# Supplementary Examples for <br> "Infinite Interval Exchange Transformations <br> FROM SHIFTS" ([LN15]) <br> Luis-Miguel Lopez Philippe Narbel 

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## A The Shifted Intervals Preservation (Lemma 3.2.4)

A technical property for the construction of the conjugate IET $T_{L}$ is given by Lemma 3.2.4, that is: for a shift $L$ with a prolongable set $\operatorname{Fact}_{L}$, then for every factor $u \in \operatorname{Fact}_{L}$ which is not left special, we have for every $w, w^{\prime} \in C y l_{L}(a u)$, with $a \in A$ :

$$
\begin{equation*}
\sigma\left(\left[w, w^{\prime}\right]\right)=\left[\sigma(w), \sigma\left(w^{\prime}\right)\right] . \tag{2}
\end{equation*}
$$

Let us give here an illustration of how this can be false when $u$ is left special, and what are the effects on the IET construction. Recall that the Thue-Morse substitution over $A=\{a, b\}$ is defined by $\theta_{t m}(a)=a b$ and $\theta_{t m}(b)=b a$, and its associated shift is denoted by $L_{t m}$. Then for instance, consider $C y l_{L_{t m}}(a b a)$ where $b a$ is left special. Let $E=\left[a b a v, a b a v^{\prime}\right] \subset C y l_{L_{t m}}(a b a)$ with $v, v^{\prime} \in A^{\mathbb{N}}$, and let $F=\left[\sigma(a b a v), \sigma\left(a b a v^{\prime}\right)\right]=$ [bav, $\left.b a v^{\prime}\right]$. The above Equality (2) does not hold whenever for some $b a v^{\prime \prime} \in F$, we have $a b a v^{\prime \prime} \notin \sigma^{-1}\left(b a v^{\prime \prime}\right)$, while $b b a v^{\prime \prime} \in \sigma^{-1}\left(b a v^{\prime \prime}\right)$, because then $b a v^{\prime \prime} \notin \sigma(E)$ (which is possible since $b a$ is left special). For instance, consider a subinterval of $E$ :

$$
E_{0}=\left[a b a a b b a a u, a b a b b u^{\prime}\right] \subset C y l_{L_{t m}}(a b a), \text { for some } u, u^{\prime} \in A^{\mathbb{N}}
$$

Note that $E_{0} \neq \emptyset$, since $L_{t m}$ is minimal, and $a b a a b b a a$ and $a b a b b$ belong to $F^{\text {act }} t_{L_{t m}}$. Let

$$
F_{0}=\left[\sigma(a b a a b b a a u), \sigma\left(a b a b b u^{\prime}\right)\right]=\left[b a a b b a a u, b a b b u^{\prime}\right] .
$$

Take any word $b a b a u^{\prime \prime}$ in $F_{0}$, which is possible since $C y l_{L_{t m}}(b a b a) \neq \emptyset$ and $C y l_{L_{t m}}(b a b a) \subset$ $F_{0}$. But there is no word $a b a b a v^{\prime \prime} \in E_{0}$, because ababa $\notin F^{a c t} L_{L_{t m}}$, and thus babav" is not in $\sigma\left(E_{0}\right)$ (in fact, $b a b a$ is not left special and $\left.\sigma^{-1}\left(b a b a v^{\prime \prime}\right)=b b a b a v^{\prime \prime}\right)$. Therefore, $\sigma\left(E_{0}\right) \neq$ $F_{0}$. And then, in the corresponding graph of the conjugate IET $T_{L_{t m}}$, we can observe that $\sigma\left(E_{0}\right)$ does not correspond to an interval, and includes a discontinuity:


A generic case on the IET $T_{L_{t m}}$ where a shifted word interval $E_{0}$ is not equal to the interval $F_{0}$ made of its shifted ends, that is, a case where $\sigma\left(\left[w, w^{\prime}\right]\right) \neq\left[\sigma(w), \sigma\left(w^{\prime}\right)\right]$.

## B Constructions of $P A R T_{L}$ (Lemma 3.2.5)

Here we give full details about Example 3.2.6 also based on the Thue-Morse shift $L_{t m}$, and illustrating the construction of the associated $P A R T_{L_{t m}}$ in the proof of Lemma 3.2.5. The left special factors of $L_{t m}$ are known to be all the prefixes of $\theta_{t m}^{n}(a b a)$ and $\theta_{t m}^{n}(b a b)$, $n \geq 0$ [BLRS09, Proposition II.2.15]. We denote by $S P_{L_{t m}}(n)$ the set of $n$-length left special factors of $L_{t m}$. Then the first steps of the construction of $P A R T_{L_{t m}}$ are:
Step 1. $\operatorname{Fact}_{L_{t m}}(2)=\{a a, a b, b a, b b\}, S P_{L_{t m}}(1)=\{a, b\}$ : No cylinders in $P A R T_{L_{t m}}$.
Step 2. $F a c t_{L_{t m}}(3)=\{a a b, a b a, a b b, b a a, b a b, b b a\}, S P_{L_{t m}}(2)=\{a b, b a\}: C y l_{L_{t m}}(a b b)$, $C y l_{L_{t m}}(b a a)$ are in $P A R T_{L_{t m}}$, so that $v^{(1)}=a b b, v^{(2)}=b a a$.
Step 3. $F_{a c t}^{L_{t m}}(4)=\{a a b a, a a b b, a b a a, a b a b, a b b a, b a a b, b a b a, b a b b, b b a a\}, S P_{L_{t m}}(3)=$ $\{a b a, a b b, b a a, b a b\}$ : No new cylinders, since $C y l_{L_{t m}}(a b b a), C y l_{L_{t m}}(b a a b)$ are already contained in the cylinders determined in Step 2.
Step 4. Fact $_{L_{t m}}(5)=\{a a b a b, a a b b a, a b a a b, a b a b b, a b b a a, a b b a b, b a a b a, b a a b b, b a b a a, b a b b a$, $b b a a b, b b a b a\}, S P_{L_{t m}}(4)=\{a b b a, b a a b\}: C y l_{L_{t m}}(a a b a b), C y l_{L_{t m}}(a b a b b)$, $C y l_{L_{t m}}(b a b a a)$, and $C y l_{L_{t m}}(b b a b a)$ are in $P A R T_{L_{t m}}$, so that $v^{(3)}=a a b a b, v^{(4)}=$ $a b a b b, v^{(5)}=b a b a a, v^{(6)}=b b a b a$.

Next, we can represent the obtained cylinders of $P A R T_{L_{t m}}$ on an abstract interval corresponding to $L_{t m}$ with its lexicographic order, in turn corresponding to intervals of $I$ when considering their images by $\phi_{\mu}$ : $\quad L_{t m}$


Below, we indicate the result of the next steps of the construction where only the $v^{(k)}$ factors are indicated (their corresponding intervals in $I$ can be observed in $T_{L_{t m}}$ - see e.g. the figure in the preceding page):


We can do the same as above for the shift $L_{f i b}$ induced by the Fibonacci substitution defined by $\theta_{f i b}(a)=a b, \theta_{f i b}(b)=a$. The shift $L_{f i b}$ is known to have complexity $p(n)=$ $n+1, \forall n>0$, thus is Sturmian [Fog02]. $L_{f i b}$ is also the natural coding of the minimal finite IET $T_{f i b}$ over two intervals, where the main discontinuity is at $\frac{1}{\rho}$, where $\rho$ is the golden ratio. The left special factors of $L_{f i b}$ are all the prefixes of $\theta_{f i b}^{n}(a), n \geq 0$ [BLRS09, Proposition 4.10.3]. Then the first steps of the construction of $P A R T_{L_{f i b}}$ are:
Step 1. $\operatorname{Fact}_{L_{f i b}}(2)=\{a a, a b, b a\}, S P_{L_{f i b}}(1)=\{a\}: C y l_{L_{f i b}}(a b)$ is in $P A R T_{L_{f i b}}$, so that $v^{(1)}=a b$.
Step 2. $\operatorname{Fact}_{L_{f i b}}(3)=\{a a b, a b a, b a a, b a b\}, S P_{L_{f i b}}(2)=\{a b\}: C y l_{L_{f i b}}(b a a)$ is in $P A R T_{L_{f i b}}$, so that $v^{(2)}=b a a$.

Step 3. $F_{a c t}^{L_{f i b}}(4)=\{a a b a, a b a a, a b a b, b a a b, b a b a\}, S P_{L_{f i b}}(3)=\{a b a\}$ : No new cylinders, since $b a b a, a a b a$ determine the only cylinders not already in $P A R T_{L_{f i b}}$.
Step 4. $F_{a c t}^{L_{f i b}}(5)=\{a a b a a, a a b a b, a b a a b, a b a b a, b a a b a, b a b a a\}, S P_{L_{f i b}}(4)=\{a b a a\}$ : $C y l_{L_{f i b}}(a a b a b)$ are in $P A R T_{L_{f i b}}$, so that $v^{(3)}=a a b a b$.
Next, the obtained cylinders of $P A R T_{L_{f i b}}$ can be put on an abstract interval corresponding to $L_{f i b}$ with its lexicographic order:

$$
L_{f i b}
$$



And below we recall the corresponding IET graph $T_{L_{f i b}}$ which corresponds to the IET $T_{f i b}$ (as expected from Corollary 3.3.4):


The approximated IET graph $T_{100}$ induced by $L_{\text {fib }}$ (with all the corresponding $v^{(k)}$ ).

## C Recodings (towards Proposition 3.3.6)

Reconsider the Thue-Morse shift $L_{t m}$. Consider some word in $L_{t m}$ like $w=\sigma\left(w^{\prime}\right)$ where $w^{\prime}$ belongs to $C y l_{L_{t m}}(a a b b a b a a b b a a b a b b a b a a b a b b a a b b a b a a b a b b a b a a b b a a b a b b a b)$, i.e. the fifth cylinder of $P A R T_{L_{t m}}$, so that

$$
w=a b b a b a a b b a a b a b b a b a a b a b b a a b b a b a a b a b b a b a a b b a a b a b b a b \ldots
$$

Now, having in mind the definitions given at the end of Section 3.3, here is how to get the corresponding symbolic orbit over the infinite alphabet $A_{\infty}$ of the natural coding by using the map $\delta_{\infty}(w)$ based on the factors $v^{(k)}$ inducing $P A R T_{L_{t m}}$. First, abb is the smallest prefix of $\sigma(w)$ with a non-special suffix $b b$, so that $a b b$ is a $v^{(k)}$ of $P A R T_{L_{t m}}$. Indeed $a b b=v^{(1)}$, so let $y_{1} \in A_{\infty}$ be such that $\zeta_{\infty}\left(y_{1}\right)=a b b$ :

$$
w=\underbrace{a b b}_{y_{1}} a b a a b b a a b a b b a b a a b a b b a a b b a b a a b a b b a b a a b b a a b a b b a b \ldots
$$

And so on for the next iterates of $\sigma$ over $w$ :

$$
\begin{aligned}
& \sigma(w)=\underbrace{b b a b a}_{y_{6}} a b b a a b a b b a b a a b a b b a a b b a b a a b a b b a b a a b b a a b a b b a b \ldots \\
& \sigma^{2}(w)=\underbrace{b a b a a}_{y_{5}} b b a a b a b b a b a a b a b b a a b b a b a a b a b b a b a a b b a a b a b b a b \ldots \\
& \sigma^{3}(w)=\underbrace{a b a a b b a a}_{y_{8}} b a b b a b a a b a b b a a b b a b a a b a b b a b a a b b a a b a b b a b \ldots \\
& \sigma^{4}(w)=\underbrace{b a a}_{y_{2}} b b a a b a b b a b a a b a b b a a b b a b a a b a b b a b a a b b a a b a b b a b \ldots
\end{aligned}
$$

As a result we obtain the word $\delta_{\infty}(w)=y_{1} y_{6} y_{5} y_{8} y_{2} \ldots$. in $A_{\infty}^{\mathbb{N}}$. In the next figure we can then follow the orbit of $\phi_{\mu}(w)$ on $T_{L_{t m}}$ visiting the intervals corresponding to each of these cylinders (see Proposition 3.3.6):


On the IET $T_{L_{t m}}$, the orbit of $\phi_{\mu}(w)$, which falls into the intervals corresponding to the cylinders induced by the $\zeta_{\infty}\left(y_{i}\right)$, i.e. $a b b$, bbaba, babaa, abaabbaa, baa, as described above.

## D IET Graph Sequences $\left\{T_{n}\right\}_{n>1}$

(Proposition 5.1.3)
Considering the Thue-Morse shift $L_{t m}$, and following Proposition 5.1.3, we show below the first terms of the corresponding map sequence $\left\{T_{n}\right\}_{n>1}$ converging to the IET $T_{L_{t m}}$ :


The first terms of $\left\{T_{n}\right\}_{n>1}$ converging to the IET $T_{L_{t m}}$. Here, each $T_{n}$ is defined as a partial function over the intervals corresponding to the cylinders determined to be in $P A R T_{L_{t m}}$ at step $n$ of the construction behind Lemma 3.2.5 (see Section B above).

## E The Thue-Morse IET Graph $T_{L_{t m}}$

Recall that in the primitive substitution case, one can directly compute the measure $\mu$ of the cylinders of the associated uniquely ergodic shift, i.e. the lengths of the corresponding intervals of the conjugate IET. A general technique relies on obtaining for each factor length the normalized Perron-Frobenius positive eigenvector of a primitive matrix associated with the substitution [Qué10, Section 5.4].

In the Thue-Morse case, the measure values can be more readily obtained. Namely by [Dek92, Theorem 1], for each $v \in \operatorname{Fact}_{L_{t m}}(n), n \geq 2, \mu\left(\operatorname{Cyl}_{L_{t m}}(v)\right)$ is either $\frac{1}{6} 2^{-m}$ or $\frac{1}{3} 2^{-m}$, where $m$ is such that $2^{m}<n \leq 2^{m+1}$, and in the specific case $n=2^{m}+1, m>0$, this value is $\frac{1}{3} 2^{-m}$. Now, when $2^{m}+1<n \leq 2^{m+1}, m>0$, the value ambiguity can be solved using $\sigma$-preservation: First, consider all the factors $\sigma^{k}(v)$ for $k=1, \ldots, n-\left(2^{m}+1\right)$ (where $\sigma$ on a finite word $v$ is $v$ minus its first letter). If there is no left special factor among them, by measure preservation the length is $\frac{1}{3} 2^{-m}$; if there is one (there can be only one since only two values are possible) the length is $\frac{1}{6} 2^{-m}$. Thus in $L_{t m}$, we have e.g. $\mu(a b b)=\mu(b a a)=1 / 6, \mu(a a b a b)=\mu(a b a b b)=1 / 12, \mu(a a b b a a b a)=\mu(a b a a b b a a)=$ $1 / 24$, etc. The next figure shows the IET graph $T_{L_{t m}}$ with its main values (also validating experimentally Proposition 5.1.3). Note that this kind of exact computations are also possible for other examples thanks to [Fri98] (see also [Saa08]).


## F Other IET Graphs

In [LN15], we chose to illustrate Theorem 1 and Proposition 5.1 .3 only with shifts coming from primitive substitutions for which minimality, unique ergodicity, and linear complexity are known [Qué10], so that Proposition 5.2 .1 applies. But we can use the construction of conjugate IETs for other shifts.

## F. 1 The Chacon Shift

First, there are substitutions which are non-primitive and nevertheless generate shifts with the needed properties for applying Proposition 5.2.1. One well-studied instance is the Chacon non-primitive substitution [Fer95, Fog02], i.e. $\theta_{c h a}(a)=a a b a, \theta_{c h a}(b)=b$. The associated shift is minimal, uniquely ergodic, and has linear complexity $p(n)=2 n-1$, for $n \geq 2$. In fact, this shift is morphic as it can be also obtained from a primitive substitution shift recoded by applying another substitution to it.

aaabaaababaabaaabaaababaababaabaaababaababaabaaababaabaaabaaababaababaabaaababaabaaabaaababaabaaabaaababaababaabaaababaabaa -
aaabaaababaabaaabaaababaababaabaaababaababaabaaababaabaaabaaababaababaabaaababaabab
aaabaaababaababaabaaababaabaaabaaababaabaa -
aaabaaababaababaabaaababaabab
aaababaabaaabaa
aaababaabab -
aabaaa
aabab
$a b-$
baabaaabaa
baabaaababaabaaabaaababaabaa
baabaaaababaabaaabaaababaababaabaaababaabaaabaaaababaabaaabaaababaababaabaaababaabaa
baabaaababaabaaabaaababaababaabaaababaababaabaaababaabaaabaaababaababaabaaababaabaaabaaababaabaaabaaababaababaabaaababaabab baabaaababaababaabaaababaabaaabaaababaabab
baababaabaaabab -
babaab

The approximated IET graph $T_{150}$ of the shift associated with the Chacon non-primitive substitution $\theta_{\text {cha }}$ (together with all the corresponding $\left.v^{(k)}\right)$.

## F. 2 S-Adic Shifts

A generalization of infinitely iterating a single substitution is to infinitely compose different substitutions. Let $\left\{\theta_{i}\right\}$ denote a set of substitutions over an alphabet $A$, and let $\left\{\theta_{i_{j}}\right\}_{j \in \mathbb{N}}$ be a sequence made of these substitutions. Then the limit

$$
w=\lim _{j \rightarrow \infty}\left(\theta_{i_{0}} \circ \theta_{i_{1}} \circ \ldots \circ \theta_{i_{j}}\right)(a), \quad a \in A,
$$

sometimes exists, and is then said to be a $s$-adic construction of $w$ (see also the DTOLsystems [KRS97]). For instance in [CN10, Section 4.11], the following infinite set of substitutions is considered over $A=\{a, b, c\}$ :

$$
\text { for each } i>0, \quad \theta_{i}(a)=b, \quad \theta_{i}(b)=c a^{i+1}, \quad \theta_{i}(c)=c a^{i} .
$$

For any composition of these substitutions applied to the letter $c$, the above limit converges to a word $w$ with linear complexity $p(n) \leq 3 n, \forall n \geq 1$, and if there is $C$ such that for all $j \in \mathbb{N}, i_{j} \leq C$, then the associated shift $L_{w}$ is uniquely ergodic. It is also minimal [Dur00, BD14]. Thus for all infinite compositions over a finite subset of $\left\{\theta_{i}\right\}$, Proposition 5.2.1 applies to $L_{w}$.


```
aac =
abbb -
acaacabbc
acabbbcabbb
acabbbcabob -
acabbocabbcaaca
acabbcaacaacabbcaacaacabbcaacabbbcabbc
acabbcaacaacabbcaacab -
acabbcaacabbbcabbbcabbcaacabbbcabbb -
acabbcaacabbbcabbbcabbcaacabbbcabbcaacaacabbcaacaacabbcaacaa
acabbcaacabbbcabbcaacaacabbcaacaacabbcaacaacabbcaacabbbcabbcaacaacabbcaacaacabbcaacab __
bbbcab
bbccab
bbc= -
bcaacaabbb - 
bcaacabob -
bcabbcaacaacabboazoa
bcabbcaacaacabbcaacaacabbcaacaa -
bcabbcaacabbbcabbbcabbcaacabbbcabbcaacaacabbcaacaacabbcaacab
caa =
```

The approximated IET graph $T_{100}$ of the $s$-adic construction $\lim _{n \rightarrow \infty}\left(\theta_{i_{0}} \circ \theta_{i_{1}} \circ \ldots \circ \theta_{i_{n}}\right)(c)$, where $i_{0} i_{1} i_{2} \ldots i_{n} \ldots$ is the fixed point $\theta_{\text {fib }}^{\omega}(1)$ of the Fibonacci substitution defined over the alphabet $\{1,2\}$ instead of $\{a, b\}$ (together with all the corresponding $v^{(k)}$ ).


The approximated IET graph $T_{100}$ of $\lim _{n \rightarrow \infty}\left(\theta_{i_{0}} \circ \theta_{i_{1}} \circ \ldots \circ \theta_{i_{n}}\right)(c)$, where $i_{0} i_{1} i_{2} \ldots i_{n} \ldots$ is the fixed point $\theta_{t m}^{\omega}(1)$ of the Thue-Morse substitution defined over $\{1,2\}$ instead of $\{a, b\}$.


The approximated IET graph $T_{100}$ of $\lim _{n \rightarrow \infty}\left(\theta_{i_{0}} \circ \theta_{i_{1}} \circ \ldots \circ \theta_{i_{n}}\right)(c)$, where $i_{0} i_{1} i_{2} \ldots i_{n} \ldots$ is the fixed point $\theta_{c h a}^{\omega}(1)$ of the Chacon substitution defined over $\{1,2\}$ instead of $\{a, b\}$.

## G Speculative Constructions of IET Graphs

The next few examples of IET graphs are about shifts which are not proved to satisfy all the assumptions to ensure the straightforward application of Proposition 5.1.3 through Proposition 5.2.1. We nevertheless show how they could look like ( $\left\{T_{n}\right\}$ always exists).

## G. 1 Toeplitz Shifts

There are other ways of describing minimal words than using substitutions. Let $\star$ be a letter not in $A$. For a finite word $v$ over $(A \cup\{\star\})$ not starting by $\star$, let $t_{0}(v)=v^{\omega}$, $t_{i+1}(v)=t_{i}(v)$, where $t_{i}(v)$ is the word obtained from $t_{i-1}(v)$ by replacing the first occurrence of $\star$ in $t_{i-1}(v)$ by the $i$ th letter of $t_{i-1}(v)$ (always different from $\star$ ). Then

$$
t(v)=\lim _{i \rightarrow \infty} t_{i}(u) \in A^{\mathbb{N}}
$$

is the Toeplitz word determined by the pattern $v$. If $p$ is the length of $v$, and $q$ is the number of $\star$ in $v$, then $t(v)$ is said to be a $(p, q)$-Toeplitz word [CK97]. Toeplitz words are minimal, and there are known to be aperiodic iff the prefix of length $p$ is not $d$-periodic, where $d=\operatorname{gcd}(p, q)$ [CK97]. Moreover, a non-periodic $(p, q)$ Toeplitz has complexity $p(n)=\Theta\left(n^{\log (p / d) /(\log (p / q))}\right)$. Thus, zero topological entropy holds, but the fact that $\left\{T_{n}\right\}_{n>1}$ converges without subsequence extraction has not been proved. Here are nevertheless the aspect of some of the corresponding graphs.


The graph $T_{70}$ for the shift associated with $t($ aaaaaab $\star \star)$ for which $p(n) \approx \Theta\left(n^{1.46}\right)$ (together with the first corresponding $v^{(k)}$ ).


The graph $T_{70}$ for the shift associated with $t($ aaaab $\star \star)$ for which $p(n) \approx \Theta\left(n^{1.55}\right)$.


The graph $T_{50}$ for the shift associated with $t(a a b \star \star)$, for which $p(n) \approx \Theta\left(n^{1.75}\right)$.

## G. 2 The Kolakoski Shift

The Kolakoski sequence $\kappa$ consists of blocks of 1 and 2, each being of length either 1 or 2 , where the length of the $i$ th block is equal to the $i$ th letter of $\kappa$ [Dek97, Fog02]:

$$
\kappa=2211212212211211221211212211211212212211 \ldots
$$

Nothing has been established about $\kappa$ in terms of complexity, recurrence, ergodicity. So no assumption can be readily checked to apply Propositions 5.1.3 or 5.2.1. We nevertheless show the corresponding approximated graph below. For $v=v_{1} \ldots v_{n} \in \operatorname{Fact}_{\kappa}, \tilde{v}$ denotes $\tilde{v_{1}} \ldots \tilde{v_{n}}$, where $\tilde{1}=2, \tilde{2}=1$, and it is also conjectured that Fact $_{\kappa}=\widetilde{\text { Fact }_{\kappa}}[D e k 97]$. The symmetry of the graph is alike the one of the Thue-Morse IET $T_{t m}$, and suggests even that: $a u \in$ Fact $_{\kappa}$ with $u$ not left special iff $\widetilde{a u} \in$ Fact $_{\kappa}$ with $\tilde{u}$ not left special $(a \in\{1,2\})$.


The graph $T_{30}$ of the shift associated with $\kappa$ (together with the first $v^{(k)}$ ).

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[^0]:    NB: The numberings of lemmas, propositions, etc. correspond to the ones in [LN15]

