Semi-groups acting on context-free graphs

Géraud Sénizergues

LaBRI
Université de Bordeaux I
351, Cours de la Libération 33405 Talence, France **

Abstract. Let Γ be some context-free graph. We give sufficient conditions on a semi-group of bisimulations H ensuring that the quotient $H\backslash\Gamma$ is context-free. Using these sufficient conditions we show that the quotient $\mathrm{Aut}(\Gamma)\backslash\Gamma$ of Γ by its full group of automorphisms is always context-free. We then give examples showing optimality (in some sense) of the above result.

Keywords: infinite graphs; ends; pushdown automata; automorphisms; bisimulations; groups; semi-groups.

1 Introduction

The study of the group of symetries (or *automorphisms*) of mathematical objects is nowadays considered as a valuable way of understanding these objects (since, in particular, it has proved to be useful in many situations, see [Die78] for example).

The automorphisms of a given *graph* Γ can be studied with respect to several points of view: one can study the general structure of *every automorphism* and classify them with respect to this structure (see [Hal73] for example), one can study the structure of the *group of automorphisms* (see [Hof82] in the case of finite graphs) and more generally the structure of any *group acting* on Γ (see [BS83] for trees, [DD90] for graphs). At last, one might study the structure of the *quotient* $H\backslash\Gamma$ of Γ by a group H acting on Γ (which seems to be a less classical subject for graphs than for varieties over $\mathbb{R}$ or $\mathbb{C}$ or for more general topological spaces).

The notion of *context-free* graph (which is closely related to the notion of *pushdown automaton*) appears in several fields of theoretical computer science: originally in the frontier between formal language theory and combinatorial group theory ([Ani71, MS83, MS85], see [BB91, p.95-100] for a survey) but also in algebraic graph theory ([Bau91, Bau92], see [Cou90a] for a survey), in model theory ([Cou89, Cou90b], see [Cou90a] for a survey) and in semantics of processes ([BBK87],[Cau90a],[CHS92],[HJM94], see [Cau95] for a survey).

The aim of this work is to study the structure of $H\backslash\Gamma$ where Γ is a context-free graph and H a group acting on Γ. The study of H itself is done in [Pel95] (while, in some sense, the fundamental article [MS83] is treating the case where Γ is exactly the Cayley graph of H).

The technical tools we are using are of two kinds: we use *geometrical* arguments about the graph Γ (leaning on results of [MS85, Sén89b, Cau90b, Pel95]) and also some classical *language*-theoretic arguments which may be applied via the characterization of context-free graphs as the computation-graphs of pda's ([MS85]). Surprizingly, we do not need any substantial result of group theory to establish the structure of $H\backslash\Gamma$: the notion of *semi-group* acting on Γ seems the easiest way of treating the action of a group on Γ (see theorem 3.5, corollary 3.12 and the proof of theorem 3.16).

** mailing adress:LaBRI and UFR Math-info, Université Bordeaux1
351 Cours de la libération -33405- Talence Cedex.
email:ges@labri.u-bordeaux.fr
fax: 56-84-66-69

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2 Preliminaries

2.1 Graphs

Let X be a finite alphabet and let $\bar{X}$ be a disjoint alphabet given with a fixed bijection $\alpha : X \to \bar{X}$. We call $X \cup \bar{X}$ a *paired* alphabet and for every x in X (resp. in $\bar{X}$) we denote by $\bar{x}$ the letter $\alpha(x)$ (resp. $\alpha^{-1}(x)$). We call *graph over* X any pair $\Gamma = (V_\Gamma, E_\Gamma)$ where V_Γ is a set and E_Γ is a subset of $V_\Gamma \times (X \cup \bar{X}) \times V_\Gamma$ fufilling the following *symetry* condition:

$$\forall v, v' \in V_\Gamma, \forall x \in X \cup \bar{X}, (v, x, v') \in E_\Gamma \Leftrightarrow (v', \bar{x}, v) \in E_\Gamma. \tag{1}$$

In the following we fix X and by *graph* we always mean *graph over* X. Let $\sim$ be some equivalence relation over the set V_Γ and let $\Pi_\sim$ denote the canonical projection from V_Γ onto $\sim\backslash V_\Gamma$. The *quotient graph* $\sim\backslash\Gamma$ is by definition:

$$\sim\backslash\Gamma = (\sim\backslash V_\Gamma, \sim\backslash E_\Gamma), \text{ where } \sim\backslash E_\Gamma = \{\Pi_\sim(v), x, \Pi_\sim(v') \mid (v,x,v') \in E_\Gamma\}.$$

Connectivity
Let Γ be some graph. By $\mathrm{d}(v,v')$ we denote the *distance* between two vertices v, v' in Γ : it is the minimal length of a path from v to v' in Γ. Let κ be an integer. We call *κ-path* in Γ any sequence $(v_0, \ldots, v_i, \ldots, v_l)$ of vertices such that,

$$\forall i \in [0, l-1], \mathrm{d}(v_i, v_{i+1}) \leq \kappa.$$

A subset $V' \subseteq V_\Gamma$ will be said *κ-connected* iff, $\forall v, v' \in V'$, there exists a κ-path in Γ from v to v', whose vertices are all in V'. Given a subset $V' \subseteq V_\Gamma$, the *frontier* $\triangle V'$ of V' is defined by :

$$\triangle V' = \{v \in V' \mid \exists v' \in V_\Gamma - V', \exists x \in X \cup \overline{X}, (v', x, v) \in E_\Gamma\},$$

and the *cofrontier* $\partial V'$ of V' is defined by :

$$\partial V' = \{(v, x, v') \in E_\Gamma \mid v \in V', x \in X \cup \overline{X}, v' \in V_\Gamma - V'\}.$$

Ends
By $\mathrm{B}(v,n)$ (resp. $\mathrm{S}(v,n)$) we denote the ball $\{v' \in V_\Gamma \mid \mathrm{d}(v,v') < n\}$ (resp. the sphere $\{v' \in V_\Gamma \mid \mathrm{d}(v,v') = n\}$). Given some vertex $v \in V_\Gamma$, we call *v-end* of Γ (relative to the ball $\mathrm{B}(v,n)$) any subset of V_Γ which is a connected component of $\Gamma - \mathrm{B}(v,n)$ for some radius n (terminology of [MS85]). More generally , given a subset $P \subseteq V_\Gamma$, we call P-end any connected component of $\Gamma - \mathrm{B}(P,n)$ for some radius n. Let E, E' be two P-ends. A map $\varphi : E \to E'$ is a *end-isomorphism* from E to E' iff φ is a graph isomorphism (for the induced subgraphs) such that $\varphi(\triangle E) = \triangle E'$.

We recall that a graph Γ is said *context-free* iff it is connected and there exists some vertex $v_0 \in V_\Gamma$ such that Γ has only finitely many classes of v_0-ends modulo end-isomorphism ([MS85]). (In fact, by [MS85, theorem 2.8], the above property turns out to be independant of the chosen v_0).

Morphisms, simulations

Definition 1 [Par81, Cau95]. Let $\Gamma = (V_\Gamma, E_\Gamma), \Gamma' = (V_{\Gamma'}, E_{\Gamma'})$ be two graphs. Let ρ be some binary relation $\rho \subseteq V_\Gamma \times V_{\Gamma'}$.
1- ρ is a *simulation* iff $\mathrm{dom}(\rho) = V_\Gamma$ and for all $v, w \in V_\Gamma, v' \in V_{\Gamma'}, \sigma \in X \cup \bar{X}$, such that $(v, \sigma, w) \in E_\Gamma$ and $v \sim v'$,

$$\text{there exists } w' \in V_{\Gamma'} \text{ such that } (v', \sigma, w') \in E_{\Gamma'} \text{ and } w \sim w'.$$

2- ρ is a *bisimulation* iff ρ and ρ^{-1} are simulations.

A map $\varphi : V_\Gamma \rightarrow V_{\Gamma'}$ is a graph *homomorphism* from Γ to Γ' iff it is a bisimulation.

Remark 2.1
1-An equivalence relation $\sim$ over V_Γ is a bisimulation over Γ iff the canonical projection $\Pi_\sim$ is an homomorphism.
2-The fact that we only consider graphs fulfilling the symetry condition(1) implies that , if ρ is a bisimulation from Γ to Γ' it is also a bisimulation from Γ^{-1} to Γ'^{-1}, which is not the case for general labelled graphs.

A binary relation $\rho \subseteq V_\Gamma \times V_{\Gamma'}$ is said (v,κ)-*connected* iff, for every $v' \in \rho v$ (i.e. v' such that $(v,v') \in \rho$), there exists some κ-path in Γ from v to v' : $(v_0,\ldots,v_i,\ldots,v_l)$ such that, $\forall i \in [0,l-1], (v_i,v_{i+1}) \in \rho$. The two following lemmas are easy.

Lemma 2.2 *Let m_1, m_2 be (v,κ)-connected bisimulations over some graph Γ. Then $m_1 \circ m_2$ is a (v,κ)-connected bisimulation. .*

Lemma 2.3 *Let $\mathcal{M}$ be a set of (v,κ)-connected bisimulations over some graph Γ. Then $\bigcup_{m\in\mathcal{M}} m$ is a (v,κ)-connected bisimulation too. .*

2.2 Automata

Finite automata A n-tape finite automaton (abbreviated n-f.a. in the sequel) is a 5-tuple:

$$\mathcal{A} = < X, Q, \delta, r, F >$$

where, as usual, X is the input-alphabet, Q is the set of states, r is the initial state, $F \subseteq Q$ is the set of final states and δ, the set of transitions, is a subset of $Q \times (X \cup \{\epsilon\})^n \times Q$.
$\mathcal{A}$ is said *letter-to-letter* iff $\delta \subseteq Q \times X^n \times Q$.
A *configuration* of $\mathcal{A}$ is any $n+1$-tuple $(q,u_1,\ldots,u_n) \in Q \times (X^*)^n$. By Config($\mathcal{A}$) we denote the set of all configurations of $\mathcal{A}$. The *one-step move relation* $\longmapsto_\mathcal{A}$ is the binary relation on Config($\mathcal{A}$) defined by:

$$(q,\mathbf{u}) \longmapsto_\mathcal{A} (q',\mathbf{u}') \Longleftrightarrow \exists \mathbf{a} \in (X \cup \{\epsilon\})^n \text{ such that } (q,\mathbf{a},q') \in \delta \text{ and } \mathbf{u} = \mathbf{a}.\mathbf{u}'$$

The language recognized by $\mathcal{A}$ is:

$$\mathrm{L}(\mathcal{A}) = \{\mathbf{u} \in X^n \mid \exists q \in F, (r,\mathbf{u}) \overset{*}{\longmapsto}_\mathcal{A} (q,(\epsilon)^n)\}.$$

A subset $L \subseteq (X^*)^n$ is said *rational* iff there exists some n-f.a. $\mathcal{A}$ such that $L = \mathrm{L}(\mathcal{A})$. A subset $L \subseteq (X^*)^n$ (equivalently , a n-ary relation over X^*) is called *length-preserving* iff, for every $(u_1,\ldots,u_i,\ldots,u_n) \in L, |u_1| = \ldots = |u_i| = \ldots = |u_n|$.

Theorem 2.4 ([Eil74], [FS93]) *Let $L \subseteq (X^*)^n$ be length-preserving. Then L is rational iff L is recognized by some* letter-to-letter *n-f.a.*

We recall that the *recognizable* subsets of ${X^*}^n$ are exactly the subsets L of the form

$$L = \bigcup_{1 \le i \le p} R_{i,1} \times \ldots \times R_{i,n}$$

where $p \in \mathbb{N}$ and all the $R_{i,j}$ are rational subsets of X^*. If L is rational and R is recognizable, then $L \cap R$ is rational too.

Pushdown automata A *pushdown automaton* on the alphabet X is a 6-tuple $\mathcal{A} = < X, Y, Q, \delta, q_0, y_0 >$ where Y is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, y_0 is the initial stack-symbol and δ, the transition function, is a mapping $\delta : YQ \times (X \cup \{\epsilon\}) \rightarrow \mathcal{P}(Y^*Q)$. Let $q, q' \in Q, \omega, \omega' \in Y^*, y \in Y, f \in X^*$ and $a \in X \cup \{\epsilon\}$; we note $(\omega y q, af) \longmapsto_{\mathcal{A}} (\omega\omega' q', f)$ if $\omega' q' \in \delta(yq, a)$. $\overset{*}{\longmapsto}_{\mathcal{A}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{A}}$.

For every $\omega q, \omega' q' \in Y^*Q$ and $f \in X^*$, we note $\omega q \xrightarrow{f}_{\mathcal{A}} \omega' q'$ iff $(\omega q, f) \overset{*}{\longmapsto}_{\mathcal{A}} (\omega' q', \epsilon)$.

$\mathcal{A}$ is said *real-time* iff, for every $yq \in YQ$, $\mathrm{Card}(\delta(yq, \epsilon)) = 0$.

2.3 Pushdown-structures

We call *transition-graph* of the pda $\mathcal{A}$, denoted $\mathcal{T}(\mathcal{A})$, the graph:
$\mathcal{T}(\mathcal{A}) = (V_{\mathcal{T}(\mathcal{A})}, E_{\mathcal{T}(\mathcal{A})})$ where $V_{\mathcal{T}(\mathcal{A})} = Y^*Q$ and

$$\begin{aligned} E_{\mathcal{T}(\mathcal{A})} = & \{(\omega q, x, \omega' q') \in Y^*Q \times X \times Y^*Q \mid \omega q \xrightarrow{x}_{\mathcal{A}} \omega' q'\} \\ & \cup \{(\omega' q', \bar{x}, \omega q) \in Y^*Q \times \bar{X} \times Y^*Q \mid \omega q \xrightarrow{x}_{\mathcal{A}} \omega' q'\}. \end{aligned}$$

We call *computation-graph* of $\mathcal{A}$,denoted $\mathcal{C}(\mathcal{A})$, the *connected component* in $\mathcal{T}(\mathcal{A})$ of $y_0 q_0$.
We call *pushdown-structure* over a graph Γ any pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a real-time pushdown-automaton over the alphabet X and $\varphi : \mathcal{C}(\mathcal{A}) \longrightarrow \Gamma$ is an isomorphism. We call *initial vertex* of the p-structure the vertex $v_0 = \varphi(y_0 q_0)$, where $y_0 q_0$ is the initial configuration of $\mathcal{A}$.

For every pair of graphs endowed with a p-structure $(\Gamma_1, \mathcal{A}_1, \varphi_1), (\Gamma_2, \mathcal{A}_2, \varphi_2)$, a binary relation $\rho \subseteq V_{\Gamma_1} \times V_{\Gamma_2}$ will be said *rational* iff $\varphi_2^{-1} \circ \rho \circ \varphi_1$ is a rational subset of $(Y_1 \cup Q_1 \cup Y_2 \cup Q_2)^* \times (Y_1 \cup Q_1 \cup Y_2 \cup Q_2)^*$ (see §2.2). The following lemma will then be useful

Lemma 2.5 *For every two real-time pushdown automata $(\mathcal{A}_1, \mathcal{A}_2)$, the computation graphs $\mathcal{C}(\mathcal{A}_1), \mathcal{C}(\mathcal{A}_2)$ are isomorphic iff there exists some* rational *isomorphism ψ from $\mathcal{C}(\mathcal{A}_1)$ to $\mathcal{C}(\mathcal{A}_2)$.*

This result was exposed at [Sén89b]. A full proof is given in [Sén89a].□

Within this terminology, the fundamental theorem of Muller and Schupp ([MS85]) can be rephrased as follows.

Theorem 2.6 *Let Γ be a graph over X. Γ is context-free if and only if Γ has some pushdown structure.*

Let us say that two p-structures $(\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)$ over the same graph Γ are *equivalent* iff $\mathrm{Id} : (\Gamma, \mathcal{A}_1, \varphi_1) \rightarrow (\Gamma, \mathcal{A}_2, \varphi_2)$ is rational.

Remark 2.7
1- A given Γ may of course have several different and even non-equivalent p-structures,
2- but, for any p-structures $((\mathcal{A}_i, \varphi_i))_{1 \leq i \leq 2}$ there always exists some graph isomorphism $\psi : (\Gamma, \mathcal{A}_1, \varphi_1) \rightarrow (\Gamma, \mathcal{A}_2, \varphi_2)$, which is rational *(by lemma 2.5 above).*

Induced p-structures
Let $\Psi : \Gamma_1 \longrightarrow \Gamma_2$ be some graph homomorphism from a connected graph Γ_1 to a connected graph Γ_2.

1. If Ψ is injective and Γ_2 has some p-structure $(\mathcal{A}_2, \varphi_2)$, there exists at most one p-structure (up to equivalence) over Γ_1 which makes Ψ rational. Such a structure $(\mathcal{A}_1, \varphi_1)$ exists iff $\mathrm{Im}\Psi$ is rational (easy remark). $(\mathcal{A}_1, \varphi_1)$ is then called the *sub-structure* induced by Ψ.
2. If Ψ is surjective and Γ_1 has some p-structure $(\mathcal{A}_1, \varphi_1)$, there exists at most one p-structure (up to equivalence) over Γ_2 which makes Ψ rational. Such a structure $(\mathcal{A}_2, \varphi_2)$ exists iff $\mathrm{Ker}\Psi$ is rational (this will be shown in corollary 3.11). $(\mathcal{A}_2, \varphi_2)$ is then called the *quotient-structure* induced by Ψ.

Canonical p-structures
The notion of *canonical automaton* associated with a context-free graph Γ has been defined in [MS85]. We use here a slight adaptation of this notion that we call a *P-canonical automaton*. Given a finite subset P of V_Γ, the P-canonical automaton describes the P-ends of Γ in the same way as the ordinary canonical (called here the v_0-canonical) automaton was describing the v_0-ends.(It is proved in [Cau90b] that Γ must have only finitely P-ends up to end-isomorphism). Instead of a completely precise definition (which would be somewhat long and technical), we just state here the key property of such a P-canonical automaton $\mathcal{A} = < X, Y, Q, \delta, q_0, y_0 >$: if $(\mathcal{A}, \varphi)$ is a p-structure over Γ where $\mathcal{A}$ is P-canonical then

$$\forall \omega q \in V_{\mathcal{C}(\mathcal{A})}, \mathrm{d}(P, \varphi(\omega q)) = \mid \omega \mid -1 \qquad (2)$$

i.e. the distance from the subset P to some vertex v is equal to the stack-height (minus 1) of the configuration representing v.
Let us remark that by (2), $\varphi^{-1}(P)$ must be of the form $y_0 Q_0$ for some $Q_0 \subseteq Q$.

Proposition 2.8 *Let Γ be a context-free graph, P a non-empty finite set of vertices of Γ and $(\mathcal{A}, \varphi)$ a p-structure over Γ. Then there exists some P-canonical p-structure $(\mathcal{A}', \varphi')$ over Γ which is equivalent to $(\mathcal{A}, \varphi)$.*

Connectivity and p-structures

Lemma 2.9 *For every rational subset K of a context-free graph Γ endowed with a p-structure $(\mathcal{A}, \varphi)$, there exists some integer κ such that K is κ-connected.*

A key-result for our investigations is the following

Theorem 2.10 ([Pel95]) *Let Γ be a context-free graph and let $\sim$ be the bisimulation induced by its group of automorphisms $Aut(\Gamma)$. Then, for every $v \in V_\Gamma$, there exists some integer κ (which may be computed from some p-structure $(\mathcal{A}, \varphi)$ over Γ) such that $\sim$ is (v, κ)-connected.*

3 Quotient of a context-free graph by a semi-group of bisimulations

We fix some context-free graph Γ throughout this section, as well as some p-structure $(\mathcal{A}, \varphi)$ over Γ.

3.1 Quotient by one bisimulation of finite order

Let us say that an equivalence relation $\sim$ over a set V has *finite order* iff every class modulo $\sim$ is finite.

Theorem 3.1 *Let $\sim$ be an equivalence relation on Γ such that:*

1. *$\sim$ is a bisimulation*
2. *$\sim$ is rational*
3. *$\sim$ has finite order*

Then the graph $\sim \backslash \Gamma$ is context-free.

Let us consider the set $P = [v_0]_\sim$. This set is finite by hypothesis (3). Let us denote by $\Pi : \Gamma \to \sim \backslash \Gamma$ the canonical projection.

Lemma 3.2 $\forall v \in V_\Gamma, \mathrm{d}(P, v) = \mathrm{d}(\Pi(v_0), \Pi(v))$.

Proof: Let $v \in V_\Gamma$.
1- Let $(u_0, u_1, \ldots, u_n)$, where $u_0 \in P$ and $u_n = v$ be some minimal path from P to v. Then $\Pi(u_0), \Pi(u_1), \ldots, \Pi(u_n)$ is some path connecting $\Pi(v_0)$ and $\Pi(v)$ in $\sim \backslash \Gamma$. Hence

$$\mathrm{d}(\Pi(v_0), \Pi(v)) \leq \mathrm{d}(P, v). \tag{3}$$

2- Let $(w_n, w_{n-1}, \ldots, w_0)$, where $w_n = \Pi(v)$ and $w_0 = \pi(v_0)$ be some minimal path in $\sim \backslash \Gamma$. As Π is a bisimulation, there exist $u_n, u_{n-1}, \ldots, u_0$ in V_Γ such that $u_n = v$, $(u_n, u_{n-1}, \ldots, u_0)$ is a path in Γ and $\Pi(u_i) = w_i$ for every $i \in [0, n]$. As $\Pi(u_0) = \Pi(v_0)$, $u_0 \in P$, hence

$$\mathrm{d}(P, v) \leq \mathrm{d}(\Pi(v_0), \Pi(v)). \tag{4}$$

By inequalities (3),(4) the lemma is proved. □

Lemma 3.3 *Let E be some P-end of Γ relative to the ball* $\mathrm{B}(P,n)$ *for some integer n. Then $\Pi(E)$ is a $\Pi(v_0)$-end of $\sim \backslash\Gamma$ relative to the ball* $\mathrm{B}(\Pi(v_0),n)$. *Moreover every $\Pi(v_0)$-end of $\sim \backslash\Gamma$ is of this form.*

Proof: **1-** Let E be a P-end of Γ relative to $\mathrm{B}(P,n)$. As E is connected and Π is a morphism, $\Pi(E)$ is connected. By lemma 3.2 $\Pi(E)$ is included in $\Gamma - \mathrm{B}(\Pi(v_0),n)$.
Let us consider some

$$(\Pi(e),\sigma,w) \in \boldsymbol{\delta}\Pi(E). \tag{5}$$

As Π is a bisimulation there exists some $v \in \Pi^{-1}(w)$ such that $(e,\sigma,v) \in E_\Gamma$. If $\mathrm{d}(P,v) \geq n$ then v must belong to E, hence $\Pi(v) \in \Pi(E)$, contradicting the hypothesis (5). Hence $\mathrm{d}(P,v) = n-1$. We have shown that

$$\Delta\Pi(E) \subseteq \mathrm{S}(\Pi(v_0),n).$$

As $\Pi(E)$ is a connected subset of $\sim \backslash\Gamma - \mathrm{B}(\Pi(v_0),n)$ whose frontier is included in $\mathrm{S}(\Pi(v_0),n)$, it must be a connected component of $\Gamma - \mathrm{B}(\Pi(v_0),n)$, i.e. a $\Pi(v_0)$-end.
2- Let F be some $\Pi(v_0)$-end of $\sim \backslash\Gamma$ relative to $\mathrm{B}(\Pi(v_0),n)$. Let E be some connected component of $\Pi^{-1}(F)$. By lemma 3.2 $E \subseteq \Gamma - \mathrm{B}(P,n)$. Let us consider some

$$(e,\sigma,v) \in \boldsymbol{\delta}E. \tag{6}$$

If $\mathrm{d}(P,v) \geq n$ then $\mathrm{d}(\Pi(v_0),\Pi(v)) \geq n$, hence the edge $(\Pi(e),\sigma,\Pi(v))$ is in $\Gamma - \mathrm{B}(\Pi(v_0),n)$, which implies that $\Pi(v) \in F$. So the edge (e,σ,v) connects two points of $\Pi^{-1}(F)$, and v must belong to E, contradicting hypothesis (6). Hence $\mathrm{d}(P,v) = n-1$. We have shown that

$$\Delta E \subseteq \mathrm{S}(P,n).$$

As E is a connected subset of $\Gamma - \mathrm{B}(P,n)$ whose frontier is included in $\mathrm{S}(P,n)$, it must be a connected component of $\Gamma - \mathrm{B}(P,n)$, i.e. a P-end.
□

Let $\psi : E \longrightarrow E'$ be a map from a P-end E to a P-end E'. We say that ψ is a end-isomorphism *preserving* $\sim$ (or is a $\sim$*-end-isomorphism*) iff ψ is a end-isomorphism and

$$\forall v,w \in E, v \sim w \Rightarrow \psi(v) \sim \psi(w).$$

Lemma 3.4 *Γ has only finitely many P-ends up to $\sim$-end-isomorphisms.*

Proof: Let us consider a *P-canonical* structure $(\mathcal{A}',\varphi')$ over Γ which is rationaly equivalent to the initial structure $(\mathcal{A},\varphi)$ (see proposition 2.8). Let us use the notation $\mathcal{A}' = < X, Y', Q', \delta', Q'_0, y'_0 >$ and $Z' = Y' \cup Q'$. $\sim$ is a rational relation over Z'^* which *preserves the length*. By theorem 2.4 $\sim$ is recognized by some *letter-to-letter* finite 2-automaton $\mathcal{S} =< X, Q_\mathcal{S}, \delta_\mathcal{S}, r_\mathcal{S}, F_\mathcal{S} >$. This automaton $\mathcal{S}$ can be choosen *deterministic* because, it can be considered as a 1-automaton over $Z' \times Z'$. Let E, E' be two P-ends and let ω (resp.ω') be the word over Z' associated with E (resp. E'). Let us set $E \equiv E'$ iff

ω, ω' have the same rightmost letter and $\delta_{\mathcal{S}}(r_{\mathcal{S}}, \omega, \omega) = \delta_{\mathcal{S}}(r_{\mathcal{S}}, \omega', \omega')$.

One can check that $\equiv$ defines only finitely many classes and that for every P-ends E, E',

$$E \equiv E' \Rightarrow E, E' \text{ are } \sim\text{-end-isomorphic} .$$

□

Proof of theorem 3.1:
Let us denote by $\mathcal{E}_P$ (resp. $\mathcal{E}_\sim$) the set of P-ends of Γ (resp. the set of $\Pi(v_0)$-ends of $\sim \backslash \Gamma$). By lemma 3.3 the map $\Pi_e : \mathcal{E}_P \to \mathcal{E}_\sim$ defined by:

$$\forall E \in \mathcal{E}_P, \Pi_e(E) = \Pi(E),$$

is well-defined and is surjective.
As $\mathcal{E}_P$ has finitely many classes modulo $\sim$-end-isomorphism (lemma 3.4), it follows that $\mathcal{E}_\sim$ has finitely many classes modulo end-isomorphism. □

3.2 Quotient by a semi-group of bisimulations

Theorem 3.5 (Main theorem) *Let H be some semi-group of bisimulations of Γ fulfilling the following conditions:*

(0) $\mathrm{Id}_\Gamma \in H$ *and* $\forall h \in H, h^{-1} \in H$
(1) H *is finitely generated*
(2) $\forall h \in H$, h *is rational*
(3) $\forall v, v' \in V_\Gamma$, *there exists some finite subset* $F(v, v') \subseteq H$ *such that*

$$\bigcup_{h \in H, (v,v') \in h} h \subseteq \bigcup_{h \in F(v,v')} h$$

.

Then $H \backslash \Gamma$ is context-free.

Construction of the set R. Let $\mathcal{G}$ be some finite set of generators of H. Hypothesis (2) of the theorem implies that each set hv_0, for $h \in \mathcal{G}$, is rational. By lemma 2.9, there exists some integer κ_0 such that, for every $h \in \mathcal{G}$, hv_0 is κ_0-connected. By lemma 2.2 every $h \in H$ is then (v_0, κ_0)-connected.
Fom now on, we fix such an integer κ_0 and we let κ_1 be the maximum diameter of the frontiers of the ends of Γ. We set

$$\kappa_2 = \kappa_0 + \kappa_1.$$

For every integer n we denote by $\mathcal{E}_n$ the set of ends of Γ relative to the ball $\mathrm{B}(v_0, n)$. We define a subset R of vertices by

$$R = \mathrm{B}(v_0, \kappa_2) \cup \bigcup_{E \in \mathcal{E}_{\kappa_2}, Hv_0 \cap E = \emptyset} E \qquad (7)$$

Lemma 3.6 *ΔR is finite.*

Proof: ΔR is included in the sphere of center v_0 and radius κ_2. □

Lemma 3.7 $\forall v \in V_\Gamma, [v]_{\sim_H} \cap R \neq \emptyset$.

Proof: Let $v \in V_\Gamma$. If $v \in R$, clearly $[v]_{\sim_H} \cap R \neq \emptyset$.
Let us suppose now that $v \notin R$.
By definition (7), there exists some end $E \in \mathcal{E}_{\kappa_2}$ such that

$$v \in E \text{ and } Hv_0 \cap E \neq \emptyset.$$

As Hv_0 is κ_0-connected and $Hv_0 \cap E \neq \emptyset$, there exists some vertex w_0 such that

$$w_0 \in Hv_0 \text{ and } \mathrm{d}(w_0, \Delta E) < \kappa_0. \tag{8}$$

Let us consider a vertex $v' \in \Delta E$ which is the intersection of some minimal path from v_0 to v with ΔE :

$$\mathrm{d}(v_0, v) = \mathrm{d}(v_0, v') + \mathrm{d}(v', v). \tag{9}$$

From (8) and the definition of κ_1 we get

$$\mathrm{d}(v', w_0) < \kappa_1 + \kappa_0 = \kappa_2 \tag{10}$$

Using (9),(10) we obtain

$$\mathrm{d}(w_0, v) \leq \mathrm{d}(w_0, v') + \mathrm{d}(v', v) < \kappa_2 + \mathrm{d}(v', v) = \mathrm{d}(v_0, v). \tag{11}$$

As $w_0 \sim_H v_0$ and $\sim_H$ is a bisimulation, there exists some $w \sim_H v$ with $\mathrm{d}(w_0, v) = \mathrm{d}(v_0, w)$ (see figure 1).
Using (11)we obtain that, for every $v \in V_\Gamma - R$, there exists some $w \sim_H v$ such that

$$\mathrm{d}(v_0, w) < \mathrm{d}(v_0, v).$$

(See figure1). By induction on $\mathrm{d}(v_0, v)$ it follows that every class $\pmod{\sim_H}$ has at least one representative in R. □

Lemma 3.8 *There exists some finite subset $K \subseteq H$ such that $\sim_H |R = \sim_K |R$.*

Proof: Let us introduce two finite subsets K_1, K of H by :

$$K_1 = \bigcup_{w,w' \in \mathrm{B}(v_0, 2\cdot\kappa_2)} F(w, w') \quad , \quad K = K_1 \circ K_1^{-1}.$$

We show below that, if $v_1, w_1 \in R$ are such that

$$\mathrm{d}(v_1, Hv_0 \cap R) = \mathrm{d}(Hv_1 \cap R, Hv_0 \cap R) \text{ and } w_1 \in Hv_1 \cap R \tag{12}$$

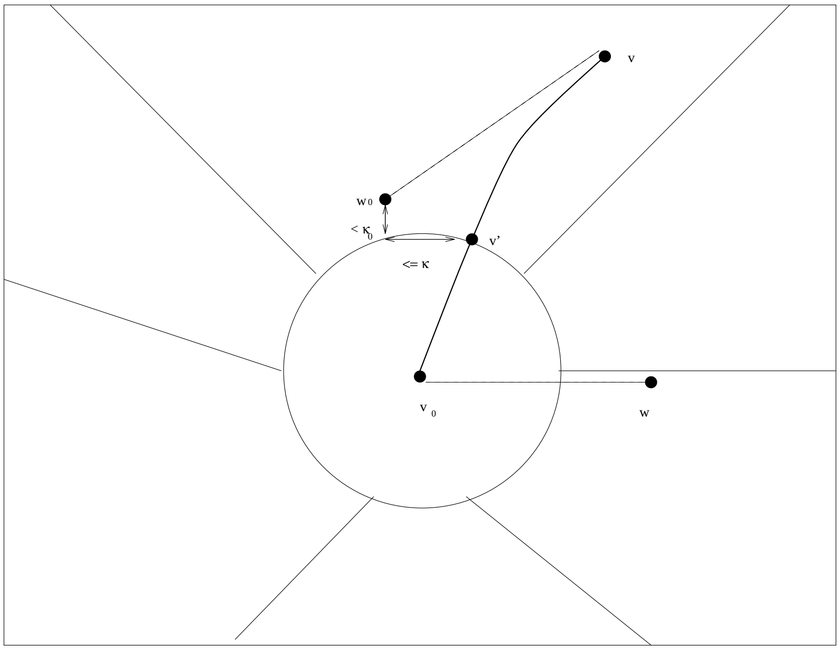

Fig. 1. R is a set of representatives

then $w_1 \in K_1 v_1$. As each class $\pmod{\sim_H}$ contains at least one such vertex v_1, it will follow that

$$\forall w_1, w_1' \in R, w_1 \sim_H w_1' \Rightarrow w_1 \in (K_1 \circ K_1^{-1}) w_1'.$$

Let v_1, w_1 fulfilling (12). Let $w_0 \in H v_0 \cap R$ such that

$$\mathrm{d}(v_1, w_0) = \mathrm{d}(H v_1 \cap R, H v_0 \cap R). \tag{13}$$

Let $h_1 \in H$ such that $w_1 \in h_1 v_1$.
As h_1 is a bisimulation, there exists some $w_0' \in h_1 w_0$ such that

$$\mathrm{d}(v_1, w_0) = \mathrm{d}(w_1, w_0') \tag{14}$$

By minimality of $\mathrm{d}(v_1, w_0)$(13) and equality (14)we have

$$\mathrm{d}(w_1, w_0') \leq \mathrm{d}(w_1, v_0) \tag{15}$$

case 1: $\mathrm{d}(w_1, v_0) < \kappa_2$.
Then by (15), $\mathrm{d}(w_1, w_0') < \kappa_2$, hence $\mathrm{d}(v_0, w_0') < 2 \cdot \kappa_2$.
case 2: $\mathrm{d}(w_1, v_0) \geq \kappa_2$ and $\mathrm{d}(w_0', v_0) < \kappa_2$.
It is clear that $\mathrm{d}(v_0, w_0') < 2 \cdot \kappa_2$.

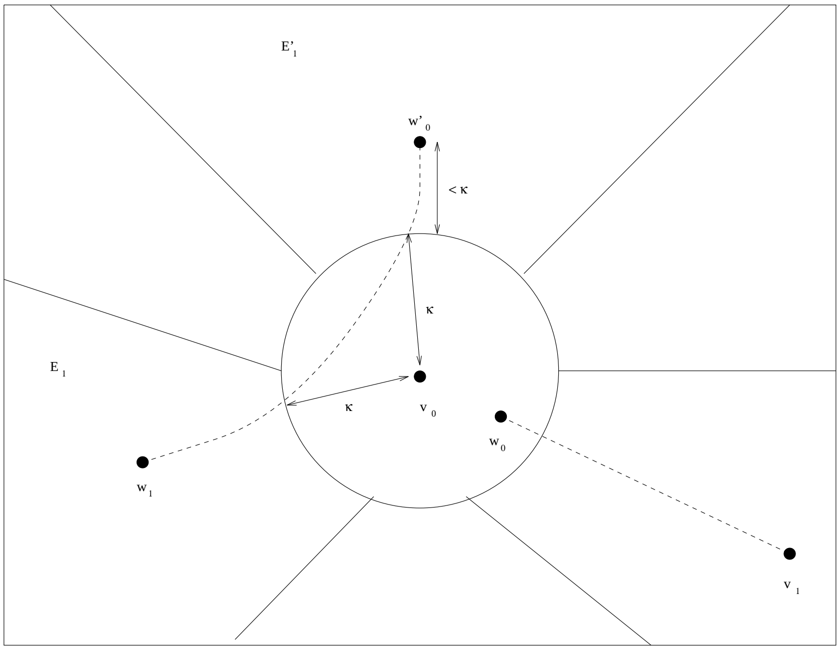

Fig. 2. Construction of K finite, case 3

case 3: $\mathrm{d}(w_1, v_0) \geq \kappa_2$ and $\mathrm{d}(w'_0, v_0) \geq \kappa_2$.

By definition of R (see equation (7)), $w'_0 \notin R$. Hence there are two distinct ends $E_1, E'_1 \in \mathcal{E}_{\kappa_2}$ such that $w_1 \in E_1, w'_1 \in E'_1$ (see figure 2).

As E_1, E'_1 must be disjoint we have:

$$\mathrm{d}(w_1, \Delta E_1) + 1 + \mathrm{d}(\Delta E'_1, w'_0) \leq \mathrm{d}(w_1, w'_0) \leq \mathrm{d}(w_1, w_0) \leq \mathrm{d}(w_1, \Delta E_1) + \kappa_2.$$

Hence $1 + \mathrm{d}(\Delta E'_1, w'_0) \leq \kappa_2$, which leads to

$$\mathrm{d}(v_0, w'_0) \leq \kappa_2 + \mathrm{d}(\Delta E'_1, w'_0) < 2 \cdot \kappa_2.$$

In every case $\mathrm{d}(v_0, w'_0) < 2 \cdot \kappa_2$, hence

$$h_1 \subseteq \bigcup_{h \in H, (w_0, w'_0) \in h} h \subseteq \bigcup_{h \in F(w_0, w'_0)} h \subseteq \bigcup_{h \in K_1} h.$$

But $w_1 \in h_1 v_1$, so that $w_1 \in K_1 v_1$ (as required). $\square$

Construction of the graph $\hat{R}$

Let us denote by Π_H (resp. Π_K) the canonical projection from Γ onto $H \backslash \Gamma$ (resp. from $\hat{R}$ onto $K \backslash \hat{R}$). Let us consider some map $\tau :\sim_H \backslash V_\Gamma \to R$ fufilling

$$\forall v \in V_\Gamma, \tau(\Pi_H(v)) \in \Pi_H(v).$$

We construct now a graph $\hat{R} = (V_{\hat{R}}, E_{\hat{R}})$ where $V_{\hat{R}} = R$ and $E_{\hat{R}}$ is defined by:

$E_{\hat{R}} = \{(v, \sigma, v') \mid v, v' \in R \text{ and } (v, \sigma, v') \in E_\Gamma\} \cup \{(v, \sigma, \tau(\Pi_H(v')) \mid v \in R, v' \notin R \text{ and } (v, \sigma, v') \in E_\Gamma\} \cup \{(\tau(\Pi_H(v)), \sigma, v') \mid v \notin R, v' \in R \text{ and } (v, \sigma, v') \in E_\Gamma\}$.

Lemma 3.9 *$\hat{R}$ is context-free.*

Proof: Let us consider the graph $\Gamma_R = (R, E_\Gamma \cap (R \times (X \cup \bar{X}) \times R))$.
Almost all the ends of Γ_R are ends of Γ, hence Γ_R is context-free. As $\hat{R}$ is obtained from Γ_R by adjunction of finitely many edges, $\hat{R}$ is context-free too. □

Lemma 3.10 *$H \backslash \Gamma \approx \quad K \backslash \hat{R}$.*

Proof: One can check that $\bar{\tau} = \Pi_K \circ \tau$ is a graph isomorphism from $H \backslash \Gamma$ to $K \backslash \hat{R}$. □

Sketch of proof of the main theorem Let H and Γ fulfill the hypotheses of theorem 3.5. By lemma 3.10 it is sufficient to prove that $K \backslash \hat{R}$ is context-free.
As each element of K is a bisimulation, $\sim_K$ is a bisimulation. As each element of K is rational and K is finite, $\sim_K$ is rational over Γ. As R is rational, the subset $\varphi^{-1}(R) \times \varphi^{-1}(R)$ of $Z^* \times Z^*$ (where Z is the configuration alphabet of $\mathcal{A}$) is recognizable . Hence $(\varphi^{-1} \circ \sim_K \circ \varphi) \cap (\varphi^{-1}(R) \times \varphi^{-1}(R))$ is a rational subset of $Z^* \times Z^*$, hence $\sim_K |R$ is a rational relation over Γ_R. It follows that $\sim_K |R$ is a rational relation over $\hat{R}$ too (for some suitable p-structure inherited from the p-structure of Γ). As $Kv_0 \cap R$ is finite and $\hat{R}$ is connected, $\sim_K |R$ has finite order over $\hat{R}$.
The three hypotheses of theorem 3.1 are met by $K, \hat{R}$, hence $K \backslash \hat{R}$ is context-free.□

Corollary 3.11 *Let $\Psi : \Gamma_1 \longrightarrow \Gamma_2$ be some surjective graph homomorphism from a context-free graph Γ_1 onto a graph Γ_2 and let $(\mathcal{A}_1, \varphi_1)$ be some p-structure over Γ_1. Ψ induces a quotient structure over Γ_2 iff* $\mathrm{Ker}\Psi$ *is rational.*

Case of groups

Corollary 3.12 *Let H be a group of automorphisms of Γ fulfilling the following conditions:*

(1) H is finitely generated
(2) $\forall h \in H$, h is rational
(3) $\mathrm{Stab}_H(v_0)$ is finite

Then $H \backslash \Gamma$ is context-free.

Proof: Let H fulfill the hypotheses of the theorem. Let us remark that every automorphism h of Γ is a binary relation over V_Γ which is a bisimulation. Hence H is a semi-group of bisimulations. It is clear that it fulfills condition (0) of theorem 3.5 and that hypothesis (1)(2) of this theorem are equivalent to hypothesis (1)(2) of theorem 3.5. For every $v, v' \in V_\Gamma$ let us define $F(v, v')$ as the set of all elements of H mapping v to v'. $F(v, v')$ is a right coset modulo $\mathrm{Stab}_H(v)$, hence $F(v, v')$ is finite. By theorem 3.5 $H \backslash \Gamma$ is context-free. □

Remark 3.13 *It is noteworthy that*
1- *In the particular case where* $H = \mathrm{Aut}(\Gamma)$ *the hypotheses of theorem 3.12 are equivalent one to each other:* (1) $\Leftrightarrow$ (3) *is stated in [Pel95, Theorem 22] while* (1) $\Leftrightarrow$ (2) *can be easily deduced from the proof of [Pel95, Theorem 22].*
2- *In the general case, the conclusion of theorem 3.12 may fail as soon as one of the three hypotheses is not met (see part 4).*

3.3 Quotient by the group of automorphisms

We denote here by $\sim$ the equivalence relation over V_Γ induced by the action of $\mathrm{Aut}(\Gamma)$ (i.e. $\sim_{\mathrm{Aut}(\Gamma)}$). For every subset $T \subseteq V_\Gamma$ we denote by $\mathrm{Aut}(\Gamma, T)$ the subgroup of elements of $\mathrm{Aut}(\Gamma)$ which preserve the set T:

$$\mathrm{Aut}(\Gamma, T) = \{h \in \mathrm{Aut}(\Gamma) \mid h(T) = T\}.$$

Lemma 3.14 $\sim_{\mathrm{Aut}(\Gamma, v_0)}$ *is a rational relation.*

Sketch of proof: Let us suppose that the p-structure $(\mathcal{A}, \varphi)$ is v_0-canonical (this can be assumed without loss of generality by proposition 2.8). For every $\omega \in Y^*, q \in Q$ we define:

$$\mathcal{A}_{\omega q} = < X, Y, Q, \delta, q, \omega >,$$

$$\mathcal{C}(\omega q) \text{ is the computation-graph of } \mathcal{A}_{\omega q}, \text{ restricted to } Y^+ Q$$

$$\Delta\mathcal{C}(\omega q) = \mathcal{C}(\omega q) \cap YQ,$$

For every $\omega_1, \omega_2 \in Y^*$ and $q \in Q$, we define the map
$\tau_{\omega_1, \omega_2 q_2} : \mathcal{C}(\omega_2 q_2) \longrightarrow \mathcal{C}(\omega_1 \omega_2 q_2)$ by:

$$\forall \omega q \in V_{\mathcal{C}(\omega_2 q_2)}, \tau_{\omega_1, \omega_2 q_2}(\omega q) = \omega_1 \omega q.$$

Property (2) of $\mathcal{A}$ implies that this map is a graph embedding. Let us define a finite 2-automaton $\mathcal{R}$ over $Z = Y \cup Q$ recognizing $\varphi^{-1} \circ \sim_{\mathrm{Aut}(\Gamma, v_0)} \circ \varphi = \sim_{\mathcal{C}(y_0 q_0)}$. Let $\mathcal{R} = < Z, Q_\mathcal{R}, \delta_\mathcal{R}, r_\mathcal{R}, F_\mathcal{R} >$ where

- $Q_\mathcal{R}$ consists of a special symbol q_t , which is the only element of $F_\mathcal{R}$, and all the 3-tuples of the form

$$(\mathcal{C}(yq), \mathcal{C}(y'q'), \alpha) \tag{16}$$

where $\alpha : \Delta\mathcal{C}(yq) \rightarrow \Delta\mathcal{C}(y'q')$ is a bijection which admits at least one extension to an isomorphism $\tilde{\alpha} : \mathcal{C}(yq) \rightarrow \mathcal{C}(y'q')$,
- $r_{\mathcal{R}} = (\mathcal{C}(y_0 q_0), \mathcal{C}(y_0 q_0), \mathrm{Id}_{\{y_0 q_0\}})$,
- $\delta_{\mathcal{R}}$ consists of the two following types of transitions
$\forall z_1, z_1', y_1, y_1', y_2, y_2' \in Y, \ \ \forall q_1, q_1', q_2, q_2' \in Q, \ \ \forall \alpha, \alpha'$,

$$(\mathcal{C}(y_1 q_1), \mathcal{C}(y_2 q_2), \alpha) \xrightarrow{(z_1, z_1')}_{\mathcal{R}} (\mathcal{C}(y_1' q_1'), \mathcal{C}(y_2' q_2'), \alpha') \text{ is a transition } \Longleftrightarrow$$
$$z_1 = y_1, z_1' = y_1', \quad y_i y_i' q_i' \in \mathcal{C}(y_i q_i)(\text{ for } i = 1, 2) \text{ and}$$
$\alpha \cup (\tau_{y_2, y_2' q_2'} \circ \alpha' \circ \tau^{-1}_{y_1, y_1' q_1'})$ admits at least one extension to an isomorhism
$$\tilde{\alpha} : \mathcal{C}(y_1 q_1) \rightarrow \mathcal{C}(y_2 q_2), \tag{17}$$

$$\forall y_1, y_2 \in Y, \forall q_1, q_1', q_2, q_2' \in Q, \forall \alpha$$
$$(\mathcal{C}(y_1 q_1), \mathcal{C}(y_2 q_2), \alpha) \xrightarrow{(q_1', q_2')}_{\mathcal{R}} q_t \text{ is a transition } \Longleftrightarrow$$
$$\alpha(y_1 q_1') = y_2 q_2'. \tag{18}$$

One can then prove by induction on n that
$\forall y_1, y_1', y_2, y_2' \in Y, \forall \omega_1, \omega_2 \in Y^n, \forall q_1, q_1', q_2, q_2' \in Q, \forall \alpha, \alpha'$,

$$(\mathcal{C}(y_1 q_1), \mathcal{C}(y_2 q_2), \alpha) \xrightarrow{(\omega_1 y_1', \omega_2 y_2')}_{\mathcal{R}} (\mathcal{C}(y_1 \omega_1 y_1' q_1'), \mathcal{C}(y_2 \omega_2 y_2' q_2'), \alpha')) \Longleftrightarrow$$
$$y_i \omega_i y_i' q_i' \in \mathcal{C}(y_i q_i)(\text{ for } i = 1, 2) \text{ and}$$
$\alpha \cup (\tau_{y_2 \omega_2, y_2' q_2'} \circ \alpha' \circ \tau^{-1}_{y_1 \omega_1, y_1' q_1'})$ admits at least one extension to an isomorphism
$$\tilde{\alpha} : \mathcal{C}(y_1 q_1) \rightarrow \mathcal{C}(y_2 q_2). \tag{19}$$

(See figure 3, where $\tau_{y_i \omega_i, y_i' q_i'}$ is simply denoted by τ_i). □

Lemma 3.15 *There exists some subgroup $K < \mathrm{Aut}(\Gamma)$ which is finitely generated, acts rationaly and transitively on $[v_0]_\sim$.*

Proof: Let κ be some integer satisfying the conclusion of theorem 2.9 for the vertex v_0. For every $v \in \mathrm{B}(v_0, \kappa) \cap [v_0]_\sim$ there exists some $\varphi_v \in \mathrm{Aut}(\Gamma)$ which is rational and maps v_0 to v (lemma 2.5). Let us consider

$$\mathcal{K} = \{\varphi_v \mid v \in \mathrm{B}(v_0, \kappa) \cap [v_0]_\sim\} \text{ and } \quad K = <\mathcal{K}>$$

(i.e. K is the subgroup generated by $\mathcal{K}$).
One can prove by induction on the integer n that : for every κ-path $(v_0, v_1, \ldots, v_n)$ where $v_i \in [v_0]_\sim$,

$$\text{there exists } k \in \mathcal{K}^n \text{ such that } k v_0 = v_n. \tag{20}$$

By theorem 2.9, $[v_0]_\sim$ is κ-connected, hence, by equation (20),

$$\forall v \in [v_0]_\sim, \exists k \in K, k v_0 = v.$$

□

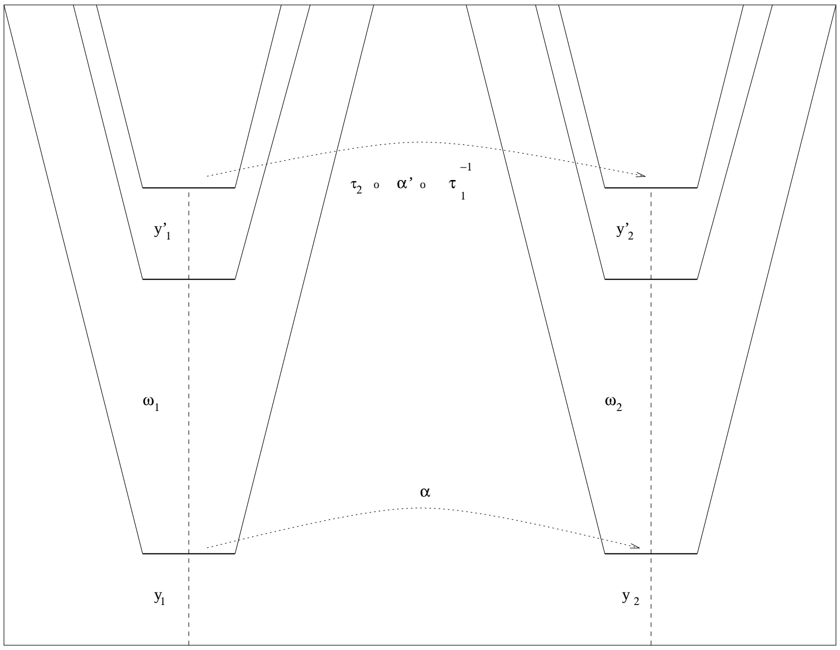

Fig. 3. $\sim_{\mathrm{Aut}(\Gamma, v_0)}$ is rational

Theorem 3.16 *Aut(*Γ*)*$\backslash\Gamma$ *is context-free.*

Proof: Let us consider the finite set of bisimulations:

$$\mathcal{H} = \{\sim_{\mathrm{Aut}(\Gamma, v_0)}\} \cup \mathcal{K}.$$

Let H be the semi-group of relations generated by $\mathcal{H}$. From lemma 3.15 one can deduce that

$$\mathrm{Aut}(\Gamma) = K \circ \mathrm{Aut}(\Gamma, v_0), \tag{21}$$

hence

$$\sim_{\mathrm{Aut}(\Gamma)} = \sim_H \ . \tag{22}$$

As $\mathcal{H}$ is finite, contains Id_Γ and is closed under inverse, H fulfills hypotheses (0)(1) of theorem 3.5. As each element of $\mathcal{H}$ is rational (by lemma 3.14 and by construction of $\mathcal{K}$), H fufills hypothesis (2) of theorem 3.5.
Let $v, v' \in V_\Gamma$. Let us consider $n = \mathrm{d}(v_0, v), n' = \mathrm{d}(v, v')$.

$$\bigcup_{\substack{h \in H \\ (v,v') \in h}} h = \bigcup_{\substack{h \in \mathrm{Aut}(\Gamma, v_0) \\ hv = v'}} h \subseteq \bigcup_{\substack{h \in \mathrm{Aut}(\Gamma, v_0) \\ hv_0 \in \mathrm{B}(v_0, 2 \cdot n + n')}} h \subseteq \bigcup_{f \in \mathcal{K}^{2 \cdot n + n'} \circ \sim_{\mathrm{Aut}(\Gamma, v_0)}} f.$$

Hence the finite set $F(v, v') = \mathcal{K}^{2 \cdot n + n'} \circ \sim_{\mathrm{Aut}(\Gamma, v_0)}$ fulfills equation (3) of theorem 3.5. Applying theorem 3.5 we obtain that $H\backslash\Gamma$ is context-free, and by equality (22) we conclude that $\mathrm{Aut}(\Gamma)\backslash\Gamma$ is context-free. $\square$

One can easily generalize this theorem as follows

Theorem 3.17 *Let Γ be a context-free graph and T be any rational subset of V_Γ (with respect to some p-structure on Γ). Then $Aut(\Gamma, T)\backslash\Gamma$ is context-free.*

Remark: When Γ is defined as the computation graph of some pda $\mathcal{A}$, and T is defined as the set of *terminal* configurations of $\mathcal{A}$, $\mathrm{Aut}(\Gamma, T)\backslash\Gamma$ can be viewed as the computation graph of a new pda $\bar{\mathcal{A}}$ with set of terminal configurations $\Pi(T)$, which recognizes the *same language* as $\mathcal{A}$. Moreover this automaton $\bar{\mathcal{A}}$ is *automorphism-free* in the sense that the group $\mathrm{Aut}(\mathcal{C}(\bar{\mathcal{A}}), \Pi(T))$ is trivial.

4 Examples

Each one of examples 1,2,3 shows that when the corresponding hypothesis of corollary 3.12 fails, the conclusion ("$H\backslash\Gamma$ is c.f.") may fail too. Example 4 is somewhat pathological. It shows that, even when hypotheses 1,2,3 are met, the canonical projection $\Pi : \Gamma \to H\backslash\Gamma$ may be non-rational.

4.1 Example 1

Case where H is not *finitely generated.*
Let $\Gamma_1 = (V_1, E_1)$ where $V_1 = \mathrm{F}_2$ (the free group over two generators a, b) and $V_1 = \{(g, x, gx) \mid w \in \mathrm{F}_2, x \in \{a, b, a^{-1}, b^{-1}\}\}$ (Γ_1 is simply the Cayley graph of F_2). Let us consider $H = \mathrm{F}_2'$ (the derived group of F_2). It is well-known that the group $\mathrm{F}_2/\mathrm{F}_2'$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Hence $H\backslash\Gamma_1$ is an infinite 2×2-grid, which is known to be not context-free (it has infinite tree-width).

4.2 Example 2

Case where H does not act *rationaly.*
Let $\Gamma_2 = (V_2, E_2)$ where $V_2 = \{0,1\}^*$ and $E_2 = \{(w, x, wx) \mid w \in \{0,1\}^*, x \in \{0,1\}\} \cup \{(wx, \bar{x}, w) \mid w \in \{0,1\}^*, x \in \{0,1\}\}$. ($\Gamma_2$ is simply the infinite complete binary tree). Given any subset $S \subseteq \mathbb{N} - \{0\}$, we define an isomorphism $h_S : \Gamma_2 \to \Gamma_2$ by : $h_S(\epsilon) = \epsilon$ and $\forall w \in \{0,1\}^*$,

$$\text{if } |w| \notin S \cup \{0\},\ h_S(w0) = h_S(w)1,\ h_S(w1) = h_S(w)0,$$
$$\text{if } |w| \in S \cup \{0\},\ h_S(w0) = h_S(w)0,\ h_S(w1) = h_S(w)1.$$

Let $H_S =< h_S >= \{\mathrm{Id}, h_S\}$. One can see that a vertex $H_S w$ of $H_S\backslash\Gamma_2$ has 3 neighbours if $|w| \in S$ while it has 2 neighbours if $|w| \notin S$. This shows that, if S is choosen non-recursive, the quotient graph $H_S\backslash\Gamma_2$ is not recursive (hence non-context-free).

4.3 Example 3

Case where the stabilizer of v_0 is *infinite*.
Let $\Gamma_3 = (V_3, E_3)$ where $V_3 = \{0,1,\#\}^*$ and $E_3 = \{(w, x, wx) \mid w \in \{0,1,\#\}^*, x \in \{0,1,\#\}\} \cup \{(wx, \bar{x}, w) \mid w \in \{0,1,\#\}^*, x \in \{0,1,\#\}\}$. ($\Gamma_3$ is simply the infinite complete ternary tree). Let us consider the rational function $\sigma : \{0,1,\#\}^* \rightarrow \{0,1,\#\}^*$ computed by the 2-automaton given in figure 4 (σ is indeed a sequential map). One can easily see that, as σ preserves the prefix ordering (i.e. for

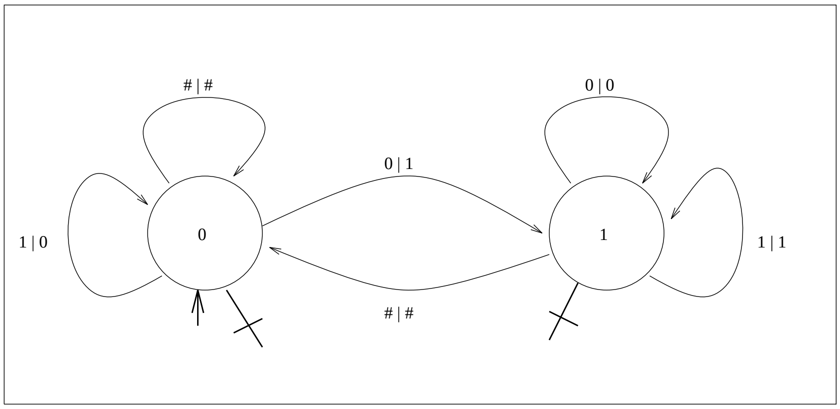

Fig. 4. the map σ.

all $w, w' \in \{0,1,\#\}^*, w \preceq w' \Rightarrow \sigma(w) \preceq \sigma(w')$), σ is an automorphism of Γ_3. Let $H = <\sigma>$ (the subgroup generated by σ). In order to compute the orbits (mod H) we introduce the map $\nu : \{0,1\}^* \rightarrow \mathbb{N}$. For every $w = b_k \dots b_j \dots b_0$, where the b_j are symbols in $\{0,1\}$,

$$\nu(w) = \Sigma_{0 \leq j \leq k} b_j 2^j,$$

For every integers n, d we denote by $r(n,d)$ the residual of n in the integral division by d. For every integer j let us consider the map $\mu_j : \mathbb{N} \rightarrow \{0,1\}^j$ defined by:

$$\forall n \in \mathbb{N}, \mu_j(n) \in \{0,1\}^j \text{ and } \nu(\mu_j(n)) = r(n, 2^j).$$

One can check that

$$\forall i, j \in \mathbb{N}, \sigma^i(0^j) = {}^t\mu_j(i). \tag{23}$$

The following equivalences follow from (23): for every $w \in \{0,1\}^*$,

$$0^j \sim_H w \Longleftrightarrow w \in \{0,1\}^j, \tag{24}$$

$$0^j\#0^k \sim_H 0^j\#w \Longleftrightarrow \begin{array}{l}(k < j \text{ and } w = 0^k) \text{ or}\\ (k \geq j \text{ and } w \in 0^j\{0,1\}^{k-j}).\end{array} \tag{25}$$

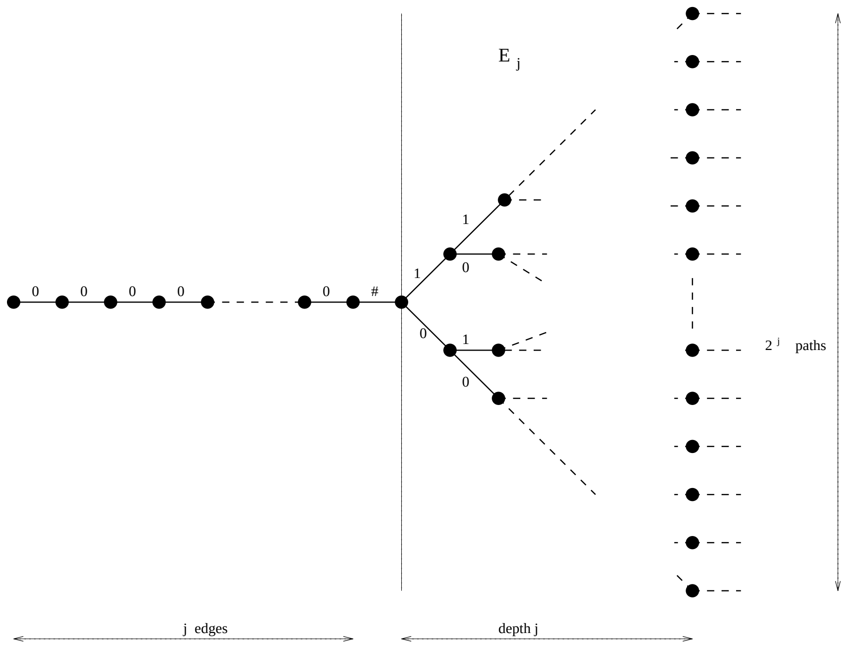

Fig. 5. the end E_j.

Let $v_0 = \epsilon$ and let us denote by E_j the v_0-end of $H\backslash\Gamma_2$ which contains $H0^j\#$. By (24)(25), E_j has exactly 2^j infinite paths whose label belongs to $\{0,1\}^\omega$. (We sketch on figure 5 the end E_j resticted to its edges labelled on $\{0,1\}$).

Hence $j \neq j' \Longrightarrow E_j, E_{j'}$ are not end-isomorphic. Hence $H\backslash\Gamma_2$ is not context-free.

4.4 Example 4

Case where the hypothesis of theorem 3.5 are met but the canonical projection Π is *not rational.*
Let $B = \{x, y, z\}$, $\hat{B} = (B \cup \bar{B})$ and $X = \{a\}$. Let $\Gamma_4 = (V_4, E_4)$ where $V_4 = \mathrm{F}(B)$ (the free group over B) and $E_4 = \{(w, e, wg) \mid w \in \mathrm{F}(B), e \in \{a, \bar{a}\}, g \in \hat{B}\}$.($\Gamma_4$ is simply the Cayley graph of $\mathrm{F}(B)$ over B, where, in some sense, the labels have been erased). As usual, we identify $\mathrm{F}(B)$ with the set $\mathrm{I} \subseteq \hat{B}^*$ of irreducible words. For every letter $t \in \hat{B}$, we define $\mathrm{I}_t = \mathrm{I} - \bar{t}\hat{B}^*$. Let us suppose that we are given a bijection $\tilde{\sigma} : \mathrm{I}_y \to \mathrm{I}_y$ fufilling the following properties:

$$\forall u, v \in \mathrm{I}_y, u \preceq v \Longrightarrow \tilde{\sigma}(u) \preceq \tilde{\sigma}(v), \tag{26}$$

For all $u \in \mathrm{I}_y$,

$$uy \in \mathrm{I} \Longrightarrow \tilde{\sigma}(uy) = \tilde{\sigma}(u)y, \tag{27}$$

$$u\bar{y} \in \mathrm{I} \Longrightarrow \tilde{\sigma}(u\bar{y}) = \tilde{\sigma}(u)\bar{y}, \tag{28}$$

$$ux \in \mathrm{I} \Longrightarrow \tilde{\sigma}(ux) \in \{\tilde{\sigma}(u)x, \tilde{\sigma}(u)z\}, \tag{29}$$
$$u\bar{x} \in \mathrm{I} \Longrightarrow \tilde{\sigma}(u\bar{x}) \in \{\tilde{\sigma}(u)\bar{x}, \tilde{\sigma}(u)\bar{z}\}, \tag{30}$$
$$uz \in \mathrm{I} \Longrightarrow \tilde{\sigma}(uz) \in \{\tilde{\sigma}(u)x, \tilde{\sigma}(u)z\}, \tag{31}$$
$$u\bar{z} \in \mathrm{I} \Longrightarrow \tilde{\sigma}(u\bar{z}) \in \{\tilde{\sigma}(u)\bar{x}, \tilde{\sigma}(u)\bar{z}\}. \tag{32}$$

By (26),$\tilde{\sigma}$ is an automorphism of the subgraph induced by I_y. We then define maps $\tau_\alpha : \Gamma_4 \rightarrow \Gamma_4$ (for $\alpha \in \{x, y, \bar{x}, \bar{y}\}$) by :

$$\forall u \in \mathrm{I} - x\hat{B}^*,\ \tau_x(\bar{x}u) = u,\ \tau_x(u) = xu, \tag{33}$$
$$\forall u \in \mathrm{I} - y\hat{B}^*,\ \tau_y(\bar{y}u) = u,\ \tau_y(u) = y\tilde{\sigma}(u), \tag{34}$$

and $\tau_{\bar{x}}$ (resp. $\tau_{\bar{y}}$) is defined by a formula analogous to (33) (resp. (34)) where $(x, \bar{x})$ are exchanged (resp. $(y, \bar{y})$ are exchanged and $\tilde{\sigma}$ is replaced by $\tilde{\sigma}^{-1}$). One can check that the τ_α are automorphisms of Γ_4 and that $\tau_\alpha, \tau_{\bar{\alpha}}$ are inverse one of each other(for $\alpha \in \{x, y\}$). Let us consider the unique product

$$< ., . >: \mathrm{F}(\{x, y\}) \times \mathrm{F}(\{x, y, z\}) \rightarrow \mathrm{F}(\{x, y, z\})$$

such that the map $\alpha \mapsto < \alpha, . >$ is a group homomorphism $\mathrm{F}(\{x, y\}) \rightarrow \mathrm{Aut}(\Gamma_4)$ and

$$\forall \alpha \in \{x, y\}, \forall \beta \in \mathrm{F}(\{x, y, z\}), < \alpha, \beta >= \tau_\alpha(\beta).$$

In other words, $< ., . >$ is a left action of $H = \mathrm{F}(\{x, y\})$ over the graph Γ_4. From properties (27-32), one can deduce that

$$\forall w \in \mathrm{I} \cap \mathrm{F}(x, y), |< w, 1 >|=| w | .$$

Hence

$$\mathrm{Stab}_H(\epsilon) = \{\epsilon\}. \tag{35}$$

One can also derive from (27-32) that

$$< y^*(yx)^*, \epsilon >=< H, \epsilon > \cap\ y^*(y(x + z))^*. \tag{36}$$

Let us choose now the particuliar map $\tilde{\sigma}$ defined by the finite 2-automaton shown on figure 6 (it is a kind of product of the 2-dfa exhibited in figure 4 with a 1-dfa recognizing I_y). We have left out of the figure all the additional transitions $\alpha\beta i \xrightarrow{(\gamma,\gamma)} \gamma\gamma 1$, for $\alpha, \beta, \gamma \in \hat{B}$ and $\gamma \in \hat{B} - \{\bar{\alpha}, \bar{\beta}\}$. The following formula then describes the set $< y^*(yx)^*, \epsilon >$:

$$\forall i, j \in \mathbb{N}, < y^i(yx)^j, \epsilon >=< y^i, (yz)^j >= y^i\theta(\ {}^t\mu_j(i - 1)), \tag{37}$$

where $\theta : \{0, 1\}^* \rightarrow \{x, y, z\}^*$ is the unique monoid homomorphism such that

$$\theta(0) = yx, \theta(1) = yz.$$

By (36,37) $H\epsilon$ is not rational (with respect to the obvious p-structure of Γ_4 obtained by considering $\mathrm{F}(B)$ as a pda with stack alphabet $\hat{B}$). But $H\epsilon = \Pi_H^{-1} \circ \Pi_H(\epsilon)$. Hence there is no p-structure over $H\backslash\Gamma_4$ making Π rational.

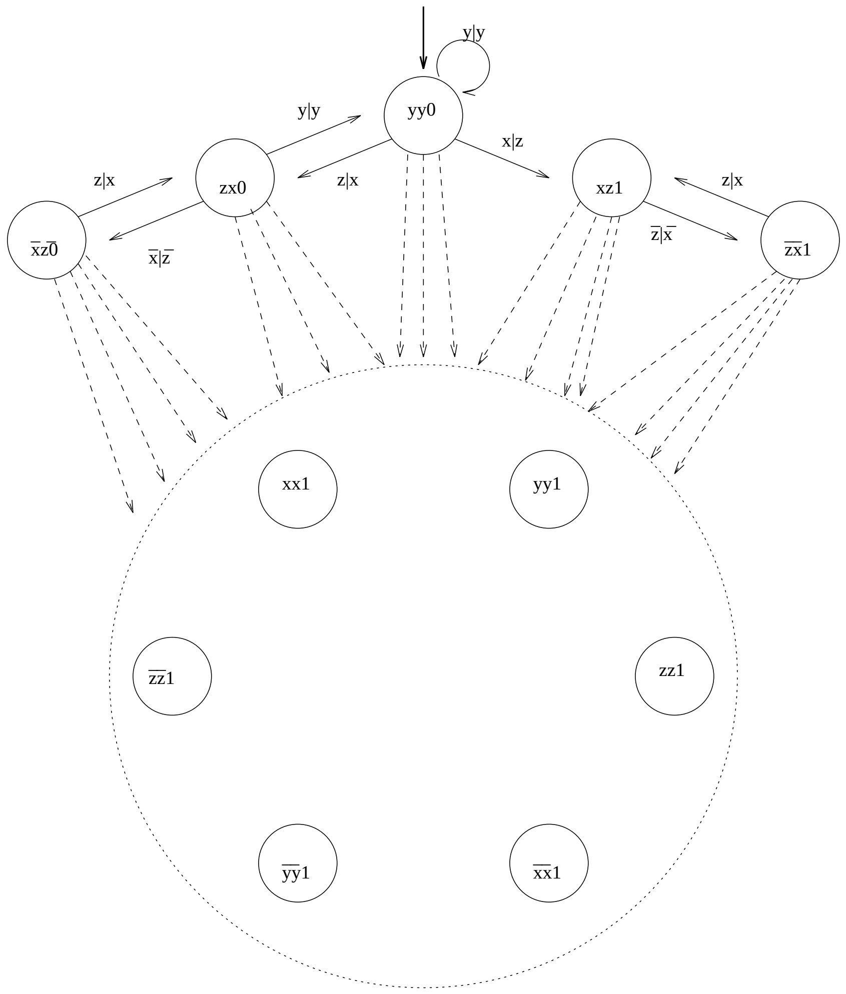

Fig. 6. The 2-dfa computing $\tilde{\sigma}$.

5 Discussion

One natural research perspective arising from this work is to treat the case of a group H acting on a connected graph Γ of bounded degree and finite tree-width. Considering the results of [BS83, MS83, Pel95], it seems natural to expect the group H to be virtually- free (provided the stabilizer of some vertex is finite). Considering our corollary 3.12 it seems natural to expect the graph $H\backslash\Gamma$ to have still finite tree-width (provided H is finitely generated).
Another natural question is whether, under the hypotheses of theorem 3.16, the canonical projection Π is rational (equivalently, whether $\sim_{\mathrm{Aut}(\Gamma)}$ is rational, by corollary 3.11)? Example 4 just shows that this property must be specific of the

full group $\mathrm{Aut}(\Gamma)$ and we expect this property to be true.

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