Submitted to the International Conference on Rewriting Techniques and Applications
http://rewriting.loria.fr/rta/

TERMINATION OF LINEAR BOUNDED TERM REWRITING SYSTEMS

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ABSTRACT. For the whole class of linear term rewriting systems and for each integer k, we define *k-bounded rewriting* as a restriction of the usual notion of rewriting. We show that the k-bounded uniform termination, the k-bounded termination, the inverse k-bounded uniform, and the inverse k-bounded problems are decidable. The *k-bounded* class (BO(k)) is, by definition, the set of linear systems for which every derivation can be replaced by a k-bounded derivation. In general, for BO(k) term rewriting systems, the uniform (respectively inverse uniform) k-bounded termination problem is not equivalent to the uniform (resp. inverse uniform) termination problem, and the k-bounded (respectively inverse k-bounded) termination problem is not equivalent to the termination (respectively inverse termination) problem. This leads us to define more restricted classes for which these problems are equivalent: the classes BOLP(k) of *k-bounded systems that have the length preservation property*. By definition, a system is BOLP(k) if every derivation of length n can be replaced by a k-bounded derivation of length n. We define the class BOLP of *bounded systems that have the length preservation property* as the union of all the BOLP(k) classes. The class BOLP contains (strictly) several already known classes of systems: the inverse left-basic semi-Thue systems, the linear growing term rewriting systems, the inverse Linear-Finite-Path-Ordering systems, the strongly bottom-up systems.

1. Introduction

General context. A Term-Rewriting System (TRS in short) $\mathcal{R}$ is said to be *terminating* on a term s when it does not admit any infinite derivation starting on s. It is said to be *inverse terminating* on s when the system $\mathcal{R}^{-1}$ terminates on s. The TRS $\mathcal{R}$ is said to be *uniformly terminating* (u-terminating in short) when it does not admit any infinite derivation, and it is said to be *inverse u-terminating* when the system $\mathcal{R}^{-1}$ u-terminates. The u-termination property is part of the definition of a *complete* TRS, which is a useful algebraic notion. These properties are also pertinent for TRSs which are models of functional programs or any kind of computational process. It is well-known that these problems are undecidable for general finite TRS ([7]) and even for quite restricted subclasses of TRS (see [2],[10] for example). Nevertheless, because of its importance, many techniques have been developed in order to prove uniform-termination (u-termination in short) and termination

1998 ACM Subject Classification: Primary: F.4.2, Secondary: F.3.2, F.4.1.
Key words and phrases: term rewriting, termination, rewriting strategy.

of TRSs (see in particular [3],[9, section 1.3], [14, chap. 6]) or even to decide automatically u-termination or termination, but for specific classes of TRS.

Contents. The present paper follows the last trend of research quoted above:
1- we show that u-termination, inverse u-termination, termination, inverse termination are decidable for a particular strategy that we call bounded rewriting,
2- we deduce from this decision procedure that the usuals u-termination, inverse u-termination, termination and inverse termination problems are decidable for some classes of TRS.

We define a new rewriting strategy for linear TRSs called *bounded* rewriting. Let $k \in \mathbb{N}$. Intuitively, a derivation is said to be k-*bounded* ($\mathsf{bo}(k)$) if when a rewriting rule is applied, the parts of the substitution located at a depth greater than k are not used further in the derivation, i.e. do not match a left-handside of a rule applied further. A TRS $\mathcal{R}$ will be said to be k-*bounded* ($\mathsf{bo}(k)$) if for any derivation $s \to_{\mathcal{R}}^{*} t$, there exists a k-bounded derivation $s \;{}_{\mathsf{bo}(k)}\!\!\to_{\mathcal{R}}^{*} t$. The class of k-bounded TRS is denoted by $\mathsf{BO}(k)$, and the class of bounded TRS BO is $\bigcup_{k\in\mathbb{N}} \mathsf{BO}(k)$. A TRS will be said to $\mathsf{bo}(k)$-terminates on a term s if there is no infinite $\mathsf{bo}(k)$-derivation starting on s. It is said to be uniformly $\mathsf{bo}(k)$-terminates (u-$\mathsf{bo}(k)$-terminates in short) if there is no infinite $\mathsf{bo}(k)$-derivation. The main result of this paper is the decidability of the u-$\mathsf{bo}(k)$-termination problem and of the $\mathsf{bo}(k)$-termination problem. We also prove in section 6 that the inverse u-$\mathsf{bo}(k)$-termination and the inverse $\mathsf{bo}(k)$ termination prolems are decidable. This rewriting strategy is closely related to the *bottom-up* strategy introduced in [4]: every bottom-up TRS is bounded, and for every bounded TRS, there is an equivalent TRS which is bottom-up. Both strategies are defined using marking tools, but the definition of the bounded strategy is simpler and more intuitive. For every linear TRS $(\mathcal{R}, \mathcal{F})$ and every integer k, there is a TRS $(\mathcal{R}', \mathcal{F})$ such that for every $s, t \in \mathcal{T}(\mathcal{F})$:

- there is a derivation of length n from s to t in $\mathcal{R}$ iff there is a derivation of length n from s to t in $\mathcal{R}'$,
- there is a $\mathsf{bo}(k)$-derivation of length n from s to t in $\mathcal{R}$ iff there is a $\mathsf{bo}(0)$-derivation of length n form s to t in $\mathcal{R}'$.

Thus, it is sufficient to prove that the u-$\mathsf{bo}(0)$-termination and the $\mathsf{bo}(0)$-termination problems are decidable to obtain the decidability of the u-$\mathsf{bo}(k)$-termination and the $\mathsf{bo}(k)$-termination problems. Following the idea developed for the bottom-up strategy, we use a ground TRS $\mathcal{S} \cup \mathcal{A}$ to simulate $\mathsf{bo}(0)$-derivations. This construction is made in such a way that the existence of an infinite $\mathsf{bo}(0)$-derivation starting on a term s in $\mathcal{R}$ is equivalent to the existence of an infinite derivation starting on s in $\mathcal{S} \cup \mathcal{A}$. It follows from the decidability of the termination and u-termination problems for ground TRS that the u-$\mathsf{bo}(0)$-termination and the $\mathsf{bo}(0)$-termination problems are decidable. The TRS $\mathcal{A}$ has rules which allow to replace any subterm of a term t located at an internal node by a leaf labeled by the constant symbol $\#$, and the TRS $\mathcal{S}$ consists of a set of rules of the form $l\sigma \to r\sigma$ where $l \to r \in \mathcal{R}$ and σ is a substitution that maps variables to an element of $\mathcal{F}_0 \cup \{\#\}$. A $\mathsf{bo}(0)$-step $C[l\sigma] \to C[r\sigma]$ in $\mathcal{R}$ is simulated in two steps : first, using $\mathcal{A}$, we reduce $C[l\sigma]$ to $C[l\sigma']$ where $l\sigma' \in \mathsf{LHS}(\mathcal{S})$, and then we apply the rule $l\sigma' \to r\sigma' \in \mathcal{S}$. We define a subclass of $\mathsf{BO}(k)$, the length preservation bottom-up class $\mathsf{BOLP}(k)$, for which:

- termination (respectively inverse termination) and k-bounded termination (resp. inverse k-bounded termination) are equivalent,
- u-termination (respectively inverse u-termination) and u-k-bounded termination (resp. inverse u-k-bounded termination) are equivalent.

A $\mathsf{BO}(k)$ TRS is $\mathsf{BOLP}(k)$ iff for every derivation $s \rightarrow_{\mathcal{R}}^{*} t$ there is a $\mathsf{bo}(k)$-derivation of same length. The class of length preservation bounded TRSs BOLP is $\bigcup_{k \in \mathbb{N}} \mathsf{BOLP}(k)$. This class contains several already known TRSs: the inverse left-basic semi-Thue systems [12], the linear growing TRS [8], the inverse Linear-Finite-Path-Overlapping TRSs [13], and the strongly bottom-up TRSs [4]. Note that a version of this article with full proofs is available at PRECISER.

2. Preliminaries

2.1. Words and Terms

The set $\mathbb{N}$ is the set of positive integers. A finite *word* over an alphabet A is a map $u : [0, \ell - 1] \rightarrow A$, for some $\ell \in \mathbb{N}$. The integer ℓ is the *length* of the word u and is denoted by $|u|$. The set of words over A is denoted by A^* and endowed with the usual *concatenation* operation $u, v \in A^* \mapsto u \cdot v \in A^*$. The *empty* word is denoted by ε. A word u is a prefix of a word v iff there exists some $w \in A^*$ such that $v = u \cdot w$. We denote by $u \preceq v$ the fact that u is a prefix of v. Assuming a total order on A, we denote by $\preceq_{Lex}$ the lexicographic order on words.

We assume the reader familiar with terms. We call *signature* a set $\mathcal{F}$ of symbols with fixed arity $ar : \mathcal{F} \rightarrow \mathbb{N}$. The subset of symbols of arity m is denoted by $\mathcal{F}_m$.

As usual, a set $P \subseteq \mathbb{N}^*$ is called a *tree-domain* (or, *domain*, for short) iff for every $u \in \mathbb{N}^*$, $i \in \mathbb{N}$:

$$(u \cdot i \in P \Rightarrow u \in P) \ \& \ (u \cdot (i+1) \in P \Rightarrow u \cdot i \in P).$$

We call $P' \subseteq P$ a *subdomain* of P iff, P' is a domain and, for every $u \in P, i \in \mathbb{N}$:

$$(u \cdot i \in P' \ \& \ u \cdot (i+1) \in P) \Rightarrow u \cdot (i+1) \in P'.$$

A (first-order) *term* on a signature $\mathcal{F}$ is a partial map $t : \mathbb{N}^* \rightarrow \mathcal{F}$ whose domain is a non-empty tree-domain and which respects the arities. We denote by $\mathcal{T}(\mathcal{F}, \mathcal{V})$ the set of first-order terms built upon the signature $\mathcal{F} \cup \mathcal{V}$, where $\mathcal{F}$ is a finite signature and $\mathcal{V}$ is a denumerable set of variables of arity 0.

The domain of t is also called its set of *positions* and denoted by $\mathcal{P}\mathsf{os}(t)$. The set of variables of t is denoted by $\mathcal{V}\mathsf{ar}(t)$. The root symbol of t, $t(\epsilon)$ is also denoted by $\mathsf{root}(t)$. The set of variable positions (resp. non variable positions) of a term t is denoted by $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$ (resp. $\mathcal{P}\mathsf{os}_{\overline{\mathcal{V}}}(t)$). The set of *leaves* of t is the set of positions $u \in \mathcal{P}\mathsf{os}(t)$ such that $u \cdot \mathbb{N} \cap \mathcal{P}\mathsf{os}(t) = \emptyset$. It is denoted by $L\mathsf{v}(t)$. A *branch* is a set of positions P satisfying: there exists $u \in L\mathsf{v}(t)$ such that $v \in P$ iff $v \preceq u$. We write $\mathcal{P}\mathsf{os}^+(t)$ for $\mathcal{P}\mathsf{os}(t) \backslash \{\epsilon\}$. Given $v \in \mathcal{P}\mathsf{os}^+(t)$, its *father* $\mathsf{fth}(v)$ is the position u such that $v = u \cdot w$ and $|w| = 1$. Given a term t and $u \in \mathcal{P}\mathsf{os}(t)$ the *subterm of t at u* is denoted by t/u and defined by $\mathcal{P}\mathsf{os}(t/u) = \{w \mid u \cdot w \in \mathcal{P}\mathsf{os}(t)\}$ and $\forall w \in \mathcal{P}\mathsf{os}(t/u)$, $t/u(w) = t(u \cdot w)$. A term which does not contain twice the same variable is called *linear*. Given a linear term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $x \in \mathcal{V}\mathsf{ar}(t)$, we shall denote by $\mathsf{pos}(t, x)$ the position of x in t. The *depth* of a term t is inductively defined by:

- $dpt(t) := 0$ if $t \in \mathcal{V}$,
- $dpt(t) := 1$ if $t \in \mathcal{F}_0$,
- $dpt(t) := 1 + \max(\{dpt(t/i)), i \in \{0, \ldots, n\}\})$ if $\mathsf{root}(t) \in \mathcal{F}_n$.

A term containing no variables is called *ground*. The set of ground terms is $\mathcal{T}(\mathcal{F})$. Among all the variables, there is a special one $\square$. A term containing exactly one occurrence of $\square$ is called a *context*. A context is usually denoted as $C[]$. If v is the position of $\square$ in $C[]$, $C[t]$ denotes the term $C[]$ where t has been substituted at position v. We also denote by $C[]_v$ such a context and by $C[t]_v$ the result of the substitution. We denote by $|t| := \mathsf{Card}(\mathcal{P}\mathsf{os}(t))$ the *size* of a term t. A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The substitution σ extends uniquely to a morphism $\sigma : \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}, \mathcal{V})$, where $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$, for each $f \in \mathcal{F}$, $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Let t be a linear term and $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t) = \{u_1, \ldots, u_n\}$, where the u_i are given in lexicographic order. The term t is said to be standardized if for all $i, 1 \leq i \leq n$, $t/u_i = x_i$.

2.2. Term rewriting systems

A *rewrite rule* built upon the signature $\mathcal{F}$ is a pair $l \to r$ of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We call l (resp. r) the *left-handside* (resp. *right-handside*) of the rule (*lhs* and *rhs* for short). A rule is *linear* if both its left and right-handsides are linear. A rule is *left-linear* (resp. *right-linear*) if its left-handside (resp. right-hanside) is linear. Given a set of rules $\mathcal{R}$, we denote by $\mathsf{LHS}(\mathcal{R})$ the set $\{l \mid l \to r \in \mathcal{R}\}$. A TRS is a pair $(\mathcal{R}, \mathcal{F})$ where $\mathcal{F}$ is a signature and $\mathcal{R}$ a set of rewrite rules built upon the signature $\mathcal{F}$. When $\mathcal{F}$ is clear from the context or contains exactly the symbols of $\mathcal{R}$, we may omit $\mathcal{F}$ and write simply $\mathcal{R}$. The TRS $\mathcal{R}$ is said to respect the *variable restriction* if for every $l \to r \in \mathcal{R}$, $\mathcal{V}\mathsf{ar}(l) \subseteq \mathcal{V}\mathsf{ar}(r)$. We denote by $(\mathcal{R}^{-1}, \mathcal{F})$ the TRS consisting of the rules $\{r \to l | l \to r \in \mathcal{R}\}$. Given a TRS $(\mathcal{R}, \mathcal{F})$, and two terms t_1, t_2, we say that there exists a $\mathcal{R}$*-rewriting step* between t_1 and t_2 in $\mathcal{R}$ and write $t_1 \to_{\mathcal{R}} t_2$ if there exists a context $C[]$, a rule $l \to r \in \mathcal{R}$, and a substitution σ such that $t_1 = C[l\sigma]$ and $t_2 = C[r\sigma]$. The term $l\sigma$ is called a *redex* of t_1, and $r\sigma$ is called a *contractum* of t_1. Given some $n \geq 0$, a derivation in $\mathcal{R}$ of length n from s to t is a sequence of the form $s = s_0 \to_{\mathcal{R}} s_1 \to_{\mathcal{R}} \ldots \to_{\mathcal{R}} s_n = t$. The relation $\to^n_{\mathcal{R}}$ is defined as follows: $s \to^n t$ if there exists a derivation of length n from s to t. The relation $\to^*_{\mathcal{R}}$ (resp. $\to^+_{\mathcal{R}}$) is defined by: $s \to^* t$ (resp. $s \to^+ t$) if there is some $n \geq 0$ (resp. $n > 0$) such that $s \to^n_{\mathcal{R}} t$. More generally, the notation defined in [9] will be used in proofs.

A TRS is *left-linear* (resp. *right-linear*) if each of its rules is left-linear (resp. right-linear). A TRS is linear if each of its rules is linear. A TRS $\mathcal{R}$ is *growing* [8] if for every rule $l \to r \in \mathcal{R}$, all variables in $\mathcal{V}\mathsf{ar}(l) \cap \mathcal{V}\mathsf{ar}(r)$ occur at depth 0 or 1 in l. Two TRSs $(\mathcal{R}, \mathcal{F})$ and $(\mathcal{R}', \mathcal{F})$ are said to be *equivalent* if for all $n \geq 0$, $\to^n_{\mathcal{R}} = \to^n_{\mathcal{R}'}$.

3. Bounded rewriting

From now on, until the end of section 5, we suppose that all the TRS are satisfying the variable restriction. In order to define *bounded rewriting* for linear TRS, we need some marking tools. In the following we assume that $\mathcal{F}$ is a signature. We shall illustrate many of our definitions with the following TRS

Example 3.1. $\mathcal{F} = \{\mathsf{a}, \mathsf{b}, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{i}\}$, $\mathcal{R}_1 = \{\mathsf{f}(x) \to \mathsf{g}(x), \mathsf{g}(\mathsf{h}(x)) \to \mathsf{i}(x), \mathsf{i}(x) \to \mathsf{a}, \mathsf{a} \to \mathsf{b}\}$.

3.1. Marking

We mark the symbols of a term using natural integers.

3.1.1. *Marked symbols.*

Definition 3.2. We define the (infinite) *signature of marked symbols*: $\mathcal{F}^{\mathbb{N}} := \{f^i \mid f \in \mathcal{F}, i \in \mathbb{N}\}$. For $j \in \mathbb{N}$, we denote by $\mathcal{F}^{\leq j}$ the signature: $\mathcal{F}^{\leq j} := \{f^i \mid f \in \mathcal{F}, i \leq j\}$. The mapping $\mathsf{m} : \mathcal{F}^{\mathbb{N}} \to \mathbb{N}$ maps every marked symbol to its mark: $\mathsf{m}(f^i) = i$.

3.1.2. *Marked terms.*

Definition 3.3. The terms in $\mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ are called *marked* terms.

The mapping m is extended to marked terms by: if $t \in \mathcal{V}$, $\mathsf{m}(t) := 0$, otherwise, $\mathsf{m}(t) := \mathsf{m}(\mathsf{root}(t))$. For every $f \in \mathcal{F}$, we identify f^0 and f; it follows that $\mathcal{F} \subset \mathcal{F}^{\mathbb{N}}$, $\mathcal{T}(\mathcal{F}) \subset \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ and $\mathcal{T}(\mathcal{F}, \mathcal{V}) \subset \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$.

We use $\mathsf{mmax}(t)$ to denote the maximal mark of a marked term t: $\mathsf{mmax}(t) := \max\{\mathsf{m}(t/u) \mid u \in \mathcal{P}\mathsf{os}(t)\}$.

Example 3.4. $\mathsf{m}(\mathsf{a}^1) = 1, \mathsf{m}(\mathsf{i}^0(\mathsf{a}^2)) = 0, \mathsf{m}(\mathsf{h}^3(\mathsf{a}^0)) = 3, \mathsf{m}(\mathsf{h}^1(x)) = 1, \mathsf{m}(x) = 0$, $\mathsf{mmax}(\mathsf{i}^0(\mathsf{a}^1)) = 1, \mathsf{mmax}(x) = 0$.

Definition 3.5. Given $t \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ and $i \in \mathbb{N}$, we define the marked term t^i whose marks are all equal to i:

if t is a variable x	$t^i := x$
if t is a constant c	$t^i := c^i$
otherwise $t = f(t_1, \ldots, t_n)$, where $n \geq 1$	$t^i := f^i({t_1}^i, \ldots, {t_n}^i)$

This marking extends to sets of terms S ($S^i := \{t^i \mid t \in S\}$) and substitutions σ ($\sigma^i : x \mapsto (x\sigma)^i$). *Notation:* in the sequel, given a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\overline{t}$ will always refer to a term of $\mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ such that $\overline{t}^0 = t$.

Definition 3.6. For every marked term $\overline{t}$, we denote by $\widehat{\overline{t}}$ the unique marked term such that:

$$\widehat{\overline{t}}^0 := \overline{t}^0, \ \ \forall u \in \mathcal{P}\mathsf{os}_{\overline{\mathcal{V}}}(t), \mathsf{m}(\widehat{\overline{t}}/u) := \max(\mathsf{m}(\overline{t}/u), |u| + 1).$$

We extend this definition to marked substitutions ($\widehat{\overline{\sigma}} : x \mapsto \widehat{x\overline{\sigma}}$) and sets of terms ($\widehat{\overline{S}} := \{\widehat{\overline{s}} \mid \overline{s} \in \overline{S}\}$).

Example 3.7. Let $\overline{t_1} = \mathsf{f}^0(\mathsf{f}^1(x))$, and $\overline{t_2} = \mathsf{f}^2(\mathsf{f}^2(\mathsf{h}^2(\mathsf{a}^2)))$. We have: $\widehat{\overline{t_1}} = \mathsf{f}^1(\mathsf{f}^2(x))$, $\widehat{\overline{t_2}} = \mathsf{f}^2(\mathsf{f}^2(\mathsf{h}^3(\mathsf{a}^4)))$.

3.2. Marked rewriting

Let $\mathcal{R}$ be a linear TRS, and let $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$. Let us suppose that $\overline{s}$ decomposes as

$$\overline{s} = \overline{C}[\overline{l}\overline{\sigma}]_v, \quad \text{with} \quad (l, r) \in \mathcal{R}, \tag{3.1}$$

for some marked context $\overline{C}[]_v$ and substitution $\overline{\sigma}$. We then write $\overline{s} \circ\!\!\longrightarrow \overline{t}$ when

$$\overline{s} = \overline{C}[\overline{l}\overline{\sigma}], \ \ \overline{t} = \overline{C}[r\widehat{\overline{\sigma}}]. \tag{3.2}$$

More precisely, an ordered pair of marked terms $(\overline{s}, \overline{t})$ is linked by the relation $\circ\!\!\longrightarrow$ iff, there exists $\overline{C}[]_v, (l, r), \overline{l}, \overline{\sigma}$ fulfilling equations (3.1-3.2). (This is illustrated by Figure 1, where the marks are noted between brackets [...]).

The map $\overline{s} \mapsto \overline{s}^0$ (from marked terms to unmarked terms) extends into a map from marked derivations to unmarked derivations: every

$$\overline{s_0} = \overline{C_0}[\overline{l_0}\overline{\sigma_0}]_{v_0} \circ\!\!\longrightarrow \overline{C_0}[r_0\widehat{\overline{\sigma_0}}]_{v_0} = \overline{s_1} \circ\!\!\longrightarrow \ldots \circ\!\!\longrightarrow \overline{C_{n-1}}[r_{n-1}\widehat{\overline{\sigma_{n-1}}}]_{v_{n-1}} = \overline{s_n} \tag{3.3}$$

is mapped to the derivation

$$s_0 = C_0[l_0\sigma_0]_{v_0} \to C_0[r_0\sigma_0]_{v_0} = s_1 \to \ldots \to C_{n-1}[r_{n-1}\sigma_{n-1}]_{v_{n-1}} = s_n. \tag{3.4}$$

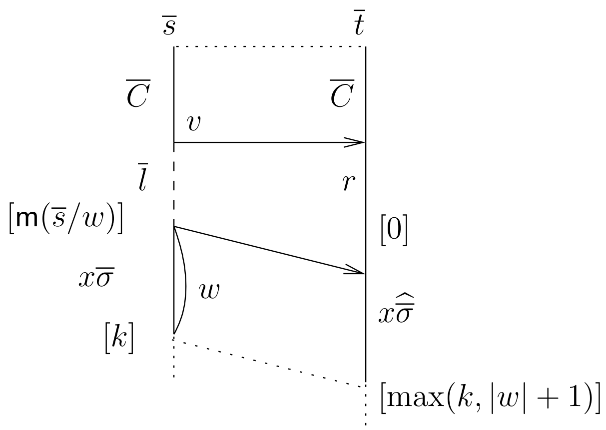

Figure 1: A marked rewriting step

The context $\overline{C_i}[]_{v_i}$, the rule (l_i, r_i), the marked version $\bar{l_i}$ of l_i and the substitution $\overline{\sigma_i}$ completely determine $\overline{s_{i+1}}$. Thus, for every fixed pair $(\overline{s_0}, s_0)$, this map is a bijection from the set of derivations (3.3), to the set of derivations (3.4).

From now on, each time we deal with a derivation $s \to^* t$ between two terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, we may implicitly decompose it as (3.4) where n is the length of the derivation, $s = s_0$ and $t = s_n$.

3.3. Bounded derivations

Definition 3.8. The marked derivation (3.3) is *k-bounded* ($\mathsf{bo}(k)$) if the following assertions hold for every $0 \leq i < n$:

- if $l_i \notin \mathcal{V}$, $\mathsf{mmax}(\overline{l_i}) \leq k$.
- if $l_i \in \mathcal{V}$, $\sup(\{\mathsf{m}(\overline{l_i}/u) | u \prec v_i\}) \leq k$

The derivation (3.4) is $\mathsf{bo}(k)$ if the corresponding marked derivation (3.3) is $\mathsf{bo}(k)$.

Example 3.9. Let us consider the following derivations in $\mathcal{R}_1$:

(1) $\mathsf{f(h(a))} \to \mathsf{g(h(a))} \to \mathsf{i(a)} \to \mathsf{a}$
(2) $\mathsf{f(h(a))} \to \mathsf{g(h(a))} \to \mathsf{g(h(b))} \to \mathsf{i(b)} \to \mathsf{a}$

The first derivation is $\mathsf{bo}(1)$ since the associated marked derivation is $\mathsf{bo}(1)$:
$\mathsf{f(h(a))} \circ\!\!\to \mathsf{g(h^1(a^2))} \circ\!\!\to \mathsf{i(a^2)} \circ\!\!\to \mathsf{a}$. The second one is $\mathsf{bo}(2)$:
$\mathsf{f(h(a))} \circ\!\!\to \mathsf{g(h^1(a^2))} \circ\!\!\to \mathsf{g(h^1(b))} \circ\!\!\to \mathsf{i(b^1)} \circ\!\!\to \mathsf{a}$.

Let $k \in \mathbb{N}$. It is clear that the composition of two $\mathsf{bo}(k)$ marked derivations is $\mathsf{bo}(k)$ too, but the composition of two unmarked $\mathsf{bo}(k)$-derivations might not be $\mathsf{bo}(k)$, as shown in the following example:

Example 3.10. The two derivations in $\mathcal{R}_1$: $\mathsf{f(h(a))} \to \mathsf{g(h(a))}$ and $\mathsf{g(h(a))} \to \mathsf{i(a)} \to \mathsf{a}$ are $\mathsf{bo}(0)$ while the derivation: $\mathsf{f(h(a))} \to \mathsf{g(h(a))} \to \mathsf{i(a)} \to \mathsf{a}$ is not $\mathsf{bo}(0)$ (but is $\mathsf{bu}(1)$).

In the following we thus (mainly) manipulate *marked* $\mathsf{bo}(k)$-derivations. Let us introduce some convenient notations.

Definition 3.11. Let $n, k \in \mathbb{N}$. The binary relation ${}_{\mathsf{bo}(k)} \circ\!\!\to^n_{\mathcal{R}}$ over $\mathcal{T}(\mathcal{F}^{\mathbb{N}})$ is defined by: $\overline{s} \;{}_{\mathsf{bo}(k)} \circ\!\!\to^n_{\mathcal{R}} \overline{t}$ iff there exists a $\mathsf{bo}(k)$-marked derivation from $\overline{s}$ to $\overline{t}$ of length n. The binary relation ${}_{\mathsf{bo}(k)}{\circ\!\!\longrightarrow}^*_{\mathcal{R}}$ is defined by: $\bar{s}\ {}_{\mathsf{bo}(k)}{\circ\!\!\longrightarrow}^*_{\mathcal{R}}\ \bar{t}$ iff there exists $m \in \mathbb{N}$ such that $\bar{s}\ {}_{\mathsf{bo}(k)}{\circ\!\!\longrightarrow}^m_{\mathcal{R}}\ \bar{t}$. The binary relation ${}_{\mathsf{bo}(k)}{\longrightarrow}^n_{\mathcal{R}}$ over $\mathcal{T}(\mathcal{F})$ is defined by: $s\ {}_{\mathsf{bo}(k)}{\longrightarrow}^n_{\mathcal{R}}\ t$ iff there exists a $\mathsf{bo}(k)$-derivation from s to t of length n. The binary relation ${}_{\mathsf{bo}(k)}{\longrightarrow}^*_{\mathcal{R}}$ is defined by: $s\ {}_{\mathsf{bo}(k)}{\longrightarrow}^*_{\mathcal{R}}\ t$ iff there exists $m \in \mathbb{N}$ such that $s\ {}_{\mathsf{bo}(k)}{\longrightarrow}^m_{\mathcal{R}}\ t$.

Next lemma shows that the study of $\mathsf{bo}(k)$-derivations can be reduced to the study of $\mathsf{bo}(0)$-derivations.

Lemma 3.12. *Let $\mathcal{R}$ be a linear TRS and let $k > 0$. There exists an equivalent linear TRS $\mathcal{R}'$ such that: for all $n \in \mathbb{N}$, ${}_{\mathsf{bo}(k)}{\longrightarrow}^n_{\mathcal{R}} = {}_{\mathsf{bo}(0)}{\longrightarrow}^n_{\mathcal{R}'}$.*

Sketch of proof. Let $\mathcal{R}'$ be the TRS consisting of the rules:

$$\{l\sigma \to r\sigma \mid l \to r \in \mathcal{R}, \sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V}), l\sigma \text{ is standardized}, \forall x \in \mathcal{V}, dpt(x\sigma) \leq k\}.$$

One can easily check that $\mathcal{R}'$ is finite, equivalent to $\mathcal{R}$ and that, for all $n \in \mathbb{N}$, ${}_{\mathsf{bo}(k)}{\longrightarrow}^n_{\mathcal{R}} = {}_{\mathsf{bo}(0)}{\longrightarrow}^n_{\mathcal{R}'}$.
■

Example 3.13. Let us consider the $\mathsf{bo}(1)$-derivation in example 3.9
$\mathsf{f}(\mathsf{h}(\mathsf{a})) \to \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{i}(\mathsf{a}) \to \mathsf{a}$ and the TRS $\mathcal{R}'$ built for $\mathcal{R}_1$ and $k = 1$. We have:

$$\begin{aligned}\mathcal{R}' =\{&\mathsf{f}(x_1) \to \mathsf{g}(x_1), \mathsf{f}(\mathsf{f}(x_1)) \to \mathsf{g}(\mathsf{f}(x_1)), \mathsf{f}(\mathsf{g}(x_1)) \to \mathsf{g}(\mathsf{g}(x_1)),\\ &\mathsf{f}(\mathsf{h}(x_1)) \to \mathsf{g}(\mathsf{h}(x_1)), \mathsf{f}(\mathsf{i}(x_1)) \to \mathsf{g}(\mathsf{i}(x_1)), \mathsf{f}(\mathsf{a}) \to \mathsf{g}(\mathsf{a}), \mathsf{f}(\mathsf{b}) \to \mathsf{g}(\mathsf{b}),\\ &\mathsf{g}(\mathsf{h}(x_1)) \to \mathsf{i}(x_1), \mathsf{g}(\mathsf{h}(\mathsf{f}(x_1))) \to \mathsf{i}(\mathsf{f}(x_1)), \mathsf{g}(\mathsf{h}(\mathsf{g}(x_1))) \to \mathsf{i}(\mathsf{g}(x_1)),\\ &\mathsf{g}(\mathsf{h}(\mathsf{h}(x_1))) \to \mathsf{i}(\mathsf{h}(x_1)), \mathsf{g}(\mathsf{h}(\mathsf{i}(x_1))) \to \mathsf{i}(\mathsf{i}(x_1)), \mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{i}(\mathsf{a}),\\ &\mathsf{g}(\mathsf{h}(\mathsf{b})) \to \mathsf{i}(\mathsf{b}), \mathsf{i}(x_1) \to \mathsf{a}, \mathsf{i}(\mathsf{f}(x_1)) \to \mathsf{a}, \mathsf{i}(\mathsf{g}(x_1)) \to \mathsf{a},\\ &\mathsf{i}(\mathsf{h}(x_1)) \to \mathsf{a}, \mathsf{i}(\mathsf{i}(x_1)) \to \mathsf{a}, \mathsf{i}(\mathsf{a}) \to \mathsf{a}, \mathsf{i}(\mathsf{b}) \to \mathsf{a}, \mathsf{a} \to \mathsf{b}\}\end{aligned}$$

and the following $\mathsf{bo}(0)$-derivation in $\mathcal{R}'$:

$$\mathsf{f}(\mathsf{h}(\mathsf{a}))\ {\circ\!\!\longrightarrow}_{\mathsf{f}(\mathsf{h}(x_1))\to\mathsf{g}(\mathsf{h}(x_1))}\ \mathsf{g}(\mathsf{h}(\mathsf{a}^1))\ {\circ\!\!\longrightarrow}_{\mathsf{h}(x_1)\to\mathsf{i}(x_1)}\ \mathsf{i}(\mathsf{a}^1)\ {\circ\!\!\longrightarrow}_{\mathsf{i}(x_1)\to\mathsf{a}}\ \mathsf{a}.$$

3.4. Bounded systems

We introduce here a hierarchy of classes of linear TRSs, based on their ability to meet the bounded restriction over derivations.

Definition 3.14. Let p be some property of derivations. A TRS $(\mathcal{R}, \mathcal{F})$ is called P if $\forall s, t \in \mathcal{T}(\mathcal{F})$ such that $s \to^*_{\mathcal{R}} t$ there exists a p-derivation from s to t.

We denote by $\mathsf{BO}(k)$ the class of $\mathsf{BO}(k)$ TRSs. One can check that, for every $k > 0$,
$\mathsf{BO}(k-1) \subsetneq \mathsf{BO}(k)$. Finally, the class of *bounded systems* BO is defined by: $\mathsf{BO} = \bigcup_{k\in\mathbb{N}} \mathsf{BO}(k)$.
The class BO contains several classes of TRS (see section 7.2).

Remark 3.15. The natural extension of BO definition to left-linear TRSs (keeping the marking process and the definitions unchanged) is not really interesting since even the TRS consisting of the rules $\{\mathsf{f}(x) \to \mathsf{g}(x, x), \mathsf{a} \to \mathsf{b}\}$ is not in BO: for every $k \in \mathbb{N}$ there is a $\mathsf{bo}(k+1)$-derivation:

$$\begin{aligned}&\mathsf{f}(\mathsf{f}(\ldots\mathsf{f}(\mathsf{a}))\ldots) \to \mathsf{g}(\mathsf{f}^1(\mathsf{f}^2(\ldots(\mathsf{f}^k(\mathsf{a}^{k+1}))\ldots), \mathsf{f}^1(\mathsf{f}^2(\ldots(\mathsf{f}^k(\mathsf{a}^{k+1}))\ldots))\\ &\to \mathsf{g}(\mathsf{f}^1(\mathsf{f}^2(\ldots(\mathsf{f}^k(\mathsf{a}^{k+1}))\ldots), \mathsf{f}^1(\mathsf{f}^2(\ldots(\mathsf{f}^k(\mathsf{b}))\ldots))\end{aligned}$$

but there is no $\mathsf{bo}(k)$-derivation from $\mathsf{f}(\mathsf{f}(\ldots\mathsf{f}(\mathsf{a}))\ldots)$ to $\mathsf{g}(\mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{a}))\ldots), \mathsf{f}(\mathsf{f}(\ldots(\mathsf{f}(\mathsf{b}))\ldots))$.

Definition 3.16. We say that the TRS $\mathcal{R}$ *u*-$\mathsf{bo}(k)$*-terminate* iff there is no infinite $\mathsf{bo}(k)$-derivation in $\mathcal{R}$.
The $\mathsf{bo}(k)$-termination problem for a linear TRS $\mathcal{R}$ is the following problem:
INSTANCE: A linear TRS $\mathcal{R}$, and an integer k.
QUESTION: Does $\mathcal{R}$ u-$\mathsf{bo}(k)$-terminate ?

Definition 3.17. We say that the TRS $\mathcal{R}$ $\mathsf{bo}(k)$*-terminate* on a term s iff there is no infinite $\mathsf{bo}(k)$-derivation starting on s in $\mathcal{R}$.
The $\mathsf{bo}(k)$-termination problem for a linear TRS $\mathcal{R}$ is the following problem:
INSTANCE: A linear TRS $\mathcal{R}$, an integer k, and a term s.
QUESTION: Does $\mathcal{R}$ $\mathsf{bo}(k)$-terminate on s ?

The main result of this paper is the decidability of the u-$\mathsf{bo}(k)$-termination and $\mathsf{bo}(k)$-termination problems. One can easily check that if there exists $l \to r \in \mathcal{R}$ such that $l \in \mathcal{V}$, then for every term s, the TRS $\mathcal{R}$ does not $\mathsf{bo}(0)$-terminate on s. Without loose of generality, we suppose from now on until the end of section 5 that a TRS $\mathcal{R}$ is such that $\mathsf{LHS}(\mathcal{R}) \cap \mathcal{V} = \emptyset$.

4. Simulation of bounded derivations by a ground rewriting system

In this section, we prove that a $\mathsf{bo}(0)$-derivation can be simulated using a ground TRS.

Definition 4.1. Let $\#$ be a constant such that $\# \notin \mathcal{F}_0$. Let $\mathcal{A}$ be the (infinite) TRS on $\mathcal{T}(\mathcal{F} \cup \{\#\}^{\mathbb{N}})$ consisting of the rules:

$$\{f^i(\overline{a_1}, \dots, \overline{a_n}) \to \#^i \mid i \in \mathbb{N}, f \in \mathcal{F}_n, n > 0, \overline{a_1}, \dots, \overline{a_n} \in (\mathcal{F}_0 \cup \{\#\})^{\mathbb{N}}\}.$$

For $j \in \mathbb{N}$, we denote by $\mathcal{A}^{\leq j}$ the restriction of $\mathcal{A}$ on $\mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq j})$ consisting of the rules:

$$\{f^i(\overline{a_1}, \dots, \overline{a_n}) \to \#^i \mid i \leq j, f \in \mathcal{F}_n, n > 0, \overline{a_1}, \dots, \overline{a_n} \in (\mathcal{F}_0 \cup \{\#\})^{\leq j}\}.$$

Lemma 4.2. *Let $\overline{s}, \overline{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}})$. If $\overline{s} \to_{\mathcal{A}}^* \overline{t}$, then $\widehat{\overline{s}} \to_{\mathcal{A}}^* \widehat{\overline{t}}$.* ∎

Definition 4.3. A marked term $\overline{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}}, \mathcal{V})$ is said to be *smoothly-increasing* (s-increasing in short) iff for every branch b, the sequence of marks on b has the form:

$$0, 0, \dots, 0, 1, 2, \dots, \ell$$

i.e. more formally: for every $w \in L\mathsf{v}(t)$, there exists some $u \preceq w$ such that,

- $\forall v \preceq u, \mathsf{m}(\overline{t}/v) = 0$,
- $\forall v \succeq u, \forall i \in \mathbb{N}$, if $v \cdot i \preceq w$ then $\mathsf{m}(\overline{t}/v \cdot i) = \mathsf{m}(\overline{t}/v) + 1$.

A substitution $\overline{\sigma}$ is said to be s-increasing if for every $x \in \mathcal{V}$, the term $x\overline{\sigma}$ is s-increasing.

Note that by definition of a s-increasing term $\overline{t}$, and since the variables are all marked by 0, for all positions $u \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$, for all $v \preceq u$, $\mathsf{m}(\overline{t}/v) = 0$.

Example 4.4. The terms $\mathsf{f}^0(\mathsf{h}^1(x))$ and $\mathsf{f}^2(\mathsf{h}^1(\mathsf{a}^2))$ are not s-increasing. The terms $\mathsf{f}^0(\mathsf{f}^0(\mathsf{h}^1(\mathsf{a}^2)))$ and $\mathsf{f}^1(\mathsf{a}^2)$ are s-increasing.

Lemma 4.5. *Let $\overline{C}[]$ and $\overline{t}$ be s-increasing. The term $\overline{C}[\overline{t}]$ is s-increasing.*

Proof. Since $\Box \in \mathcal{V}$, it is an immediate consequence of the definition of s-increasing. ∎

Lemma 4.6. *Let $\overline{s}$ be a s-increasing term and $\overline{s}\ {}_{\mathsf{bo}(0)}{\circ\!\!\longrightarrow}_{\mathcal{R}}^* \overline{t}$. The marked term $\overline{t}$ is s-increasing.* ∎

Definition 4.7. Let $\overline{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}}, \mathcal{V})$ be a marked term and P be a subdomain of $\mathcal{P}\mathsf{os}(t)$ such that $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t) \subseteq P$. We define $\mathsf{Red}(\overline{t}, P)$ as the unique term such that $\mathcal{P}\mathsf{os}(\mathsf{Red}(\overline{t}, P)) = P$ and such that $\overline{t} \rightarrow^*_{\mathcal{A}} \mathsf{Red}(\overline{t}, P)$.

The term $\mathsf{Red}(\overline{t}, P)$ is obtained from $\overline{t}$ by substituting the subtree $\overline{t}/u$ by the symbol $\#^{\mathsf{m}(\overline{t}/u)}$, for every position $u \in P \backslash L\mathsf{v}(t)$ such that $\forall i \in \mathbb{N}, u \cdot i \notin P$.

Lemma 4.8. *Let $\overline{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}}, \mathcal{V})$ and P be a subdomain of $\mathcal{P}\mathsf{os}(t)$ such that $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t) \subseteq P$. We have* $\mathsf{Red}(\widehat{\overline{t}}, P) = \widehat{\mathsf{Red}(\overline{t}, P)}$. ■

4.0.1. Top *of a term.*

Definition 4.9 (Top domain of a term)**.** Let $\overline{t}$ be a s-increasing term. We define the top domain of $\overline{t}$, denoted by $\mathsf{Topd}(\overline{t})$ as: $u \in \mathsf{Topd}(\overline{t})$ iff $u \in \mathcal{P}\mathsf{os}(\overline{t}) \wedge \mathsf{m}(\overline{t}/u) \leq 1$.

Note that by definition of a s-increasing term, $\mathsf{Topd}(\overline{t})$ is a subdomain of t and since for every $u \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$, $\mathsf{m}(\overline{t}/u) = 0$, we have $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t) \subseteq \mathsf{Topd}(\overline{t})$.

Definition 4.10 (Top of a term)**.** Let $\overline{t}$ be a s-increasing term. We denote by $\mathsf{Top}(\overline{t})$ the term $\mathsf{Red}(\overline{t}, \mathsf{Topd}(\overline{t}))$.

Example 4.11. Let $\overline{t_1} = \mathsf{f}^0(\mathsf{h}^1(\mathsf{a}^2)), \overline{t_2} = \mathsf{f}^0(\mathsf{h}^0(\mathsf{a}^1))$. We have: $\mathsf{Topd}(\overline{t_1}) = \{\epsilon, 0\}$,
$\mathsf{Topd}(\overline{t_2}) = \mathcal{P}\mathsf{os}(t_2), \mathsf{Top}(\overline{t_1}) = \mathsf{f}(\#^1), \mathsf{Top}(\overline{t_2}) = \overline{t_2}$.

Intuitively, the top of a term $\overline{t}$ will be the only part of $\overline{t}$ which could be used in a $\mathsf{bo}(0)$-derivation starting on $\overline{t}$. We extend this definition to sets of s-increasing terms ($\mathsf{Top}(\overline{S}) := \{\mathsf{Top}(\overline{t}) \mid \overline{t} \in \overline{S}\}$) and to s-increasing marked substitutions ($\mathsf{Top}(\overline{\sigma}) : x \mapsto \mathsf{Top}(x\overline{\sigma})$).

Lemma 4.12. *Let $\overline{C}[]_v, \overline{t_1}$ be s-increasing and let $\overline{t} = C[\overline{t_1}]$. We have:*

$$\mathsf{Top}(\overline{t}) = \mathsf{Top}(\overline{C}[]_v)[\mathsf{Top}(\overline{t_1})]_v.$$

■

Lemma 4.13. *Let $\overline{t}$ and $\overline{\sigma}$ be s-increasing. We have:* $\mathsf{Top}(\overline{t}\overline{\sigma}) = \mathsf{Top}(\overline{t})\mathsf{Top}(\overline{\sigma})$.

Proof. This lemma is obtained by applying lemma 4.12 several times at each position $v \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$. ■

4.0.2. *The ground system $\mathcal{S}$.*

Definition 4.14. For a linear TRS $\mathcal{R}$, we consider the following ground TRS $\mathcal{S}$ over
$\mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$ consisting of all the rules of the form: $l\overline{\sigma} \rightarrow r\widehat{\overline{\sigma}}$, where $l \rightarrow r$ is a rule of $\mathcal{R}$, and $\overline{\sigma} : \mathcal{V} \rightarrow (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$.

Note that since $\overline{\sigma} : \mathcal{V} \rightarrow (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$, by definition of $\widehat{\ }$, $\widehat{\overline{\sigma}} : \mathcal{V} \rightarrow (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$. The TRS $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ will be used to simulate the $\mathsf{bo}(0)$-derivations in $\mathcal{R}$.

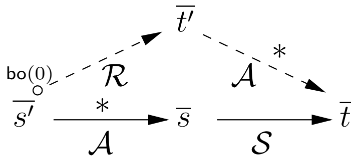

Figure 2: Lemma 4.15

4.0.3. *Lifting lemma.*

Lemma 4.15. *Let* $\overline{s'} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}})$, $\overline{s}, \overline{t} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$*. Assume that* $\overline{s'} \to^*_{\mathcal{A}} \overline{s} \to_{\mathcal{S}} \overline{t}$*. There exists a term* $\overline{t'} \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\mathbb{N}})$ *such that* $\overline{s'} \;{}_{\mathsf{bo}(0)}\!\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{t'} \to^*_{\mathcal{A}} \overline{t}$*.*

Proof. We have $\overline{s} \to_{\mathcal{S}} \overline{t}$. This means that $\overline{s} = \overline{C}[l\overline{\sigma}]_v$, $\overline{t} = \overline{C}[r\widehat{\overline{\sigma}}]_v$, for some rule $l \to r \in \mathcal{R}$, marked context $\overline{C}[]_v$, and marked substitution $\overline{\sigma} : \mathcal{V} \to (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$. Since $\overline{s'} \to^*_{\mathcal{A}} \overline{s}$, and since $\mathcal{A}$ goes from bottom to top, there exists a context $\overline{C}'[]_v$, a substitution $\overline{\sigma'}$ such that $\overline{s'}$ is of the form $\overline{s'} = \overline{C'}[l\overline{\sigma'}]_v$, with $\overline{C}'[]_v$ such that $\overline{C}'[] \to^*_{\mathcal{A}} \overline{C}[]$, and $\overline{\sigma'}$ such that for every $x \in \mathcal{V}\mathsf{ar}(l)$, $x\overline{\sigma'} \to^*_{\mathcal{A}} x\overline{\sigma}$. By definition of ${}_{\mathsf{bo}(0)}\!\!\circ\!\!\longrightarrow$, $\overline{s'} = \overline{C'}[l\overline{\sigma'}]_v \;{}_{\mathsf{bo}(0)}\!\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{t'} = \overline{C}'[r\widehat{\overline{\sigma}}]_v$. By Lemma 4.2, for every $x \in \mathcal{V}ar(r)$, $x\widehat{\overline{\sigma'}} \to^*_{\mathcal{A}} x\widehat{\overline{\sigma}}$. Hence, $\overline{t'} = \overline{C'}[r\widehat{\overline{\sigma'}}] \to^*_{\mathcal{A}} \overline{C}[r\widehat{\overline{\sigma'}}] \to^*_{\mathcal{A}} \overline{C}[r\widehat{\overline{\sigma}}] = \overline{t}$. We have built a derivation: $\overline{s'} \;{}_{\mathsf{bo}(0)}\!\!\circ\!\!\longrightarrow \overline{t'} \to^*_{\mathcal{A}} \overline{t}$. The result holds. ∎

Example 4.16. Let us consider the TRS $\mathcal{S}$ built from the TRS $\mathcal{R}_1$.

$$\begin{aligned}\mathcal{S} = \{&\mathsf{f}(\#) \to \mathsf{g}(\#^1), \mathsf{f}(\#^1) \to \mathsf{g}(\#^1), \mathsf{f}(\mathsf{a}) \to \mathsf{g}(\mathsf{a}^1), \mathsf{f}(\mathsf{a}^1) \to \mathsf{g}(\mathsf{a}^1),\\ &\mathsf{f}(\mathsf{b}) \to \mathsf{g}(\mathsf{b}^1), \mathsf{f}(\mathsf{b}^1) \to \mathsf{g}(\mathsf{b}^1), \mathsf{g}(\mathsf{h}(\#)) \to \mathsf{i}(\#^1), \mathsf{g}(\mathsf{h}(\#^1)) \to \mathsf{i}(\#^1),\\ &\mathsf{g}(\mathsf{h}(\mathsf{a})) \to \mathsf{i}(\mathsf{a}^1), \mathsf{g}(\mathsf{h}(\mathsf{a}^1)) \to \mathsf{i}(\mathsf{a}^1), \mathsf{g}(\mathsf{h}(\mathsf{b})) \to \mathsf{i}(\mathsf{b}^1), \mathsf{g}(\mathsf{h}(\mathsf{b}^1)) \to \mathsf{i}(\mathsf{b}^1),\\ &\mathsf{i}(\#) \to \mathsf{a}, \mathsf{i}(\#^1) \to \mathsf{a}, \mathsf{i}(\mathsf{a}) \to \mathsf{a}, \mathsf{i}(\mathsf{a}^1) \to \mathsf{a}, \mathsf{i}(\mathsf{b}) \to \mathsf{a}, \mathsf{i}(\mathsf{b}^1) \to \mathsf{a}, \mathsf{a} \to \mathsf{b}\}.\end{aligned}$$

We have the following derivation:

$$\mathsf{g}(\mathsf{h}(\mathsf{a})) \to_{\mathcal{A},\mathsf{a}\to\#} \mathsf{g}(\mathsf{h}(\#)) \to_{\mathcal{S},\mathsf{f}(\mathsf{h}(\#))\to\mathsf{i}(\#^1))} \mathsf{i}(\#^1)).$$

In the proof of lemma 4.15, we build the derivation:

$$\mathsf{g}(\mathsf{h}(\mathsf{a})) \;{}_{\mathsf{bo}(0)}\!\!\circ\!\!\longrightarrow_{\mathcal{R}_1,\mathsf{g}(\mathsf{h}(x))\to\mathsf{i}(x)} \mathsf{i}(\mathsf{a}^1)) \to_{\mathcal{A},\mathsf{a}^1\to\#^1} \mathsf{i}(\#^1).$$

4.0.4. *Projecting lemma.*

Lemma 4.17 (projecting lemma)**.** *Let* $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ *be s-increasing, and* $\overline{s} \;{}_{\mathsf{bo}(0)}\!\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{t}$*. There is a derivation:* $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}^{\leq 1}}\to_{\mathcal{S}} \mathsf{Top}(\overline{t})$*.*

Proof. By definition of ${}_{\mathsf{bo}(0)}\!\!\circ\!\!\longrightarrow$, there exists a context $\overline{C}[]_v$, a marked substitution $\overline{\sigma}$, and a rule $l \to r \in \mathcal{R}$ such that $\overline{s} = \overline{C}[l\overline{\sigma}]_v$ and $\overline{t} = \overline{C}[r\widehat{\overline{\sigma}}]_v$. Since $\overline{s}$ is s-increasing, by lemma 4.6, $\overline{t}$ is s-increasing, and $\mathsf{Top}(\overline{t})$ is well defined. Moreover, the marked context $\overline{C}[]_v$, the substitution $\overline{\sigma}$, and the terms r and l are s-increasing. So, by lemmas 4.12 and 4.13: $\mathsf{Top}(\overline{s}) = \mathsf{Top}(\overline{C}[]_v)[l\mathsf{Top}(\overline{\sigma})]_v$, and, $\mathsf{Top}(\overline{t}) = \mathsf{Top}(\overline{C}[]_v)[r\mathsf{Top}(\widehat{\overline{\sigma}})]_v$. By definition of Top, $\mathsf{Top}(\overline{s}) \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$. Let us define the substitution $\overline{\tau}$ by $\overline{\tau} : x \mapsto \mathsf{Red}(x\mathsf{Top}(\overline{\sigma}), \mathsf{Topd}(x\mathsf{Top}(\widehat{\overline{\sigma}})))$. By definition of Red, $\mathsf{Top}(\overline{s}) \to_{\mathcal{A}} \mathsf{Top}(\overline{C}[]_v)[l\overline{\tau}]$. Moreover, $\mathsf{Top}(\overline{s}) \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$. Thus, we have $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}^{\leq 1}}$

$$\begin{tikzcd}
\overline{s} \arrow[rr, "\mathcal{R}", "\mathsf{bo}(0)"'] \arrow[d, dashed, "\mathcal{A}", "*"'] & & \overline{t} \arrow[d, dashed, "\mathcal{A}", "*"'] \\
\mathsf{Top}(\overline{s}) \arrow[r, dashed, "*", "\mathcal{A}^{\leq 1}"'] & {} \arrow[r, dashed, "\mathcal{S}"'] & \mathsf{Top}(\overline{t})
\end{tikzcd}$$

Figure 3: Projecting lemma

$\mathsf{Top}(\overline{C}[]_v)[l\overline{\tau}]_v$. Let $x \in \mathcal{V}\mathsf{ar}(l)$. Let us prove that $x\overline{\tau} \in (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$. Let $u \in \mathcal{P}\mathsf{os}(x\sigma)$. If $|u| \geq 1$, by definition of $\widehat{\ }$, we have $\mathsf{m}(x\widehat{\overline{\sigma}}/u) \geq 2$, and $u \notin \mathsf{Topd}(x\mathsf{Top}(\widehat{\sigma}))$. Thus, $x\overline{\tau}$ is reduced to a constant, and since $\mathsf{Top}(\overline{s}) \in \mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$, $x\overline{\tau} \in (\mathcal{F}_0 \cup \{\#\})^{\leq 1}$. Hence, the rule $l\overline{\tau} \to r\widehat{\overline{\tau}}$ belongs to $\mathcal{S}$ and $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}^{\leq 1}} \mathsf{Top}(\overline{C})[l\overline{\tau}] \to_{\mathcal{S}} \mathsf{Top}(\overline{C})[r\widehat{\overline{\tau}}]$. By lemma 4.8, for all $x \in \mathcal{V}\mathsf{ar}(r)$,

$$x\widehat{\overline{\tau}} = \mathsf{Red}(x\mathsf{Top}(\overline{\sigma})\widehat{), \mathsf{Topd}}(x\mathsf{Top}(\widehat{\sigma}))) = \mathsf{Red}(\widehat{x\mathsf{Top}\overline{\sigma}}), \mathsf{Topd}(x\mathsf{Top}(\widehat{\sigma})) = x\mathsf{Top}(\widehat{\sigma}).$$

So, $\widehat{\overline{\tau}} = \mathsf{Top}(\widehat{\sigma})$, and $\mathsf{Top}(\overline{t}) = \mathsf{Top}(\overline{C}[]_v)[r\mathsf{Top}(\widehat{\sigma})]_v = \mathsf{Top}(\overline{C}[]_v)[r\widehat{\overline{\tau}}]_v$. We have built a derivation: $\mathsf{Top}(\overline{s}) \to^*_{\mathcal{A}^{\leq 1}} \to_{\mathcal{S}} \mathsf{Top}(\overline{t})$. The result holds. ■

Example 4.18. Let us consider the TRS $\mathcal{R}_1$, $\mathcal{S}$ built for this TRS, and the following $\mathsf{bo}(0)$ rewriting step: $\overline{s} = \mathsf{f}(\mathsf{f}(\mathsf{g}^1((\mathsf{a}^2)))) \circ\!\!\to_{\mathcal{R}_1, \mathsf{f}(x) \to \mathsf{g}(x)} \overline{t} = \mathsf{g}(\mathsf{f}^1(\mathsf{g}^2(\mathsf{a}^3)))$.
We have $\mathsf{Top}(\overline{s}) = \mathsf{f}(\mathsf{f}(\#^1))$, $\mathsf{Top}(\overline{t}) = \mathsf{g}(\#^1)$, and the following derivation:
$\mathsf{f}(\mathsf{f}(\#^1)) \to_{\mathcal{A}^{\leq 1}} \mathsf{f}(\#) \to_{\mathcal{S}, \mathsf{f}(\#) \to \mathsf{g}(\#^1)} \mathsf{g}(\#^1)$.

Definition 4.19. Let us define the relation $\preceq_{\mathsf{m}}$ on marked terms by:

$$\overline{s} \preceq_{\mathsf{m}} \overline{t} \Leftrightarrow s = t \wedge \forall u \in \mathcal{P}\mathsf{os}(s), \mathsf{m}(\overline{s}/u) < \mathsf{m}(\overline{t}/u).$$

Lemma 4.20. *Let $\overline{s} \to^*_{\mathcal{S} \cup \mathcal{A}^{\leq 1}} \overline{t}$. For every term $\overline{s'} \preceq_{\mathsf{m}} \overline{s}$ there exists a term $\overline{t'} \preceq_{\mathsf{m}} \overline{t}$ such that:*

$$\overline{s'} \to^*_{\mathcal{S} \cup \mathcal{A}^{\leq 1}} \overline{t'}.$$

5. Decidability of termination problems

In this section, we prove that the u-$\mathsf{bo}(k)$-termination and the $\mathsf{bo}(k)$-termination problems are decidable.

Proposition 5.1. *Let $\overline{s_0} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$. If the TRS $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ does not terminate on $\overline{s_0}$, then $\mathcal{R}$ does not $\mathsf{bo}(0)$-terminate on s_0.*

Assume that $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ does not terminate on $\overline{s_0}$. Then, by lemma 4.20, there exists an infinite rewriting sequence of terms in $\mathcal{T}((\mathcal{F} \cup \{\#\})^{\leq 1})$ starting on s_0. The TRS $\mathcal{A}^{\leq 1}$ is obviously u-terminating. Thus, such an infinite derivation contains an infinite number of steps in $\mathcal{S}$ and is of the form:

$$s_0 \to^*_{\mathcal{A}^{\leq 1}} \overline{s}_1 \to_{\mathcal{S}} \overline{s}_2 \to^*_{\mathcal{A}^{\leq 1}} \overline{s}_3 \to_{\mathcal{S}} \overline{s}_4 \to^*_{\mathcal{A}^{\leq 1}} \dots \to_{\mathcal{S}} \overline{s}_{2n} \to^*_{\mathcal{A}^{\leq 1}} \dots .$$

We now show that repeated applications of lemma 4.15 yields an infinite marked $\mathsf{bo}(0)$-derivation in $\mathcal{R}$: first, consider $s_0 \to^*_{\mathcal{A}^{\leq 1}} \overline{s_1} \to_{\mathcal{S}} \overline{s_2}$. By lemma 4.15 there exists $\overline{t_1}$ such that $s_0 \ {}_{\mathsf{bo}(0)}\!\circ\!\!\to_{\mathcal{R}} \overline{t_1} \to^*_{\mathcal{A}} \overline{s_2}$. Since $\overline{t_1} \to^*_{\mathcal{A}} \overline{s_2}$, we can apply lemma 4.15 to $\overline{t_1} \to^*_{\mathcal{A}} \overline{s_3} \to_{\mathcal{S}} \overline{s_4}$. We obtain a term $\overline{t_2}$ such that $s_0 \ {}_{\mathsf{bo}(0)}\!\circ\!\!\to_{\mathcal{R}} \overline{t_1} \ {}_{\mathsf{bo}(0)}\!\circ\!\!\to_{\mathcal{R}} \overline{t_2} \to^*_{\mathcal{A}} \overline{s_4}$. Following this process, we obtain an infinite sequence such that $\overline{s_0}\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{t_1}\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{t_2}\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \cdots\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{t_n} \ldots$. We conclude that $\mathcal{R}$ does not $\mathsf{bo}(0)$-terminate on s_0.

■

Proposition 5.2. *Let $s_0 \in \mathcal{T}(\mathcal{F})$. If $\mathcal{R}$ does not $\mathsf{bo}(0)$-terminate on s_0, then $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ does not terminate on s_0.*

Proof. If $\mathcal{R}$ does not $\mathsf{bo}(0)$-terminate on s_0, there is an infinite derivation:

$$s_0 = \overline{s}_0\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{s}_1\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \ldots \overline{s}_n\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \cdots .$$

The term s_0 is s-increasing since it has no mark. Moreover, the step $s_0\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{s}_1$ is $\mathsf{bo}(0)$. By lemma 4.17, $s_0 = \mathsf{Top}(s_0) \to^*_{\mathcal{A}^{\leq 1}} \to_{\mathcal{S}} \mathsf{Top}(\overline{s_1})$. Another application of lemma 4.17 on $\overline{s_1}\ {}_{\mathsf{bo}(0)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{s_2}$ leads to a derivation: $\mathsf{Top}(s_0) \to^+_{\mathcal{S}\cup\mathcal{A}^{\leq 1}} \mathsf{Top}(\overline{s_1}) \to^+_{\mathcal{S}\cup\mathcal{A}^{\leq 1}} \mathsf{Top}(\overline{s_2})$. Following this process, we obtain an infinite derivation:

$$\mathsf{Top}(s_0) \to^+_{\mathcal{S}\cup\mathcal{A}^{\leq 1}} \mathsf{Top}(\overline{s_1}) \to^+_{\mathcal{S}\cup\mathcal{A}^{\leq 1}} \ldots \mathsf{Top}(\overline{s_n}) \to^+_{\mathcal{S}\cup\mathcal{A}^{\leq 1}} \cdots$$

and $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ does not terminate on s_0. ■

Theorem 5.3. *The $\mathsf{bo}(0)$-termination and u-$\mathsf{bo}(0)$-termination problems are decidable.*

Proof. By propositions 5.1 and 5.2, a linear TRS $\mathcal{R}$ $\mathsf{bo}(0)$-terminates on a term s_0 iff the TRS $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ terminates on s_0. If $\mathcal{R}$ does not u-$\mathsf{bo}(0)$-terminate, then by proposition 5.2, the system $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ does not terminate. Reciprocally, if $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ does not u-terminate, then there exists an infinite derivation starting on a term $\overline{s_0}$. By proposition 5.1, the system $\mathcal{R}$ does not $\mathsf{bo}(0)$ terminate on s_0. So, $\mathcal{R}$ u-$\mathsf{bo}(0)$-terminates iff the ground TRS $\mathcal{S} \cup \mathcal{A}^{\leq 1}$ u-terminates. It is well known that the termination and the u-termination problems are decidable for ground TRS (see *e.g.*[1]). Hence, the $\mathsf{bo}(0)$-termination and u-$\mathsf{bo}(0)$-termination problems are decidable. ■

Corollary 5.4. *The $\mathsf{bo}(k)$-termination and the u-$\mathsf{bo}(k)$-termination problem are decidable.*

Proof. This is just a consequence of theorem 5.3 and lemma 3.12. ■

Note that in general, for a $\mathsf{BO}(0)$ TRS, the u-$\mathsf{bo}(0)$-termination property (respectively the $\mathsf{bo}(k)$ termination property) and the u-termination (resp. termination) property are not equivalent.

Definition 5.5. Let $\mathcal{R}$ be a $\mathsf{BO}(k)$ TRS. We say that $\mathcal{R}$ has the $\mathsf{bo}(k)$ length preservation property if for every $n \in \mathbb{N}$: $\to^n_{\mathcal{R}} = {}_{\mathsf{bo}(k)}\!\to^n_{\mathcal{R}}$.

We denote by $\mathsf{BOLP}(k)$ the class of $\mathsf{BO}(k)$ TRSs that have the $\mathsf{bo}(k)$ length preservation property. Finally, the class of *bounded systems with the length preservation property* is denoted by BOLP. One can check that for every $k > 0$, $\mathsf{BOLP}(k-1) \subsetneq \mathsf{BO}(k)$.

Example 5.6. Let $\mathcal{R}_2 = \{\mathsf{f}(x) \to \mathsf{g}(x), \mathsf{g}(\mathsf{a}) \to \mathsf{f}(\mathsf{a})\}$. This TRS is $\mathsf{BO}(0)$ but does not have the $\mathsf{bo}(0)$ length preservation property. There is a derivation of length 2: $\mathsf{f}(\mathsf{a}) \to \mathsf{g}(\mathsf{a}) \to \mathsf{f}(\mathsf{a})$, but there is no $\mathsf{bo}(0)$-derivation of length 2 from $\mathsf{f}(\mathsf{a})$ to $\mathsf{f}(\mathsf{a})$ (there is one of length 0). Moreover, this TRS does not u-terminate but u-$\mathsf{bo}(0)$-terminates.

Corollary 5.7. *Termination and u-termination problems for TRSs in $\mathsf{BOLP}(k)$ are decidable.*

Proof. Let us prove that, for a TRS $\mathcal{R} \in \mathsf{BOLP}(k)$, the $\mathsf{bo}(k)$-termination and termination are equivalent properties. Clearly, if $\mathcal{R}$ does not $\mathsf{bo}(k)$-terminate on s_0, then the TRS $\mathcal{R}$ does not terminate on s_0. Conversely, let us suppose that there is an infinite derivation starting on s_0: $s_0 \to_{\mathcal{R}} s_1 \to_{\mathcal{R}} s_2 \to_{\mathcal{R}} \ldots \to_{\mathcal{R}} s_n \ldots$. Since $\mathcal{R}$ has the $\mathsf{bo}(k)$ length preservation property, there is for each $m \in \mathbb{N}$ a marked $\mathsf{bo}(k)$-derivation $\overline{D_m}$ such that $\overline{D_m} = s_0 \ {}_{\mathsf{bo}(k)}\!\circ\!\!\longrightarrow^m_{\mathcal{R}} \overline{s_m}$. The TRS $\mathcal{R}$ has a finite number of rules, so there is only a finite number of possible one step rewriting starting on s_0. Hence, there exists a term $\overline{s'_1}$ such that the set $\{m' \mid \overline{D_{m'}} = \overline{s_0} \ {}_{\mathsf{bo}(k)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{s'_1} \ {}_{\mathsf{bo}(k)}\!\circ\!\!\longrightarrow^{m'-1}_{\mathcal{R}} \overline{s_{m'}}\}$ is infinite. Repeating this process, we obtain an infinite derivation:

$$\overline{s_0} \ {}_{\mathsf{bo}(k)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{s'_1} \ {}_{\mathsf{bo}(k)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \cdots \ {}_{\mathsf{bo}(k)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{s'_n} \ {}_{\mathsf{bo}(k)}\!\circ\!\!\longrightarrow \cdots .$$

Hence, the TRS $\mathcal{R}$ does not $\mathsf{bo}(k)$-terminate. We have established that, for all $s_0 \in \mathcal{T}(\mathcal{F})$:

$$\mathcal{R} \ \mathsf{bo}(k)\text{-terminates on } s_0 \Leftrightarrow \mathcal{R} \text{ terminates on } s_0.$$

So, for $\mathcal{R}$, termination problem is equivalent to $\mathsf{bo}(k)$-termination problem, and u-termination problem is equivalent to u-$\mathsf{bo}(k)$-termination problem. By corollary 5.4, $\mathsf{bo}(k)$-termination and u-$\mathsf{bo}(k)$ termination problems are decidable. Hence, termination and u-termination problems for TRSs in $\mathsf{BOLP}(k)$ are decidable. ■

6. Decidability of inverse termination problems

Definition 6.1. Let $\mathcal{R}$ be a linear TRS satisfying the variable restriction. The system $\mathcal{R}$ is said to inverse $\mathsf{bo}(k)$-terminate on a term s when $\mathcal{R}^{-1}$ does not admit any infinite derivation $s_0 \to_{\mathcal{R}^{-1}} s_1 \to_{\mathcal{R}^{-1}} \ldots \to_{\mathcal{R}^{-1}} s_n \ldots$ such that for all m $\in \mathbb{N}$ there exists $\overline{s_m}$ such that $\overline{s_{m+1}} \ {}_{\mathsf{bo}(k)}\!\circ\!\!\longrightarrow_{\mathcal{R}} \overline{s_m}$. It is said to inverse u-*bo*(k)-terminate when for all terms $s \in \mathcal{T}(\mathcal{F})$, $\mathcal{R}$ $\mathsf{bo}(k)$-terminates on s.

In the previous definition, if $\mathcal{R}$ does not $\mathsf{bo}(0)$-terminate on s, we can suppose that all the $\overline{s_i}$ are s-increasing.

Definition 6.2. For a term $s \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$, we denote by $N_{\overline{0}}(s)$ the number of positions u in s such that $\mathsf{m}(s/u) \neq 0$:

$$N_{\overline{0}}(s) = \mathsf{Card}(\{u \in \mathcal{P}\mathsf{os}(s) | \mathsf{m}(s/u) \neq 0\}).$$

Lemma 6.3. *Let $\mathcal{R}$ be a linear TRS such that for all $l \to r \in \mathcal{R}$, $\mathcal{V}\mathsf{ar}(l) = \mathcal{V}\mathsf{ar}(r)$ and let s,t be s-increasing. If $s \ {}_{\mathsf{bo}(0)}\!\circ\!\!\to_{\mathcal{R}} t$ then $N_{\overline{0}}(s) \leq N_{\overline{0}}(t)$. Moreover, if $N_{\overline{0}}(s) = N_{\overline{0}}(t)$, then $\mathsf{Top}(s) \to_{\mathcal{S}} \mathsf{Top}(t)$ (where $\mathcal{S}$ is the ground TRS defined in 4.14).* ■

Proposition 6.4. *The inverse u-$\mathsf{bo}(k)$-termination problem is decidable.*

Sketch of proof. By lemma 3.12, we only have to prove this result for the inverse u-$\mathsf{bo}(0)$-termination problem. Let $\mathcal{R}$ be a linear TRS. If there exists a rule $l \to r$ such that $\mathcal{V}\mathsf{ar}(r) \subset \mathcal{V}\mathsf{ar}(l)$, one can easily check that there exists an infinite inverse-$\mathsf{bo}(0)$ derivation in $\mathcal{R}^{-1}$ using only the rule $r \to l$. Thus, we can suppose that $\mathcal{V}\mathsf{ar}(r) = \mathcal{V}\mathsf{ar}(l)$. Let us prove that $\mathcal{R}$ inverse u-$\mathsf{bo}(0)$-terminate iff the ground TRS S^{-1} u-terminates. Clearly, if there is an infinite derivation in S^{-1}, $\mathcal{R}^{-1}$ does not inverse $\mathsf{bo}(0)$-terminate. Reciprocally $s_0 \to_{\mathcal{R}^{-1}} s_1 \to_{\mathcal{R}^{-1}} \ldots \to_{\mathcal{R}^{-1}} s_n \to_{\mathcal{R}^{-1}} \ldots$ be an infinite inverse-$\mathsf{bo}(0)$ derivation. By lemma 6.3, there is an integer N such that for all $m \geq N$, : $N_{\overline{0}}(\overline{s_m}) = N_{\overline{0}}(\overline{s_N})$. By lemma 6.3, for all $m \geq N$, $\mathsf{Top}(\overline{s_{m+1}}) \to_{\mathcal{S}} \mathsf{Top}(\overline{s_m})$. Hence, there is an infinite derivation in $\mathcal{S}^{-1}$: $\mathsf{Top}(\overline{s_N}) \to_{\mathcal{S}^{-1}} \mathsf{Top}(\overline{s_{N+1}}) \to_{\mathcal{S}^{-1}} \ldots$. Since the u-termination problem for ground TRS is decidable, the result holds. ■

Proposition 6.5. *The inverse $\mathsf{bo}(k)$-termination problem is decidable.* ■

Proposition 6.6. *Let $\mathcal{R}$ be a $\mathsf{BOLP}(k)$ TRS. The system $\mathcal{R}^{-1}$ u-terminates (respectively terminates on s) iff $\mathcal{R}$ inverse u-$\mathsf{bo}(k)$-terminates (resp. $\mathsf{bo}(k)$-terminates).* ■

Corollary 6.7. *Let $\mathcal{R}$ be a $\mathsf{BOLP}(k)$ TRS. The termination, u-termination, inverse termination, and inverse u-termination problems are decidable.* ∎

6.1. Bottom-up derivations

We now release the hypothesis that every TRS $\mathcal{R}$ satisfying $\mathsf{LHS}(\mathcal{R}) \cap \mathcal{V} = \emptyset$. All the rewriting systems in this section are satisfying the variable restriction. The class of BO linear TRSs is closely related to the class of bottom-up TRSs BU introduced in [4] in the following sense: every BU TRS is BO, and for every BO TRS, there is an equivalent TRS which is BU. The BU TRSs are also defined using marking tools. The marked derivation used to defined BU TRS will be denoted by $\rhd\!\!\longrightarrow$. Let us recall some of the definitions given in [4].

The right-action $\odot$ of the monoid $(\mathbb{N}, \max, 0)$ over the set $\mathcal{F}^{\mathbb{N}}$ consists in applying the operation $\max$ on every mark: for every $\bar{t} \in \mathcal{F}^{\mathbb{N}}, n \in \mathbb{N}$,

$$\mathcal{P}\mathsf{os}(\bar{t} \odot n) := \mathcal{P}\mathsf{os}(\bar{t}), \quad \forall u \in \mathcal{P}\mathsf{os}(\bar{t}), \mathsf{m}((\bar{t} \odot n)/u) := \max(\mathsf{m}(\bar{t}/u), n),$$
$$(\bar{t} \odot n)^0 = \bar{t}^0$$

For every linear marked term $\bar{t} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}}, \mathcal{V})$ and variable $x \in \mathcal{V}\mathsf{ar}(\bar{t})$, we define:

$$M(\bar{t}, x) := \sup\{\mathsf{m}(\bar{t}/w) \mid w < \mathsf{pos}(\bar{t}, x)\} + 1. \tag{6.1}$$

Let $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ and $t \in \mathcal{T}$, and let us suppose that $\overline{s} \in \mathcal{T}(\mathcal{F}^{\mathbb{N}})$ decomposes as

$$\overline{s} = \overline{C}[\bar{l}\overline{\sigma}]_v, \quad \text{with} \quad (l, r) \in \mathcal{R}, \tag{6.2}$$

for some marked context $\overline{C}[]_v$ and substitution $\overline{\sigma}$. We define a new marked substitution $\overline{\overline{\sigma}}$ (such that $\overline{\overline{\sigma}}^0 = \overline{\sigma}^0$) by: for every $x \in \mathcal{V}\mathsf{ar}(r)$,

$$x\overline{\overline{\sigma}} := (x\overline{\sigma}) \odot M(\bar{l}, x). \tag{6.3}$$

We then write $\overline{s} \rhd\!\!\longrightarrow \bar{t}$ when

$$\overline{s} = \overline{C}[\bar{l}\overline{\sigma}], \ \ \bar{t} = \overline{C}[r\overline{\overline{\sigma}}]. \tag{6.4}$$

The map $\overline{s} \mapsto \overline{s}^0$ (from marked terms to unmarked terms) extends into a map from marked derivations to unmarked derivations: every derivation d:

$$\overline{s}_0 = \overline{C}_0[\bar{l}_0\overline{\sigma}_0]_{v_0} \rhd\!\!\longrightarrow \overline{C}_0[r_0\overline{\overline{\sigma}}_0]_{v_0} = \overline{s}_1 \rhd\!\!\longrightarrow \ldots \rhd\!\!\longrightarrow \overline{C}_{n-1}[r_{n-1}\overline{\overline{\sigma}}_{n-1}]_{v_{n-1}} = \overline{s}_n \tag{6.5}$$

is mapped to the derivation $\overline{d}$:

$$s_0 = C_0[l_0\sigma_0]_{v_0} \to C_0[r_0\sigma_0]_{v_0} = s_1 \to \ldots \to C_{n-1}[r_{n-1}\sigma_{n-1}]_{v_{n-1}} = s_n. \tag{6.6}$$

Definition 6.8 ([4]). The marked derivation (6.5) is *weakly bottom-up* if, for every $0 \leq i < n$, $l_i \notin \mathcal{V} \Rightarrow \mathsf{m}(\overline{l_i}) = 0$, and $l_i \in \mathcal{V} \Rightarrow \sup\{\mathsf{m}(\overline{s_i}/u) \mid u < v_i\} = 0$.

Definition 6.9 ([4]). The derivation (6.6) is *weakly bottom-up* if the corresponding marked derivation (6.5) starting on the same term $\overline{s} = s$ is *weakly bottom-up*.

We shall abbreviate "weakly bottom-up" to wbu.

Definition 6.10 ([4]). A derivation is $\mathsf{bu}(k)$ if it is wbu and, in the corresponding marked derivation $\forall 0 \leq i \leq n$, $\mathsf{mmax}(\overline{s_i}) \leq k$.

7. Equivalence between bounded rewriting and bottom-up rewriting

7.1. Bottom-up systems

We denote by $\mathsf{BU}(k)$ the class of $\mathsf{bu}(k)$ TRSs. We define the class of *bottom-up systems*, denoted BU, by: $\mathsf{BU} = \bigcup_{k\in\mathbb{N}} \mathsf{BU}(k)$.

A TRS is said to be strongly $\mathsf{bu}(k)$ iff every wbu derivation is $\mathsf{bu}(k)$. The class of strongly $\mathsf{BU}(k)$ TRSs is denoted by $\mathsf{SBU}(k)$. We define *strongly bottom-up systems*, denoted SBU by:

$\mathsf{SBU} = \bigcup_{k\in\mathbb{N}} \mathsf{SBU}(k)$.

Lemma 7.1. *Let $\mathcal{R}$ be a TRS and let $\mathsf{e} = \max(\{dpt(l) | l \to r \in \mathcal{R}\})$. The following assertions hold:*

(1) if $\mathcal{R}$ is $\mathsf{BU}(k)$, then $\mathcal{R}$ is $\mathsf{BO}(k \cdot \mathsf{e})$,
(2) if $\mathcal{R}$ is $\mathsf{SBU}(k)$ then $\mathcal{R}$ is $\mathsf{BOLP}(k \cdot \mathsf{e})$,
(3) if $\mathcal{R}$ is BO, there is an equivalent TRS $\mathcal{R}'$ in $\mathsf{BU}(1)$.

7.2. Classes of systems in BOLP

The class $\mathsf{SBU}(1)$ contains several classes of TRSs [4]. Among them, there are:

- the inverse left-basic semi-Thue systems (viewed as unary term rewriting systems) [12],
- the linear growing term rewriting systems [8],
- the inverse Linear-Finite-Path-Overlapping TRSs [13],
- the strongly bottom-up TRSs [4].

By corollary 5.7 and lemma 7.1, for all these TRSs, the termination problem is decidable.

7.3. Some other properties of bounded systems

Let us give some properties which directly follow from the equivalence between BU and BO and the results presented in [4].

Definition 7.2. A TRS $(\mathcal{R}, \mathcal{F})$ is said to *inverse-preserves rationality* if for every recognizable set $T \subseteq \mathcal{T}(\mathcal{F})$, the set $(\to^*_{\mathcal{R}})[T] := \{s \in \mathcal{T}(\mathcal{F}) \mid \exists t \in T, s \to^*_{\mathcal{R}} t\}$ is recognizable too.

Since every BU TRS inverse-preserves rationality [4], by lemma 7.1 the following proposition holds:

Proposition 7.3. *Every BO TRS inverse-preserves rationality.* ■

Definition 7.4. The $\mathsf{BO}(k)$ (respectively $\mathsf{BU}(k)$) membership problem is the following:
INSTANCE : An integer k and a linear rewriting TRS $\mathcal{R}$.
QUESTION : Does $\mathcal{R}$ belong to $\mathsf{BO}(k)$ (resp. $\mathsf{BU}(k)$) ?

Since the $\mathsf{BU}(1)$ membership problem is undecidable [4], by lemma 7.1, the following proposition holds:

Corollary 7.5. *The $\mathsf{BO}(k)$ membership problem is undecidable.* ■

8. Related works and perspectives

Related works. We borrowed from [4] the idea of simulating derivations according to a special strategy by some ground TRS. Note however, that the class $\mathsf{BO}(k)$ itself is new. Its advantages over the class $\mathsf{BU}(k)$ is that its definition is simpler, it allows a simpler proof of the projecting lemma and it makes lemma 3.12 true, while this lemma, mutatis mutandis, does not hold for the class $\mathsf{BU}(k)$.

The principle of replacing the original rewriting relation over a signature $\mathcal{F}$ by some other binary relation over a marked-alphabet $\mathcal{F}^{\mathbb{N}}$ was already used in [5] in order to get an algorithm for termination. However, the two marking mechanisms turn out to be different:
- in the case of *word rewriting* systems, the marked derivation used here is not generated by a semi-Thue system while the marked derivation of [5] is generated by an (infinite) semi-Thue system;
- the direct image of a rational set R by a system which is match-bounded over R is rational while the direct image of a rational set by a $\mathsf{BO}(0)$ system needs not be rational; from this point of view our $\mathsf{BO}(0)$-semi-Thue systems resemble the inverses of match-bounded systems (though, they are not comparable for inclusion);
- the marking process used here extends naturally to terms while the notion of [5] seems more difficult to extend to terms (although interesting ways of doing such an extension have been studied in [6] and successfully implemented).

Perspectives. Let us mention some natural perspectives of development for this work:

- it is tempting to extend the notion of *bounded* rewriting (resp. system) to left-linear systems. This class would extend the class of growing systems studied in [11];
- we think that the direct image of a context-free language through *bounded* rewriting is context-free;
- the whole class of *bounded* systems (at least semi-Thue) should have a decidable termination problem;
- one should try to devise a class of semi-Thue systems that includes both the class of $\mathsf{BO}(k)$ systems and the class of inverses of match-bounded systems, and still possesses the interesting algorithmic properties of these classes.

Some work in these directions has been undertaken by the authors.

Acknowledgment. We thank the anonymous referees for their useful comments, which improved the presentation of our results.

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