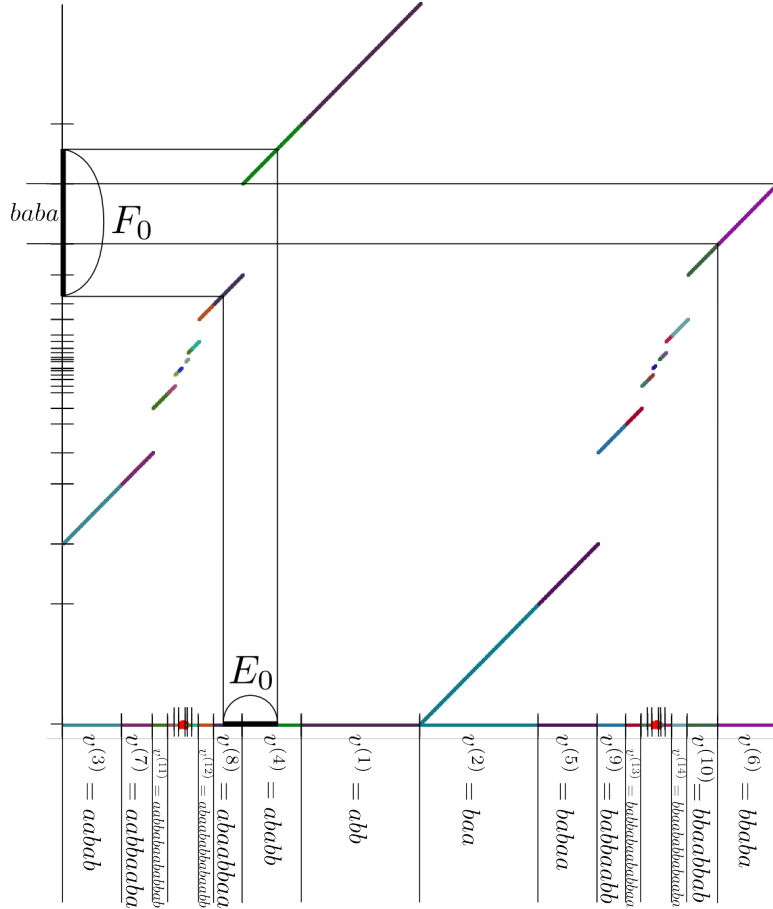
Let us give here an illustration of how this can be false when u is left special, and what are the effects on the IET construction. Recall that the **Thue-Morse substitution** over $A = \{a, b\}$ is defined by $\theta_{tm}(a) = ab$ and $\theta_{tm}(b) = ba$, and its associated shift is denoted by L_{tm} . Then for instance, consider $Cyl_{L_{tm}}(aba)$ where ba is left special. Let $E = [abav, abav'] \subset Cyl_{L_{tm}}(aba)$ with $v, v' \in A^{\mathbb{N}}$, and let $F = [\sigma(abav), \sigma(abav')] = [bav, bav']$. The above Equality (2) does not hold whenever for some $bav'' \in F$, we have $abav'' \notin \sigma^{-1}(bav'')$, while $bbav'' \in \sigma^{-1}(bav'')$, because then $bav'' \notin \sigma(E)$ (which is possible since ba is left special). For instance, consider a subinterval of E :

$$E_0 = [abaabbaau, ababbu'] \subset Cyl_{L_{tm}}(aba), \text{ for some } u, u' \in A^{\mathbb{N}}.$$

Note that $E_0 \neq \emptyset$, since L_{tm} is minimal, and $abaabbaa$ and $ababb$ belong to $Fact_{L_{tm}}$. Let

$$F_0 = [\sigma(abaabbaau), \sigma(ababbu')] = [baabbaau, babbu'].$$

Take any word $babau''$ in F_0 , which is possible since $Cyl_{L_{tm}}(baba) \neq \emptyset$ and $Cyl_{L_{tm}}(baba) \subset F_0$. But there is no word $ababav'' \in E_0$, because $ababa \notin Fact_{L_{tm}}$, and thus $babav''$ is not in $\sigma(E_0)$ (in fact, $baba$ is not left special and $\sigma^{-1}(babav'') = bbabav''$). Therefore, $\sigma(E_0) \neq F_0$. And then, in the corresponding graph of the conjugate IET $T_{L_{tm}}$, we can observe that $\sigma(E_0)$ does not correspond to an interval, and includes a discontinuity:



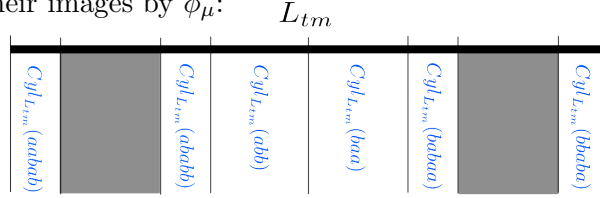
A generic case on the IET $T_{L_{tm}}$ where a shifted word interval E_0 is not equal to the interval F_0 made of its shifted ends, that is, a case where $\sigma([w, w']) \neq [\sigma(w), \sigma(w')]$.

B Constructions of $PART_L$ (Lemma 3.2.5)

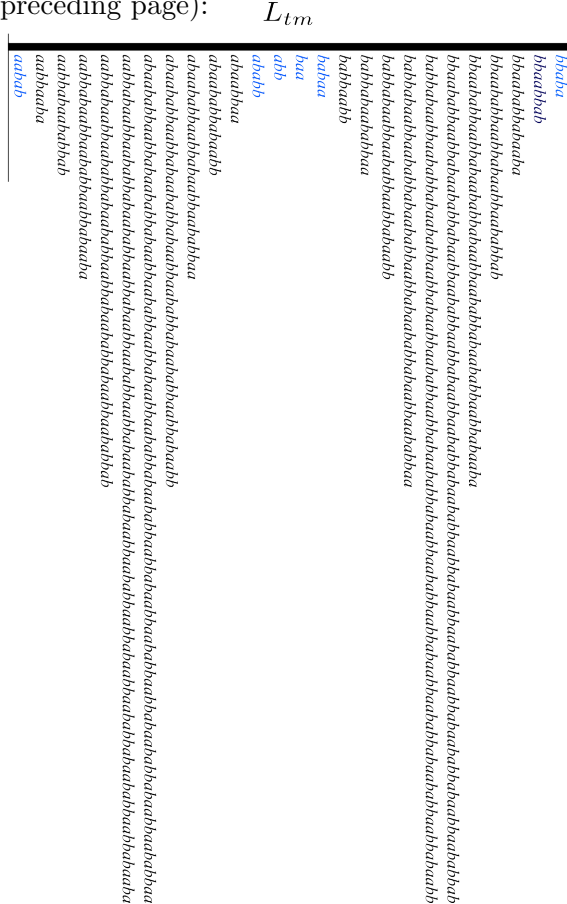
Here we give full details about Example 3.2.6 also based on the Thue-Morse shift L_{tm} , and illustrating the construction of the associated $PART_{L_{tm}}$ in the proof of Lemma 3.2.5. The left special factors of L_{tm} are known to be all the prefixes of $\theta_{tm}^n(aba)$ and $\theta_{tm}^n(bab)$, $n \geq 0$ [BLRS09, Proposition II.2.15]. We denote by $SP_{L_{tm}}(n)$ the set of n -length left special factors of L_{tm} . Then the first steps of the construction of $PART_{L_{tm}}$ are:

- Step 1. $Fact_{L_{tm}}(2) = \{aa, ab, ba, bb\}$, $SP_{L_{tm}}(1) = \{a, b\}$: No cylinders in $PART_{L_{tm}}$.
- Step 2. $Fact_{L_{tm}}(3) = \{aab, aba, abb, baa, bab, bba\}$, $SP_{L_{tm}}(2) = \{ab, ba\}$: $Cyl_{L_{tm}}(abb)$, $Cyl_{L_{tm}}(baa)$ are in $PART_{L_{tm}}$, so that $v^{(1)} = abb$, $v^{(2)} = baa$.
- Step 3. $Fact_{L_{tm}}(4) = \{aaba, aabb, abaa, abab, abba, baab, baba, babb, bbaa\}$, $SP_{L_{tm}}(3) = \{aba, abb, baa, bab\}$: No new cylinders, since $Cyl_{L_{tm}}(abba)$, $Cyl_{L_{tm}}(baab)$ are already contained in the cylinders determined in Step 2.
- Step 4. $Fact_{L_{tm}}(5) = \{aabab, aabba, abaab, ababb, abbaa, abbab, baaba, baabb, babaa, babba, bbaab, bbaba\}$, $SP_{L_{tm}}(4) = \{abba, baab\}$: $Cyl_{L_{tm}}(aabab)$, $Cyl_{L_{tm}}(ababb)$, $Cyl_{L_{tm}}(babaa)$, and $Cyl_{L_{tm}}(bbaba)$ are in $PART_{L_{tm}}$, so that $v^{(3)} = aabab$, $v^{(4)} = ababb$, $v^{(5)} = babaa$, $v^{(6)} = bbaba$.

Next, we can represent the obtained cylinders of $PART_{L_{tm}}$ on an abstract interval corresponding to L_{tm} with its lexicographic order, in turn corresponding to intervals of I when considering their images by ϕ_μ :



Below, we indicate the result of the next steps of the construction where only the $v^{(k)}$ factors are indicated (their corresponding intervals in I can be observed in $T_{L_{tm}}$ – see e.g. the figure in the preceding page):



C Recodings (towards Proposition 3.3.6)

Reconsider the Thue-Morse shift L_{tm} . Consider some word in L_{tm} like $w = \sigma(w')$ where w' belongs to $Cyl_{L_{tm}}(aabbabaabbaababbabaabbaabbaababbabaabbaababbab)$, i.e. the fifth cylinder of $PART_{L_{tm}}$, so that

$$w = abbabaabbaababbabaabbaabbaabbaababbabaabbaababbab\dots$$

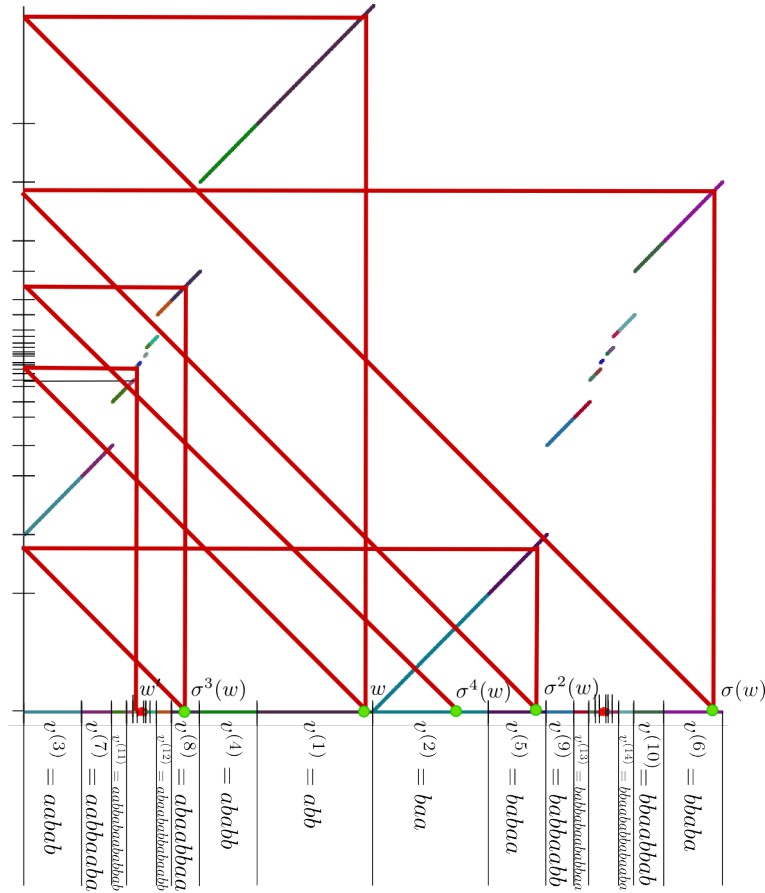
Now, having in mind the definitions given at the end of Section 3.3, here is how to get the corresponding symbolic orbit over the infinite alphabet A_∞ of the natural coding by using the map $\delta_\infty(w)$ based on the factors $v^{(k)}$ inducing $PART_{L_{tm}}$. First, abb is the smallest prefix of $\sigma(w)$ with a non-special suffix bb , so that abb is a $v^{(k)}$ of $PART_{L_{tm}}$. Indeed $abb = v^{(1)}$, so let $y_1 \in A_\infty$ be such that $\zeta_\infty(y_1) = abb$:

$$w = \underbrace{abb}_{y_1} abaabbaababbabaabbaabbaabbaababbabaabbaababbab\dots$$

And so on for the next iterates of σ over w :

$$\begin{aligned} \sigma(w) &= \underbrace{bbaba}_{y_6} abbaababbabaabbaabbaabbaababbabaabbaababbab\dots \\ \sigma^2(w) &= \underbrace{babaa}_{y_5} bbaababbabaabbaabbaabbaababbabaabbaababbab\dots \\ \sigma^3(w) &= \underbrace{abaabbaa}_{y_5} babbabaabbaabbaabbaababbabaabbaababbab\dots \\ \sigma^4(w) &= \underbrace{baa}_{y_2} bbaababbabaabbaabbaabbaabbaabbaabbaababbab\dots \end{aligned}$$

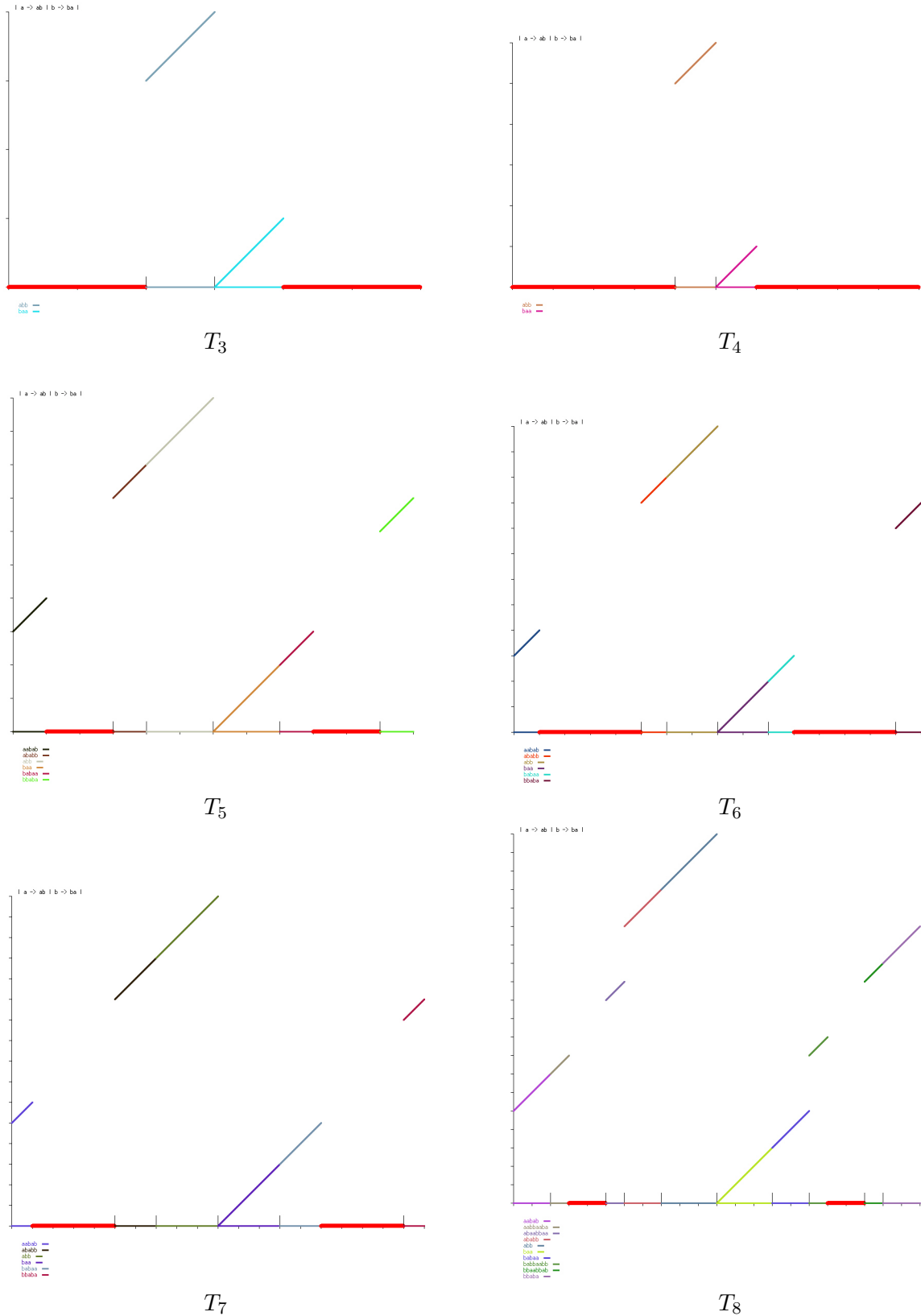
As a result we obtain the word $\delta_\infty(w) = y_1 y_6 y_5 y_8 y_2 \dots$ in $A_\infty^{\mathbb{N}}$. In the next figure we can then follow the orbit of $\phi_\mu(w)$ on $T_{L_{tm}}$ visiting the intervals corresponding to each of these cylinders (see Proposition 3.3.6):



On the IET $T_{L_{tm}}$, the orbit of $\phi_\mu(w)$, which falls into the intervals corresponding to the cylinders induced by the $\zeta_\infty(y_i)$, i.e. $abb, bbaba, babaa, abaabbaa, baa$, as described above.

D IET Graph Sequences $\{T_n\}_{n>1}$ (Proposition 5.1.3)

Considering the Thue-Morse shift L_{tm} , and following Proposition 5.1.3, we show below the first terms of the corresponding map sequence $\{T_n\}_{n>1}$ converging to the IET $T_{L_{tm}}$:

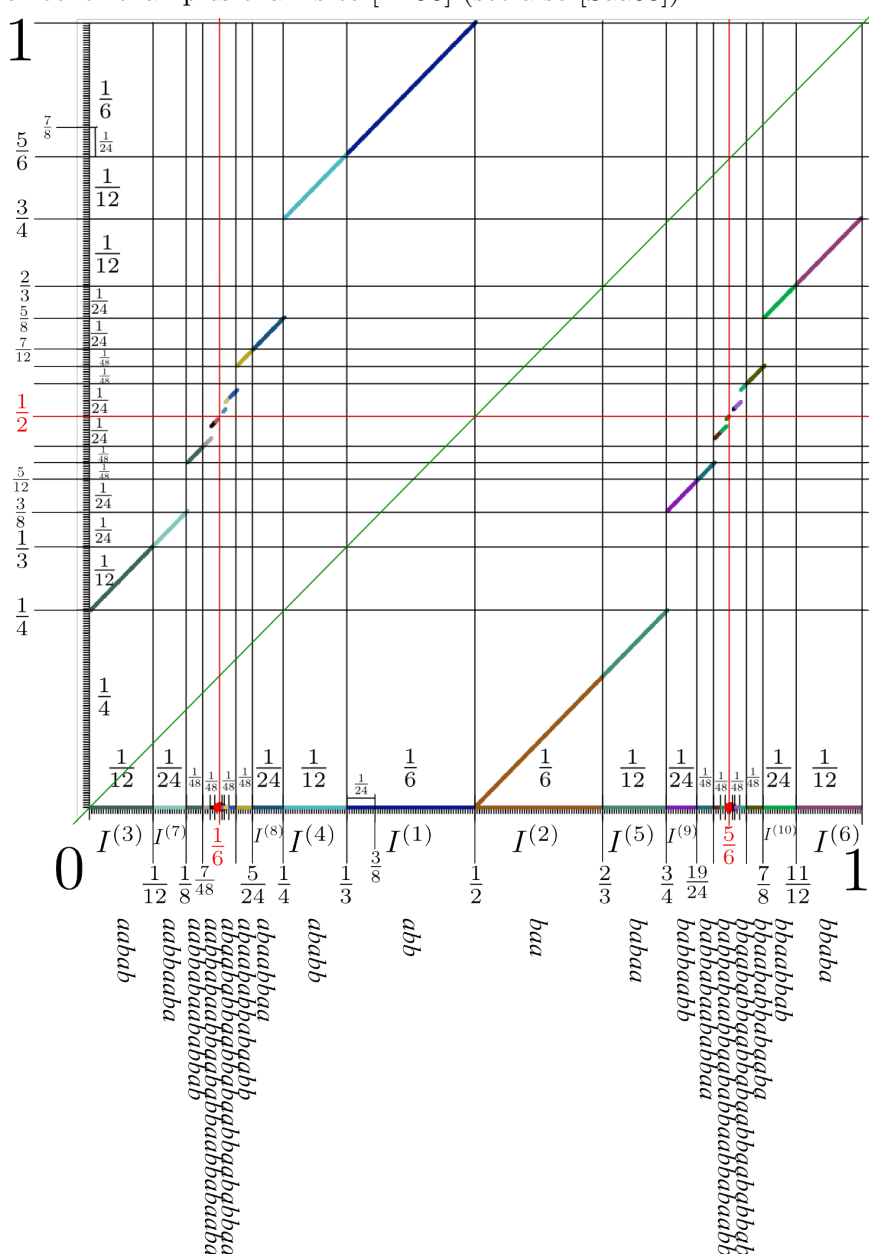


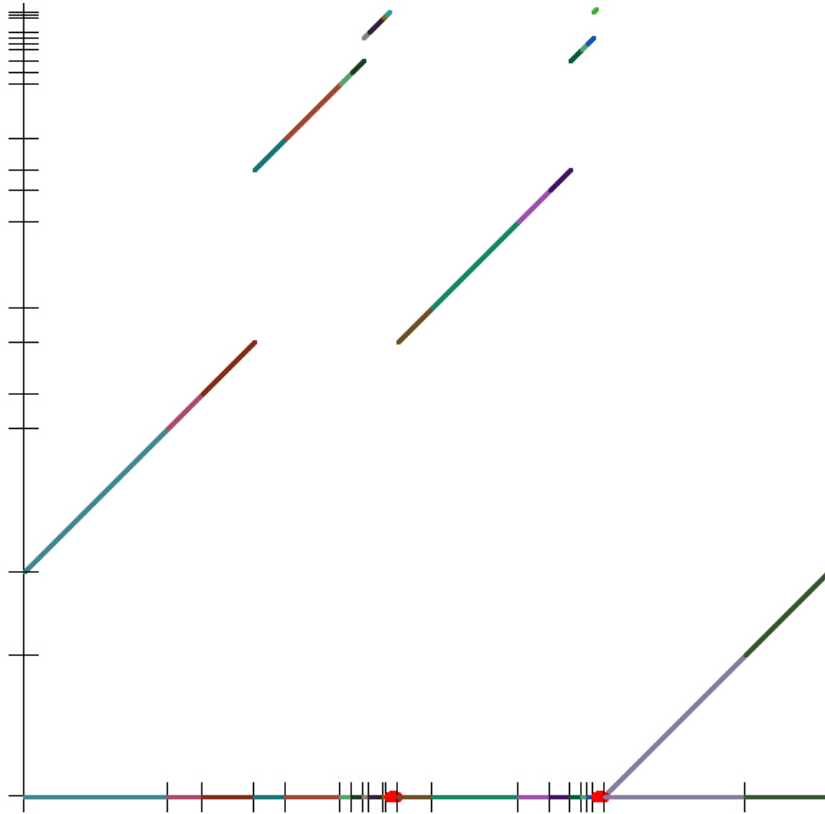
The first terms of $\{T_n\}_{n>1}$ converging to the IET $T_{L_{tm}}$. Here, each T_n is defined as a partial function over the intervals corresponding to the cylinders determined to be in $PART_{L_{tm}}$ at step n of the construction behind Lemma 3.2.5 (see Section B above).

E The Thue-Morse IET Graph $T_{L_{tm}}$

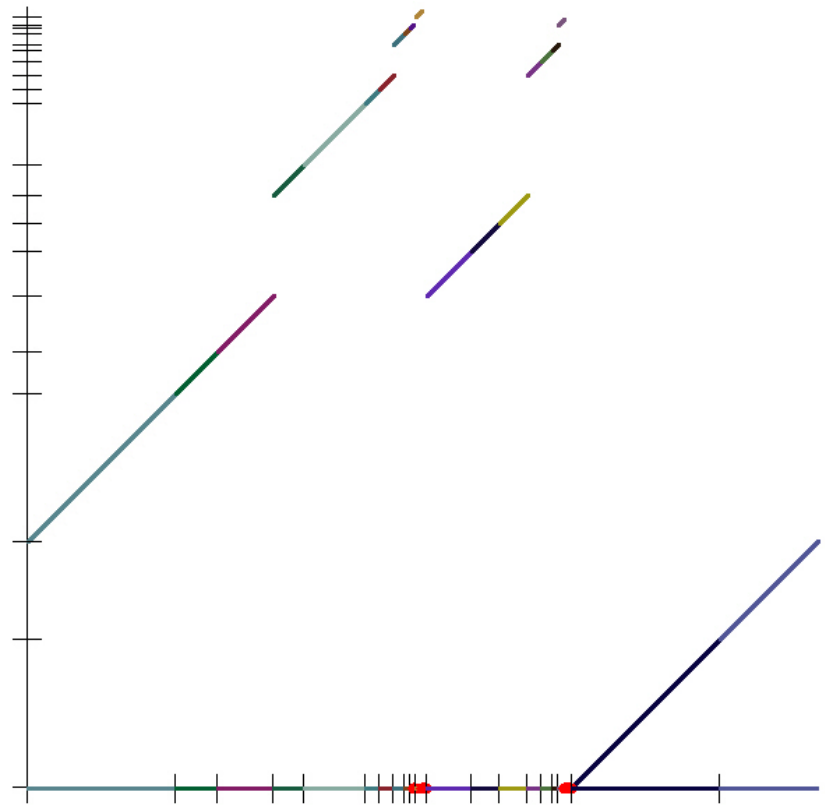
Recall that in the primitive substitution case, one can directly compute the measure μ of the cylinders of the associated uniquely ergodic shift, i.e. the lengths of the corresponding intervals of the conjugate IET. A general technique relies on obtaining for each factor length the normalized Perron-Frobenius positive eigenvector of a primitive matrix associated with the substitution [Qué10, Section 5.4].

In the Thue-Morse case, the measure values can be more readily obtained. Namely by [Dek92, Theorem 1], for each $v \in \text{Fact}_{L_{tm}}(n)$, $n \geq 2$, $\mu(\text{Cyl}_{L_{tm}}(v))$ is either $\frac{1}{6}2^{-m}$ or $\frac{1}{3}2^{-m}$, where m is such that $2^m < n \leq 2^{m+1}$, and in the specific case $n = 2^m + 1$, $m > 0$, this value is $\frac{1}{3}2^{-m}$. Now, when $2^m + 1 < n \leq 2^{m+1}$, $m > 0$, the value ambiguity can be solved using σ -preservation: First, consider all the factors $\sigma^k(v)$ for $k = 1, \dots, n - (2^m + 1)$ (where σ on a finite word v is v minus its first letter). If there is no left special factor among them, by measure preservation the length is $\frac{1}{3}2^{-m}$; if there is one (there can be only one since only two values are possible) the length is $\frac{1}{6}2^{-m}$. Thus in L_{tm} , we have e.g. $\mu(abb) = \mu(baa) = 1/6$, $\mu(aabab) = \mu(ababb) = 1/12$, $\mu(aabbaaba) = \mu(abaabbaa) = 1/24$, etc. The next figure shows the IET graph $T_{L_{tm}}$ with its main values (also validating experimentally Proposition 5.1.3). Note that this kind of exact computations are also possible for other examples thanks to [Fri98] (see also [Saa08]).





The approximated IET graph T_{100} of $\lim_{n \rightarrow \infty} (\theta_{i_0} \circ \theta_{i_1} \circ \dots \circ \theta_{i_n})(c)$, where $i_0 i_1 i_2 \dots i_n \dots$ is the fixed point $\theta_{tm}^\omega(1)$ of the Thue-Morse substitution defined over $\{1, 2\}$ instead of $\{a, b\}$.



The approximated IET graph T_{100} of $\lim_{n \rightarrow \infty} (\theta_{i_0} \circ \theta_{i_1} \circ \dots \circ \theta_{i_n})(c)$, where $i_0 i_1 i_2 \dots i_n \dots$ is the fixed point $\theta_{cha}^\omega(1)$ of the Chacon substitution defined over $\{1, 2\}$ instead of $\{a, b\}$.

G Speculative Constructions of IET Graphs

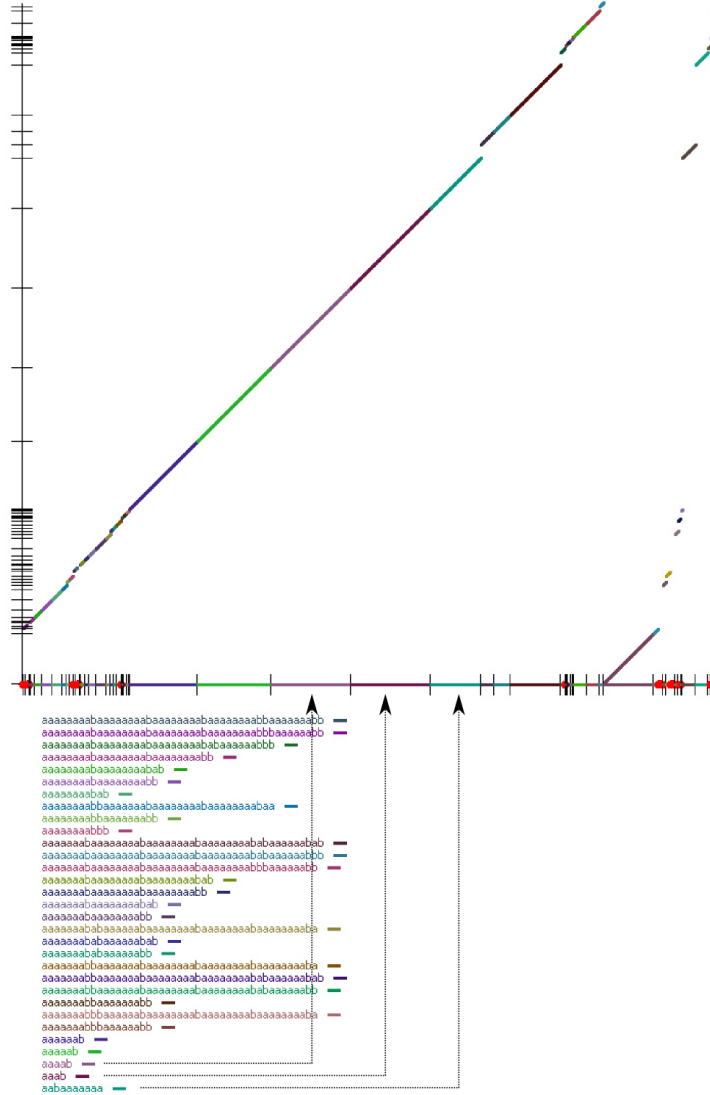
The next few examples of IET graphs are about shifts which are not proved to satisfy all the assumptions to ensure the straightforward application of Proposition 5.1.3 through Proposition 5.2.1. We nevertheless show how they could look like ($\{T_n\}$ always exists).

G.1 Toeplitz Shifts

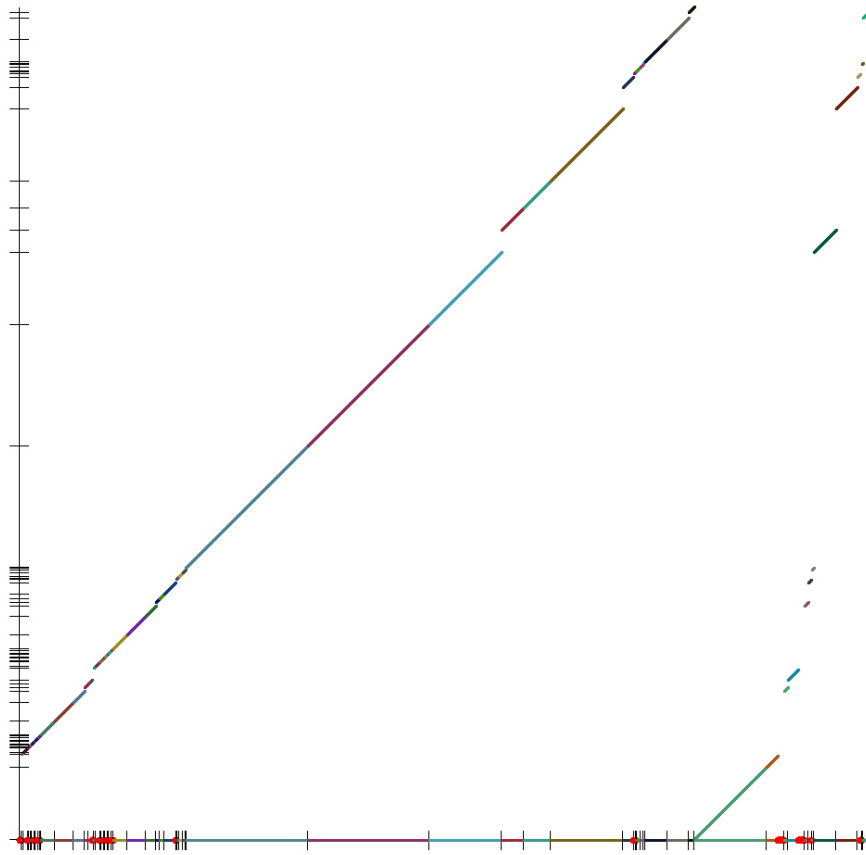
There are other ways of describing minimal words than using substitutions. Let \star be a letter not in A . For a finite word v over $(A \cup \{\star\})$ not starting by \star , let $t_0(v) = v^\omega$, $t_{i+1}(v) = t_i(v)$, where $t_i(v)$ is the word obtained from $t_{i-1}(v)$ by replacing the first occurrence of \star in $t_{i-1}(v)$ by the i th letter of $t_{i-1}(v)$ (always different from \star). Then

$$t(v) = \lim_{i \rightarrow \infty} t_i(v) \in A^{\mathbb{N}}$$

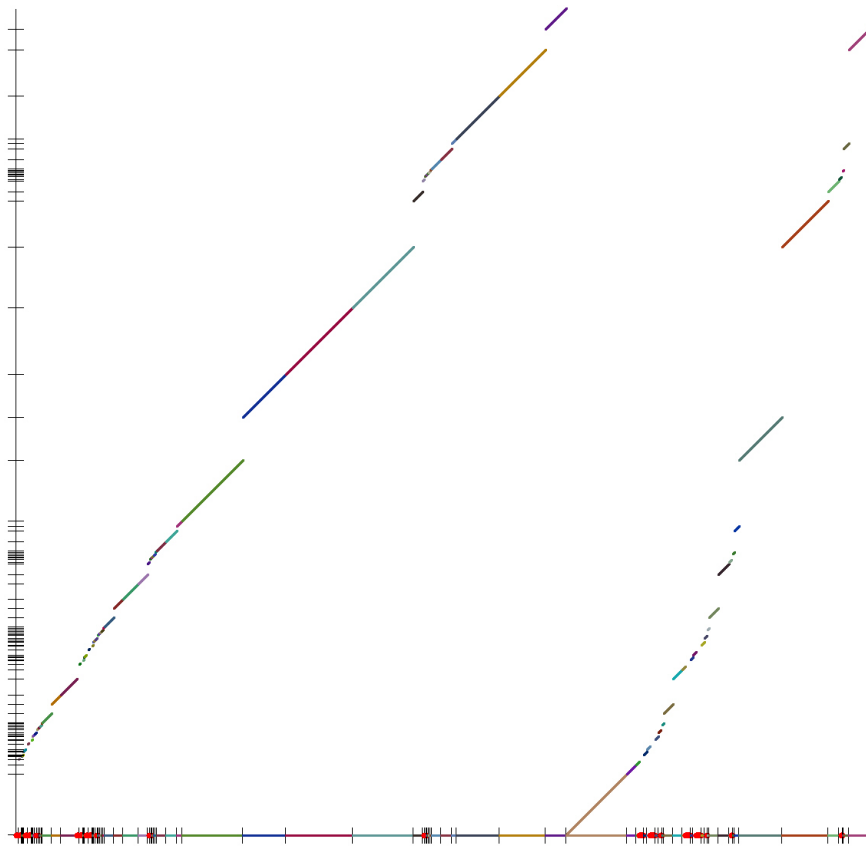
is the **Toeplitz word** determined by the **pattern** v . If p is the length of v , and q is the number of \star in v , then $t(v)$ is said to be a (p, q) -**Toeplitz word** [CK97]. Toeplitz words are minimal, and there are known to be aperiodic iff the prefix of length p is not d -periodic, where $d = \gcd(p, q)$ [CK97]. Moreover, a non-periodic (p, q) Toeplitz has complexity $p(n) = \Theta(n^{\log(p/d)/(\log(p/q))})$. Thus, zero topological entropy holds, but the fact that $\{T_n\}_{n>1}$ converges without subsequence extraction has not been proved. Here are nevertheless the aspect of some of the corresponding graphs.



The graph T_{70} for the shift associated with $t(aaaaaab \star \star)$ for which $p(n) \approx \Theta(n^{1.46})$ (together with the first corresponding $v^{(k)}$).



The graph T_{70} for the shift associated with $t(aaaab \star \star)$ for which $p(n) \approx \Theta(n^{1.55})$.



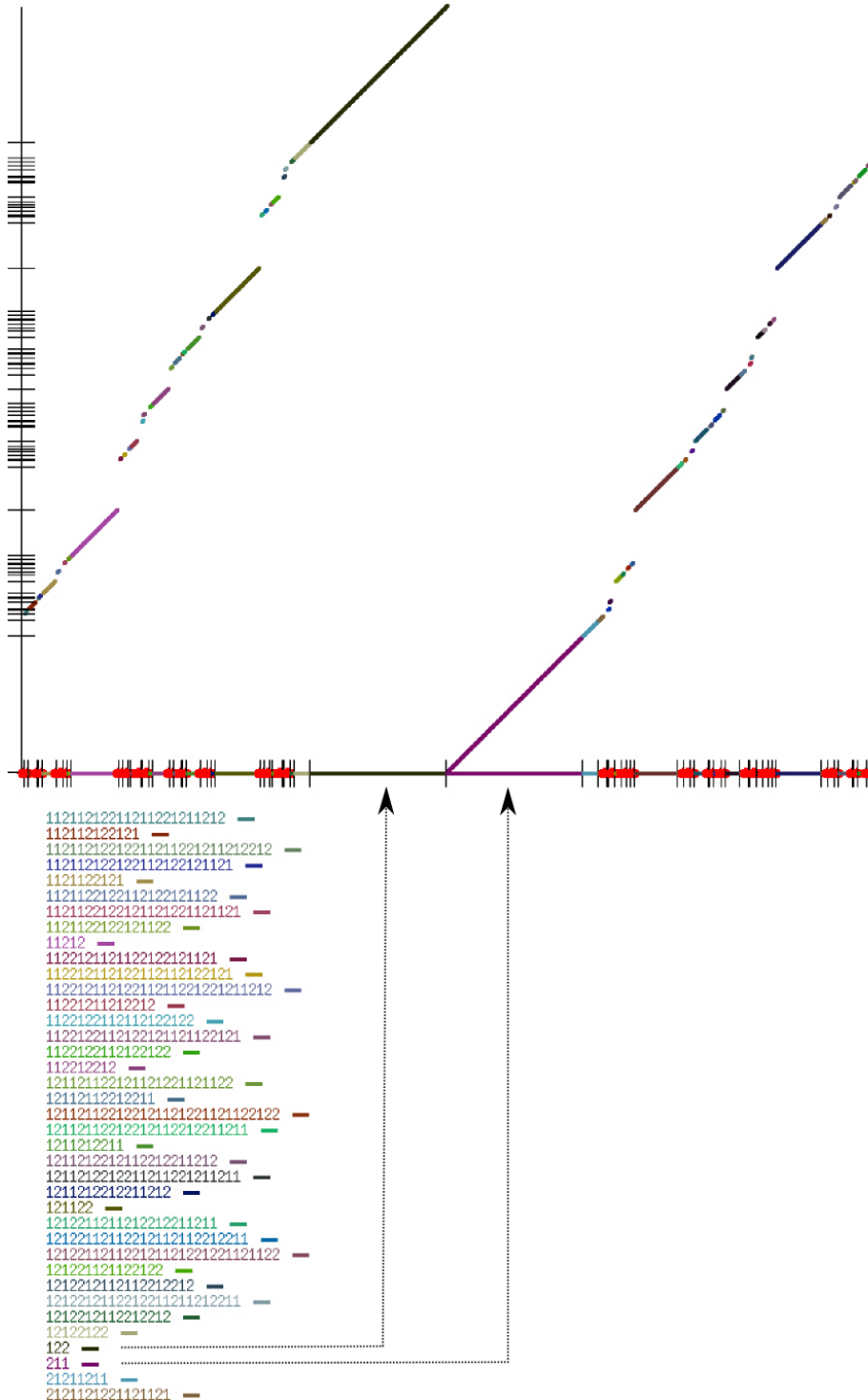
The graph T_{50} for the shift associated with $t(aab \star \star)$, for which $p(n) \approx \Theta(n^{1.75})$.

G.2 The Kolakoski Shift

The **Kolakoski sequence** κ consists of blocks of 1 and 2, each being of length either 1 or 2, where the length of the i th block is equal to the i th letter of κ [Dek97, Fog02]:

$$\kappa = 2211212212211211221211212211211212212211\dots$$

Nothing has been established about κ in terms of complexity, recurrence, ergodicity. So no assumption can be readily checked to apply Propositions 5.1.3 or 5.2.1. We nevertheless show the corresponding approximated graph below. For $v = v_1\dots v_n \in \text{Fact}_\kappa$, \tilde{v} denotes $\tilde{v}_1\dots\tilde{v}_n$, where $\tilde{1} = 2$, $\tilde{2} = 1$, and it is also conjectured that $\text{Fact}_\kappa = \widetilde{\text{Fact}_\kappa}$ [Dek97]. The symmetry of the graph is alike the one of the Thue-Morse IET T_{tm} , and suggests even that: $au \in \text{Fact}_\kappa$ with u not left special iff $\tilde{a}\tilde{u} \in \text{Fact}_\kappa$ with \tilde{u} not left special ($a \in \{1, 2\}$).



The graph T_{30} of the shift associated with κ (together with the first $v^{(k)}$).

References

- [BD14] V. Berthé and V. Delecroix. Beyond substitutive dynamical systems: S -adic expansions. In *Numeration and substitution 2012*, RIMS Kôkyûroku Bessatsu, B46, pages 81–123. Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.
- [BLRS09] J. Berstel, A. Lauve, Ch. Reutenauer, and F. V. Saliola. *Combinatorics on words*, volume 27 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2009. Christoffel words and repetitions in words.
- [CK97] J. Cassaigne and J. Karhumäki. Toeplitz words, generalized periodicity and periodically iterated morphisms. *European J. Combin.*, 18(5):497–510, 1997.
- [CN10] J. Cassaigne and F. Nicolas. Factor complexity. In V. Berthé and M. Rigo, editors, *Combinatorics, automata and number theory*, volume 135 of *Encyclopedia Math. Appl.*, pages 163–247. Cambridge University Press, Cambridge, 2010.
- [Dek92] F. M. Dekking. On the Thue-Morse measure. *Acta Univ. Carolin. Math. Phys.*, 33(2):35–40, 1992.
- [Dek97] F. M. Dekking. What is the long range order in the Kolakoski sequence? In *The mathematics of long-range aperiodic order (Waterloo, ON, 1995)*, volume 489 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 115–125. Kluwer Acad. Publ., Dordrecht, 1997.
- [Dur00] F. Durand. Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *Ergodic Theory Dynam. Systems*, 20(4):1061–1078, 2000. Corrigendum and addendum in *Ergodic Theory Dynam. Systems* **23** (2003), no. 2, 663–669.
- [Fer95] S. Ferenczi. Les transformations de Chacon: Combinatoire, structure géométrique, lien avec les systèmes de complexité $2n + 1$. *Bull. Soc. Math. France*, 123(2):271–292, 1995.
- [Fog02] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.
- [Fri98] A. E. Frid. On the frequency of factors in a D0L word. *J. Autom. Lang. Comb.*, 3(1):29–41, 1998.
- [KRS97] L. Kari, G. Rozenberg, and A. Salomaa. L systems. In *Handbook of formal languages, Vol. 1*, pages 253–328. Springer, Berlin, 1997.
- [LN15] L.-M. Lopez and Ph. Narbel. Infinite interval exchange transformations from shifts. To appear in *Ergodic Theory Dynam. Systems* (preprint available in arXiv:1506.06683 [mathDS]), 2015.
- [Qué10] M. Quéffelec. *Substitution dynamical systems—spectral analysis (2nd Edition)*, volume 1294 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.
- [Saa08] K. Saari. *On the frequency of letters in morphic sequences*. PhD thesis, University of Turku, 2008.