Iterated pushdown automata and sequences of rational numbers.

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Abstract. We introduce a link between automata of level k and tree-structures. This method leads to new decidability results about integer sequences. We also reduce some equality problems for sequences of rational numbers to the equivalence problem for deterministic automata of level k.

Keywords: Iterated pushdown automata; tree-structure; integer sequences; weak arithmetics; equivalence problems.
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1 Introduction

The class of pushdown automata of level k (for $k \geq 1$) has been introduced in [Gre70, Mas74] as a generalisation of the automata and grammars of [Aho68, Aho69, Fis68] and has been the object of many further studies: see [Mas76, Dam82, Eng83, ES84, EV86, DG86], and more recently [Cau02, KNU02].

We focus here on some links between these automata and, on one hand, some results in mathematical logics, on the other hand some new classes of sequences of numbers.

We show that the structure of the memory of any pushdown automata of level k with pushdown alphabet Γ, is logically definable inside the k-fold expansion of the finite structure Γ. This remark enables one to make use of the powerful generalisation of Rabin's tree-theorem ([Rab69]) over arbitrary tree-structures due to A. Muchnik ([MS92, Sem84, Wal96, BB03]). We thus re-obtain the known decidability properties of this class of automata and also obtain some new ones.

We focus then on a class of *integer* sequences recognised (in a suitable sense) by such automata (we denote by $\mathbb{S}_k$ the class of integer sequences recognised by deterministic pushdown automata of level k). This class enjoys nice closure

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properties and seems quite wide. Level 2 contains the classical *rational* sequences of integers (see [BR88]).
The decidability results obtained above lead to extensions of the well-known result of Büchi establishing the decidability of the Monadic Second-order Theory of $\langle \mathbb{N}, S \rangle$, the set of natural integers endowed with just the successor function S ([Büc60]).

Next, we consider the class $\mathcal{F}(S_k)$ consisting of all the sequences of *rational* numbers which can be decomposed as $\frac{a_n - b_n}{a'_n - b'_n}$ for sequences $a, b, a', b' \in \mathbb{S}_k$.
This class enjoys nice closure properties too and generalizes some well-known classes of recurrent sequences (or formal power series). The level 3, for example, contains all the so-called P-recurrent sequences of rational numbers, corresponding also to the D-finite formal power series (see [Sta80] for a survey). As a corollary of the above closure properties, the equality problem for two sequences in $\mathcal{F}(S_k)$ reduces to the equivalence problem for deterministic pushdown automata of level k. This establishes a bridge between the algorithmic problems about sequences (treated in [PWZ96], for example) and the decision problems about automata (treated in [Sén02], for example).

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2 Preliminaries

We introduce here some notation and basic definitions which will be used throughout the text.

2.1 Words

If A is a set, A^* denotes the set of words (finite sequences) over A, ε is the empty word and $A^+ = A^* \setminus \{\varepsilon\}$. The symbol a_ε will always denote an element of $A \cup \{\varepsilon\}$.

For a given word u we denote by $|u|$ the length of u.

For $n \geq 0$ we define $A^n = \{u \in A^*. \, |u| = n\}$, $A^{(n)} = \{u \in A^*. \, |u| \leq n\}$ and [n]={1,. . . ,n}.

2.2 Logics

Monadic Second Order Logic (MSOL) Let $Sig = \{r_1, \ldots\}$ be a signature containing relational symbols only and
$Var = \{x, y, z, \ldots, X, Y, Z \ldots\}$ be a set of variables, where $x, y, \ldots$ denote first order variables and $X, Y, \ldots$ second order variables.

The set of MSOL-formulas over Sig is the smallest set such that :

- $x \in X$ et $Y \subset X$ are MSOL-formulas for every $x, Y, X \in Var$
- $r(x_1, \ldots x_n, X_1 \ldots X_m)$ is a MSOL-formula for every $r \in Sig$, of arity (n, m) and every $x_1, \ldots x_n, X_1 \ldots X_m \in Var$
- if α, β are MSOL-formulas then $\neg\alpha$, $\alpha \vee \beta$, $\exists x.\alpha$ and $\exists X.\alpha$ are MSOL-formulas.

Let $M = \langle D_M, r_1, \ldots \rangle$ be a structure over the signature Sig, and val a function $val : Var \to D_M \cup \mathcal{P}(D_M)$ such that for every $x, X \in Var$, $val(x) \in D_M$ and $val(X) \subset D_M$.

The satisfiability of a MSOL-formula in a structure M with valuation val is defined by induction on the structure of the formula :

- $M, val \models x \in X$ iff $val(x) \in val(X)$
- $M, val \models Y \subset X$ iff $val(Y) \subset val(X)$
- $M, val \models r(x_1, \ldots, X_1, \ldots)$ iff $r(val(x_1), \ldots, val(X_1), \ldots)$
- $M, val \models \neg\alpha$ iff $M, val \models \alpha$ is false
- $M, val \models \alpha \vee \beta$ iff $M, val \models \alpha$ or $M, val \models \beta$
- $M, val \models \exists x.\alpha$ ssi there exists d in D_M such that $M, val[x \leftarrow d] \models \alpha$ where $val[x \leftarrow d]$ denotes the valuation satisfying : $val[x \leftarrow d](y) = val(y)$ if $y \neq x$ and $val[x \leftarrow d](x) = d$
- $M, val \models \exists X.\alpha$ iff there exists $S \subset D_M$ such that $M, val[Y \leftarrow S] \models \alpha$

Substructures We consider $M = \langle D_M, r_1, \ldots \rangle$ a structure containing relational symbols only and L the set of MSOL-formulas over M

Definition 1. *Let $D(x)$ be an unary formula in L.*

If F is in L, the MSOL-formula F^D obtained from F by relativizing quantifiers of F to D, is defined inductively as follows :

- *if F is quantifier-free, then $F^D = F$*
- *if $F = E \vee G$, then $F^D = E^D \vee G^D$*
- *if $F = \neg E$, then $F^D = \neg E^D$*
- *if $F = \exists x.G$, then $F^D = \exists x.\ (D(x) \wedge G^D)$*
- *if $F = \exists X.G$, then $F^D = \exists X.\ [(\forall x.\ (x \in X \Rightarrow D(x)) \wedge G^D]$*

We give now the definition of a substructure *induced by formulas*. For sake of simplicity of the notation, let us restrict ourselves to an unary formula $D(x) \in L$ and a binary formula $E(u, v) \in L$

Definition 2. *The structure* $M(D,E) = \langle C, R\rangle$ *induced by* $D(x)$ *and* $E(u,v)$ *has the domain* $C = \{a.\ a \in D_M,\ M \models D(a)\}$ *and the binary relation* $R = \{(a,b).\ a,b \in C,\ M \models E(a,b)\}$.

Let now $F(x_1,\dots,x_n)$ be a formula in a language with binary predicate symbol P and define $F^{D,E}$ be the formula obtained by first forming F^D then replacing in F^D all atomics formulas $P(z_1,z_2)$ by $E(z_1,z_2)$. Note that the quantifiers in $E(u,v)$ are not relativized to D.

Theorem 1 (Rabin [Bar]).
For $c_1,\dots,c_n \in C$,

$$M(D,E) \models F(c_1,\dots,c_n) \text{ iff } M \models F^{D,E}(c_1,\dots,c_n)$$

In particular, if the MSO-theory of M is decidable then the MSO-theory of $M(D,E)$ is decidable too.

2.3 Automata

The level-k languages have been first introduced by Maslov [Mas74] by means of an extension of indexed grammars [Aho68] at level k. Indexed languages defined by Aho admit a characterization by automata : the *nested stack* automata [Aho69]. Maslov gives also a characterization of level-k languages by means of *multilevel stack* automata [Mas76]. Since this time, many other grammars and automata have been devised to define this class of languages. One can cite “level-k grammars” studied by Engelfriet [ES84] and Damm [Dam82] which are an extension of “macro grammars “ of Fischer [Fis68] and “iterated-pushdown automata” studied by Damm and Goerdt [DG86], by Engelfriet [Eng83] and more recently by Knapik, Niwinski and Urzyczyn [KNU02].

General definitions Here we shall use the definition of [DG86] and stick to their notation.
Iterated pushdown-automata are an extension of classical pushdown-automata to a storage structure built iteratively. This storage structure - originally defined by Maslov [Mas76]- can be described as :

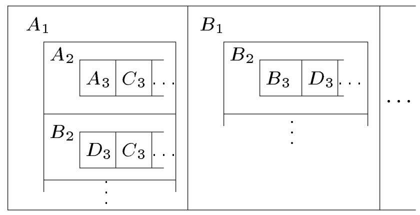
- a 1-iterated pushdown store consists of a classical pushdown-list of symbols
- a $(k+1)$-iterated pushdown store consists of a pushdown-list of pairs (pushdown-symbol, k-iterated pushdown store).

Fig 2.3 gives a typical $3-\mathrm{pds}$ (3-iterated pushdown store).
Let us give now a mathematical definition of $k-\mathrm{pds}$.

Definition 3 (k-iterated pushdown store). *Let Γ be a set of pushdown symbols. We define by induction on k the set of* k-iterated pushdown-stores over *Γ,*

$0\text{-pds}(\Gamma) = \{\varepsilon\} \quad (k+1)\text{-pds}(\Gamma) = (\Gamma[k\text{-pds}(\Gamma)])^*$.
We denote by $\mathrm{it}-\mathrm{pds}(\Gamma)$ *(iterated pushdown store) the union of all* $k-\mathrm{pds}(\Gamma)$.

Fig. 1. a $3 - \text{pds}$: *ex-pds*



Remark 1.
1- The empty word,ε, belongs to every set $k - \text{pds}(\Gamma)$.
2- It follows that, for every symbol $A \in \Gamma$, $A[\varepsilon]$ belongs to $(k+1) - \text{pds}(\Gamma)$.
3- More generally,
$0 - \text{pds}(\Gamma) \subseteq 1 - \text{pds}(\Gamma) \subseteq \ldots k - \text{pds}(\Gamma) \subseteq (k+1) - \text{pds}(\Gamma) \subseteq \ldots$.
4- In the rest of the paper we will often denote by $\cdots AB \cdots$ what should be denoted by $\cdots A[\varepsilon]B \cdots$ (where A, B are letters from Γ). More precisely: inside a word denoting a $k - \text{pds}$, every letter $A \in \Gamma$ followed by a symbol other than "[", means the pds $A[\varepsilon]$.
5- For every $k \in \mathbb{N}$, the set $k - \text{pds}(\Gamma)$ endowed woth the concatenation operation is a monoid.

Definition 4. *Let* $\omega \in k - \text{pds}(\Gamma)$*. We say that* ω *is* atomic *iff, for every* $\omega_1, \omega_2 \in k - \text{pds}(\Gamma)$*,* $\omega = \omega_1 \cdot \omega_2 \Rightarrow (\omega_1 = \varepsilon$ *or* $\omega_2 = \varepsilon)$*.*
In that case we also say that ω *is an atom.*

One can check that $(k - \text{pds}(\Gamma), \cdot)$ is a free monoid. The set $\{\omega \in k - \text{pds}(\Gamma) \mid \omega is$ atomic $\}$ is thus the base of this free monoid. Notice that, if ω is an atomic k-pds, it is also an atomic k'-pds for every $k' \geq k$. Every nonempty $\omega \in (k+1) - \text{pds}(\Gamma)$ has a unique decomposition as:

$$\omega = A[flag] \cdot rest$$

with $A \in \Gamma$, $flag \in k - \text{pds}(\Gamma)$, and $rest \in (k+1) - \text{pds}(\Gamma)$.

Example 1 *Let us denote by ex-pds the* $3 - \text{pds}$ *given in fig 2.3 without the dots. According to definition 3, we should write:*

$$ex - pds = A_1[A_2[A_3[\varepsilon]C_3[\varepsilon]]B_2[D_3[\varepsilon]C_3[\varepsilon]]]B_1[B_2[B_3[\varepsilon]D_3[\varepsilon]]].$$

According to remark 1,point 4, we shall (abusively) write:

$$ex - pds = A_1[A_2[A_3C_3]B_2[D_3C_3]]B_1[B_2[B_3D_3]].$$

The decomposition of ex-pds as a product of atoms is:

$$ex - pds = A_1[A_2[A_3C_3]B_2[D_3C_3]] \cdot B_1[B_2[B_3D_3]].$$

We now formalize operations allowed on the store.

Definition 5 (The reading operation).
$topsym : \mathrm{it}-\mathrm{pds}(\Gamma) \to \Gamma^*$.

- $topsyms(\varepsilon) = \varepsilon$,
- $topsyms(A[flag]rest) = A.topsyms(flag)$.

Example 2 *The reading operation :*
$topsyms(ex\text{-}pds) = A_1A_2A_3$

Definition 6 (The *pop* operation at level j).
$pop_j : \mathrm{it}\text{-}pds(\Gamma) \to \mathrm{it}\text{-}\mathrm{pds}(\Gamma)$.

- $pop_j(\varepsilon)$ *is undefined,*
- $pop_1(A[flag]rest) = rest$,
- $pop_{j+1}(A[flag]rest) = A[pop_j(flag)]rest$.

Example 3 *The pop operation :*
$pop_1(ex\text{-}pds) = B_1[B_2[B_3D_3]]$
$pop_2(ex\text{-}pds) = A_1[B_2[D_3C_3]]B_1[B_2[B_3D_3]]$
$pop_3(ex\text{-}pds) = A_1[A_2[C_3]B_2[D_3C_3]]B_1[B_2[B_3D_3]]$

Definition 7 (The *push* operation at level j). *For* $\alpha = A_1 \dots A_r \in \Gamma^+$, $push_j(\alpha) : \mathrm{it}\text{-}\mathrm{pds}(\Gamma) \to \mathrm{it}\text{-}\mathrm{pds}(\Gamma)$.

- $push_1(\alpha)(\varepsilon) = \alpha$,
- $push_{j+1}(\alpha)(\varepsilon)$ *is undefined for* $j \geq 1$,
- $push_1(\alpha)(A[flag]rest) = A_1[flag] \dots A_r[flag]rest$,
- $push_{j+1}(\alpha)(A[flag]rest) = A[push_j(\alpha)(flag)]rest$

Example 4 *The push operation :*
$push_2(AB)(ex\text{-}pds) = A_1[A[A_3C_3]B[A_3C_3]B_2[D_3C_3]]B_1[B_2[B_3D_3]]$
$push_3(AB)(ex\text{-}pds) = A_1[A_2[ABC_3]B_2[D_3C_3]]B_1[B_2[B_3D_3]]$

Definition 8 (Syntax of k-*pdas*). *Let* $k \geq 1$ *:*

1. *Let* $POP = \{pop_j | j \in [k]\}$, $PUSH(\Gamma) = \{push_j(\alpha) | \alpha \in \Gamma^+, j \in [k]\}$, *and* $TOPSYMS(\Gamma) = \Gamma^{(k)} \backslash \{\varepsilon\}$.
2. *A k-iterated pushdown automaton over a terminal alphabet* Σ *is a 6-tuple* $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, Z)$ *with*
   - Q *is a finite set of states,* $q_0 \in Q$ *denoting the initial symbol,*
   - Γ *is a finite set of pushdown-symbols with* $Z \in \Gamma$ *as initial symbol,*
   - *the transition function* δ *is a map from* $Q \times (\Sigma \cup \{\varepsilon\}) \times TOPSYMS(\Gamma)$ *into the set of finite subsets of* $Q \times (PUSH(\Gamma) \cup POP)$ *such that : if* $(q, push_j(\alpha)) \in \delta(p, a_\varepsilon, topsyms)$ *then* $j \leq |topsyms|+1$ *and if* $(q, pop_j) \in \delta(p, a_\varepsilon, topsyms)$ *then* $j \leq |topsyms|$. *These conditions avoid cases where operations are undefined.*
3. *The class of k-pdas over* Σ *will be denoted by* $k\text{-}PDA(\Sigma)$.

Definition 9 (Semantics of k-pdas). *Let $\mathcal{A} \in k\text{-}PDA(\Sigma)$ as above :*

1. *The set of* configurations *of $\mathcal{A}$ is $Con_{\mathcal{A}} = Q \times \Sigma^* \times k\text{-pds}(\Gamma)$.*
2. *The* single step relation $\vdash_{\mathcal{A}} \subseteq Con_{\mathcal{A}} \times Con_{\mathcal{A}}$ *of $\mathcal{A}$ is defined by* $(p, w, \omega) \vdash_{\mathcal{A}} (q, v, \omega')$ *iff* $(q, f) \in \delta(p, a_\varepsilon, topsyms(\omega))$,$a_\varepsilon v = w$ *and* $\omega' = f(\omega)$.
3. *We denote by $\vdash_{\mathcal{A}}^*$ the reflexive, transitive closure of $\vdash_A$*
4. *The* language accepted by $\mathcal{A}$ *(with empty store) is defined by* $L(\mathcal{A}) = \{w \in \Sigma^* | \exists q \in Q, (q_0, w, Z) \vdash_{\mathcal{A}}^* (q, \varepsilon, \varepsilon)\}$.
5. *The class of languages accepted by k-pdas over Σ is denoted by $k\text{-}\mathcal{PDA}(\Sigma)$.*

The usual techniques for pushdown-automata can be used to prove that acceptance by empty store and acceptance by (final states and empty store) define the same class of languages.

Example 5 *The following 2-pda $\mathcal{A}$ fulfills : $L(\mathcal{A}) = \{a^{f(n)} | n \geq 0\}$, where f denotes the Fibonacci's sequence.*

$\mathcal{A} = (\{q_0, q_1, q_2\}, \{a\}, \{Z, X_1, X_2, F\}, \delta, q_0, Z)$ *with*

$\delta(q_0, \varepsilon, Z) = \{(q_0, push_2(F)),\ (q_0, push_1(X_2))\}$,
$\delta(q_0, \varepsilon, ZF) = \{(q_0, push_2(FF)),\ (q_0, push_1(X_2))\}$,
$\delta(q_0, \varepsilon, X_1F) = \{(q_1, pop_2)\}$, $\delta(q_0, \varepsilon, X_2F) = \{(q_2, pop_2)\}$,
$\delta(q_0, a, X_1) = \{(q_0, pop_1)\}$, $\delta(q_0, a, X_2) = \{(q_0, pop_1)\}$,
$\delta(q_1, \varepsilon, X_1F) = \{(q_0, push_1(X_1X_2))\}$, $\delta(q_2, \varepsilon, X_2F) = \{(q_0, push_1(X_1)\}$,
$\delta(q_1, \varepsilon, X_1) = \{(q_0, push_1(X_1X_2))\}$, $\delta(q_2, \varepsilon, X_2) = \{(q_0, push_1(X_1)\}$.

We give an accepting configurations sequence for $a^{f(3)} = a^3$:

$(q_0, a^3, Z[\varepsilon]) \vdash (q_0, a^3, Z[F]) \vdash (q_0, a^3, Z[FF]) \vdash (q_0, a^3, Z[FFF]) \vdash$
$(q_0, a^3, X_2[FFF]) \vdash (q_2, a^3, X_2[FF]) \vdash (q_0, a^3, X_1[FF]) \vdash (q_1, a^3, X_1[F]) \vdash (q_0, a^3, X_1[F]X_2[F]) \vdash (q_1, a^3, X_1[\varepsilon]X_2[F]) \vdash (q_0, a^3, X_1[\varepsilon]X_2[\varepsilon]X_2[F]) \vdash$
$(q_0, a^2, X_2[\varepsilon]X_2[F]) \vdash (q_0, a, X_2[F]) \vdash (q_2, a, X_2[\varepsilon]) \vdash (q_0, a, X_1[\varepsilon]) \vdash (q_0, \varepsilon, \varepsilon)$

Some basic tools Let $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, Z)$ be some k-dpda. A *total state* of $\mathcal{A}$ is any pair $(q, \omega) \in Q \times k-\text{pds}(\Gamma)$. A *mode* is a pair $(q, \omega) \in Q \times (\Gamma^{(k)} \cup \{\varepsilon\})$. Given a configuration $c = (q, u, \omega) \in Con_{\mathcal{A}}$, the total-state of c is (q, ω) and the mode of (q, ω) (and of c, as well) is $(q, topsyms(\omega))$.

Derivation We associate with $\mathcal{A}$ an infinite "alphabet"

$$V_{\mathcal{A}} = \{(p, \omega, q) \mid p, q \in Q, \omega \in k-\text{pds}(\Gamma) - \{\varepsilon\}\}.$$

The set of *productions* associated with $\mathcal{A}$, denoted by $P_{\mathcal{A}}$ is made of the set of all the following rules:
the *transition* rules:

$$(p\omega q)\to_{\mathcal{A}} a_\varepsilon(p'\omega' q)$$

if $(p, a_\varepsilon, \omega)\vdash_{\mathcal{A}} (p', \varepsilon, \omega')$ and $q \in Q$ is arbitrary,

$$(p\omega q)\to_{\mathcal{A}} a_\varepsilon$$

if $(p, a_\varepsilon, \omega)\vdash_{\mathcal{A}} (q, \varepsilon, \varepsilon)$.
the *decomposition* rule:

$$(p\omega q)\to_{\mathcal{A}} (p\eta r)(r\eta' q)$$

if $\omega = \eta \cdot \eta', \eta \neq \varepsilon, \eta' \neq \varepsilon$ and $r \in Q$ is arbitrary. The one-step *derivation* generated by $\mathcal{A}$, denoted by $\to_{\mathcal{A}}$, is the smallest subset of $(V \cup \Sigma)^* \times (V \cup \Sigma)^*$ which contains $P_{\mathcal{A}}$ and is compatible with left-product and right-product. Finally, the *derivation* generated by $\mathcal{A}$, denoted by $\to^*_{\mathcal{A}}$, is the reflexive and transitive closure of $\to_{\mathcal{A}}$. These notions correspond to the usual notion of *context-free grammar* associated with the pushdown automaton $\mathcal{A}_1$: this automaton has the pushdown alphabet $\Gamma_1 = \{A[\omega] \mid A \in \Gamma, \omega \in (k-1) - \mathrm{pds}(\Gamma)\}$ and has the transition function

$$\delta_1(q, a_\varepsilon, A[\omega]) = \{(q', \eta') \in Q \times \Gamma_1 \mid (q, a_\varepsilon, A[\omega])\vdash_{\mathcal{A}} (q', \eta')\}.$$

Of course, as soon as $k \geq 2$, this pushdown alphabet is infinite, but all the usual properties of the relation $\to_{\mathcal{A}} \ = \to_{\mathcal{A}_1}$ and its links with $\vdash_{\mathcal{A}} \ = \vdash_{\mathcal{A}_1}$ remain true in this context. In particular, for every $u \in \Sigma^*, p, q \in Q, \omega \in \Gamma_1$

$$(p, \omega, q)\to^*_{\mathcal{A}} u \Leftrightarrow (p, u, \omega)\vdash^*_{\mathcal{A}} (q, \varepsilon, \varepsilon).$$

The following lemma is useful

Lemma 1. *Let $p_i, q_i \in Q, \omega_i \in \Gamma_1$ for $i \in 1, 2, 3$. The following properties are equivalent:*
(1) $(p_1, \omega_1, q_1)\to^*_{\mathcal{A}} (p_2, \omega_2, q_2)(p_3, \omega_3, q_3)$
(2) There exist $\omega', \omega'_2, \omega'', \omega''_3 \in \Gamma_1$, such that:
$(p_1, \varepsilon, \omega_1)\vdash^*_{\mathcal{A}} (p_2, \varepsilon, \omega')$; $\omega' = \omega_2\omega'_2$; $(q_2, \varepsilon, \omega'_2)\vdash^*_{\mathcal{A}} (p_3, \varepsilon, \omega'')$; $\omega'' = \omega_3\omega''_3$; $(q_3, \varepsilon, \omega''_3)\vdash^*_{\mathcal{A}} (q_1, \varepsilon, \varepsilon)$.

We usually assume that Γ and Q are disjoint, therefore, omitting the commas in (p, ω, q) does not lead to any confusion.

Determinism The automaton $\mathcal{A}$ is said *deterministic* iff, for every $q \in Q, \omega \in \Gamma^{(k)}, \sigma \in \Sigma$

$$\mathrm{Card}(\delta(q, \epsilon, \omega)) \leq 1 \text{ and } \mathrm{Card}(\delta(q, \sigma, \omega)) \leq 1. \tag{1}$$

$$\mathrm{Card}(\delta(q, \epsilon, \omega)) = 1 \Rightarrow \mathrm{Card}(\delta(q, \sigma, \omega)) = 0, \tag{2}$$

Undeterminates We often introduce some letters $\Omega, \Omega', \Omega''$ which do not belong to Γ. One can extend the relations $\rightarrow^*_{\mathcal{A}}, \vdash^*_{\mathcal{A}}$ to the pushdown alphabet $\Gamma \cup \{\Omega, \Omega', \Omega''\}$. These new symbols are called *undeterminates*. We often denote by $\alpha[\Omega]$ some element of $\mathrm{pds}(\Gamma \cup \{\Omega\})$ where Γ is a"leaf" i.e. every occurrence of Ω in α is followed by $[\varepsilon]$ (if we use the rigorous notation). We denote by $\alpha[\omega]$ the pds obtained by substituting ω to Ω in $\alpha[\Omega]$. One can notice that, if

$$(p\alpha[\Omega]q) \rightarrow^*_{\mathcal{A}} (r\beta[\Omega]s)$$

then , for every $\omega \in k - \mathrm{pds}(\Gamma \cup \{\Omega\})$,

$$(p\alpha[\omega]q) \rightarrow^*_{\mathcal{A}} (r\beta[\omega]s).$$

This "substitution-principle" will be widely used in our proofs.

Normalized automata We say that $\mathcal{A}$ is *level-partitioned* iff Γ is the disjoint union of subsets $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ such that, in every transition of $\mathcal{A}$, every occurrence of a letter from Γ_i is at level i. It is easy to transforme any k-pushdown automaton into another one wich recognizes the same language and is *level-partitioned*.

2.4 Sequences

Let $(\mathbb{Q}, +, \cdot)$ be the field of rational numbers. A *sequence* of rational numbers is any map $u : \mathbb{N} \rightarrow \mathbb{Q}$. We denote by $u(n)$ (sometimes also by u_n) the image of the integer n by the map u. Such a sequence u can be also viewed as a formal power series

$$u(X) = \sum_{n=0}^{\infty} u_n X^n.$$

The following operators on series are classical:
E: the *shift* operator

$$(\mathsf{E}u)(n) = u(n+1); (\mathsf{E}u)(X) = \frac{u(X) - u(0)}{X}$$

Δ: the difference operator

$$(\Delta u)(n) = u(n+1) - u(n); (\Delta u)(X) = \frac{u(X)(1-X) - u(0)}{X}$$

Σ: the summation operator

$$(\Sigma u)(n) = \sum_{j=0}^{n} u(j); (\Sigma u)(X) = \frac{u(X)}{1-X}$$

$+$: the sum operator

$$(u+v)(n) = u(n) + v(n); (u+v)(X) = u(X) + v(X)$$

$\cdot$: the external product, for every $r \in \mathbb{Q}$

$$(r \cdot u)(n) = r \cdot u(n);$$

$\odot$: the Hadamard product, (also called the ordinary product)

$$(u \odot v)(n) = u(n) \cdot v(n);$$

$\times$: the convolution product

$$(u \times v)(n) = \sum_{k=0}^{n} u(k) \cdot v(n-k); (u \times v)(X) = u(X) \cdot v(X)$$

$\circ$: the sequence composition

$$(u \circ v)(n) = u(v(n));$$

$\bullet$: the series composition

$$(u \bullet v)(X) = \sum_{n=0}^{\infty} u(n) \cdot v(X)^n$$

(this last operation is defined as soon as $v(0) = 0$, which ensures that the family of series $(u(n) \cdot v(X)^n)_{n \geq 0}$ is summable).
The set of all sequences of rational numbers is also denoted by $\mathbb{Q}[[X]]$. The structure $(\mathbb{Q}[[X]], +, \times)$ is a ring, with unit $\mathbb{1} = (1, 0, \ldots, 0, \ldots)$. Given some number $r \in \mathbb{Q}$, we use also the same notation r for the sequence (or series):

$$r = r \cdot \mathbb{1} = (r, 0, \ldots, 0, \ldots)$$

while we use the notation $\frac{r}{1-X}$ for the constant sequence (or series):

$$\frac{r}{1-X} = (r, r, \ldots, r, \ldots).$$

3 Tree-structures, words and pushdowns

We construct here a connection between the notion of k-iterated pushdown store (recalled in definition 3) and the structure $\Gamma^{<k>}$ obtained from the alphabet Γ by iterating k times the *tree-structure* operation. This connection allows to obtain a general decidability result for the computation-graphs of k-pushdown automata (theorem 4). This prepares the ground for section 6 where we define a wide class of unary predicates P for which the structure $\langle \mathbb{N}, S, P \rangle$ admits a decidable second order monadic theory (theorem 26).

3.1 Tree-structures

Definition 10 (Tree-structure). *Let $Sig = (r_1, \ldots)$ be a signature containing only relational symbols. For a structure $M = \langle D_M, r_1, \ldots\rangle$ over the signature Sig one constructs the tree-structure $M^* = \langle D_M^*, son, clone, r_1^*, \ldots\rangle$ over the extended signature $Sig^* = Sig \cup \{son, clone\}$, where D_M^* is the set of all finite sequences of elements of D_M and the relations are defined by:*

$$son = \{(w, dw) : \quad w \in D_M^*, \quad d \in D_M\}$$

$$clone = \{ddw : \quad w \in D_M^*, \quad d \in D_M\}$$

$$r^* = \{(d_1 w, \ldots, d_k w) : \quad w \in D_M^*, \quad (d_1, \ldots, d_k) \in r_M\},$$

($r \in Sig$, of arity k).

The following theorem, known as Muchnik's theorem, appeared in a weaker form in [She75, Stu75] and was stated in [Sem84, MS92]. Complete written proofs have been given in [Wal96, BB03].

Theorem 2. *For every MSOL sentence φ over the signature Sig^* one can effectively find a MSOL sentence $\widehat{\varphi}$ over the signature Sig such that, for every structure M:*

$$M \models \widehat{\varphi} \quad \textit{iff} \quad M^* \models \varphi$$

A direct corollary follows this theorem :

Corollary 1. *The* MSO*-theory of M is decidable iff the* MSO*-theory of M^* is decidable.*

By iterating the tree-structure we obtain some structures defined over some words of words of word ...

For instance, if $M = \langle\{a, b\}, r_a, r_b\rangle$ is the complete binary tree,
then $M^* = \langle\{a, b\}^*, r_a^*, r_b^*, son, clone\rangle$,
and $(M^*)^* = \langle(\{a, b\}^*)^*, (r_a^*)^*, (r_b^*)^*, son^*, clone^*, son, clone\rangle$, where $(\{a, b\}^*)^*$ denotes the set of words of words over $\{a, b\}$, *i.e* the set of finite sequences of elements of $\{a, b\}^*$.

In order to formalize structures obtained by this process, let us first define the type of their domain.

Definition 11 (Star iteration). *Let Γ be a finite set of symbols. For all $k \geq 0$, one denotes by $\Gamma^{<k>}$ the set defined inductively by :*

$$\Gamma^{<0>} = \Gamma$$
$$\Gamma^{<k>} = (\Gamma^{<k-1>})^*, \ k > 0.$$

In other words, $\Gamma^{<k+1>}$ denotes the set of finite sequences over $\Gamma^{<k>}$.

Now, one can define the k-iteration of the tree-structure of some structure M:

Definition 12 (k-iteration of the tree-structure). *Let M be a structure containing only relational symbols, the k-iteration of the tree-structure of M, $k \geq 0$ is the structure M_k defined by :*

$$M_0 = M \quad \text{defined over the signature } Sig_0$$
$$M_k = M_{k-1}^*,\ k > 0,\ \text{defined over the signature } Sig_k = Sig_{k-1} \cup \{clone, son\}$$

Then, for all $k \geq 0$, the MSOL-theory of M_k is decidable if and only if the MSOL-theory of M is decidable, and :

$$M_0 = \langle D_M, (r_i^0)_{i\in I}\rangle$$
$$M_1 = \langle D_M^{<1>}, (r_i^1)_{i\in I}, son_{1,0}, clone_{1,0}\rangle$$
$$M_k = \langle D_M^{<k>}, (r_i^k)_{i\in I}, (son_{k,i}, clone_{k,i})_{0\leq i<k}\rangle$$

where:
$r^0 = r = \{(d_1, ..d_n),\ d_1, \ldots, d_n \in D_M \mid r(d_1, ..d_k)\}$
$r^k = (r^{k-1})^*$

and $son_{k,i}$ and $clone_{k,i}$ are defined inductively by:
$son_{k,0} = \{(w, dw) : d \in D_M^{(k-1)},\ w \in D_M^{(k)}\}$
$son_{k+1,i+1} = (son_{k,i})^*$

$clone_{k,0} = \{ddw : d \in D_M^{(k-1)},\ w \in D_M^{(k)}\}$
$clone_{k+1,i+1} = (clone_{k,i})^*$.
We will often abbreviate $son_{k,0}$ by son_k or son if the level is understood (idem for $clone_{k,0}$).

In order to express the computation of a k-*pda* by MSOL-formulas, we need to connect k-*pds*(Γ) with the set $\Gamma^{<k>}$.

We shall do so by mapping injectively the set k-*pds*(Γ) into $\Gamma^{<k>}$. Notice that definition of $\Gamma^{<k>}$ allows the *letters* (*i.e.* element of $\Gamma^{<k-1>}$) that constitutes a word to be the empty word. The image of our mapping will consist of the elements of $\Gamma^{<k>}$ that do not use such empty components.

Let us introduce the notion of *level-k words*

Definition 13 (Level-k words).
For all $k \geq 0$, the set $\Gamma^{[k]}$ of level-k words is defined as follows :

- $\Gamma^{[0]} = \Gamma$
- $\Gamma^{[1]} = \{[a_1, \ldots, a_n] \mid a_1, \ldots a_n \in \Gamma,\ n \geq 0\}$, *and we denote* $\varepsilon_1 = [\,]$
- *if* $k > 0$, $\Gamma^{[k+1]} = \{[u_1, \ldots, u_m] \mid u_1, \ldots, u_m \in \Gamma^{[k]} \setminus \{\varepsilon_k\},\ m \geq 0\}$, *and* $\varepsilon_{k+1} = [\,]$

Remark 2.
1- The level of a non empty word is given by the deepness of brackets appearing in the word.
2- There is only one empty word: $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_k$.

Definition 14 (Concatenation). *For all $k > 0$, we define the concatenation of two level-k words as follows :*

$$\begin{aligned} \bullet_k : \Gamma^{[k]} \times \Gamma^{[k]} &\to \Gamma^{[k]} \\ ([u_1, \ldots, u_n], [v_1, \ldots, v_m]) &\mapsto [u_1, \ldots, u_n, v_1, \ldots, v_m],\ n, m \geq 0 \end{aligned}$$

If the level is understood, we denote by $\bullet$ the concatenation.

Example 6 *Here is a level-3 word : $u = [[[a, b, c, d], [g, e]], [[a]]]$ composed of two level-2 words : $u_1 = [[a, b, c, d], [g, e]]$, $u_2 = [[a]]$*

Notice that ε_3 is a level-3 word, but $[[[a, b, c], [g, e]], \varepsilon_2, [[a]]]$ is not a level-3 word.
Then, we have for all alphabet Γ, for all $k \geq 0$, $\Gamma^{[k]} \subseteq \Gamma^{<k>}$, (and for $k \geq 2$, $\Gamma^{[k]} \subsetneq \Gamma^{<k>}$).

We can now define a monoïd isomorphism φ_k, $k \geq 1$ between k-$pds(\Gamma)$ and $\Gamma^{[k]}$:

$$\begin{aligned} \varphi_1 : (\text{1-pds}(\Gamma), .) &\to (\Gamma^{[1]}, \bullet_1) \\ A[\varepsilon] &\mapsto [A] \\ \varepsilon &\mapsto [\,] = \varepsilon_1 \end{aligned}$$

and for $k \geq 1$

$$\begin{aligned} \varphi_{k+1} : ((k+1)\text{-pds}(\Gamma), .) &\to (\Gamma^{[k+1]}, \bullet_{k+1}) \\ A[flag] &\mapsto [insert_A(\varphi_k(flag))] \\ \varepsilon &\mapsto \varepsilon_k \end{aligned}$$

where for all $k \geq 1$, $insert_a : \Gamma^{[k]} \mapsto \Gamma^{[k]}$ is defined inductively by :

- if $u \in \Gamma^{[1]}$, $insert_a(u) = [a] \bullet_1 u$
- if $u \in \Gamma^{[k+1]}, k \geq 1$,
  - if $u = \varepsilon_{k+1}$, $insert_a(u) = [insert_a(\varepsilon_k)]$
  - if $u = [u_1] \bullet_{k+1} v$, $insert_a(u) = [insert_a(u_1)] \bullet_{k+1} v$

In other words, the $insert_a(u)$ operation adds the letter a at the beginning of the first word of level 1 appearing in the word u.

Example 7 *The bijection φ.*

1. *Let u be the level-3 word given in the example above, then* $\varphi_3^{-1}(u) = a[b[cd]\,g[e]]a[\varepsilon]$
2. $\varphi_3(\textit{ex-pds}) = [[[A_1, A_2, A_3, C_3], [B_2, D_3, C_3]], [[B_1, B_2, B_3, D_3]]]$

Now, in order to work with a structure with domain $A^{[k]}$, we have to consider substructures of the structure M_k.

3.2 The structure $\mathcal{M}_k(A)$.

We define here a structure having a decidable monadic theory for which the domain is exactly the set $A^{[k]}$.

We define the structure $\mathcal{M}_k(\Gamma)$, $k \geq 1$ in the following way :
we consider first the structure $\mathcal{M}_1(\Gamma)$ defined by
$\mathcal{M}_1(\Gamma) = \langle \Gamma^*, \, son, \, clone, \, (s_a^1)_{a \in \Gamma} \rangle$, with $s_a^1 = \{(u, \, au)\}$.

This structure admits a decidable monadic second order theory.
Then, we construct the structure $\mathcal{M}_2(\Gamma)$, which is the structure $(\mathcal{M}_1(\Gamma))^*$ restricted to the domain $\Gamma^{[2]}$, then we iterate this process.

For that purpose, we define inductively the structure $\mathcal{M}_k(\Gamma)$ and the MSOL-formulas Σ_k and $\widetilde{\Sigma}_{k+1}$ satisfying :

$$\mathcal{M}_k(\Gamma) \models \Sigma_k(u) \text{ iff } u = \varepsilon_k \text{ and } (\mathcal{M}_k(\Gamma))^* \models \widetilde{\Sigma}_{k+1}(u) \text{ iff } u \in \Gamma^{[k+1]}.$$

Let us set for all $k \geq 1$,

$\Sigma_k(x) == \neg(\exists y. \, son(y, x))$ defined over the structure $\mathcal{M}_k(\Gamma)$,
$\widehat{\Sigma}_{k+1}(X) == \forall x \in X. \, \exists y. \, [son(y, x) \Rightarrow \neg(\Sigma_k)^* \wedge y \in X]$,
$\widehat{\widehat{\Sigma}}_{k+1}(X) == \widehat{\Sigma}_{k+1}(X) \wedge \forall Y. \, [\widehat{\Sigma}_{k+1}(Y) \Rightarrow Y \subset X]$,
$\widetilde{\Sigma}_{k+1}(x) == \exists X. \, [(x \in X \wedge \widehat{\widehat{\Sigma}}_{k+1}(X)]$, and

$\mathcal{M}_{k+1}(\Gamma) = (\mathcal{M}_k(\Gamma))^*(\widetilde{\Sigma}_{k+1})$, substructure of $(\mathcal{M}_k(\Gamma))^*$ induced by $\widetilde{\Sigma}_{k+1}$.

According to Rabin's theorem 1, the monadic second order theory of $\mathcal{M}_k(\Gamma)$ is decidable.

3.3 Pushdowns expressed inside $\mathcal{M}_k(A)$

We express here the operations allowed on k-$pds(A)$ by MSOL-formulas over the structure $\mathcal{M}_k$.

First, we construct the $\Phi_\omega^k(u)$ formula, representing the *topsyms* function.

Lemma 2. *For all $\omega \in \Gamma^+$, $0 \leq |\omega| \leq k$, one can find a MSOL-formula Φ_ω^k satisfying for all $u \in \Gamma^{[k]}$:*

$$M_k(\Gamma) \models \Phi_\omega^k(u) \textit{ iff } \omega = topsyms(\varphi_k^{-1}(u))$$

Construction:

- if $k = 1$, then $\omega = A : \Phi^1_A(u) == \exists y|\ s^1_A(y, u)$
- $k \geq 1$
  - if $\omega = A : \Phi^{k+1}_A(u) == (\Phi^k_A)^*(u) \wedge [\exists y|\ son_{k+1,1}(y, u) \Rightarrow (\Sigma_k)^*(y)]$
  - $\omega = A_1 \dots A_n$, $2 \leq n \leq k : \Phi^{k+1}_\omega(u) == \exists y|\ s^{k+1}_{A_1}(y, u) \wedge (\Phi^k_{A_2 \dots A_n})^*(y)$

Proof:

By induction on k :

- if $k = 1$ then $\omega = A$

$$\mathcal{M}_1(\Gamma) \models \Phi^1_A(u)$$

$$\begin{aligned} &\text{iff } \ \exists v \in \Gamma^{[1]}|\ \ u = [A] \bullet v \\ &\text{iff } \ A = topsym(\varphi_1^{-1}(u)) \end{aligned}$$

- let us assume the property true for $k \geq 1$. We show that the property is true for $k + 1$ by induction on the length of ω :
  - if $|\omega| = 1$, $\omega = A$, we need to remark that for all k, for all $u \in \Gamma^{[k]}$: $A = topsym(\varphi_k^{-1}(u))$ iff $u = insert_A(\varepsilon_{k+1}) \bullet_{k+1} u'$. Then,

$$\mathcal{M}_k(\Gamma) \models \Phi^k_A(u)$$

$$\begin{aligned} &\text{iff } \ u = [u_1] \bullet u',\ u_1 = insert_A(\varepsilon_k) \bullet_k u'_1,\ u'_1 = \varepsilon_k \\ &\text{iff } \ u = insert_A(\varepsilon_{k+1}) \bullet_{k+1} u' \end{aligned}$$

  - let us assume the property true for $|\omega| = n$, then if $\omega = A_1 \dots A_{n+1}$:

$$\mathcal{M}_{k+1}(\Gamma) \models \Phi^{k+1}_\omega(u)$$

$$\begin{aligned} &\text{iff there exists } v \in \Gamma^{[k]}, u' \in \Gamma^{[k+1]} \text{ such that} \\ &\quad u = [insert_{A_1}(v)] \bullet u' \text{ and } A_2 \dots A_{n+1} = topsym(\varphi_k^{-1}(v)) \\ &\text{iff } \omega = topsym(\varphi_k^{-1}(u)) \end{aligned}$$

□

One constructs now the Π^k_i formula, representing the pop_i operation.

Lemma 3. *For all $1 \leq i \leq k$ one can find a MSOL-formula verifying for all $u, v \in \Gamma^{[k]}$:*

$$\mathcal{M}_k(\Gamma) \models \Pi^k_i(u, v) \ \ \textit{iff} \ \ \varphi_k^{-1}(v) = pop_i(\varphi_k^{-1}(u))$$

Construction:

- $i = 1 \leq k$: $\Pi_1^k(u,v) == son(v,u)$
- $1 \leq i \leq k$: $\Pi_{i+1}^{k+1}(u,v) == \exists x,\, y| \bigvee_{A \in \Gamma} [s_A^k(x,u) \wedge s_A^k(y,v)] \wedge (\Pi_i^k)^*(x,y)$

Proof: By induction on i, $1 \leq i \leq k$:

- if $i = 1$:

$$\mathcal{M}_k(\Gamma) \models \Pi_1^k(u,v)$$

$$\begin{array}{ll} \text{iff} & \text{there exists } u_1 \in \Gamma^{[k-1]} \text{ such that } u = [u_1] \bullet_k v \\ \text{iff} & \varphi_k^{-1}(v) = pop_1(\varphi_k^{-1}(u)) \end{array}$$

- let us assume the property true for $i \leq k$, then for $i + 1 \leq k + 1$ we have :

$$\mathcal{M}_{k+1}(\Gamma) \models \Pi_{i+1}^{k+1}(u,v)$$

$$\begin{array}{ll} \text{iff} & \text{there exists } u_1,\, v_1 \in \Gamma^{[k]},\, t \in \Gamma^{[k+1]},\, A \in \Gamma, \text{ such that} \\ & u = [insert_A(u_1)] \bullet_{k+1} t,\ v = [insert_A(v_1)] \bullet_{k+1} t \text{ and} \\ & \varphi_k^{-1}(v_1) = pop_i(\varphi_k^{-1}(u_1)) \\ \text{iff} & \varphi_{k+1}^{-1}(u) = A[\varphi_k^{-1}(u_1)].\varphi_{k+1}^{-1}(t),\ \varphi_{k+1}^{-1}(v) = A[\varphi_k^{-1}(v_1)].\varphi_{k+1}^{-1}(t) \\ & \text{and } \varphi_k^{-1}(v_1) = pop_i(\varphi_k^{-1}(u_1)) \\ \text{iff} & \varphi_k^{-1}(v) = pop_i(\varphi_k^{-1}(u)) \end{array}$$

□

Finally, one constructs the $\Psi_i^k(\omega)$ formulas, representing the $push_i$ operation.

Lemma 4. *For all $1 \leq i \leq k$, $\omega \in \Gamma^+$, one can find a MSOL-formula verifying for all $u, v \in \Gamma^{[k]}$:*

$$\mathcal{M}_k(\Gamma) \models \Psi_i^k(\omega)(u,v) \quad \textit{iff} \quad \varphi_k^{-1}(v) = push_i(w)(\varphi_k^{-1}(u))$$

Construction:

- $i = 1$:
  - $\omega = A$:

$$\Psi_1^k(A)(u,v) == [\Sigma^k(u) \Rightarrow s_A^k(u,v)] \wedge \neg(\Sigma^k(u)) \Rightarrow [\exists x|\ son_{k,k-1}(x,u) \wedge s_A^k(x,v)]$$

  - $\omega = A_1 \dots A_{n+1}$, $n \geq 1$, $k \geq 1$:

$$\Psi_1^k(\omega)(u,v) ==$$

$$\exists y, z|\ \Psi_1^k(A_{n+1})(u,y)\ \wedge\ son(y,z)\ \wedge\ clone(z)\ \wedge\ \Psi_1^k(A_1 \dots A_n)(z,v)$$

- $1 \leq i \leq k$:

$$\Psi_{i+1}^{k+1}(\omega)(u,v) == \exists x,\, y| \bigvee_{A\in\Gamma} [s_A^{k+1}(x,u) \wedge s_A^{k+1}(y,u)] \wedge (\Psi_k^i(\omega))^*(x,y)$$

Proof: By induction on i:

- if $i = 1$ we show the property by induction on the length of ω :
  - if $\omega = A$:

$$\mathcal{M}_k(\Gamma) \models \Psi_1^k(\omega)(u,v)$$

$$\begin{aligned} &\text{iff if } u = \varepsilon_k \text{ then } v = insert_A(u) \text{ else} \\ &\quad \exists t \in \Gamma^{[k]},\, B \in \Gamma : u = insert_B(t),\, v = insert_A(t) \\ &\text{iff } \varphi_k^{-1}(v) = push_1(A)(\varphi_k^{-1}(u)) \end{aligned}$$

  - if $\omega = A_1 \ldots A_{n+1}$:

$$\mathcal{M}_{k+1}(\Gamma) \models \Psi_1^{k+1}(\omega)(u,v)$$

$$\begin{aligned} &\text{iff there exists } s,t,t' \in \Gamma^{[k+1]},\ t_1 \in \Gamma^{[k]},\ A \in \Gamma, \text{ such that :} \\ &\quad \varphi_{k+1}^{-1}(t) = push_i(A_{n+1})(\varphi_{k+1}^{-1}(u)),\ t = [t_1] \bullet t' \\ &\quad \varphi_{k+1}^{-1}(v) = push_i(A_1 \ldots A_n)(\varphi_{k+1}^{-1}(s)) \text{ and } s = [t_1] \bullet t \\ &\text{iff there exists } w,w' \in (k+1) - pds(\Gamma), \text{ such that :} \\ &\quad w = push_i(A_{n+1})(\varphi_{k+1}^{-1}(u)),\ \varphi_{k+1}^{-1}(v) = push_i(A_1 \ldots A_n)(w') \\ &\quad w = A_{n+1}[flag]rest \text{ and } w' = A_{n+1}[flag]A_{n+1}[flag]rest \\ &\text{iff } \varphi_k^{-1}(v) = push_i(\omega)(\varphi_k^{-1}(u)) \end{aligned}$$

- let us assume the property true for $i \geq 1$, then for $i+1$ we have :

$$\mathcal{M}_{k+1}(\Gamma) \models \Psi_{i+1}^{k+1}(\omega)(u,v) \text{ iff } \varphi_{k+1}^{-1}(v) = push_{i+1}(\omega)(\varphi_{k+1}^{-1}(u))$$

  as in the proof above.

□

3.4 Computation graph

We generalize the notion of computation graph of a 1-pda [MS85] to k-pdas (for arbitrary $k \geq 1$), and we show that the monadic second-order theory of these graphs is decidable.
We recall a *total state* of $\mathcal{A}$ consists of a pair (q, pds), where $q \in Q$ and $pds \in k - \text{pds}(\Gamma)$.

Definition 15. *Let* $\mathcal{A} \in k - PDA(\Sigma)$*, the computation graph* $\mathbf{G}(\mathcal{A}) = (V, E)$ *of* $\mathcal{A}$ *is defined by :*
$V = Q \times k\text{-}pds(\Gamma)$*,* $E \subset V \times \Sigma \cup \{\varepsilon\} \times V$ *and*
$E = \{((q, pds), a_\varepsilon, (p, pds'))| \ (q, a_\varepsilon, pds) \vdash_{\mathcal{A}} (p, \varepsilon, pds')\}$

Definition 16. *The structure induced by the computation graph* $\mathbf{G}$ *is* $\mathcal{M}_{\mathbf{G}} = \langle Q \times k\text{-}pds, (R_\alpha)_{\alpha \in \Sigma \cup \{\varepsilon\}} \rangle$ *with*

$$R_\alpha = \{((p, u), (q, v))| \ ((p, u), \alpha, (q, v)) \in E\}$$

Theorem 3. *The* MSO*-theory of* $\mathcal{M}_{\mathbf{G}}$ *is decidable.*

Proof: One considers first the structure $\mathcal{M}_k(\Gamma, Q) = \langle Q \times \Gamma^{[k]}, son_k^Q, clone_k^Q, \ldots \rangle$ defined over the signature Sig_k and such that for all relation $r \in \mathcal{M}_k(\Gamma, Q)$, if $(u_1, u_2, \ldots) \in r$ then $((q_1, u_1), (q_2, u_2), \ldots) \in r^Q$ for all $q_1, q_2, \ldots \in Q$.

The MSO-theory of $\mathcal{M}_k(\Gamma, Q)$ is decidable.

One constructs now the formula $\Theta_{\mathcal{A}}(a_\varepsilon)$, $a_\varepsilon \in \Sigma \cup \{\varepsilon\}$ satisfying for all (p, u), $(q, v) \in Q \times A^{[k]}$

$$\mathcal{M}_k(\Gamma, Q) \models \Theta_{\mathcal{A}}(a_\varepsilon)((p, u), (q, v)) \text{ iff } (p, a_\varepsilon, \varphi^{-1}(u)) \vdash_A (q, \varepsilon, \varphi^{-1}(v)).$$

Let us set :

$$\Theta_{\mathcal{A}}(a_\varepsilon)((p, u), (q, v)) == \bigvee_{(q, push_i(\omega)) \in \delta(p, \alpha, w)} [\Phi_w^Q((p, u)) \wedge \Psi_i^Q(\omega)((p, u), (q, v))] \vee$$
$$\bigvee_{(q, pop_i) \in \delta(p, \alpha, w)} [\Phi_w^Q((p, u)) \wedge \Pi_i^Q((p, u), (q, v))]$$

The substructure of $\mathcal{M}_k(\Gamma, Q)$ induced by formulas $\Theta_{\mathcal{A}}(a_\varepsilon)$ admits a decidable monadic theory (theorem 1) and is isomorphic to $\mathcal{M}_{\mathbf{G}}$. $\Box$

Definition 17. *Let* $\mathcal{A} \in k\text{-}PDA(\Sigma)$*, and* $c = (q, u)$ *be a total state of* $\mathcal{A}$*. The computation graph* $\mathbf{G}(\mathcal{A})_c = (V_c, E_c)$ *of* $\mathcal{A}$ *rooted in* c *is defined from the computation graph* $\mathbf{G}(\mathcal{A}) = (V, E)$ *by :*

$$V_c = \{(p, v) \in V | \ c \vdash_{\mathcal{A}}^* (p, v)\}; \ \ E_c = \{(c_1, a_\varepsilon, c_2) \in E | \ c_1, c_2 \in V_c\}.$$

Definition 18. *The structure* $\mathcal{M}_{\mathbf{G}_c}$ *induced by* $\mathbf{G}(\mathcal{A})_c$ *is defined by :*
$\mathcal{M}_{\mathbf{G}_c} = \langle V_c, (R_{c,\alpha})_{\alpha \in \Sigma \cup \{\varepsilon\}} \rangle$ *with*

$$R_{c,\alpha} = R_\alpha \cap E_c.$$

Theorem 4. *The* MSO*-theory of* $\mathcal{M}_{\mathbf{G}_c}$ *is decidable.*

Proof: One considers the following formulas:

$$\Lambda(X) == c \in X \ \wedge \ \forall y, z. \ [(\bigvee_{\alpha \in \Sigma \cup \{\varepsilon\}} R_\alpha(y, z) \ \wedge \ y \in X) \Rightarrow z \in X]$$

$$\Lambda_0(X) == \Lambda(X) \ \wedge \ \forall Y. \ [\Lambda(Y) \Rightarrow X \subset Y] \ \text{ and } \ D_0(x) == \exists X. [\Lambda_0(X) \wedge x \in X]$$

For all s in V, $\mathcal{M}_{\mathbf{G}} \models D_0(s)$ iff $s \in V_c$, hence $\mathcal{M}_{\mathbf{G}_c} = \mathcal{M}_{\mathbf{G}}(D_0, (R_\alpha))$. By theorem 1 and theorem 3, the MSO-theory of $\mathcal{M}_{\mathbf{G}_c}$ is thus decidable. □

Corollary 2 (Membership problem). *Let $\mathcal{A} \in$ it-PDA(Σ) and $w \in \Sigma^*$, then one can decide whether $w \in L(\mathcal{A})$.*

Proof: The structure $\mathcal{M}_{\mathbf{G}_{(q_0,Z)}}$ has a decidable MSO theory. Thus it is suffcient to construct a MSO-formula Υ_w satisfying :

$$\mathcal{M}_\Gamma \models \Upsilon_w \text{ iff } (q_0, w, Z) \vdash^*_{\mathcal{A}} (q_F, \varepsilon, \varepsilon).$$

One sets:

$$\Lambda(x, X) == x \in X \quad \wedge \quad \forall y, z. \ [(R_\varepsilon(y, z) \quad \wedge \quad y \in X) \Rightarrow z \in X]$$

and

$$\Upsilon(x, X) == \Lambda(x, X) \quad \wedge \quad \forall Y. \ [\Lambda(x, Y) \Rightarrow X \subset Y].$$

Then:

$$\widetilde{\Upsilon}_\varepsilon(x, y) == \exists X. \ [\Upsilon(x, X) \ \wedge \ (y \in X)]$$

$$\widetilde{\Upsilon}_\alpha(x, y) == \exists x', y'. \ [\widetilde{\Upsilon}_\varepsilon(x, x') \quad \wedge \quad R_\alpha(x', y') \quad \wedge \quad \widetilde{\Upsilon}_\varepsilon(y', y)]$$

and

$$\widetilde{\Upsilon}_{\alpha_1 \ldots \alpha_n}(x, y) ==$$

$$\exists x_1, \ldots x_{n-1}. \ [\widetilde{\Upsilon}_{\alpha_1}(x, x_1) \quad \wedge \quad \bigwedge_{i=2 \ldots n-1} \widetilde{\Upsilon}_{\alpha_i}(x_{i-1}, x_i) \quad \wedge \quad \widetilde{\Upsilon}_{\alpha_n}(x_{n-1}, y)].$$

Finally

$$\Upsilon_w == \widetilde{\Upsilon}_w((q_0, Z), (q_F, \varepsilon))$$

□

Corollary 3 (Emptiness problem).
Let $\mathcal{A} \in$ it-PDA(Σ), then one can decide whether $L(\mathcal{A}) = \emptyset$.

Corollary 4 (Finiteness problem).
Let $\mathcal{A} \in$ it-PDA(Σ), then one can decide whether $L(\mathcal{A})$ is finite.

4 Integer sequences

4.1 Sequences defined by automata

We define here a class of *integer sequences* by means of k-pushdown automata. Specifically, we use a slightly restrictive class of k-pdas, the *counter k-pda*. These are an extension of the classical *counter pda* which recognize some words with a memory consisting of natural integers only. We show that the class of integer sequences thus defined is closed under many natural operations (theorem 23).

Definition 19 (Counter k-pushdown store). *Let Γ be an alphabet with a distinguish symbol $F \in \Gamma$. The set of k-*counter *pushdown stores over Γ, with counter F, is denoted by $k - \mathrm{pds}_c(\Gamma)$ and defined by:*

$$1 - pds_c(\Gamma) = (F[\varepsilon])^* \quad (k+1) - pds_c(\Gamma) = ((\Gamma - \{F\}) \cdot [k - pds_c(\Gamma)])^*.$$

This means that the symbol F can appear at level k only and that no other symbol can occur at level k.

Definition 20 (Counter k-*pdas*). *A k-pda $\mathcal{A} = (Q, X, \Gamma, \delta, q_0, Z)$ is said to be a* counter *k-pdas*, with counter F, if $\mathcal{A}$ is a k-pda over a pushdown alphabet $\Gamma \supseteq \{F\}$, such that the set of counter pushdown stores over Γ is closed under the computation relation i.e. for every $q, q' \in Q, \omega, \omega' \in k - \mathrm{pds}(\Gamma), u, u' \in X^*$, if $\omega \in k - \mathrm{pds}_c(\Gamma)$ and $(q, u, \omega) \vdash_{\mathcal{A}} (q', u', \omega')$ then $\omega' \in k - \mathrm{pds}_c(\Gamma)$.

In the rest of the paper we abbreviate "deterministic counter k-pushdown automaton" by k-dcpa.

Example 8 *Here is a 3-pds_c :*

$$A[B[FF]C[F]]B[E[FFF]].$$

Definition 21 (k-computable sequences). *A sequence of natural integers f is called a k-computable sequence iff there exists a k-dcp $\mathcal{A}$, over a pushdown-alphabet Γ containing at least k different symbols $A_1, A_2, \ldots, A_{k-1}, F$, such that, for all $n \geq 0$:*

$$(q_0, a^{f(n)}, A_1[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \vdash_{\mathcal{A}}^* (q_0, \varepsilon, \varepsilon).$$

One denotes by $\mathbb{S}_k$ the set of all k-computable sequences of natural integers.

We show in next lemmas that from any counter automaton $\mathcal{A}$ *computing* a sequence $n \mapsto f(n)$, in the sense of definition 21, one can derive some other k-pda accepting the languages $L = \{a^{f(n)},\ n \geq 0\}$ and $L = \{\prod_{n \geq 0}(a^{f(n)}b)\}$.

Lemma 5. *Let f be a level-k sequence, then one can construct a counter k-pda $\mathcal{A}'$ such that $L(\mathcal{A}') = \{a^{f(n)},\ n \geq 0\}$.*

Proof: Let us suppose that $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ and

$$(q_0, a^{f(n)}, A_1[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \twoheadrightarrow_{\mathcal{A}} (q_0, \varepsilon, \varepsilon)$$

Let us set $\mathcal{A}' = (Q, \{a\}, \Gamma \cup \{Z'\}, \delta \cup \delta', q_0, Z')$ with :

$\delta'(q_0, \varepsilon, Z') = \{(q_0, push_2(A_2))\}$
for all $2 \leq i \leq k-2$ $\delta'(q_0, \varepsilon, Z'A_2 \ldots A_i) = \{(q_0, push_{i+1}(A_{i+1}))\}$
$\delta'(q_0, \varepsilon, Z'A_2 \ldots A_{k-1}) = \{(q_0, push_k(F)), (q_0, push_1(A_1))\}$
$\delta'(q_0, \varepsilon, Z'A_2 \ldots A_{k-1}F) = \{(q_0, push_k(FF)), (q_0, push_1(A_1))\}$.

Then, each accepting computation has the form :
$(q_0, a^{f(n)}, Z') \vdash^*_{\mathcal{A}'} (q_0, a^{f(n)}, Z'[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \vdash_{\mathcal{A}'}$
$(q_0, a^{f(n)}, A_1[\ldots [A_{k-1}[F^n]] \ldots]) \vdash^*_{\mathcal{A}'} (q_0, \varepsilon, \varepsilon)$ □

Lemma 6. *Let f be a k-computable sequence, then one can construct a k-dcp $\mathcal{A}' = (Q', \{a, b\}, \Gamma', \delta', q_0, Z')$ such that for all $n \geq 0$, there exists $\omega_n \in k\text{-}pds(\Gamma')$ satisfying :*

$$(q_0, \prod_{1 \leq i \leq n} (a^{f(i)} b), Z') \vdash^*_{\mathcal{A}'} (q_0, \varepsilon, \omega_n).$$

Proof: Let us suppose that $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ and

$$(q_0, a^{f(n)}, A_1[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon)$$

We define $\mathcal{A}' = (Q \cup \{q_c\}, \{a\}, \Gamma \cup \{Z', C\}, \delta \cup \delta', q_0, Z')$ with :

$\delta'(q_0, \varepsilon, Z') = (q_0, push_2(A_2))$
for all $2 \leq i \leq k-2$ $\delta'(q_0, \varepsilon, Z'A_2 \ldots A_i) = (q_0, push_{i+1}(A_{i+1}))$
$\delta'(q_0, \varepsilon, Z'A_2 \ldots A_{k-1}) = (q_0, push_1(A_1C))$
$\delta'(q_0, b, CA_2 \ldots A_{k-1}) = (q_c, push_k(F))$
$\delta'(q_0, b, CA_2 \ldots A_{k-1}F) = (q_c, push_k(FF))$
$\delta'(q_c, \varepsilon, CA_2 \ldots A_{k-1}F) = (q_0, push_1(A_1C))$.

One can check, by induction on n, that $\mathcal{A}'$ has always a computation of the form:

$$\begin{aligned}(q_0, \prod_{1 \leq i \leq n} (a^{f(i)} b), Z) &\vdash^* (q_0, a^{f(n)}b, A_1[\ldots A_{k-1}[F^n] \ldots] \cdot C[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \\ &\vdash^* (q_0, b, C[A_2 \ldots [A_{k-1}[F^n]] \ldots]) \\ &\vdash^* (q_0, \varepsilon, A_1[A_2 \ldots [A_{k-1}[F^{n+1}]] \ldots] \cdot C[A_2 \ldots [A_{k-1}[F^{n+1}]] \ldots])\end{aligned}$$

□

4.2 Some k-computable sequences.

We show in this section that $\mathbb{N}$-rational sequences are 2-computable and sequences solution of a system of polynomial recurrence equations with integer coefficients are 3-computable.

Definition 22 ($\mathbb{N}$-rational sequences). *A sequence $(u_n)_{n\geq 0}$ is $\mathbb{N}$-rational iff there is a matrix M in $\mathbb{N}^{d\times d}$ and two vectors L in $\mathbb{B}^{1\times d}$ and C in $\mathbb{B}^{d\times 1}$ such that $u_n = L \cdot M^n \cdot C$.*

An equivalent definition consists of considering a finite graph G, with multiple edges, and sets I and F of vertices such that M is the adjacency matrix of G and L and C are the characteristic vectors of I and F. Then u_n is the number of paths through G of length n from a vertex of I to a vertex of F.

Example 9 *Let $(u_n)_{n\geq 0}$ be the sequence having the following representation of dimension 2 :*

$$L = \begin{pmatrix} 1\,1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We set
$M^n = \begin{pmatrix} u_{1,1}(n) & u_{1,2}(n) \\ u_{2,1}(n) & u_{2,2}(n) \end{pmatrix}$ *then* $M^{n+1} = \begin{pmatrix} u_{1,1}(n) + u_{1,2}(n) & 2.u_{1,1}(n) \\ u_{2,1}(n) + u_{2,2}(n) & 2.u_{2,1}(n) \end{pmatrix}$
and $(u_n)_{n\leq 0}$ is the sequence defined by :

$$u_n = u_{1,1}(n) + u_{1,2}(n)$$

with :
$u_{1,1}(0) = 1 \;\; u_{1,1}(n+1) = u_{1,1}(n) + u_{1,2}(n) \quad u_{1,2}(0) = 0 \;\; u_{1,2}(n+1) = 2.u_{1,1}(n)$
$u_{2,1}(0) = 0 \;\; u_{1,1}(n+1) = u_{2,1}(n) + u_{2,2}(n) \quad u_{2,2}(0) = 1 \;\; u_{1,2}(n+1) = 2.u_{2,1}(n)$

Proposition 5 *If $(u_n)_{n\geq 0}$ is a $\mathbb{N}$-rational sequence, then $(u_n)_{n\geq 0} \in \mathbb{S}_2$.*

Let us assume u is defined from $L \in \mathbb{B}^{1\times d}, M \in \mathbb{N}^{d\times d}, C \in \mathbb{B}^{d\times 1}$.
Construction :
Let $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ with : $Q = \{q_0\} \cup \{q_{i,j}, 1 \leq i,j \leq d\}$, $\Gamma = \{F\} \cup \{U_{i,j}, 1 \leq i,j \leq d\}$, and the transition function δ is defined by:

1 $\delta(q_0, a, U_{i,i}) = (q_0, pop_1)$ and if $i \neq j$, $\delta(q_0, \varepsilon, U_{i,j}) = (q_0, pop_1)$,

2 $\delta(q_0, \varepsilon, U_{i,j}F) = (q_{i,j}, pop_2)$,

3 $\delta(q_{i,j}, \varepsilon, U_{i,j}F) = \delta(q_{i,j}, \varepsilon, U_{i,j}) = (q_0, push_1(\prod_{1\leq l\leq d} U_{i,l}^{m_{l,j}}))$

4 $\delta(q_0, \varepsilon, UF) = \delta(q_0, \varepsilon, U) = (q_0, push_1(\prod_{1\leq j\leq d} (\prod_{1\leq i\leq d} U_{i,j}^{l_i})^{c_j}))$.

Proof: Let us consider the sequences $u_{i,j}$ defined by the following recurrence relations:

$$u_{i,j}(n+1) = \sum_{1 \le l \le d} u_{i,l}(n).m_{l,j},$$

$$u_{i,i}(0) = 1, \quad \text{and } u_{i,j}(0) = 0 \text{ for } i \neq j.$$

Let us show by induction on $n \ge 0$ the following auxiliary property $P(n)$:
$\forall 1 \le i, j \le d$,

$$(q_0 U_{i,j}[F^n] q_0) \rightarrow^*_{\mathcal{A}} a^{u_{i,j}(n)}. \tag{3}$$

Basis:$n = 0$. Transitions (1) ensure $P(0)$ is true.
Induction step: Let $n \ge 0$ and let us assume $P(n)$.
By a transition (2) followed by a transition (3) and by a decomposition rule:

$$\begin{aligned}(q_0 U_{i,j}[F^{n+1}] q_0) &\rightarrow_{\mathcal{A}} (q_{i,j} U_{i,j}[F^n] q_0) \\ &\rightarrow_{\mathcal{A}} (q_0 \prod_{1 \le l \le d} ((U_{i,l}[F^n])^{m_{l,j}} q_0) \\ &\rightarrow^*_{\mathcal{A}} \prod_{1 \le l \le d} (q_0 U_{i,l}[F^n] q_0)^{m_{l,j}}\end{aligned}$$

By induction hypothesis:

$$(q_0 U_{i,l}[F^n] q_0) \rightarrow^*_{\mathcal{A}} a^{u_{i,l}(n)}.$$

Composing the above derivations we obtain:

$$(q_0 U_{i,j}[F^{n+1}] q_0) \rightarrow^*_{\mathcal{A}} \prod_{1 \le l \le d} (a^{u_{i,l}(n)})^{m_{l,j}} = a^{u_{i,j}(n+1)}.$$

Hence $P(n+1)$ is proved.
Property (3) is thus established.
Let us examine now the sequence u_n. For every $n \ge 0$:

$$u(n) = \sum_{1 \le i \le d} \sum_{1 \le j \le d} \ell_i \cdot u_{i,j}(n) \cdot c_j. \tag{4}$$

Applying transition (4) of $\mathcal{A}$ followed by decompositions, we see that:

$$\begin{aligned}(q_0 U[F^n] q_0) &\rightarrow^*_{\mathcal{A}} (q_0 (\prod_{1 \le j \le d} (\prod_{1 \le i \le d} (U_{i,j}[F^n]^{l_i})^{c_j})) q_0) \\ &\rightarrow^*_{\mathcal{A}} \prod_{1 \le j \le d} (\prod_{1 \le i \le d} (q_0 U_{i,j}[F^n] q_0)^{l_i \cdot c_j}))\end{aligned} \tag{5}$$

and from $P(n)$ we deduce that

$$\prod_{1 \le i \le d} (q_0 U_{i,j}[F^n] q_0)^{l_i \cdot c_j}) \rightarrow^*_{\mathcal{A}} \prod_{1 \le i \le d} a^{u_{i,j}(n) \cdot l_i \cdot c_j} \tag{6}$$

Combining derivations (5) and (6) we obtain, by formula (4):

$$(q_0 U[F^n] q_0) \to^*_{\mathcal{A}} a^{u(n)}.$$

□

Lemma 7. *Let $(u_n)_{n\geq 0}$ be the sequence defined by $u_{n+1} = (u_n)^d$, $d \geq 1$ and $u_0 = c \in \mathbb{N}$. Then $(u_n)_{n\geq 0} \in \mathbb{S}_3$.*

Construction: Let us set $\mathcal{A} = (\{q_0, q_F, q\}, \{a\}, \{F, A\}, \delta, q_0, Z)$ with :

1.1 $\delta(q_0, \varepsilon, AAF) = (q_F, pop_3)$,
1.2 $\delta(q_F, \varepsilon, AAF) = \delta(q_F, \varepsilon, AA) = (q_0, push_2(A^d))$,

2.1 $\delta(q_0, \varepsilon, AA) = (q, pop_2)$,
2.2 for all $\omega \in \Gamma^{(3)}$, $\delta(q, \varepsilon, \omega) = (q_0, push_1(A^c))$,

3 $\delta(q_0, a, A) = (q_0, pop_1)$.

Proof:
Let Ω be an undeterminate for the automaton $\mathcal{A}$ (we recall it means that $\Omega \notin \{F, A\}$). We prove by induction on n, the following property $\mathbf{P}(n)$:

$$(q_0 A[A[F^n]\Omega] q_0) \to^*_{\mathcal{A}} (q_0 A[\Omega] q_0)^{u_n}$$

Basis:$n = 0$
By transition (2.1)

$$(q_0 A[A[\varepsilon]\Omega] q_0) \to_{\mathcal{A}} (q A[\Omega] q_0)$$

using then a transition (2.2) and the definition of $\to$ we get:

$$(q A[\Omega] q_0) \to_{\mathcal{A}} (q_0 (A[\Omega])^c q_0) \to^*_{\mathcal{A}} (q_0 A[\Omega] q_0)^c. \tag{7}$$

Thus $\mathbf{P}(0)$ is proved.
Induction step: Let $n \geq 0$ and let us assume $\mathbf{P}(n)$.
Using transition (1.1) we get:

$$(q_0 A[A[F^{n+1}]\Omega] q_0) \to_{\mathcal{A}} (q_F A[A[F^n]\Omega] q_0) \tag{8}$$

using then transition (1.2) we get:

$$(q_F A[A[F^n]\Omega] q_0) \to (q_0 A[(A[F^n])^d \Omega] q_0) \tag{9}$$

Let $i \in \mathbb{N}$ and $\omega_i = (A[F^n])^i \Omega$. Substituting this $3-$pds ω to the undeterminate Ω in $P(n)$ we obtain:

$$(q_0 A[(A[F^n])^i \Omega] q_0) \to^*_{\mathcal{A}} (q_0 A[(A[F^n])^{i-1} \Omega] q_0)^{u_n}.$$

Composing all these derivations (for $1 \leq i \leq d$) together we obtain:

$$(q_0 A[(A[F^n])^d \Omega] q_0) \to^*_{\mathcal{A}} (q_0 A[\Omega] q_0)^{(u_n)^d}. \tag{10}$$

Composing derivations (8,9,10) we obtain

$$(q_0 A[A[F^{n+1}]\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{(u_n)^d} = (q_0 A[\Omega]q_0)^{u_{n+1}}$$

i.e. $\mathbf{P}(n+1)$. Transition (3) expressed as a derivation gives:

$$q_0 A[\varepsilon]q_0 \rightarrow_{\mathcal{A}} a$$

we can then conclude that, for every $n \in \mathbb{N}$

$$(q_0 A[A[F^n]]q_0) \rightarrow^*_{\mathcal{A}} a^{u_n}.$$

□

Proposition 6 *Let $P(X) = \sum_{0 \leq i \leq d} a_i X^i$ be a polynomial with coefficients $a_i \in \mathbb{N}$ and $(u_n)_{n \geq 0}$ be the sequence defined by $u_{n+1} = P(u_n)$ and $u_0 = c \in \mathbb{N}$. Then $(u_n)_{n \geq 0} \in \mathbb{S}_3$.*

We use the same ideas as in the proof above.
Construction: Let us set $\mathcal{A} = (\{q_0, q, q_F\}, \{a\}, \{F, A, A_0, \ldots A_d\}, \delta, q_0, Z)$ with :

1.1 $\delta(q_0, \varepsilon, AAF) = (q_F, pop_3)$,
1.2 $\delta(q_F, \varepsilon, AAF) = \delta(q_F, \varepsilon, AA) = (q_0, push_1(A_0^{a_0} A_1))$,
2 $\delta(q_0, \varepsilon, A_0 AF) = \delta(q_0, \varepsilon, A_0 A) = (q_0, pop_2)$
3 $\delta(q_0, \varepsilon, A_1 AF) = \delta(q_0, \varepsilon, A_1 A) = (q_0, push_1(A^{a_1} A_2))$,
4 for all $2 \leq i \leq d$, $\delta(q_0, \varepsilon, A_i AF) = \delta(q_0, \varepsilon, A_i A) = (q_0', push_2(AA))$,
5 for all $2 \leq i < d$, $\delta(q_0', \varepsilon, A_i AF) = \delta(q_0', \varepsilon, A_i A) = (q_0, push_1(A^{a_i} A_{i+1}))$,
6 $\delta(q_0', \varepsilon, A_d AF) = \delta(q_0', \varepsilon, A_d A) = (q_0, push_1(A^{a_d}))$,
7.1 $\delta(q_0, \varepsilon, AA) = (q_0, push_1(B^c))$
7.2 $\delta(q_0, \varepsilon, BA) = (q_1, push_1(A))$
7.3 $\delta(q_1, \varepsilon, AA) = (q_0, pop_2)$,
8 $\delta(q_0, a, A) = (q_0, pop_1)$.

Proof: Let Ω be an undeterminate. Let us show that, for every $n \geq 0$, the following property $\mathbf{P}(n)$ holds:

$$(q_0 A[A[F^n]\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 \Omega q_0)^{u(n)}.$$

We first check that, for every $n \geq 0$

$$(q_0 A[A[F^{n+1}]\Omega]q_0) \rightarrow^*_{\mathcal{A}} \prod_{i=0}^{d} (q_0 A[(A[F^n])^i \Omega]q_0)^{a_i} \qquad (11)$$

Such a derivation can be detailed as:

$$\begin{array}{llll}
(q_0 A[A[F^{n+1}]\Omega]q_0) \rightarrow_{\mathcal{A}} & (q_F A[A[F^n]\Omega]q_0) & \text{(by a transition (1.1))} \\
\rightarrow_{\mathcal{A}} & (q_0(A_0[A[F^n]\Omega])^{a_0} A_1[A[F^n]\Omega]q_0) & \text{(by a transition (1.2))} \\
\rightarrow^*_{\mathcal{A}} & (q_0(A_0[A[F^n]\Omega]q_0)^{a_0}(q_0 A_1[A[F^n]\Omega]q_0) & \text{(by decomposition rule)} \\
\rightarrow_{\mathcal{A}} & (q_0 A[\Omega]q_0)^{a_0}(q_0 A_1[A[F^n]\Omega]q_0) & \text{(by transition (2))} \\
\rightarrow^*_{\mathcal{A}} & (q_0 A[\Omega]q_0)^{a_0}(q_0(A[A[F^n]\Omega])^{a_1} A_2[A[F^n]\Omega]q_0) & \text{(by (3))} \\
\rightarrow^*_{\mathcal{A}} & (q_0 A[\Omega]q_0)^{a_0}(q_0(A[A[F^n]\Omega]q_0)^{a_1}(q_0 A_2[A[F^n]\Omega]q_0) & \text{(by decomposition)}
\end{array}$$

$\rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{a_0} \prod_{i=1}^{d}(q_0(A[(A[F^n])^i\Omega]q_0)^{a_i}$(using $d-2$ times $(4,5)$ and finally $(4,6)$).
Let us prove by induction $\mathbf{P}(n)$:
Basis:$n = 0$
The following derivation is valid:

$$\begin{array}{lll}
q_0 A[A[\varepsilon]\Omega]q_0 \rightarrow_{\mathcal{A}} & q_0(B[A[\varepsilon]\Omega])^c q_0 & \text{(by (7.1))} \\
\rightarrow^*_{\mathcal{A}} & (q_0(B[A[\varepsilon]\Omega]q_0)^c & \text{(by decomposition rules)} \\
\rightarrow^*_{\mathcal{A}} & (q_1 A[A[\varepsilon]\Omega]q_0)^c & \text{(by (7.2))} \\
\rightarrow^*_{\mathcal{A}} & (q_0 A[\Omega]q_0)^c & \text{(by (7.3))}
\end{array}$$

Inductive step:let us assume $\mathbf{P}(n)$.
By induction on i, using $\mathbf{P}(n)$ and the substitution $\Omega \leftarrow (A[F^n])^{i-1}\Omega$, we can show that:

$$(q_0 A[(A[F^n])^i \Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{u(n)^i}.$$

These derivations allow to obtain:

$$\prod_{i=0}^{d}(q_0 A[(A[F^n])^i\Omega]q_0)^{a_i} \rightarrow^*_{\mathcal{A}} (q_0 A[\Omega]q_0)^{P(u(n))} = (q_0 A[\Omega]q_0)^{u(n+1)}.$$

The combination of derivation (11) with the derivation above proves $\mathbf{P}(n+1)$.
Substituting ε to Ω in property $\mathbf{P}(n)$ and applying transition (8), we may conclude that

$$(q_0 A[A[F^n]]q_0) \rightarrow^*_{\mathcal{A}} a^{u(n)}.$$

□

Proposition 7 *Let $P_i(X_1, \ldots, X_p)$, $(1 \le i \le p)$ be polynomials with coefficients in $\mathbb{N}$, $c_1, \ldots, c_i, \ldots c_p \in \mathbb{N}$ and , u_i, $(1 \le i \le p)$ be the sequence defined by $u_i(n+1) = P_i(u_1(n), \ldots, u_p(n))$, and $u_i(0) = c_i$. Then $u_1 \in \mathbb{S}_3$.*

Sketch of proof: Let us suppose that for every $1 \le i \le p$,

$$P_i(X_1, \ldots X_p) = \sum_{j=0}^{\nu_i} a_{i,j} X_1^{d_{i,j,1}} \ldots X_p^{di,j,p}.$$

Following the same lines as in the proof of proposition 6, it is possible to construct a 3-dcp automaton satisfying for all $1 \le i \le p$

$$(q_0 A[U_i[F^{n+1}]\Omega]q_0) \to^* \prod_{j=0}^{\nu_i} (q_0 A[(\prod_{\ell=1}^{p} (U_\ell[F^n])^{d_{i,j,\ell}}) \cdot \Omega]q_0)^{a_{i,j}}$$

and also

$$(q_0 A[U_i[\varepsilon]\Omega]q_0) \to^* (q_0 A[\Omega]q_0)^{c_i}.$$

These derivations imply, by induction on n, that:

$$(q_0, A[U_i[F^n]]q_0) \to^* a^{u_i(n)}.$$

□

4.3 Operations over sequences/automata.

In this section, we investigate the closure properties of the classes $\mathbb{S}_k$. It turns out that the union of all the $\mathbb{S}_k$ is closed under classical operations like sum, product, composition, convolution product, and by more complex operations like resolution of system of polynomial recurrence equations with coefficients in subsets of $\mathbb{N}^{\mathbb{N}}$.
We conclude the section by theorem 23 which summarizes the closure properties established so far.

The following technical lemma will be useful for these constructions.

Lemma 8. *Let $f \in \mathbb{S}_{k+1}$, $k \ge 2$ and $\mathcal{A}$ be a $(k+1)$-dcpa over a pushdown alphabet $\Gamma \supseteq \{A_1, A_2, \ldots, A_k, F\}$ fulfilling: $\forall n \ge 0$,*
(P0) $(q_0, a^{f(n)}, A_1[\ldots[A_k[F^n]]\ldots]) \vdash_{\mathcal{A}}^* (q_0, \varepsilon, \varepsilon)$.
Then, one can construct a $(k+1)$-dcp $\mathcal{A}'$ defined on a pushdown-symbols set $\Gamma' \supseteq \Gamma$, containing a special symbol $\bar{A}_1$ and a set of states Q', such that:
(P1) *$\mathcal{A}'$ is level-partitioned*
(P2) *for every letter $\Omega \notin \Gamma'$, $(q_0, A_1[A_2[\ldots[A_k[F^n]]\ldots]\Omega], q_0) \to_{\mathcal{A}'}^* (q_0, \bar{A}_1[\Omega], q_0)^{f(n)}$.*
(P3) *δ' has no lefthand-side of the form $(q_0, \bar{A}_1 \cdot \omega)$ for any $\omega \in \Gamma'^{(k)} \cup \{\varepsilon\}$.*

Remark 8
1- Let us add to the transitions of $\mathcal{A}'$ the transition: $\delta(q_0, a, \bar{A}_1) = (q_0, pop_1)$. We thus obtain an automaton $\mathcal{A}''$ fulfilling properties **(P0)**, **(P1)**, **(P2)**.
2- Property **(P3)** *makes the automaton $\mathcal{A}'$ "open" to a combination with another automaton: it suffices to add transitions starting from $q_0\bar{A}_1[\omega]$ for some well-chosen ω, and leading to a total state from another deterministic automaton. Property* **(P3)** *guaranties that the compound automaton thus constructed will be deterministic.*

Construction:
First step

From $\mathcal{A}$ it is possible to build another $(k+1)$-dcpa $\mathcal{B}$ fulfilling conditions $(\mathbf{P0}), (\mathbf{P1})$ and the additional condition
$(\mathbf{P4})$: $\forall q \in Q, \forall A \in \Gamma$, there is no transition with lefthand-side (q, A).
The automaton $\mathcal{A}$ can be transformed into a level-partitioned automaton $\mathcal{B}_0$, as explained in §2.3. One can then transform $\mathcal{B}_0$ by means of adding a "bottom symbol" at level two allowing to simulate the transitions starting with an empty second level. The resulting automaton $\mathcal{B}$ meets conditions $(\mathbf{P0}), (\mathbf{P1}), (\mathbf{P4})$

Second step

Let us suppose now that $\mathcal{A}$ fulfills $(\mathbf{P0}), \mathbf{P1}), (\mathbf{P4})$.
Let us set $\mathcal{A}' = (Q', \emptyset, \Gamma', \delta', r_0, Z)$ with
$Q' = Q \cup \{\bar{q} \mid q \in Q\} \cup \{r_0, r_1, r_2, r_3\}$ and $\Gamma' = \Gamma \cup \{\bar{A}_1, B_1, B_2\} \cup \{A^q \mid A \in \Gamma, q \in Q\}$
and δ' consists of the following transitions:

- for the precise symbols $A_1, A_2, \ldots A_k, F$ used in $(\mathbf{P1})$
  0.1 $\delta'(r_0, A_1 A_2 \cdots A_k F) = (r_1, push_1(A_1 B_1))$
  0.2 $\delta'(r_1, A_1 A_2 \cdots A_k F) = (q_0, push_2(A_2 B_2))$

- for all $\delta(q, \varepsilon, \omega) = (p, f)$
  1 $\delta'(q, \varepsilon, \omega) = (p, f)$

- for all $A \in \Gamma, \delta(q, a, A\omega) = (p, f)$
  2.1 $\delta'(q, \varepsilon, A\omega) = (r_2, push_1(A_1 A^q)$
  2.2 $\delta'(r_0, \varepsilon, A^q\omega) = (\bar{q}, push_1(A)$
  2.3 $\delta'(\bar{q}, \varepsilon, A\omega) = (p, f)$

- for all $L_1, L_2 \in \Gamma, \omega \in \Gamma^{(k-1)}$
  2.4 $\delta'(r_2, \varepsilon, L_1 L_2 \omega) = (r_2, pop_2)$

- for all $\omega \in \Gamma^{(k-1)}$
  2.5 $\delta'(r_2, \varepsilon, A_1 B_2 \omega) = (r_0, push_1(\bar{A}_1)$
  2.6 $\delta'(r_3, \varepsilon, \bar{A}_1 B_2 \omega) = (r_0, pop_2)$

- for all $A \in \Gamma, \omega \in \Gamma^{(k-1)}$
  3 $\delta'(q_0, \varepsilon, B_1 A \omega) = (r_0, pop_1)$

Informal explanations
By (1) $\mathcal{A}'$ mimics all ε-moves of $\mathcal{A}$. Let $\mathcal{A}$ perform a reading-move: $(q, a, \omega_1) \vdash (p, \varepsilon, \eta_1)$. The automaton $\mathcal{A}'$ pushes a symbol A_1, thus reaching a pds $A_1[\omega_2 \Omega] \cdot \omega_1'$. Then $\mathcal{A}'$ erases ω_2 and substitutes the special symbol $\bar{A}_1$ to A_1, thus reaching a pds $\bar{A}_1[\Omega] \cdot \omega_1'$. Rules (2.1,2.2,2.3) are devised so that, starting from state r_0 and pds ω_1', it can simulate the reading move of $\mathcal{A}$, thus reaching (p, ε, η_1).

Determinism
Let us check that $\mathcal{A}'$ is deterministic: this amounts to check that the modes (i.e

the pairs (state, pushdown-word)) of two different types of transitions are disjoint. For the pair (1),(2.1), there is no common mode since $\mathcal{A}$ is deterministic. For the pair (1),(3), there is no common mode since B_1 is a new pushdown symbol.
For the pair (2.1),(3), the same argument applies.
For the pair (2.4),(2.5), there is no common mode since B_2 is a new pushdown symbol, hence $B_2 \neq L_2$.
For all other pairs, the sets of states on which they apply are disjoint.

Conditions **P1**, **P3**
As the initial automaton $\mathcal{A}$ is level-partitionned, so is $\mathcal{A}'$.
The only lefthand-side of transition where $\bar{A}_1$ occurs, is the l.h.s of (2.6). As its state is different from r_0, condition **P3** is fulfilled.
In order to prove that the automaton $\mathcal{A}'$ fulfills property (**P2**)we establish the sequence of facts 9, 10,11,12,13,14.

Fact 9 *For every* $p, p' \in Q, \omega, \omega' \in (k+1) - \mathrm{pds}(\Gamma')$,
if $(p, \varepsilon, \omega) \vdash^*_{\mathcal{A}} (p', \varepsilon, \omega')$ *then* $(p, \varepsilon, \omega) \vdash^*_{\mathcal{A}'} (p', \varepsilon, \omega')$

This fact can be deduced from transitions (1). Notice that we consider the possibility that ω, ω' contain occurences of letters from $\Gamma' - \Gamma$. The relation $\vdash^*_{\mathcal{A}}$ is defined from the transitions of $\mathcal{A}$, but applied to total states in $Q \times (k{+}1) - \mathrm{pds}(\Gamma')$).
We define the pds:

$$\alpha_n = A_1[A_2[A_3[\cdots[F^n]\cdots]]]; \quad \beta_n = A_2[A_3[\cdots[F^n]\cdots]]; \quad \gamma_n = A_3[\cdots[F^n]\cdots].$$

(if $k = 2$ $\gamma_n = F^n$). We define a map τ :(k+1)$-$ pds(Γ) $\to$(k+1)$-$ pds(Γ') by: for every atom $A[\omega_2]$, with $\omega_2 \in k - \mathrm{pds}(\Gamma')$

$$\tau(A[\omega_2]) = A[\omega_2 B_2[\gamma_n]\Omega]$$

and for every $\omega = \eta_1\eta_2\cdots\eta_\ell$ where η_i are atoms of $(k+1) - \mathrm{pds}(\Gamma)$,

$$\tau(\omega) = \tau(\eta_1)\tau(\eta_2)\cdots\tau(\eta_\ell)B_1[\beta_n].$$

From now on, we call *special* the *pds* which have the form $\tau(\omega)$ for some $\omega \in (k+1) - \mathrm{pds}(\Gamma)$.

Fact 10 *For every* $p, p' \in Q, u, u' \in a^*, \omega, \omega' \in k+1 - \mathrm{pds}(\Gamma)$,
if $(p, u, \omega) \vdash^*_{\mathcal{A}} (p', u', \omega')$ *then* $(p, u, \tau(\omega)) \vdash^*_{\mathcal{A}} (p', u', \tau(\omega'))$

This fact holds because, as $\mathcal{A}$ fulfills (**P4**), the first computation cannot use the information that the list of top-symbols has length one. Hence the first computation is mapped by τ into the second one.

Fact 11 *For every* $p, p' \in Q, q \in Q', \omega, \omega'$ *special pds,*
if $(p, a, \omega) \vdash_{\mathcal{A}} (p', \varepsilon, \omega')$ *then* $(p, \omega, q) \to^*_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0)(p', \omega', q)$.

Let us prove this fact.The pds ω is special, hence, there exist $\omega_1 \in (k+1)-\mathrm{pds}(\Gamma), \omega_2 \in (k)-\mathrm{pds}(\Gamma), \omega_3 \in (k-1)-\mathrm{pds}(\Gamma)$ such that:

$$\omega = A[\omega_2 B_2[\omega_3]\Omega] \cdot \omega_1.$$

The hypothesis of the fact shows that, by a transition (2.1),

$$(p, \varepsilon, \omega) \vdash_{\mathcal{A}'} (r_2, \varepsilon, A_1[\omega_2 B_2[\omega_3]\Omega] \cdot A^q[\omega_2 B_2[\omega_3]\Omega] \cdot \omega_1),$$

hence, for every $q \in Q$,

$$(p, \omega, q) \rightarrow_{\mathcal{A}'} v(r_2, A_1[\omega_2 B_2[\omega_3]\Omega], r_0) \cdot (r_0, A^q[\omega_2 B_2[\omega_3]\Omega]\omega_1, q) \tag{12}$$

By transitions (2.4)(2.5)(2.6)

$$(r_2, A_1[\omega_2 B_2[\omega_3]\Omega], r_0) \rightarrow^*_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0) \tag{13}$$

and by transitions (2.2)(2.3)

$$(r_0, A^q[\omega_2 B_2[\omega_3]\Omega]\omega_1, q) \rightarrow^*_{\mathcal{A}'} (p', \omega', q) \tag{14}$$

Composing the three derivations (12,13,14), we obtain the conclusion of fact 11.

Fact 12 $(q_0, A_1[\beta_n B_2[\gamma_n]\Omega] B_1[\beta_n], r_0) \rightarrow^*_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0)^{f(n)} \cdot (q_0, B_1[\beta_n], r_0)$.

Let us show this fact, reformulated as:

$$(q_0, \tau(\alpha_n), r_0) \rightarrow^*_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0)^{f(n)} \cdot (q_0, \tau(\varepsilon), r_0) \tag{15}$$

By hypothesis

$$(q_0, a^{f(n)}, \alpha_n) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon)$$

Fact 10 allows to deduce that

$$(q_0, a^{f(n)}, \tau(\alpha_n)) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \tau(\varepsilon)).$$

This computation can be factorised into $f(n)+1$ subcomputations:

$$(p_i, \varepsilon, \tau(\eta_i)) \vdash^*_{\mathcal{A}} (s_i, \varepsilon, \tau(\omega_i));\;\; (s_i, a, \tau(\omega_i)) \vdash_{\mathcal{A}} (p_{i+1}, \varepsilon, \tau(\eta_{i+1})) \tag{16}$$

for $0 \leq i \leq f(n)-1$ with $p_0 = q_0$ (the initial state of $\mathcal{A}$), and

$$(p_{f(n)}, \varepsilon, \tau(\eta_{f(n)})) \vdash^*_{\mathcal{A}} (s_{f(n)}, \varepsilon, \tau(\varepsilon)) \tag{17}$$

with $s_{f(n)} = q_0$. Via facts (9,11), for every $q \in Q'$, the above computations translate into the derivations:

$$(p_i, \tau(\eta_i), q) \rightarrow^*_{\mathcal{A}'} (s_i, \tau(\omega_i), q) \rightarrow_{\mathcal{A}'} (r_0, A_1[\Omega], r_0)(p_{i+1}, \tau(\eta_{i+1}), q) \tag{18}$$

(for $0 \leq i \leq f(n)-1$ with $p_0 = q_0$) and

$$(p_{f(n)}, \tau(\eta_{f(n)}, q)) \rightarrow^*_{\mathcal{A}'} (s_{f(n)}, \tau(\varepsilon), q) = (q_0, \tau(\varepsilon), q). \tag{19}$$

The composition of all derivations (18,19), for $q = r_0$, gives derivation (15).

Fact 13 $(r_0, A_1[\beta_n \cdot \Omega], r_0) \to^*_{\mathcal{A}'} (q_0, A_1[\beta_n]B_2[\gamma_n]\Omega]B_1[\beta_n], r_0)$.

This is just the grammatical counterpart of transitions (0.1)(0.2).

Fact 14 $(q_0, B_1[\beta_n], r_0) \to_{\mathcal{A}'} \quad \varepsilon$.

This translates transition (3).
Proof: Let us combine into one derivation the derivations given in Fact 13, Fact 12 and finally Fact 14. We obtain the required derivation:

$$(r_0, A_1[\beta_n \cdot \Omega], r_0) \to^*_{\mathcal{A}'} (r_0, \bar{A}_1[\Omega], r_0)^{f(n)}.$$

Up to a renaming of the states, (**P2**) holds for $\mathcal{A}'$. □

Proposition 15 (Sum) *If* $f, g \in \mathbb{S}_{k+1}$, $k \geq 2$ *then* $f + g \in \mathbb{S}_{k+1}$.

Proof: Let $\mathcal{A}, \mathcal{A}'$ be two $k+1$-dcp computing respectively f and g. We assume

$$q_0 A_1[\beta_n] q_0 \to^*_{\mathcal{A}} a^{f(n)}; \quad q_0 A'_1[\beta_n] q_0 \to^*_{\mathcal{A}'} a^{g(n)}.$$

It suffices to construct a $k+1$-dcp $\mathcal{B}$ performing the following computation: starting from total-state $r_0 D_1[\beta_n]$, by a $push_1$ operation it moves to $q_0 A_1[\beta_n] A'_1[\beta_n]$, where q_0 is the common starting (and final) state of $\mathcal{A}, \mathcal{A}'$. Then it simulates $\mathcal{A}$ on $q_0 A_1[\beta_n]$ and finally, it simulates $\mathcal{A}'$ on $q_0 A'_1[\beta_n]$. □

Proposition 16 (Ordinary product) *If* $f, g \in \mathbb{S}_{k+1}$, $k \geq 2$ *then* $f \odot g \in \mathbb{S}_{k+1}$.

Construction: By lemma 8, after a suitable choice for the concrete sets of states and pushdown alphabets, we obtain two $(k+1)$-dcp $\mathcal{A}_1$ and $\mathcal{A}_2$ fulfilling conditions:
(**P1**) $\mathcal{A}_1, \mathcal{A}_2$ are level-partitioned
(**P2.1**) for every letter $\Omega \notin \Gamma_1, (q_0, A_1[A_2[\ldots[A_k[F^n]]\ldots]\Omega], q_0) \to^*_{\mathcal{A}} (q_0, \bar{A}_1[\Omega], q_0)^{f(n)}$.
(**P2.2**) for every letter $\Omega \notin \Gamma_2, (q_0, \bar{A}_1[A_2[\ldots[A_k[F^n]]\ldots]\Omega], q_0) \to^*_{\mathcal{A}} (q_0, A'_1[\Omega], q_0)^{g(n)}$.
(**P3.1**) δ_1 has no lefthand-side of the form $(q_0, \bar{A}_1 \cdot \omega)$ for any $\omega \in \Gamma_1^{(k)} \cup \{\varepsilon\}$.
(**P3.2**) δ_2 has no lefthand-side of the form $(q_0, A'_1 \cdot \omega)$ for any $\omega \in \Gamma_2^{(k)} \cup \{\varepsilon\}$.
(**P4**) $Q_1 \cap Q_2 = \{q_0\}$.
(**P5**) $\Gamma_1 \cap \Gamma_2 = \{\bar{A}_1, A_2, A_3 \ldots A_k, F\}$.
We consider the $(k+1)$-dcp $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ where: $Q = Q_1 \cup Q_2 \cup \{r_0\}$, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{D_1\}$ and δ is the union of $\delta_1 \cup \delta_2$ with the additional transitions:

1.1 $\delta(q_0, \varepsilon, D_1 A_2 \ldots A_k) = (r_0, push_2(A_2))$
1.2 $\delta(r_0, \varepsilon, D_1 A_2 \ldots A_k) = (q_0, push_1(A_1))$
2 $\delta(q_0, \varepsilon, A'_1) = (q_0, pop_1)$

Proof: The automaton $\mathcal{A}$ is deterministic:
let us consider $(q_1, a_\varepsilon, \omega_1)$ l.h.s. of a rule of δ_1, $(q_2, b_\varepsilon, \omega_2)$ l.h.s. of a rule of δ_2. Both can apply on the same total state only if $q_1 = q_2$ and $\omega_1 = \omega_2$. In that

case $q_1 = q_2 = q_0$ and $\omega_1 = \omega_2 \in \bar{A}_1 A_2 \cdots A_k \cdot \{F, \varepsilon\}$. But condition **P3.1** makes impossible such a l.h.s. for δ_1. Transition (1.2) is the only one starting from state r_0. No transition from δ_1 uses symbol A'_1 and, by **P3.2**, no transition from δ_2 starts from (q_0, A'_1).
Let us show now that it computes $f \odot g$. For every $n \geq 0$, we note $\beta_n = A_2[\cdots[A_k[F^n]]\cdots]$. The following derivation holds:

$$\begin{array}{rcl} (q_0 D_1[\beta_n] q_0) \rightarrow_{\mathcal{A}} & (q_0 A_1[\beta_n \beta_n] q_0) & (\text{ by } (1.1,1.2)) \\ \rightarrow^*_{\mathcal{A}} & (q_0 \bar{A}_1[\beta_n] q_0)^{f(n)} & (\text{ by } \mathbf{P2.1}) \\ \rightarrow^*_{\mathcal{A}} & (q_0 A'_1[\varepsilon] q_0)^{f(n) \cdot g(n)} & (\text{ by } \mathbf{P2.2}) \\ \rightarrow^*_{\mathcal{A}} & a^{f(n) \cdot g(n)} & (\text{ by } (2)) \end{array}$$

□

Proposition 17 *If $f \in \mathbb{S}_{k+1}$, $k \geq 2$, and g is the sequence defined by $g(0) = c$ and $g(n+1) = f(n).g(n)^d$, $d \geq 1$, then $g \in \mathbb{S}_{k+1}$.*

Proof: There exists a $(k+1)$-dcp $\mathcal{A}_1 = (Q_1, \{a\}, \Gamma_1, \delta_1, q_0, Z)$ fulfilling conditions $(\mathbf{P1}), (\mathbf{P2}), (\mathbf{P3})$ stated in lemma 8. We consider the $(k+1)$-dcp $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ where: $Q = Q_1 \cup \{r_0\}$, $\Gamma = \Gamma_1 \cup \{D_1\}$ and δ is the union of δ_1 with the additional transitions:

0.1 $\delta(q_0, \varepsilon, D_1 A_2 \ldots A_k F) = (r_0, pop_{k+1})$
0.2 $\delta(r_0, \varepsilon, D_1 A_2 \ldots A_k) = (q_0, push_2(A_2^{d+1}))$
0.3 $\delta(q_0, \varepsilon, D_1 A_2 \ldots A_k) = (q_0, pop_2)$
1 $\delta(q_0, \varepsilon, \bar{A}_1 A_2 \ldots A_k F) = \delta(q_0, \varepsilon, \bar{A}_1 A_2 \ldots A_k) = (q_0, push_1(D_1))$
2 $\delta(q_0, a, D_1) = (q_0, pop_1)$

This automaton is deterministic: δ_1 is a deterministic transition map and, by condition **P3**, transition (1) does not break this determinism. Moreover transitions (0.i) and (2) cannot interfer with δ_1 (since D_1 is a new letter), and cannot interfer one with each other.
In order to show that $\mathcal{A}$ does compute g, we summarize some interesting basic derivations:
By (**P2**):

$$(q_0 A_1[\beta_n \Omega] q_0) \rightarrow^*_{\mathcal{A}} (q_0 A_1[\beta_n \Omega] q_0)^{f(n)}$$

Starting rules: using transitions (0.1,0.2)

$$(q_0 D_1[\beta_{n+1} \Omega] q_0) \rightarrow^*_{\mathcal{A}} (q_0 A_1[\beta_n^{d+1} \Omega] q_0)$$

and using transition (0.3)

$$(q_0 D_1[\beta_0 \Omega] q_0) \rightarrow^*_{\mathcal{A}} (q_0 D_1[\Omega] q_0)^c$$

Gluing rule: using transition (1), for every $n \geq 0$

$$(q_0 \bar{A}_1[\beta_n \Omega] q_0) \rightarrow^*_{\mathcal{A}} (q_0 D_1[\beta_n \Omega] q_0)$$

Ending rule: using transition (2)

$$(q_0 D_1[\varepsilon]q_0) \rightarrow_{\mathcal{A}} a$$

Let us prove, by induction over $n \geq 0$, the following property $\mathbf{P}(n)$:

$$(q_0 D_1[\beta_n \Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 D_1[\Omega]q_0)^{g(n)}$$

Basis:$n = 0$.
The second starting rule proves $\mathbf{P}(0)$.
Induction step
We exhibit the derivation:

$$\begin{array}{rlll} (q_0 D_1[\beta_{n+1}\Omega]q_0) \rightarrow^*_{\mathcal{A}} & (q_0 A_1[\beta_n^{d+1}\Omega]q_0) & (\text{ by first starting rule}) & \\ \rightarrow^*_{\mathcal{A}} & (q_0 \bar{A}_1[\beta_n^d \Omega]q_0)^{f(n)} & (\text{ by } \mathbf{P2}) & \\ \rightarrow^*_{\mathcal{A}} & (q_0 D_1[\beta_n^d \Omega]q_0)^{f(n)\cdot g(n)} & (\text{ by the gluing rule}) & (20) \end{array}$$

Applying d times $\mathbf{P}(n)$ (with accurate substitutions to the undeterminate Ω) we get that:

$$(q_0 D_1[\beta_n^d \Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 D_1[\Omega]q_0)^{g(n)^d} \qquad (21)$$

Composing derivations (20) and (21) we obtain:

$$(q_0 D_1[\beta_{n+1}\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 D_1[\Omega]q_0)^{f(n)\cdot g(n)^d} = (q_0 D_1[\Omega]q_0)^{g(n+1)}$$

Hence $\mathbf{P}(n+1)$ is proved.
Combining $\mathbf{P}(n)$ with the ending rule, we deduce that, for every $n \geq 0$,

$$(q_0 D_1[\beta_n]q_0) \rightarrow^*_{\mathcal{A}} a^{g(n)}$$

□
Let us notice that,by proposition 15 and proposition 16, $(\mathbb{S}_k, +, \cdot)$ is a semi-ring.
We denote by $P(n, X_1, \ldots, X_j, \ldots, X_p)$ any element of the semi-ring $\mathbb{S}_k[X_1, \ldots, X_j, \ldots, X_p]$ to emphasize the fact that the coefficients of P are functions of the integer argument n.

Proposition 18 *Let $k \geq 2$. Let $P_i(n, X_1, \ldots, X_p)$, $1 \leq i \leq p$ be polynomials with coefficients in $\mathbb{S}_{k+1}$ and u_i, for $1 \leq i \leq p$, be sequences defined by $u_i(n+1) = P_i(n, u_1(n), \ldots, u_p(n))$, and $u_i(0) = c_i$. Then $u_1 \in \mathbb{S}_{k+1}$.*

Sketch of proof: Let us suppose that for every $1 \leq i \leq p$,

$$P_i(X_1, \ldots X_p) = \sum_{j=0}^{\nu_i} a_{i,j}(n) X_1^{d_{i,j,1}} \ldots X_p^{d_{i,j,p}}.$$

Every coefficient $a_{i,j}(n)$ is computed by some $(k+1)$-dcp $\mathcal{A}_{i,j} = (Q_{i,j}, \{a\}, \Gamma_{i,j}, \delta_{i,j}, q_0, Z_{i,j})$ fulfilling conditions $(\mathbf{P1}), (\mathbf{P2}), (\mathbf{P3})$ stated in lemma 8. A suitable choice of the sets involved in the definition of these automata can be made so that, for every $(i,j) \neq (i',j')$

$$Q_{i,j} \cap Q_{i',j'} = \{q_0\}; \;\; \varGamma_{i,j} \cap \varGamma_{i',j'} = \{A_2, \ldots, A_k, F\}.$$

We assume that, for every (i,j):

$$Q \supseteq Q_{i,j}; \;\; \varGamma \supseteq \varGamma_{i,j}; \;\; \delta \supseteq \delta_{i,j};$$

and $\varGamma$ posesses some additional symbols $U_1, U_2, \ldots, U_p$ of level 2, and A of level 1. Suppose that the transitions allow the following basic derivations:
Coefficient rules:

$$(q_0 A_{i,j}[\beta_n \varOmega] q_0) \rightarrow^*_{\mathcal{A}_{i,j}} (q_0 \bar{A}_{i,j}[\varOmega] q_0)^{a_{i,j}(n)}$$

(this is just condition (**P2**) for the automata $\mathcal{A}_{i,j}$)
Starting rules:

$$(q_0 A[U_i[\gamma_{n+1}]\varOmega] q_0) \rightarrow^*_{\mathcal{A}} (q_0 A_{i,0}[\beta_n^2 \varOmega] A_{i,1}[\beta_n^2 \varOmega] \cdots A_{i,\nu_i}[\beta_n^2 \varOmega] q_0)$$

and

$$(q_0 A[U_i[\gamma_0]\varOmega] q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[\varOmega] q_0)^{c_i}$$

Gluing rules: for every $n \geq 0$, the gluing rule (**Gij**) is:

$$(q_0 \bar{A}_{i,j}[\beta_n \varOmega] q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[(\prod_{\ell=1}^{p} U_\ell[\gamma_n]^{d_{i,j,\ell}}) \varOmega] q_0)$$

Ending rule:

$$(q_0 A[\varepsilon] q_0) \rightarrow_{\mathcal{A}} a$$

Let us consider property $\mathbf{P}(n)$ defined by:

$$\forall i \in [1,p], (q_0 A[U_i[\gamma_n]\varOmega] q_0) \rightarrow^*_{\mathcal{A}} (q_0 A[\varOmega] q_0)^{u_i(n)}.$$

Property $\mathbf{P}(n)$ can be proved by induction on n, under the assumption that the coefficient rules, the starting rules , the gluing rules and the ending rules are valid. Leaning on the normalisation property (**P3**), it is possible to add transitions to the union of the $\delta_{i,j}$ in such a way that all these rule are made valid and the automaton $\mathcal{A}$ remains deterministic:
- the different modes of the different lefthand-sides of the rules given above are distinct
- it suffices thus to decompose each of these rules into a finite sequence of elementary moves, using disjoint sets of states for the intermediate total states of the different rules, to obtain a *deterministic* dcp. □

Proposition 19 *Let $f \in \mathbb{S}_{k+1}, g \in \mathbb{S}_k$, $k \geq 3$, Then, the sequence h defined for all $n \geq 0$ by $h(n) = f(n)^{g(n)}$ belongs to $\mathbb{S}_{k+1}$.*

Proof: Let us proceed as in the proof of proposition 17 : we expose, in a first step, a list of particular derivations (that we call "rules") and prove that these rules are sufficient to compute the required sequence; in a second step, we explain how to construct an automaton which makes these rules available.

First step

Let $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ be some $(k+1)$-dcpa fulfilling condition (**P1**) stated in lemma 8. We suppose that $\Gamma \supseteq \{A_1, \bar{A}_1, A_2, \ldots, A_k, F\} \cup \{B_2, \bar{B}_2\}$, where the levels are given by the indices. For every operation f on $k - \mathrm{pds}(\Gamma)$, we define the operation $A_1 \cdot f$ as:

$$A_1 \cdot push_i(L) = push_{i+1}(L), A_1 \cdot pop_i = pop_{i+1}.$$

Let us define: $A_1^{-1}\mathcal{A} = (Q, \{a\}, \Gamma, A_1^{-1}\delta, q_0, Z)$, where

$$A_1^{-1}\delta = \{(q, a_\varepsilon, \omega, q', \omega') \in Q \times \{a, \varepsilon\} \times k{-}\mathrm{pds}(\Gamma) \times Q \times (k{+}1){-}\mathrm{pds}(\Gamma) \mid (q, a_\varepsilon, A_1\omega, q', A_1 f) \in \delta\}.$$

We suppose that $A_1^{-1}\mathcal{A}$ fulfills condition **P3** of lemma 8, for the state r_0 and the letter of level 1, $\bar{B}_2$ (notice $\bar{B}_2$ has level 2 in $\mathcal{A}$). Let us suppose that $\mathcal{A}$ allows the following basic derivations (where Ω is an undeterminate):

f-computation, D1:

$$(q_0 A_1[A_2[\gamma_n]\Omega]q_0) \rightarrow^*_{\mathcal{A}} (q_0 \bar{A}_1[\Omega]q_0)^{f(n)}$$

g-computation, D2:

$$(r_0 B_2[\gamma_n \Omega]r_0) \rightarrow^*_{A_1^{-1}\mathcal{A}} (r_0 \bar{B}_2[\Omega]r_0)^{g(n)}$$

Gluing rule, G21: $\forall \omega_3 \in (k-1) - \mathrm{pds}(\Gamma)$,

$$(r_0 A_1[\bar{B}_2[\omega_3]\Omega]q_0) \rightarrow_{\mathcal{A}} (q_0 A_1[A_2[\omega_3]\Omega]q_0)$$

Gluing rule, $G^{(0)}21$:

$$(r_0 A_1[\varepsilon]q_0) \rightarrow_{\mathcal{A}} (q_0 \bar{A}_1[\varepsilon]q_0)$$

Gluing rule, G12: for every $\omega_3 \in (k-1) - \mathrm{pds}(\Gamma)$ and $L_2 \in \Gamma, L_2 \neq \bar{B}_2$,

$$(q_0 \bar{A}_1[L_2[\omega_3]\Omega]q_0) \rightarrow_{\mathcal{A}} (r_0 A_1[L_2[\omega_3]\Omega]q_0).$$

Ending rule, E: for every $\omega_3 \in (k-1) - \mathrm{pds}(\Gamma)$ and $L_2 \in \Gamma, L_2 \neq \bar{B}_2$,

$$(q_0 \bar{A}_1[\varepsilon]q_0) \rightarrow_{\mathcal{A}} a.$$

The intuition behind these rules is that the gluing rule Gij allows to connect the end of a computation Di with the beginning of a computation Dj. The special gluing rule, $G^{(0)}21$ handles the case where the computation D2 results in the number 0, leading to the value $f(n)^0 = 1.(*)$[1]

[1] we adopt the convention that $0^0 = 1$ in the definition of $h = f^g$

Let us prove by induction over $p \geq 1$ the following property $\mathcal{P}(p)$:
for every $H_p \in k-\mathrm{pds}(\Gamma)$, which does not have $\bar{B}_2$ as leftmost head-symbol, if

$$(r_0 H_p r_0) \rightarrow^*_{A_1{}^{-1}\mathcal{A}} (r_0 \bar{B}_2[\gamma_n] r_0)^p \tag{22}$$

then

$$(q_0 \bar{A}_1[H_p] q_0) \rightarrow^*_{\mathcal{A}} (q_0 \bar{A}_1[\varepsilon] q_0)^{f(n)^p}. \tag{23}$$

Basis:$p = 0$
We suppose that (22) holds. We then exhibit the derivation:

$$\begin{aligned}(q_0 \bar{A}_1[H_0] q_0) &\rightarrow_{\mathcal{A}} (r_0 A_1[H_0] q_0) \text{ (by rule G12, notice } \bar{B}_2 \text{ is not the leftmost head-symbol)}\\ &\rightarrow^*_{\mathcal{A}} (r_0 A_1[\varepsilon] q_0) \text{ (by hypothesis (22))}\\ &\rightarrow^*_{\mathcal{A}} (q_0 \bar{A}_1[\varepsilon] q_0) \text{ (by rule } G^{(0)}21).\end{aligned}$$

Induction step:
We suppose that hypothesis (22) is fulfilled by $p+1$ and that $\mathcal{P}(p)$ holds. By means of lemma 1 we can translate hypothesis (22) into: there exists some $H_p \in k-\mathrm{pds}(\Gamma)$ such that

$$(r_0, \varepsilon, H_{p+1}) \vdash^*_{A_1{}^{-1}\mathcal{A}} (r_0, \varepsilon, \bar{B}_2[\gamma_n] H_p) \text{ and } (r_0 H_p r_0) \rightarrow^*_{A_1{}^{-1}\mathcal{A}} (r_0 \bar{B}_2[\gamma_n] r_0)^p$$

We exhibit the derivation:

$$\begin{aligned}(q_0 \bar{A}_1[H_{p+1}] q_0) &\rightarrow^*_{\mathcal{A}} (r_0 A_1[H_{p+1}] q_0) \text{ (by rule G12)}\\ &\rightarrow^*_{\mathcal{A}} (r_0 A_1[\bar{B}_2[\gamma_n] H_p] q_0) \text{ (by above translation)}\\ &\rightarrow^*_{\mathcal{A}} (q_0 A_1[A_2[\gamma_n] H_p] q_0) \text{ (by rule G21)}\\ &\rightarrow^*_{\mathcal{A}} (q_0 \bar{A}_1[H_p] q_0)^{f(n)} \text{ (by D1)}\end{aligned} \tag{24}$$

Combining this derivation with $\mathcal{P}(p)$, we get:

$$(q_0 \bar{A}_1[H_{p+1}] q_0) \rightarrow^*_{\mathcal{A}} (q_0 \bar{A}_1[\varepsilon] q_0)^{f(n)^{p+1}},$$

(end of the induction).
Let us consider $H = B_2[\gamma_n \gamma_n]$. By derivation $D2$, H fulfills hypothesis (22) for the integer $p = g(n)$. Hence, by $\mathcal{P}(p)$,

$$(q_0 \bar{A}_1[B_2[\gamma_n \gamma_n]] q_0) \rightarrow^*_{\mathcal{A}} (q_0 \bar{A}_1[\varepsilon] q_0)^{f(n)^{g(n)}}. \tag{25}$$

Second step
Let us construct such an automaton $\mathcal{A}$. The sequence $f(n)$ is computed by some $(k+1)$-dcp $\mathcal{A}_1 = (Q_1, \{a\}, \Gamma_1, \delta_1, q_0, Z)$ fulfilling conditions $(\mathbf{P1}), (\mathbf{P2}), (\mathbf{P3})$ stated in lemma 8. As well The sequence $g(n)$ is computed by some k-dcp $\mathcal{A}_2 = (Q_2, \{a\}, \Gamma_2, \delta_2, r_0, Z)$ fulfilling the same conditions. We suppose that $\Gamma_1 \cap \Gamma_2 = \{A_3, \ldots, A_k, F\}$. where A_i has level i for $\mathcal{A}_1$ (resp. level $i-1$ for $\mathcal{A}_2$) and F has level $k+1$ for $\mathcal{A}_1$ (resp. level k for $\mathcal{A}_2$). Let us define $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, s_0, Z)$ where

$$Q = Q_1 \cup Q_2 \cup \{s_1\}; \quad \Gamma = \Gamma_1 \cup \Gamma_2 \cup \{D_1\};$$

δ is the union of $\delta_1 \cup (A_1 \cdot \delta_2)$ with the following rules:

0.1 $\delta(q_0,\varepsilon,D_1A_2\dots A_k)=\delta(q_0,\varepsilon,A_1A_2\dots A_kF)=(s_1,push_3(B_2B_2))$
0.2 $\delta(s_1,\varepsilon,D_1A_2\dots A_k)=\delta(s_1,\varepsilon,D_1A_2\dots A_kF)=(q_0,push_1(\bar{A}_1))$
2.1.0 $\delta(r_0,\varepsilon,A_1)=(q_0,\bar{A}_1)$
2.1 $\delta(r_0,\varepsilon,A_1\bar{B}_2\omega)=(q_0,push_1(A_2))$ for every $\omega\in\Gamma^{(k-1)}\cup\{\varepsilon\}$
1.2 $\delta(q_0,\varepsilon,\bar{A}_1L_2\omega)=(r_0,push_1(A_1))$ for every $\omega\in\Gamma^{(k-1)}\cup\{\varepsilon\},L_2\neq\bar{B}_2$
3 $\delta(q_0,a,\bar{A}_1)=(q_0,\varepsilon)$.

Due to conditions **P3** for the initial automata $\mathcal{A}_i$, transitions (2.1),(1.2),(3) do not introduce any non-determinism. Transition (2.1.0) uses a mode (r_0,A_1) which is not used in $A_1\cdot\delta_2$. Transitions (0.1),(0.2) use a new pushdown symbol. Thus $\mathcal{A}$ is deterministic.
The transitions are choosen so as to make the rules (described in first step) available: D_1 holds by the choice of δ_1, D_2 holds by the choice of δ_2, $G21$ holds by transitions (2.1), $G^{(0)}21$ holds by transition (2.1.0), $G12$ holds by transitions (1.2), E holds by rule (3).
From the initial rules (0.1,0.2), property 25 and the ending rule, we get:

$$(q_0D_1[\beta_n]q_0)\rightarrow^*_{\mathcal{A}}(q_0\bar{A}_1[B_2[\gamma_n\gamma_n]]q_0)\rightarrow^*_{\mathcal{A}}a^{f(n)^{g(n)}}.$$

□

Proposition 20 (convolution-product) *Let $f\in\mathbb{S}_{k+1}$ and $g\in\mathbb{S}_k$, for $k\geq 3$. Then $f\times g\in\mathbb{S}_{k+1}$ where $f\times g$ denotes the convolution-product:*

$$(f\times g)(n)=\sum_{m=0}^{n}f(n-m).g(m)\ \textit{for all}\ n\in\mathbb{N}$$

Proof: The proof of this proposition uses the same kind of argument as proposition 16 concerning the product. We just have to combine the construction given there with a set of rules generating the sequence of pairs $(0,n),(1,n-1)\dots(n,0)$.
First step
Let us suppose we are given a $(k+1)$-dcpa $\mathcal{A}=(Q,\{a\},\Gamma,\delta,q_0,Z)$ fulfilling condition (**P1**) stated in lemma 8. We suppose that $\Gamma\supseteq\{D_1,A_1,A_2,\dots,A_k,G,F,E\}$, where the levels are given by the indices, and G,E have level k, F has level $k+1$. As usual the letters Ω,Ω',Ω'' used below are undeterminates. The letters G,E will constitute the counters for the sequence g while the letter F will be used in the counters for the sequence f. Let us use the notations: for every $0\leq m\leq n$,

$$\beta_n=A_2[\cdots[A_k[F^n]]\cdots];\ \ \gamma_{m,n}=G[F^m](\prod_{j=m+1}^{n}E[F^j])\cdot E[F^n]$$

(notice that $|\gamma_{m,n}|=n-m+2$),

$$\omega_{k-2}[\Omega]=A_2[\cdots[A_{k-1}[\Omega]]\cdots];$$

Let us suppose that $\mathcal{A}$ allows the following basic derivations:
Initial derivation, D0: for every $n \geq 0$

$$(q_0 D_1[\beta_n \Omega] q_0) \to^*_{\mathcal{A}} (q_0 A_1[(\omega_{k-2}[\gamma_{n,n}])^2 \Omega] q_0)$$

f-computation, D1: for every $\ell \geq 0$

$$(q_0 A_1[\omega_{k-2}[G[F^n]E[\Omega']\Omega'']\Omega]q_0 \to^*_{\mathcal{A}} (q_0 \bar{A}_1[\Omega]q_0)^{f(n)}$$

g-computation, D2: for every $\ell \geq 2, \eta_1, \eta_2, \ldots, \eta_\ell \in F^*$,

$$(q_0 \bar{A}_1[\omega_{k-2}[G[\eta_1]E[\eta_2]\cdots E[\eta_\ell]]\Omega]q_0 \to^*_{\mathcal{A}} (q_0 A_1'[\Omega]q_0)^{g(\ell-2)}$$

pairs-generation, D3: for every $1 \leq m \leq n$

$$(q_0 A_1[\omega_{k-2}[\gamma_{m,n}]\Omega]q_0) \to^*_{\mathcal{A}} (q_0 A_1[\omega_{k-2}[\gamma_{m-1,n}]\Omega]q_0)(q_0 A_1[\omega_{k-2}[\gamma_{m-1,n}]\Omega]q_0)(q_0 A_1[\omega_{k-2}[\gamma_{m,n}]\Omega]q_0)$$

ending rule, D4:

$$(q_0 A_1'[\varepsilon]q_0) \to^*_{\mathcal{A}} a.$$

From these rules the following derivations would follow:

$$\begin{array}{rcl}
(q_0 D_1[\beta_n]q_0) \to^*_{\mathcal{A}} & (q_0 A_1[(\omega_{k-2}[\gamma_{n,n}])^2]q_0) & (\text{ by rule D0 }) \\
\to^*_{\mathcal{A}} & (q_0 \bar{A}_1[\gamma_{n,n}]q_0)^{f(n)} & (\text{ by D1 }) \\
\to^*_{\mathcal{A}} & (q_0 A_1'[\varepsilon]q_0)^{f(m)\cdot g(n-m)} & (\text{ by D2}).
\end{array}$$

Using $(n+1)$ times derivation D3 we get:

$$\begin{array}{rcl}
(q_0 A_1[\omega_{k-2}[\gamma_{n,n}]\omega_{k-2}[\gamma_{n,n}]]q_0) \to^*_{\mathcal{A}} & \prod_{m=0}^{n}(q_0 A_1[(\omega_{k-2}[\gamma_{m,n}])^2]q_0) & \\
\to^*_{\mathcal{A}} & \prod_{m=0}^{n}(q_0 A_1'[\varepsilon]q_0)^{f(m)\cdot g(n-m)} & (\text{ by the above derivation }) \\
= & (q_0 A_1'[\varepsilon]q_0)^{(f\times g)(n)} & \\
\to^*_{\mathcal{A}} & a^{(f\times g)(n)} & (\text{ by D4 }).
\end{array}$$

Second step
Let us construct such an automaton $\mathcal{A}$. The sequence $f(n)$ is computed by some $(k+1)$-dcp $\mathcal{A}_1' = (Q_1, \{a\}, \Gamma_1, \delta_1, q_0, Z)$, where $\Gamma_1 \supseteq \{A_1, \bar{A}_1, A_2, \ldots, A_{k-1}, G, E, F\}$ fulfilling conditions $(\mathbf{P1}), (\mathbf{P2}), (\mathbf{P3})$ stated in lemma 8. By a variation of the construction given for lemma 8, we can build $\mathcal{A}_1$ fulfilling $(\mathbf{P1}), (D1), (\mathbf{P3})$: the main idea is to treat symbol E as if it was a bottom symbol of level k; G plays the role of A_k in lemma 8.
As well the sequence $g(n)$ is computed by some k-dcp $\mathcal{A}_2 = (Q_2, \{a\}, \Gamma_2, \delta_2, r_0, Z)$,where $\Gamma_2 \supseteq \{\bar{A}_1, A_1', A_2, \ldots A_{k-1}, G, E, F\}$, fulfilling conditions $(\mathbf{P1}), (D2), (\mathbf{P3})$. The main idea here, is to start with the automaton $\mathcal{A}_2'$ fulfilling $(\mathbf{P1}), (\mathbf{P2}), (\mathbf{P3})$, and to replace the single symbol F by two symbols E, G, of level k, playing the same role as F did. We then add a symbol F of level $k+1$ and just "ignore it": every occurrence of F can be changed into ε without any effect on derivation

(D2). We choose the alphabets so that $\Gamma_1 \cap \Gamma_2 = \{\bar{A}_1, A_2, \ldots, A_{k-1}, G, E, F\}$. Let us define $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, s_0, Z)$ where

$$Q = Q_1 \cup Q_2 \cup \{r_1, r_2, r_3, s_1, s_2\}; \quad \Gamma = \Gamma_1 \cup \Gamma_2 \cup \{D_1\};$$

δ is the union of $(\delta_1 \cup \delta_2)$ with the following rules:

0.1 $\delta(q_0, \varepsilon, D_1 A_2 \ldots A_k) = \delta(q_0, \varepsilon, D_1 A_2 \ldots A_k F) = (s_1, push_k(GE))$
0.2 $\delta(s_1, \varepsilon, D_1 A_2 \ldots A_{k-1} G) = \delta(s_1, \varepsilon, D_1 A_2 \ldots A_{k-1} GF) = (s_2, push_2(A_2 A_2))$
0.3 $\delta(s_2, \varepsilon, D_1 A_2 \ldots A_{k-1} G) = \delta(s_2, \varepsilon, D_1 A_2 \ldots A_{k-1} GF) = (q_0, push_1(A_1))$
3.1 $\delta(q_0, \varepsilon, A_1 A_2 \ldots A_{k-1} GF) = (r_1, push_1(A_1 A_1))$
3.2 $\delta(r_1, \varepsilon, A_1 A_2 \ldots A_{k-1} GF) = (r_2, push_k(GE))$
3.3 $\delta(r_2, \varepsilon, A_1 A_2 \ldots A_{k-1} GF) = (r_3, pop_{k+1})$
3.4 $\delta(r_3, \varepsilon, A_1 A_2 \ldots A_{k-1} G) = \delta(r_3, \varepsilon, A_1 A_2 \ldots A_{k-1} GF) = (q_0, push_1(A_1 A_1)$
4 $\delta(q_0, a, A_1') = (q_0, pop_1)$.

Due to conditions (**P3**) for the initial automata $\mathcal{A}_i$, this new automaton $\mathcal{A}$ is still deterministic.
The transitions are choosen so as to make the rules (described in first step) available: (D0) holds by the choice of rules (0.i), (D1) by the choice of δ_1, (D2) by the choice of δ_2, (D3) by transitions (3.j), (D4) by transition (4). We can conclude that $\mathcal{A}$ computes $f \times g$. $\Box$

Proposition 21 (convolution-inverse) *Let $g \in \mathbb{S}_k$, $k \geq 3$, and f be the sequence defined by $f(0) = 1$ and for all $n \geq 0$, $f(n+1) = \sum_{m=0}^{n} f(m) g(n-m)$. Then $f \in \mathbb{S}_{k+1}$.*

Sketch of proof: We use the same notation and follow the same lines as for proposition 20.
First step Let us suppose we are given a $(k+1)$-dcpa $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$, which is level-partitioned. We suppose that $\Gamma \supseteq \{D_1, A_1, A_2, \ldots, A_k, G, F, E, \bar{E}\}$, where this new symbol $\bar{E}$ plays the role of a bottom symbol for the automaton computing g. We call here "blocking pds" every 2-pds $\bar{U}$ of the form $\bar{E}[\omega_1] \cdot \omega_2$, for some ω_i which is a i-pds. Let us suppose that $\mathcal{A}$ allows the following basic derivations:
Initial derivation, D0: for every $n \geq 0$

$$(q_0 D_1[\beta_n] q_0) \rightarrow^*_{\mathcal{A}} (q_0 A_1[(\omega_{k-2}[\gamma_{n,n}])^2] q_0)$$

g-computation, D2 : for every $\ell \geq 2, \eta_1, \eta_2, \ldots, \eta_\ell \in F^*$, and $\bar{U}$, blocking pds

$$(q_0 \bar{A}_1[\omega_{k-2}[G[\eta_1] E[\eta_2] \cdots E[\eta_\ell] \bar{U}] \Omega] q_0 \rightarrow^*_{\mathcal{A}} (q_0 A_1'[\Omega] q_0)^{g(\ell-2)}$$

pairs-generation, D3 : for every $n \geq 0$

$$(q_0 A_1[\omega_{k-2}[\gamma_{n+1,n+1} \Omega'] \Omega] q_0) \rightarrow^*_{\mathcal{A}} \prod_{m=0}^{n} (q_0 \bar{A}_1[\omega_{k-2}[\gamma_{m,n} \Omega'] \Omega] q_0)$$

starting pairs, D30 :

$$(q_0 A_1[(\omega_{k-2}[\gamma_{0,0}\Omega'])^2\Omega]q_0) \to^*_{\mathcal{A}} (q_0 \bar{A}_1[\Omega]q_0)$$

gluing rule , *G23* : for every $0 \leq m \leq n$, there exists some blocking pds $\bar{U}$ such that,

$$(q_0 A'_1[\omega_{k-2}[\gamma_{m,n}\Omega']\Omega]q_0) \to^*_{\mathcal{A}} (q_0 A_1[(\omega_{k-2}[\gamma_{m,m}\bar{U}\Omega'])^2]\Omega]q_0).$$

ending rule , *E* :

$$(q_0 \bar{A}_1[\varepsilon]q_0) \to^*_{\mathcal{A}} a.$$

Let us prove by induction the following property $\mathbf{P}(n)$: for every $n \geq 0$ and every blocking pds $\bar{U}$

$$(q_0 A_1[(\omega_{k-2}[\gamma_{n,n}\bar{U}])^2\Omega]q_0) \to^*_{\mathcal{A}} (q_0 \bar{A}_1[\Omega]q_0)^{f(n)}.$$

Basis: $\mathbf{P}(0)$ follows from (D30), where ε by substituting $\bar{U}$ for Ω'.
Induction step:

$$\begin{array}{lll}
(q_0 A_1[(\omega_{k-2}[\gamma_{n+1,n+1}\bar{U}])^2]\Omega]q_0) \to^*_{\mathcal{A}} & \prod_{m=0}^{n}(q_0 \bar{A}_1[(\omega_{k-2}[\gamma_{m,n}\bar{U}])^2\Omega]q_0) & (\text{ by (D3)}) \\
\to^*_{\mathcal{A}} & \prod_{m=0}^{n}(q_0 A_1[\omega_{k-2}[\gamma_{m,n}\bar{U}]\Omega]q_0)^{g(n-m)} & (\text{ by D2)} \\
\to^*_{\mathcal{A}} & \prod_{m=0}^{n}(q_0 A'_1[(\omega_{k-2}[\gamma_{m,m}\bar{U}_m])^2\Omega]q_0)^{g(n-m)} & (\text{ by G23)} \\
\to^*_{\mathcal{A}} & \prod_{m=0}^{n}(q_0 \bar{A}_1[\Omega]q_0)^{f(m)\cdot g(n-m)} & (\text{ by induction hypothesis)} \\
= & (q_0 \bar{A}_1[\Omega]q_0)^{f(n+1)} &
\end{array}$$

(where all the $\bar{U}_m$ are blocking pds). Using $(D0)$ and $(D4)$ we finally obtain:

$$(q_0 D_1[\beta_n]q_0) \to^*_{\mathcal{A}} a^{f(n)}.$$

Second step:
One can construct an automaton $\mathcal{A}_2$ realizing (D2) and fulfilling also conditions $(\mathbf{P1}, \mathbf{P3})$ of lemma 8. The other rules can be made valid by a set of transitions, in a way similar to that used in the proof of proposition 20. □

Remark 3. Let us see the sequence g as a formal power series

$$g = \sum_{n=0}^{\infty} g(n)X^n.$$

Proposition 21 asserts that the series $\frac{1}{1-Xg}$ belongs to $\mathbb{S}_{k+1}$. In other words, the convolution inverse of every formal power series of the form $1 - Xg$, where $g \in \mathbb{S}_k$, belongs to $\mathbb{S}_{k+1}$.

Proposition 22 (Sequence composition) *Let* $k_1 \geq 2, k_2 \geq 2, f \in \mathbb{S}_{k_1}$ *and* $g \in \mathbb{S}_{k_2}$, *then* $f \circ g \in \mathbb{S}_{k_1+k_2-1}$.

This proposition is a direct corollary of the following lemma :

Lemma 9. *Let $\mathcal{A}_1$ be a (k_1+1)-dcpa (with $k_1 \geq 1$) fulfilling $(q_0^1, a^{f(n)}, A_1[\ldots A_{k_1}[F^n]]) \vdash^*_{\mathcal{A}_1} (q_0^1, \varepsilon, \varepsilon)$ and $\mathcal{A}_2$ be a deterministic k_2-pda (with $k_2 \geq 2$), then one can construct a deterministic (k_1+k_2)-pda $\mathcal{A}$ such that: $\forall flag \in k_2 - \mathrm{pds}(\Gamma_2)$ if $(q_0^2, a^{n_0}, flag) \vdash^*_{\mathcal{A}_2} (q_0^2, \varepsilon, \varepsilon)$ then $(q_0, a^{f(n_0)}, A_1[\ldots A_{k_1}[flag]\ldots]) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon)$. If in addition $\mathcal{A}_2$ is a counter automaton, then $\mathcal{A}$ is a counter automaton too.*

Construction: One supposes that $F \notin \Gamma_2$ and that $\mathcal{A}_1$ satisfies the following normalisation conditions:
(**NC1**) for all $k \leq k_1+1$, if $\delta_1(q, a_\varepsilon, \omega) = \{(p, pop_k)\}$, then $|\omega| = k$.
(**NC'1**) if $\delta_1(q, a_\varepsilon, \omega) = \{(p, push_i(\eta)\}$, then $|\eta| = 2$.
(For all k-dcpa one can construct an equivalent k-dcpa fulfilling these two conditions). We define $\mathcal{A}$ as follows : $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, (q_0^1, q_0^2), Z)$ with $Q = (Q_1 \cup \{q_F, q \in Q_1\}) \times Q_2$,
$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{A^p, A \in \Gamma_1, p \in Q_2\}$ and δ is the union of the two following lists of transitions.

Transitions inherited from $\mathcal{A}_1$

- for all $\delta_1(q, a_\varepsilon, \omega) = \{(p, f)\}$, $\omega \in \Gamma_1^{(k_1)}$ and $q' \in Q_2$,
  1.1 $\delta((q, q'), a_\varepsilon, \omega) = \{((p, q'), f)\}$
- for all $\delta_1(q, a_\varepsilon, \omega F) = \{(p, push_i(C_1C_2))\}$ (where $1 \leq i \leq k_1$), $\omega \in \Gamma_1^{k_1}$ and $s \in Q_2, \omega_2 \in \Gamma_2^{(k_2)}$
  1.2 $\delta((q, s), a_\varepsilon, \omega F \omega_2) = \{((p, s), push_i(C_1 C_2^s))\}$
- for all $\delta_1(q, a_\varepsilon, \omega F) = \{(p, f)\}, \omega \in \Gamma_1^{k_1}$, where f is not of the form $push_i(C_1C_2)$ for any $1 \leq i \leq k_1$ and $s \in Q_2, \omega_2 \in \Gamma_2^{(k_2)}$
  1.3 $\delta((q, s), a_\varepsilon, \omega F \omega_2) = \{((p, s), f)\}$

Notice that transitions (1.3) may include some $\delta((q, s), a_\varepsilon, \omega F \omega_2) = \{((p, s), push_{k_1+1}(FF))\}$ as soon as $\delta_1(q, a_\varepsilon, \omega F) = \{(p, push_{k_1+1}(FF))\}$

Transitions inherited from $\mathcal{A}_2$

- for all $\delta_2(q, \varepsilon, \omega_2) = \{(p, f)\}$, $q' \in Q_1$, $\omega_1 \in \Gamma_1^{k_1}$,
  2.1 $\delta((q', q), \varepsilon, \omega_1 \omega_2) = \{((q', p), f_{+k_1})\}$
- for all $\delta_2(q, a, \omega_2) = \{(p, f)\}$, $q' \in Q_1$, $\omega_1 \in \Gamma_1^{k_1}$
  2.2 $\delta((q', q), \varepsilon, \omega_1 \omega_2) = \{((q'_F, p), f_{+k_1})\}$,
  and for all $\omega'_2 \in \Gamma_2^{(k_2)}$,such that F_2 is the first letter of ω'_2
  2.3 $\delta((q'_F, p), \varepsilon, \omega_1 \omega'_2) = ((q', p), push_{k_1+1}(FF_2))$
- for all $\omega_1, \eta_1 \in \Gamma_1^*$, $B^s \in \Gamma$ such that $|\omega_1 B^s \eta_1| = k_1$ and $|\omega_1| = k$ and for all $(q, p) \in Q_1 \times Q_2, \omega_2 \in \Gamma_2^{(k_2)}$,
  2.4 $\delta((q, p), \varepsilon, \omega_1 B^s \eta_1 F \omega_2) = ((q, s), push_{k+1}(B))$

In the definition above, for every integer k and every pushdown operation f, the operation f_{+k} is defined by:

- if $f = pop_i$ then $f_{+k} = pop_{i+k}$
- if $f = push_i(\omega)$ then $f_{+k} = push_{i+k}(\omega)$

Informal explanations Let us describe informally what is the expected behaviour of this automaton $\mathcal{A}$.
The total states of $\mathcal{A}$ are obtained by taking a pds ω over Γ_1 and substituting to the ε of level k some total states of $\mathcal{A}_2$, (q_2, ω_2) (suitably encoded) and to every symbol F of level $(k+1)$ some $F[\eta_2]$ where η_2 is a pds of level k_2 over Γ_2. Those η_2 will be ignored by $\mathcal{A}$ but carried along the computations. The transitions of $\mathcal{A}$ have two different types:
-the transitions of type 1 (numbered (1.i)) mimic transitions from $\mathcal{A}_1$and memorize the state of the upwards component from $\mathcal{A}_2$.
-the transitions of type 2 (numbered (2.j)) mimic transitions from $\mathcal{A}_2$; every reading-transition of $\mathcal{A}_2$ is replaced by an ε-transition producing an extra symbol F.
Proof: We use below an undeterminate $\Omega \notin \Gamma$. For every word $\omega = C_1 \cdot C_2 \cdots C_{k_1}$ with $C_i \in \Gamma_1$, we note $\omega[\Omega] = C_1[C_2 \cdots C_{k_1}[\Omega] \cdots]$. We prove by induction the following property $P(m)$, $m \geq 1$:

$$
\begin{array}{rccr}
\text{for all} & n \geq n_0 \geq 0, \omega \in \Gamma_1^{k_1}, & \gamma \in k_2 - \mathrm{pds}(\Gamma_2), & \\
\text{if} & (p_1, u, \omega[F^n]) \vdash^m_{\mathcal{A}_1} & (q_1, \varepsilon, \varepsilon) & (26)\\
\text{and} & (p_2, a^{n_0}, flag) \vdash^*_{\mathcal{A}_2} & (q_2, \varepsilon, \varepsilon) & (27)\\
\text{then} & ((p_1,p_2), u, \omega[F[\gamma]^{n-n_0} flag]) \vdash^*_{\mathcal{A}} & ((q_1,q_2), \varepsilon, \varepsilon) & (28)
\end{array}
$$

Basis: $m = 0$.
Hypothesis (26) cannot be true since $\omega[F^n] \neq \varepsilon$, so that the implication ((26) and (27)) $\Rightarrow$ (28) holds: $P(0)$ is true.
Induction step:
Let us suppose that $m \geq 0$ and for every $\ell \leq m, P(\ell)$ is true. Let us prove that $P(m+1)$ is true.
We distinguish 8 cases, according to the the values of n, n_0 and to the first transition used in computation (26).
Case 0:$n = n_0 = 0$.
The automaton $\mathcal{A}$ may mimic computation (27), by means of transitions (2.1):

$$((p_1,p_2), \epsilon, \omega[flag]) \vdash^*_{\mathcal{A}} ((p_1,q_2), \varepsilon, \omega[\varepsilon]) \tag{29}$$

by means of transitions (1.1)(1.2)(1.3)(2.4), the automaton $\mathcal{A}$ may also mimic computation (26):

$$((p_1,q_2), \varepsilon, \omega[\varepsilon] \vdash^*_{\mathcal{A}} ((q_1,q_2), \varepsilon, \varepsilon). \tag{30}$$

The sequence of computations (29),(30), prove $P(m+1)$.
Case 1: $n = n_0 \neq 0$ and the first transition applied in (26) is $\delta_1(p_1, a_\epsilon, C_1 \ldots C_{k_1} F) =$

(p, pop_{k_1+1}).
The two hypothesis of $P(m+1)$ take, in this case the following form:

$$(p_1, u, \omega[F^n]) \to^1_{\mathcal{A}_1} (p, u', \omega[F^{n-1}]) \vdash^m_{\mathcal{A}_1} (q_1, \varepsilon, \varepsilon) \tag{31}$$

and

$$(p_2, a^{n_0}, flag) \vdash^*_{\mathcal{A}_2} (r, a^{n_0}, flag') \vdash_{\mathcal{A}_2} (s, a^{n_0-1}, flag'') \vdash^*_{\mathcal{A}_2} (q_2, \varepsilon, \varepsilon) \tag{32}$$

The automaton $\mathcal{A}$ mimics a first part of computation (32): using transitions (2.1)

$$((p_1, p_2), u, \omega[flag]) \vdash^*_{\mathcal{A}} ((p_1, r), u, \omega[flag']) \tag{33}$$

using then a transition (2.2)

$$((p_1, r), u, \omega[flag']) \vdash_{\mathcal{A}} ((p_{1,F}, s), u, \omega[flag'']) \tag{34}$$

and finally a transition (2.3):

$$((p_{1,F}, s), u, \omega[flag'']) \vdash_{\mathcal{A}} ((p_1, s), u, \omega[F[\alpha]flag'']) \tag{35}$$

where α is some $(k_2-1)-$pds (namely the one which is the "flag" of the leftmost atom of $flag$). By a transition (1.3), $\mathcal{A}$ mimics the first step of computation (31):

$$((p_1, s), u, \omega[F[\alpha]flag'']) \vdash_{\mathcal{A}} ((p, s), u', \omega[flag'']) \tag{36}$$

Let us use now the facts that:

$$(p, u', \omega[F^{n-1}] \vdash^m_{\mathcal{A}_1} (q_1, \varepsilon, \varepsilon) \text{ and } (s, a^{n_0-1}, flag'') \vdash^*_{\mathcal{A}_2} (q_2, \varepsilon, \varepsilon).$$

By induction hypothesis, these two computations ensure that

$$((p, s), u', \omega[flag''] \vdash^*_{\mathcal{A}} ((q_1, q_2), \varepsilon, \varepsilon) \tag{37}$$

Putting together computations (33,34,35,36,37) we obtain

$$((p_1, p_2), u, \omega[flag]) \vdash^*_{\mathcal{A}} ((q_1, q_2), \varepsilon, \varepsilon)$$

which establishes $P(m+1)$.
Case 2:$n = n_0 \neq 0$ and the first transition applied in (26) is
$\delta_1(p_1, a_\epsilon, C_1 \dots C_k F) = (p, push_1(D_1 D_2))$.
The two hypothesis of $P(m+1)$ take, in this case the following form:

$$\begin{aligned}(p_1, u, \omega[F^n]) \vdash^1_{\mathcal{A}_1} & \ (p, u', D_1[C_2 \cdots C_{k_1}[F^n] \cdots] D_2[C_2 \cdots C_{k_1}[F^n] \cdots]) \\ \vdash^{m'}_{\mathcal{A}_\infty} & \ (q, u'', D_2[C_2 \cdots C_{k_1}[F^n] \cdots]) \\ \vdash^{m''}_{\mathcal{A}_1} & \ (q_1, \varepsilon, \varepsilon)\end{aligned} \tag{38}$$

and

$$(p_2, a^{n_0}, flag) \vdash^*_{\mathcal{A}_2} (r, a^{n_0}, flag') \vdash_{\mathcal{A}_2} (s, a^{n_0-1}, flag'') \vdash^*_{\mathcal{A}_2} (q_2, \varepsilon, \varepsilon). \tag{39}$$

As in case 1, $\mathcal{A}$ can mimic $\mathcal{A}_2$ along computations (33,34,35):

$$((p_1, p_2), u, \omega[flag]) \vdash^*_{\mathcal{A}} ((p_1, s), u, \omega[F[\alpha] flag'']) \tag{40}$$

By a transition (1.2):

$$((p_1, s), u, \omega[F[\alpha] flag'']) \vdash_{\mathcal{A}} ((p, s), u', D_1[\cdots C_{k_1}[F[\alpha] flag''] D_2^s[\cdots C_{k_1}[F[\alpha] flag'']) \tag{41}$$

As $m' + m'' = m$, the induction hypothesis applies on both parts of computation (38):

$$((p, s), u', D_1[\cdots C_{k_1}[F[\alpha] flag''] \cdots]) \vdash^*_{\mathcal{A}} ((q, q_2), u'', \varepsilon) \tag{42}$$

$$((q, s), u'', D_2[\cdots C_{k_1}[F[\alpha] flag''] \cdots]) \vdash^*_{\mathcal{A}} ((q_1, q_2), \varepsilon, \varepsilon) \tag{43}$$

Concatenating some fixed pds on the right side of all total states of computation (42) we get:

$$\begin{aligned} ((p, s), u', D_1[\cdots C_{k_1}[F[\alpha] flag''] \cdots] D_2^s[\cdots C_{k_1}[F[\alpha] flag''] \cdots]) \vdash^*_{\mathcal{A}} \\ ((q, q_2), u'', D_2^s[\cdots C_{k_1}[F[\alpha] flag''] \cdots]) \end{aligned} \tag{44}$$

By a transition (2.4):

$$((q, q_2), u'', D_2^s[\cdots C_{k_1}[F[\alpha] flag''] \cdots]) \vdash_{\mathcal{A}} ((q, s), u'', D_2[\cdots C_{k_1}[F[\alpha] flag'']) \tag{45}$$

Putting together computations (40,41,44,45,43), we obtain that

$$((p_1, p_2), u, \omega[flag]) \vdash^*_{\mathcal{A}} ((q_1, q_2), \varepsilon, \varepsilon),$$

which establishes $P(m+1)$.
Case 3: $n = n_0 \neq 0$ and the first transition applied in (26) is $\delta_1(p_1, a_\epsilon, C_1 \ldots C_k F) = (p, push_i(D_1 D_2))$, with $2 \leq i \leq k_1$.
We follow the same lines as for case 2, the only difference is that the push operation is performed at a level $i > 1$ which leads to somewhat heavy notation.
The two hypothesis of $P(m+1)$ take the following form:

$$\begin{aligned} (p_1, u, \omega[F^n]) \vdash^1_{\mathcal{A}_1} & \ (p, u', C_1[\cdots[C_{i-1}[D_1[C_{i+1} \cdots C_{k_1}[F^n] \cdots] D_2[C_{i+1} \cdots C_{k_1}[F^n] \cdots]]] \cdots]) \\ \vdash^{m'} & \ (q, u'', E_1[\cdots[E_{i-1}[D_2[C_{i+1} \cdots C_{k_1}[F^n] \cdots]]] \cdots]) \\ \vdash^{m''}_{\mathcal{A}_1} & \ (q_1, \varepsilon, \varepsilon) \end{aligned} \tag{46}$$

and

$$(p_2, a^{n_0}, flag) \vdash^*_{\mathcal{A}_2} (r, a^{n_0}, flag') \vdash_{\mathcal{A}_2} (s, a^{n_0 - 1}, flag'') \vdash^*_{\mathcal{A}_2} (q_2, \varepsilon, \varepsilon). \tag{47}$$

(Such a decomposition (46) does exist because of condition (**NC1**)). As in case 2, $\mathcal{A}$ can mimic $\mathcal{A}_2$ for a while:

$$((p_1, p_2), u, \omega[flag]) \vdash^*_{\mathcal{A}} ((p_1, s), u, \omega[F[\alpha] flag'']) \tag{48}$$

By a transition (1.2):

$$((p_1,s),u,\omega[F[\alpha]flag'']) \vdash_{\mathcal{A}}$$
$$((p,s),u',C_1[\cdots[C_{i-1}[D_1[C_{i+1}\cdots C_{k_1}[F[\alpha]flag'']]D_2^s[C_{i+1}\cdots C_{k_1}[F[\alpha]flag'']]]]\cdots]). \quad (49)$$

As $m'+m''=m$, the induction hypothesis applies on the two last steps of computation (46):

$$((p,s),u',C_1[\cdots[C_{i-1}[D_1[C_{i+1}\cdots C_{k_1}[F[\alpha]flag'']\cdots]]]\cdots])\vdash_{\mathcal{A}}^{*} ((q,q_2),u'',E_1[\cdots[E_{i-1}[\varepsilon]]\cdots]) \quad (50)$$

$$((q,s),u'',E_1[\cdots[E_{i-1}[D_2[C_{i+1}\cdots C_{k_1}[F[\alpha]flag'']\cdots]]]\cdots])\vdash_{\mathcal{A}}^{*} ((q_1,q_2),\varepsilon,\varepsilon) \quad (51)$$

Concatenating, at level i, some fixed pds on the right side of all total states of computation (50) we get:

$$((p,s),u',C_1[\cdots[C_{i-1}[D_1[\cdots C_{k_1}[F[\alpha]flag'']\cdots]D_2^s[\cdots C_{k_1}[F[\alpha]flag'']\cdots]]]\cdots]) \vdash_{\mathcal{A}}^{*}$$
$$((q,q_2),u'',E_1[\cdots[E_{i-1}[D_2^s[\cdots C_{k_1}[F[\alpha]flag'']\cdots]]]\cdots]) \quad (52)$$

By a transition (2.4):

$$((q,q_2),u'',E_1[\cdots[E_{i-1}[D_2^s[\cdots C_{k_1}[F[\alpha]flag'']\cdots]]]\cdots]) \vdash_{\mathcal{A}}$$
$$((q,s),u'',E_1[\cdots[E_{i-1}[D_2[\cdots C_{k_1}[F[\alpha]flag'']\cdots]]]\cdots]) \quad (53)$$

Putting together computations (48,49,52,53,51), we obtain that

$$((p_1,p_2),u,\omega[flag]) \vdash_{\mathcal{A}} ((q_1,q_2),\varepsilon,\varepsilon),$$

which establishes $P(m+1)$.
Case 4: $n=n_0\neq 0$ and the first transition applied in (26) is $\delta_1(p_1,a_\epsilon,C_1\ldots C_kF)=(p,push_i(D))$,with $2\leq i\leq k_1$.
This case is quite similar to case 3. Notice that no transition (1.2) is used in the simulation of the first transition of (26), and correspondingly no transition (2.4) is used in this simulation. The proof would follow the same lines as the above proof for case 3, but would be easier. We leave it to the reader.
Case 5: $n=n_0\neq 0$ and the first transition applied in (26) is $\delta_1(p_1,a_\epsilon,C_1\ldots C_kF)=(p,push_{k_1+1}(FF))$.
The two hypothesis of $P(m+1)$ take, in this case the following form:

$$(p_1,u,\omega[F^n])\vdash_{\mathcal{A}_1}^{1} (p,u',\omega[F^{n+1}])\vdash_{\mathcal{A}_1}^{m} (q_1,\varepsilon,\varepsilon) \quad (54)$$

and

$$(p_2,a^{n_0},flag)\vdash_{\mathcal{A}_2}^{*} (r,a^{n_0},flag')\vdash_{\mathcal{A}_2} (s,a^{n_0-1},flag'')\vdash_{\mathcal{A}_2}^{*} (q_2,\varepsilon,\varepsilon). \quad (55)$$

As in case 1, $\mathcal{A}$ can mimic $\mathcal{A}_2$ for a while:

$$((p_1,p_2),u,\omega[flag])\vdash_{\mathcal{A}}^{*} ((p_1,s),u',\omega[F[\alpha]flag'']) \quad (56)$$

By a transition (1.3):

$$((p_1,s),u',\omega[F[\alpha]flag''])\vdash_{\mathcal{A}} ((p,s),u',\omega[F[\alpha]F[\alpha]flag'']) \tag{57}$$

Taking into account the last steps of computations (54,55) and applying $P(m)$, we know that:

$$((p,s),u',\omega[F[\alpha]^2flag''])\vdash^*_{\mathcal{A}} ((q_1,q_2),\varepsilon,\varepsilon) \tag{58}$$

Putting together computations (56,57,58) we obtain

$$((p_1,p_2),u,\omega[flag])\vdash^*_{\mathcal{A}} ((q_1,q_2),\varepsilon,\varepsilon)$$

which establishes $P(m+1)$.
Case 6: $n=n_0\neq 0$ and the first transition applied in (26) is $\delta_1(p_1,a_\epsilon,C_1\ldots C_kF)=(p,push_{k_1+1}(F))$.
The proof is almost the same as for case 5.
Case 7: $n>n_0$.
The two hypothesis of $P(m+1)$ take the following general form:

$$(p_1,u,\omega[F^n])\vdash^1_{\mathcal{A}_1} (p',u',\omega'[F^{n'}])\vdash^m_{\mathcal{A}_1} (q_1,\varepsilon,\varepsilon) \tag{59}$$

and

$$(p_2,a^{n_0},flag)\vdash^*_{\mathcal{A}_2} (q_2,\varepsilon,\varepsilon). \tag{60}$$

As $n-n_0\geq 1$, whatever first move is performed in (59), either a transition (1.2) or a transition (1.3) allows $\mathcal{A}$ to mimic that move:

$$((p_1,p_2),u,\omega[F[\gamma]^{n-n_0}flag])\vdash_{\mathcal{A}} ((p',p_2),u',\omega'[F[\gamma]^{n'}]flag]) \tag{61}$$

By induction hypothesis

$$((p',p_2),u',\omega'[F[\gamma]^{n'}]flag])\vdash^*_{\mathcal{A}} ((q_1,q_2),\varepsilon,\varepsilon). \tag{62}$$

Putting together computations (61, 62), we obtain the conclusion of property $P(m+1)$. $\square$
Let us summarize the closure properties demonstrated in this section.

Theorem 23.
0- For every $f\in\mathbb{S}_{k+1}$, $k\geq 1$, and every integer $c\in\mathbb{N}$,
the sequences Ef (the shift of f), $f+\frac{c}{1-X}$, belong to $\mathbb{S}_{k+1}$;
if $\forall n\in\mathbb{N}, f(n)\geq c$ then $f-\frac{c}{1-X}$ belongs to $\mathbb{S}_{k+1}$;
the sequence defined by $0\mapsto c, n+1\mapsto f(n)$ belongs to $\mathbb{S}_{k+1}$.
1- For every $f,g\in\mathbb{S}_{k+1}$, $k\geq 1$, the sequences $f+g, f\odot g$ (the ordinary product), belong to $\mathbb{S}_{k+1}$. For every $f'\in\mathbb{S}_{k+2}$, f'^g belong to $\mathbb{S}_{k+2}$.
2-For every $f\in\mathbb{S}_{k+1}, g\in\mathbb{S}_k$, $k\geq 2$, $f\times g$ (the convolution product) belongs to $\mathbb{S}_{k+1}$.
3-For every $g\in\mathbb{S}_k$, $k\geq 2$, the sequence f defined by: $f(0)=1$ and $f(n+1)=\sum_{m=0}^{n} f(m)\cdot g(n-m)$ (the convolution inverse of $X\times f$) belongs to $\mathbb{S}_{k+1}$.
4- For every $f\in\mathbb{S}_k, g\in\mathbb{S}_\ell$, $k,l\geq 2$, $f\circ g$ (the sequence composition) belongs to

$\mathbb{S}_{k+\ell-1}$.
5- For every $k \geq 2$ and for every system of recurrent equations expressed by polynomials in $\mathbb{S}_{k+1}[X_1, \ldots, X_p]$, with initial conditions in $\mathbb{N}$, every solution belongs to $\mathbb{S}_{k+1}$.

Proof: Point (0) is obvious. Point (4) has been proved in proposition 22. Points (1),(2),(3),(5) have been proved in previous propositions, but with the restriction that all sequences involved have a level $k \geq 3$. This is due to the fact that the normal form given in lemma 8 is proved for level $k \geq 3$ only. For sequences of level 2 one could state a slightly weaker lemma, where **P1**, **P3** are maintained but **P2** is replaced by:

$$(q_0, A_1[F^n \bar{F} \Omega], q_0) \rightarrow^*_{\mathcal{A}} (q_0, \bar{A}_1[\Omega], q_0)^{f(n)},$$

(the idea is that $\bar{F}$ is a blocking symbol, acting as if it was marking the bottom of the pushdowns at level 2). Owing to this complementary version of lemma 8, one can adapt the proofs to the case where some sequences have level 2. □

5 Integer double sequences

We introduce here a notion of *k-computable multiple sequence* $u(n_1, n_2, \ldots, n_r)$. We focus on a particular kind of double sequences $f(m,n)$ wich is needed in the study of simple sequences of rational numbers (see section 7).

A sytematic and thorough study of multiple sequences would certainly be useful but we have left it for future work.

Definition 23. *Let $k \geq 2$. The double sequence $f(m,n)$, $0 \leq m \leq n$ is called a k-*computable *double sequence iff, there exists a k-dcp $\mathcal{A}$ such that, for all $0 \leq m \leq n$,*

$$(q_0, a^{f(m,n)}, A_1[\ldots A_{k-1}[\gamma_{m,n}]\ldots]) \vdash^*_{\mathcal{A}} (q_0, \varepsilon, \varepsilon),$$

*where $\gamma_{m,n}$ denotes the 2-*pds $(\prod_{j=m}^{n-1} A_k[F^j]) \cdot \bar{A}_k[F^n]$.

We denote by $\mathbb{S}_k^{(2)}$ the set of level k double sequences.

Lemma 10. *Let $k \geq 2$. Let $b_{1,1}, b_{1,2}, b_{2,1}, b_{2,2}, c_1, c_2 \in \mathbb{S}_{k+1}$. Let us consider the double sequences f_1, f_2 defined by: for every $0 \leq \ell < n$,*

$$\begin{pmatrix} f_1(\ell,n) \\ f_2(\ell,n) \end{pmatrix} = \begin{pmatrix} b_{1,1}(\ell) & b_{1,2}(\ell) \\ b_{2,1}(\ell) & b_{2,2}(\ell) \end{pmatrix} \cdot \begin{pmatrix} f_1(\ell+1,n) \\ f_2(\ell+1,n) \end{pmatrix}$$

and

$$f_1(n,n) = c_1(n), \quad f_2(n,n) = c_2(n).$$

Then $f_1, f_2 \in \mathbb{S}_{k+1}^{(2)}$.

Proof: We use the notation:

$$\gamma_{m,n} = (\prod_{j=m}^{n-1} A_k[F^j]) \cdot \bar{A}_k[F^n]; \;\; \beta_n = A_2[\ldots A_{k-1}[A_k[F^n]]\ldots];$$

$$\omega_{k-2}[\Omega] = A_2[\cdots[A_{k-1}[\Omega]]\cdots];$$

First step:
Let us suppose we are given a $(k+1)$-dcpa $\mathcal{A} = (Q, \{a\}, \Gamma, \delta, q_0, Z)$ which is level-partitioned. The alphabet Γ contains at least the following symbols: U_1, U_2 (symbols for f_1, f_2), $D_1, \bar{D}_1, D_2, \bar{D}_2$ (symbols for c_1, c_2), $A_{i,j}, \bar{A}_{i,j}$ (symbols for $b_{i,j}$) and $\{A_1, A_2, \ldots, A_{k-1}, A_k, \bar{A}_k, F\}$. Suppose that the following rules are valid:
Starting rule , D0:

$$(q_0 U_i[\omega_{k-2}[\gamma_{n,n}]\Omega]q_0) \to^*_{\mathcal{A}} (q_0 D_i[\beta_n \Omega]q_0)$$

Coefficients rule, D1:

$$(q_0 A_{i,j}[\beta_n \Omega]q_0) \to^*_{\mathcal{A}} (q_0 \bar{A}_{i,j}[\Omega]q_0)^{b_{i,j}(n)}$$

Simple sequences rules , D2:

$$(q_0 D_i[\beta_n \Omega]q_0) \to^*_{\mathcal{A}} (q_0 \bar{D}_i[\Omega]q_0)^{c_i(n)}$$

Gluing rules, G(i,(i,j)): for $0 \leq m < n$

$$(q_0 U_i[\omega_{k-2}[\gamma_{m,n}]\Omega]q_0) \to^*_{\mathcal{A}} (q_0 A_{i,1}[\beta_n \omega_{k-2}[\gamma_{m+1,n}]\Omega]q_0)(q_0 A_{i,2}[\omega_{k-2}[\gamma_{m+1,n}]\Omega]q_0)$$

Gluing rules, G((i,j),j):

$$(q_0 \bar{A}_{i,j}[\Omega]q_0) \to^*_{\mathcal{A}} (q_0 U_j[\Omega]q_0)$$

Ending rule, E:

$$(q_0 \bar{D}_i[\varepsilon]q_0) \to^*_{\mathcal{A}} a.$$

Let us prove, by descending induction over m, that, for every $0 \leq m \leq n$, the following propert $\mathbf{P}(m,n)$ holds:

$$(q_0 U_i[\omega_{k-2}[\gamma_{m,n}]]q_0) \to^*_{\mathcal{A}} a^{f_i(m,n)}.$$

Basis: $m = n$.
We get the derivation:

$$\begin{array}{llll} (q_0 U_i[\omega_{k-2}[\gamma_{n,n}]]q_0) & \to^*_{\mathcal{A}} & (q_0 D_i[\beta_n]q_0) & (\text{ by rule D0 }) \\ & \to^*_{\mathcal{A}} & (q_0 \bar{D}_i[\varepsilon]q_0)^{f_i(n,n)} & (\text{ by D2 }) \\ & \to^*_{\mathcal{A}} & a^{f_i(n,n)} & (\text{ by rule E }). \end{array}$$

which establishes $\mathbf{P}(n,n)$.
Induction step: $0 \le m < n$ and $\mathbf{P}(m+1,n)$ is assumed true. We get the derivation:

$$\begin{array}{lcl}
 & (q_0 U_i[\omega_{k-2}[\gamma_{m,n}]]q_0) & \\
\to_{\mathcal{A}}^{*} & (q_0 A_{i,1}[\beta_n \omega_{k-2}[\gamma_{m+1,n}]]q_0)(q_0 A_{i,2}[\omega_{k-2}[\gamma_{m+1,n}]]q_0) & (\text{ by rule G(i(i,j)) }) \\
\to_{\mathcal{A}}^{*} & (q_0 \bar{A}_{i,1}[\beta_n \omega_{k-2}[\gamma_{m+1,n}]]q_0)^{b_{i,1}(n)}(q_0 \bar{A}_{i,2}[\omega_{k-2}[\gamma_{m+1,n}]]q_0)^{b_{i,2}(n)} & (\text{ by D1 }) \\
\to_{\mathcal{A}}^{*} & (q_0 U_1[\omega_{k-2}[\gamma_{m+1,n}]]q_0)^{b_{i,1}(n)}(q_0 U_2[\omega_{k-2}[\gamma_{m+1,n}]]q_0)^{b_{i,2}(n)} & (\text{ by G((i,j)j) }) \\
\to_{\mathcal{A}}^{*} & (a^{f_1(m+1,n)})^{b_{i,1}(n)}(a^{f_2(m+1,n)})^{b_{i,2}(n)} & (\text{ by } \mathbf{P}(m+1,n))
\end{array}$$

which establishes $\mathbf{P}(m,n)$.
Second step:
By lemma 8 there exist automata $\mathcal{B}_{i,j}$ (resp. $\mathcal{C}_i$) computing the sequences $b_{i,j}$ (resp. c_i), and fulfilling conditions ($\mathbf{P1}, \mathbf{P2}, \mathbf{P3}$). These automata furnish the sets of rules allowing (D1)(D2). As usual we choose the concrete sets of states and pushdown alphabets, in such a way that the only common state between two of them is $\{q_0\}$ and the only common pushdown symbols between two of them are $\{A_2, A_3, \ldots, A_k, \bar{A}_k, F\}$. Each of the rules G((i,j),j) and (D3) can be easily reduced to a single transition. The rules D0 and G(i,(i,j)) can be decomposed into a finite number of transitions, with distinct states for all the intermediate total-states. One can check that the modes of the initial total-states of the different rules are all distinct. The existence of the deterministic $(k+1)$-dcpa $\mathcal{A}$ follows. □

Lemma 11. *Let $k \ge 2$. Let $b_{1,1}, b_{1,2}, b_{2,1}, b_{2,2}, c, d \in \mathbb{S}_{k+1}$ and $\alpha, \beta \in \mathbb{N}$. Let us consider the double sequences $\overline{f}, \overline{g}$ defined by:*

$$\overline{f}(m,n) = f(0,m,n), \;\; \overline{g}(m,n) = g(0,m,n)$$

where the triple sequences f, g fulfill, for every $0 \le \ell \le n, 0 \le m \le n$

$$\begin{aligned}
f(n,n,n) &= \alpha \\
g(n,n,n) &= \beta \\
f(\ell,m,n) &= f(\ell+1,m,n) \text{ if } \ell = m, \ell < n \\
g(\ell,m,n) &= g(\ell+1,m,n) \text{ if } \ell = m, \ell < n \\
f(n,m,n) &= c(n) \text{ if } 0 \le m < n \\
g(n,m,n) &= d(n) \text{ if } 0 \le m < n \\
f(\ell,m,n) &= b_{1,1}(\ell) f(\ell+1,m,n) + b_{1,2}(\ell) g(\ell+1,m,n) \text{ if } \ell \ne m, \ell < n \\
g(\ell,m,n) &= b_{2,1}(\ell) f(\ell+1,m,n) + b_{2,2}(\ell) g(\ell+1,m,n) \text{ if } \ell \ne m, \ell < n
\end{aligned}$$

Then, $\overline{f}, \overline{g} \in \mathbb{S}_{k+1}^{(2)}$

Sketch of proof: Let us represent a triple (ℓ, m, n) where $0 \le \ell \le n, 0 \le m \le n$ by the 2-pds:

$$B_k^+[F^{\ell-m}] \cdot \gamma_{m,n} \text{ if } \ell > m; \;\; B_k^-[F^{m-\ell}] \cdot \gamma_{m,n} \text{ if } \ell \le m,$$

where B_k^+, B_k^- are two symbols of level k. The recurrence defining f, g follows essentially the same pattern as in lemma 10: it is a linear recurrence scheme, where the two variables are ℓ, n while m can be seen as a parameter. These recurrence relations can be translated into rules and finally into a $(k+1)$-dcpa. $\square$

Lemma 12. *If $f(m,n) \in \mathbb{S}_k^{(2)}$, then for every fixed $m_0 \in \mathbb{N}$, $f(m_0, n)$, $f(n,n) \in \mathbb{S}_k$.*

The proof is obvious.

Lemma 13. *If $f(n) \in \mathbb{S}_k$, then the double sequence F defined by: for every $0 \leq m \leq n, F(m,n) = f(n)$ belongs to $\mathbb{S}_k^{(2)}$.*

Lemma 14 (ordinary-product). *Let $f(m,n)$, $g(m,n) \in \mathbb{S}_{k+1}^{(2)}$, $k \geq 2$, then $(f(m,n).g(m,n))_{0 \leq m \leq n} \in \mathbb{S}_{k+1}^{(2)}$*

The proof is analogous with that of proposition 16.

Lemma 15 (pseudo-convolution). *Let $f(m,n) \in \mathbb{S}_k^{(2)}$, $g(m,n) \in \mathbb{S}_{k+1}^{(2)}, k \geq 2$. Then the sequence h defined for all $n \geq 0$ by $h(n) = \sum_{0 \leq m \leq n} f(m,n).g(n-m,n)$ belongs to $\mathbb{S}_{k+1}$.*

The proof is analogous with that of proposition 20.

6 Application to weak arithmetics

In [ER66], Elgot and Rabin devise a method for constructing unary predicates P such that the MSO theory of $\langle \mathbb{N}, S, P \rangle$ is decidable (here S denotes the successor relation). Further results in this direction have been established in ([Sem84],[Mae99],[CT00]). This kind of problem takes place in the more general perspective of studying “weak”arithmetical theories, which posess interesting decidability properties ([Bès02]).

We use here decidability results on k-pdas in order to demonstrate the decidability of the monadic theory of structures $\langle \mathbb{N}, S, P \rangle$, for a large class of predicates P (theorem 24 and theorem 26).

6.1 Extensions of $\langle \mathbb{N}, S, P \rangle$

We first consider some graphs having a particular form.

Definition 24 (Infinite linear graphs). *A graph $\Gamma_0 = (V_0, E_0)$ labelled over an alphabet $\Sigma \cup \{\varepsilon\}$ with origin v_0 is infinite linear if for all $u \in V_0$:*

- *there is one and only one $v \in V_0$, $\alpha \in \Sigma \cup \{\varepsilon\}$ such that $(u, \alpha, v) \in E_0$,*

– *if $u \neq v_0$, there is one and only one $v \in V_0$, $\beta \in \Sigma \cup \{\varepsilon\}$ such that $(v, \beta, u) \in E_0$*

(Notice that a graph satisfying this definition looks like an infinite straight line). One can remark that if there exists $a \in \Sigma$ labelling an infinity of paths, i.e.,

$$v_0 \twoheadrightarrow^{\omega_1} v_1 \rightarrow^a u_1 \twoheadrightarrow^{\omega_2} v_2 \rightarrow^a u_2 \ldots \twoheadrightarrow^{\omega_2} v_n \rightarrow^a u_n \ldots$$

with $|\omega_i|_a = 0$, where $|\omega|_a$ is the number of a occurring in ω. Let us define a bijection φ between $\mathbb{N}$ and $\{u|\ (v, a, u) \in E_0\} \cup \{v_0\}$, as follows :

$\varphi(0) = v_0$

$\varphi(n+1)$ is the only vertex such that there exists v in V_0 and $\omega \in \Sigma^*$, $|\omega|_a = 0$ such that $\varphi(n) \twoheadrightarrow^{\omega} v \rightarrow^a \varphi(n+1)$.

One considers now $\mathcal{A} \in it{-}PDA(\{a, b\})$ and the structure $\mathcal{M}_{\Gamma_0} = (V, R_a, R_b, R_\varepsilon)$ induced by its computation graph issue from (q_0, Z).

One defines MSOL-formulas D, $\widehat{S}$, $\widehat{P}$:

$D(x) == (x = (q_0, Z))) \vee (\exists z.\ R_a(z, x))$,
$\widehat{S}(x, y) == \widetilde{\Upsilon}_a(x, y) \vee \widetilde{\Upsilon}_{ba}(x, y)$,
$\widehat{P}(x) == \exists z.\ [\widetilde{\Upsilon}_{ba}(x, z)]$

with $\widetilde{\Upsilon}_\omega$ as defined in corollary 2.

Then, for all u in V,

$$\mathcal{M}_{\Gamma_0} \models D(u) \text{ iff there exists } n \geq 0 \text{ such that } \varphi(n) = u$$

and for all u, v such that $D(u)$ and $D(v)$

$$\mathcal{M}_{\Gamma_0} \models \widehat{S}(u, v) \text{ iff there exists } n \geq 0 \text{ such that } \varphi(n) = u \text{ and } \varphi(n+1) = v,$$

$$\mathcal{M}_{\Gamma_0} \models \widehat{P}(u) \text{ iff there exists } n \geq 0 \text{ such that } \varphi(n) = u \text{ and } \varphi(n) \twoheadrightarrow^{ba} \varphi(n+1)$$

Then, vertices u such that $\widehat{P}(u)$ and $D(u)$ denote particular integers.

Theorem 24. *Let f be an increasing sequence of integer, and $\mathbf{G}$ be the computation graph of a k-pda over $\{a, b\}$, rooted in (q_0, Z) such that $\mathbf{G}$ is infinite linear and has the form:*

$$(q_0, Z) \twoheadrightarrow^{a^{f(0)}} \rightarrow^b \ \twoheadrightarrow^{a^{f(1)}} \rightarrow^b \cdots \twoheadrightarrow^{a^{f(n)}} \rightarrow^b \cdots$$

Then, the monadic second order theory of $\langle \mathbb{N}, S, P_f \rangle$
where $S = \{(n, n+1)|\ n \in \mathbb{N}\}$ and $P_f = \{\sum_{0 \leq i \leq n} f(i)|\ n \in \mathbb{N}\}$ is decidable.

Proof: The bijection φ defined above is an isomorphism from the structure $\mathcal{M}_{\Gamma_{v_0}}(D, \widehat{S}, \widehat{P}) = \langle D, R_S, R_P \rangle$ to the structure $\langle \mathbb{N}, S, P_f \rangle$. By theorem 1 and theorem18, the MSO-theory of $\langle D, R_S, R_P \rangle$ is decidable. It follows that the MSO-theory of $\langle \mathbb{N}, S, P_f \rangle$ is decidable. □

6.2 Differentiably, k-computable sequences

The particular form of the predicates P_f considered in theorem 24 leads naturally to the following class of sequences.

Definition 25. *Let $k \geq 2$. We define the class $\Sigma\mathbb{S}_k \subseteq \mathbb{N}^{\mathbb{N}}$ as the set*

$$\Sigma\mathbb{S}_k = \{\Sigma v \mid v \in \mathbb{S}_k\}.$$

Fact 25 *Let $u \in \mathbb{N}^{\mathbb{N}}$. The sequence u belongs to $\Sigma\mathbb{S}_k$ if and only if Δu belongs to $\mathbb{S}_k$.*

This follows easily from point (0) of theorem 23. After this fact we name "differentiably k-computable sequences" the elements of $\Sigma\mathbb{S}_k$.

Lemma 16. *Let $k \geq 1$ and $U \in \Sigma\mathbb{S}_{k+1}$. Then $\mathsf{E}U \in \Sigma\mathbb{S}_{k+1}$.*

Proof: Suppose that $U \in \Sigma\mathbb{S}_{k+1}$. We notice that $\Delta\mathsf{E}U = \mathsf{E}\Delta U$. Using fact 25 and stability of $\mathbb{S}_{k+1}$ by shift, we obtain that $\mathsf{E}U \in \Sigma\mathbb{S}_{k+1}$. □

Lemma 17. *Let $k \geq 1$ and $U, V \in \Sigma\mathbb{S}_{k+1}$. Then $U + V \in \Sigma\mathbb{S}_{k+1}$.*

Lemma 18. *Let $k \geq 1$ and $U, V \in \Sigma\mathbb{S}_{k+1}$. Then $U \odot V \in \Sigma\mathbb{S}_{k+1}$.*

Proof: Let $U, V \in \Sigma\mathbb{S}_{k+1}$. The following identity is well-known:

$$\Delta(U \odot V) = \Delta(U \odot \mathsf{E}V) + U \odot (\Delta V).$$

By theorem 23, the sequences $U, V, \mathsf{E}V$ all belong to $\mathbb{S}_{k+1}$, and the righthand side of the above identity must belong to $\mathbb{S}_{k+1}$. By fact 25 $U \times V \in \Sigma\mathbb{S}_{k+1}$. □

Lemma 19. *Let $k \geq 1$ and $U, V \in \Sigma\mathbb{S}_{k+1}$. Then $U \times V \in \Sigma\mathbb{S}_{k+1}$.*

Proof: Let $U = \Sigma u$ and $V = \Sigma v$ for some $u, v \in \mathbb{S}_{k+1}$. Let us transform the expression $\Delta(U \times V)$ into an expression which does not use the operator Δ any more.

$$\begin{aligned}
\Delta(\Sigma u \times \Sigma v) &= \frac{(\Sigma u \times \Sigma v)(1 - X) - \Sigma u(0) \cdot \Sigma v(0)}{X} \\
&= \frac{\frac{u \times v}{1-X} - \frac{u(0) \cdot v(0)}{1-0}}{X} \\
&= \mathsf{E}(\frac{u \times v}{1 - X}) \\
&= \mathsf{E}\Sigma(u \times v).
\end{aligned}$$

By theorem 23, the final expression obtained belongs to $\mathbb{S}_{k+1}$, hence $\Sigma u \times \Sigma v$ belongs to $\Sigma\mathbb{S}_{k+1}$. □

Lemma 20. *Let $V \in \Sigma\mathbb{S}_k$, $k \geq 2$, such that $V(0) \geq 1$. Let U be the sequence defined by*

$$U(0) = 1 \text{ and for all } n \geq 0, U(n+1) = \sum_{k=0}^{n} U(k) \cdot V(n-k).$$

Then $U \in \Sigma\mathbb{S}_{k+1}$.

Proof: Let $v \in \mathbb{S}_{k+1}$ such that $V = \Sigma v$. As asserted in remark 3

$$U = \frac{1}{1 - \frac{Xv}{1-X}}.$$

Let us compute the series ΔU.

$$\Delta U = \Delta(\frac{1}{1 - \frac{Xv}{1-X}}) = \Delta(\frac{1-X}{1-X-Xv}) = \frac{1}{X}[\frac{(1-X)^2}{1-X-Xv} - 1]$$

hence

$$\Delta U = \frac{1}{X}[\frac{(-X + X^2 + Xv}{1-X-Xv}] \tag{63}$$

Let us compute the series $U \times v - 1$:

$$U \times v - 1 = \frac{(1-X)v}{1-X-Xv} - 1 = [\frac{(-X + X^2 + Xv)}{1-X-Xv}] \tag{64}$$

From equations (63),(64) we get the identity:

$$\Delta U = U \times v - 1. \tag{65}$$

By the stability properties established in theorem 23, U belongs to $\mathbb{S}_{k+1}$ and $U \times v$ belongs to $\mathbb{S}_{k+1}$ too. The hypothesis that $U(0) = 1$ and $V(0) \geq 1$ ensure that $U \times v - 1$ belongs to $\mathbb{S}_{k+1}$. Formula (65) shows that $U \in \Sigma\mathbb{S}_{k+1}$. □

Lemma 21. *Let $k_1 \geq 1, k_2 \geq 1, U \in \Sigma\mathbb{S}_{k_1+1}$ and $V \in \Sigma\mathbb{S}_{k_2+1}$. Then $U \circ V \in \Sigma\mathbb{S}_{k_1+k_2+1}$.*

Sketch of proof: Let $U = \Sigma u, V = \Sigma v$ for some $u \in \mathbb{S}_{k_1}, v \in \mathbb{S}_{k_2}$. Then $U \circ V = \sum_{k=0}^{k=V(n)} u(k)$. Hence

$$(\Delta(U \circ V))(n) = \sum_{k=V(n)+1}^{k=V(n+1)} u_k.$$

Some $(k_1 + k_2)$-dcp computing $U \circ V$ can be constructed along the following lines.
1- From a $(k_1 + 1)$-dcpa computing ΔU, one can construct a $(k_1 + 1)$-dpda $\mathcal{A}'_1$ allowing derivations

$$(q_0, A_1[\omega_{k_1-1}[(\prod_{i=0}^{m} GF^{\ell_i})\bar{G}\Omega']\Omega], q_0) \to^*_{\mathcal{A}'_1} \prod_{i=0}^{m} (q_0 \bar{A}_1[\Omega]q_0)^{u(\ell_i)} \tag{66}$$

(as usual $\omega_{k_1-1}[\Omega]$ denotes $A_2[A_3[\ldots[A_{k_1}[\Omega]]\ldots]]$) and $F, G, \bar{G}$ have the same level k_1+1.
2- From a (k_2+1)-dcpa computing ΔV, one can construct a (k_2+1)-dpda $\mathcal{A}_2'$ allowing derivations

$$(r_0, B_1[B_2[\ldots B_{k_2}[F^n\Omega]\ldots]], r_0) \to^*_{\mathcal{A}_2'} (r_0(\bar{B}_1[\prod_{i=0}^{n+1}(\prod_{\ell=V(i)+1}^{V(i+1)} GF^\ell)\bar{G})\Omega]r_0) \quad (67)$$

3- A "compound" automaton $\mathcal{A}$, with level k_1+k_2+1 can then be constructed from $\mathcal{A}_1'$, $\mathcal{A}_2'$, along the ideas of the proof of lemma 9. This automaton $\mathcal{A}$ will lead to a "composition" of derivations (66,67) along the scheme:

$$\begin{aligned}(s_0, A_1[\omega_{k_1-1}[B_1[B_2[\ldots B_{k_2}[F^n]\ldots]]]], s_0) \to^*_{\mathcal{A}} \\ (s_0, A_1[\omega_{k_1-1}[\bar{B}_1[\prod_{i=0}^{n+1}(\prod_{\ell=V(i)+1}^{V(i+1)} GF^\ell)\bar{G})]]], s_0) \to^*_{\mathcal{A}} \\ \prod_{\ell=V(n)+1}^{V(n+1)} (q_0\bar{A}_1[\varepsilon]q_0)^{u(\ell)}\end{aligned}$$

(We do not assert that such a derivation will really exist; in fact $\mathcal{A}$ performs alternatively simulations of $\mathcal{A}_1'$ and simulations of $\mathcal{A}_2'$, in such a way that, globally, it simulates the above sequence of two derivations, see the proof of lemma 9). □

Lemma 22. *Let $k \geq 2$. Let $U_1, U_2, \ldots, U_p$ be sequences of integers, $P_1, P_2, \ldots, P_p$ be polynomials in $\Sigma\mathbb{S}_{k+1}[X_1, X_2, \ldots, X_p]$, $c_1, c_2, \ldots, c_p \in \mathbb{N}$ such that:for every $1 \leq i \leq p$*

$$U_i(n+1) = P_i(n, U_1(n), U_2(n), \ldots, U_p(n)) \text{ and } c_i = U_i(0) \leq U_i(1).$$

Then $U_1 \in \Sigma\mathbb{S}_{k+1}$.

Proof: Let U_i, P_i, c_i fulfill the hypothesis of the lemma. Let $a_0(n), a_1(n), \ldots, a_q(n)$ be a sequence enumerating all the coefficients of the polynomials $P_1, P_2, \ldots, P_p$. There exist some polynomials $Q_i \in \mathbb{N}[X_0, \ldots, X_q, X_{q+1}, \ldots, X_{q+p}]$, such that, for every $1 \leq i \leq p$:

$$P_i(n, U_1(n), U_2(n), \ldots, U_p(n)) = Q_i(a_0(n), \ldots, a_q(n), U_1(n), \ldots, U_p(n)).$$

The Euler-Mac-Laurin formula applied to polynomials Q_i expresses the difference $Q_i(X_0, \ldots, X_{q+p}) - Q_i(Y_0, \ldots, Y_{q+p})$ under the form:

$$\sum_{\bar{k}} \frac{1}{\bar{k}!} \frac{\partial^{\bar{k}} Q_i}{(\partial X_0)^{k_0} \ldots (\partial X_{q+p})^{k_{q+p}}}(Y_0, \ldots, Y_{q+p}) \cdot (X_0 - Y_0)^{k_0} \cdots (X_{q+p} - Y_{q+p})^{k_{q+p}}, \quad (68)$$

where $\bar{k} = (k_1, k_2, \ldots, k_{q+p})$ varies over all the $(q+p)$ tuples with sum $k_1 + k_2 + \ldots + k_{q+p}$ smaller or equal to the degree of Q_i. For every monomial $M = X_0^{d_0} X_1^{d_1} \cdots X_{q+p}^{d_{q+p}}$ the partial derivative

$$\frac{1}{\bar{k}!} \frac{\partial^{\bar{k}} M}{(\partial X_0)^{k_0} \ldots (\partial X_{q+p})^{k_{q+p}}} (Y_0, \ldots, Y_{q+p}),$$

is equal to

$$C_{d_0}^{k_0} C_{d_1}^{k_1} \cdots C_{d_{q+p}}^{k_{q+p}} \cdot Y_0^{d_0 - k_0} Y_1^{d_1 - k_1} \cdots Y_{q+p}^{d_{q+p} - k_{q+p}}. \tag{69}$$

Every partial derivative

$$R_{i,\bar{k}} = \frac{\partial^{\bar{k}} Q_i}{(\partial X_0)^{k_0} \ldots (\partial X_{q+p})^{k_{q+p}}} (Y_0, \ldots, Y_{q+p})$$

is a linear combination, with coefficients in $\mathbb{N}$, of monomials of the form (69), hence it has only non-negative integer coefficients:

$$R_{i,\bar{k}} \in \mathbb{N}[Y_0, \ldots, Y_{q+p}].$$

Let us apply the following substitution to the undeterminates $X_0, \ldots, X_{q+p}, Y_0, \ldots, Y_{q+p}$,

$$X_j \leftarrow a_j(n+1) \text{ for } 0 \leq j \leq q; \;\; X_{q+\ell} \leftarrow U_\ell(n+1) \text{ for } 0 \leq \ell \leq p,$$

$$Y_j \leftarrow a_j(n) \text{ for } 0 \leq j \leq q; \;\; Y_{q+\ell} \leftarrow U_\ell(n) \text{ for } 0 \leq \ell \leq p.$$

We obtain: $(\Delta U_i)(n+1) =$

$$\sum_{\bar{k}} R_{i,\bar{k}}(a_0(n+1), \ldots, a_q(n+1), U_1(n), \ldots, U_p(n)) \cdot (\Delta \bar{a}(n))^{\bar{k}} \cdot (\Delta \bar{U}(n))^{\bar{k}} \tag{70}$$

where the expression $(\Delta \bar{a})^{\bar{k}}(n)$ means:

$$(\Delta a_0)^{k_0}(n) \cdots (\Delta a_q)^{k_q}(n)$$

and the expression $(\Delta \bar{U})^{\bar{k}}(n)$ means:

$$(\Delta U_1)^{k_{q+1}}(n) \cdots (\Delta U_p)^{k_{q+p}}(n).$$

By the closure properties established in theorem 23, every sequence $R_{i,\bar{k}}(a_0(n+1), \ldots, a_q(n+1), U_1(n), \ldots, U_p(n))$ belongs to $\mathbb{S}_{k+1}$. Equation (70) is thus a system of polynomial equations, with coefficients in $\mathbb{S}_{k+1}$, with initial conditions $U_i(1) - U_i(0) \in \mathbb{N}$ and whose vector of solutions is:

$$(\Delta U_1)(n), \ldots, (\Delta U_p)(n)).$$

By theorem 23, point (4), all the $(\Delta U_i)(n)$ belong to $\mathbb{S}_{k+1}$, which proves that all the $U_i(n)$ belong to $\Sigma \mathbb{S}_{k+1}$. $\square$

Let us summarize the closure properties demonstrated in this subsection.

Theorem 26.
0- For every $U \in \Sigma\mathbb{S}_{k+1}$, $k \geq 1$, and every integer $c \in \mathbb{N}$, the sequences EU (the shift of U), $U + \frac{c}{1-X}$ (adding number c to every term), belong to $\mathbb{S}_{k+1}$;
if every $U(n) \geq c$ then $U - \frac{c}{1-X}$ (substracting number c to every term) belongs to $\Sigma\mathbb{S}_{k+1}$;
if the number $U(0)$ is greater or equal to c, then the sequence defined by $0 \mapsto c, n+1 \mapsto U(n)$ belongs to $\Sigma\mathbb{S}_{k+1}$.
1- For every $U, V \in \Sigma\mathbb{S}_{k+1}$, $k \geq 1$, the sequences $U + V, U \odot V$ (the ordinary product) belong to $\Sigma\mathbb{S}_{k+1}$.
2-For every $U \in \Sigma\mathbb{S}_{k+1}, V \in \Sigma\mathbb{S}_k$, $k \geq 2$, $U \times V$ (the convolution product) belongs to $\Sigma\mathbb{S}_{k+1}$.
3-For every $V \in \Sigma\mathbb{S}_k$,$k \geq 2$,such that $V(0) \geq 1$, the sequence U defined by: $U(0) = 1$ and $U(n+1) = \sum_{m=0}^{n} U(m) \cdot V(n-m)$ (the convolution inverse of XV) belongs to $\Sigma\mathbb{S}_{k+1}$.
4- For every $U \in \Sigma\mathbb{S}_k, V \in \Sigma\mathbb{S}_\ell$, $k, l \geq 2$, $U \circ V$ (the sequence composition) belongs to $\Sigma\mathbb{S}_{k+\ell-1}$.
5- For every $k \geq 2$, if $U_1(n), \ldots U_p(n)$ is the vector of solutions of a system of recurrent equations expressed by polynomials in $\Sigma\mathbb{S}_{k+1}[X_1, \ldots, X_p]$, with initial conditions $U_i(0), U_i(1) \in \mathbb{N}$, with $U_i(0) \leq U_i(1)$, then $U_1 \in \Sigma\mathbb{S}_{k+1}$.

Let us recall that, by theorem 24, for every sequence $U \in \Sigma\mathbb{S}_{k+1}$, the predicate $P = \{U(n) \mid n \in \mathbb{N}\}$ leads to a structure $\langle \mathbb{N}, S, P \rangle$ which has a decidable Monadic Second Order theory.

7 Sequences of rational numbers

We define here a class of sequences of *rational* numbers that can be described by k-level automata. The results of section 4 showing that many natural operations over sequences can be translated as operations over automata are carried over this more general situation.

Definition 26. *Let $\mathbb{S}$ be a set of sequences of natural integers. We denote by $\mathcal{D}(\mathbb{S})$ the set of sequences $(u_n)_{n \geq 0}$ of the form:*

$$u_n = a_n - b_n \quad \text{for all } n \geq 0,$$

for some sequences $a, b \in \mathbb{S}$. We denote by $\mathcal{F}(\mathbb{S})$ the set of sequences $(r_n)_{n \geq 0}$ of the form:

$$r_n = \frac{a_n - b_n}{a'_n - b'_n} \quad \text{for all } n \geq 0,$$

for some sequences $a, b, a', b' \in \mathbb{S}$.

Theorem 27. *Let u, v be sequences of rational numbers in $\mathcal{F}(\mathbb{S}_k)$ (resp. $\mathcal{D}(\mathbb{S}_k)$) for some integer $k \geq 3$. Then the sequences of rational numbers $u+v, u-v, u \odot v$*

are in $\mathcal{F}(\mathbb{S}_k)$ (resp. $\mathcal{D}(\mathbb{S}_k)$).
If v does not vanish, then $\frac{u}{v}$ is in $\mathcal{F}(\mathbb{S}_k)$ too.

Proof: This theorem 27 is clear since, by theorem 23, $\mathbb{S}_k$ is closed under sum and ordinary product. □
Let us define the equality problem for sequences in $\mathcal{F}(\mathbb{S}_k)$ as the following algorithmic problem:

INPUT: two sequences $u, v \in \mathcal{F}(\mathbb{S}_k)$,
QUESTION: $u = v$?

i.e. is it true that, $\forall n \in \mathbb{N}, u_n = v_n$?

Corollary 5. *Let $k \geq 3$. The equality problem for sequences in $\mathcal{F}(\mathbb{S}_k)$ reduces to the equivalence problem for deterministic counter k-pushdown automata.*

Proof: Just notice that $\frac{a-b}{a'-b'} = \frac{c-d}{c'-d'}$ iff

$$ac' + bd' + a'd + b'c = ad' + bc' + a'c + b'd. \tag{71}$$

By theorem 23 each side of this equation is recognized by a single k-dcp that can be computed from the eight automata defining $a, b, c, d, a', b', c', d'$. Equation (71) can thus be considered as an instance of the equivalence problem for k-dcp. □

Let us notice that, by theorem 27, $(\mathcal{D}(\mathbb{S}_k), +, \cdot)$ and $(\mathcal{F}(\mathbb{S}_k), +, \cdot)$ are rings. We denote by $P(n, X_1, \ldots, X_j, \ldots, X_p)$ any element of $\mathcal{F}(\mathbb{S}_k)[X_1, \ldots, X_j, \ldots, X_p]$ to emphasize the fact that the coefficients of P are functions of the integer argument n.

Theorem 28. *Let $P_i(n, X_1, \ldots, X_j, \ldots, X_p)$, for $1 \leq i \leq p$, be polynomials with coefficients in $\mathcal{F}(\mathbb{S}_k)$ ($k \geq 3$) and $c_1, c_2, \ldots, c_p \in \mathbb{Q}$. Let u_i, for $1 \leq i \leq p$, be sequences defined by*
$u_i(n+1) = P_i(n, u_1(n), \ldots, u_j(n), \ldots u_p(n))$, and $u_i(0) = c_i$. Then $u_1 \in \mathcal{F}(\mathbb{S}_k)$.

Proof: We suppose the sequences $u_1(n), u_2(n), \ldots, u_p(n)$ fulfill the recurrence

$$u_i(n+1) = P_i(n, u_1(n), \ldots, u_j(n), \ldots u_p(n)) \tag{72}$$

for $1 \leq i \leq p, n \in \mathbb{N}$ and

$$u_i(0) = c_i. \tag{73}$$

We prove the theorem in three steps.
Step 1: Case where the $P_i \in \mathcal{D}(\mathbb{S}_k)[X_1, \ldots, X_p], c_i \in \mathbb{Z}$.
Let us consider the polynomial

$$Q_i(n, X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p) = P_i(n, X_1 - Y_1, X_2 - Y_2, \ldots, X_p - Y_p).$$

It can be decomposed as a sum of monomials of the form

$$\epsilon \cdot u_n \cdot X_1^{\alpha_1} Y_1^{\beta_1} X_2^{\alpha_2} Y_2^{\beta_2} \cdots X_p^{\alpha_p} Y_p^{\beta_p}$$

where $\epsilon \in \{+1, -1\}, \alpha_i, \beta_i \in \mathbb{N}, (u_n)_{n\in\mathbb{N}} \in \mathbb{S}_k$. The polynomials Q_i can thus be decomposed as

$$Q_i(n, X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p) =$$
$$Q_i^+(n, X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p) - Q_i^-(n, X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p)$$

with $Q_i^+, Q_i^- \in \mathbb{S}_k[X_1, Y_1, X_2, Y_2, \ldots, X_p, Y_p]$. As well,

$$c_i = c_i^+ - c_i^-$$

for some $c_i^+, c_i^- \in \mathbb{N}$. Let us define new sequences $u_i^+(n), u_i^-(n)$ by:

$$u_i^\epsilon(n+1) = Q_i^\epsilon(n, u_1^\epsilon(n), \ldots, u_j^\epsilon(n), \ldots u_p^\epsilon(n)) \tag{74}$$

for $1 \le i \le p, \epsilon \in \{+, -\}, n \in \mathbb{N}$, and

$$u_i^\epsilon(0) = c_i^\epsilon \tag{75}$$

for $1 \le i \le p, \epsilon \in \{+, -\}$.
From the recurrence (74), the initial conditions (75) and the definition of Q_i, Q_i^ϵ one can see that the sequences $u_i^+ - u_i^-$ are fulfilling recurrence (72) and initial condition (73). Hence $u_1 = u_1^+ - u_1^-$ where $u_1^\epsilon \in \mathbb{S}_k$, which shows that $u_1 \in \mathcal{D}(\mathbb{S}_k)$.
Step 2: Case where the $P_i \in \mathcal{F}(\mathbb{S}_k)[X_1, \ldots, X_p]$ and are all homogeneous of the same degree $d \in \mathbb{N}$, $c_i \in \mathbb{Q}$.
As the set $\mathcal{D}(\mathbb{S}_k)$ is closed under Hadamard-product, we can suppose that all the coefficients of the polynomials P_i are sequences of the form

$$\frac{A(n)}{D(n)}$$

for different sequences $A \in \mathcal{D}(\mathbb{S}_k)$ and a single sequence $D \in \mathcal{D}(\mathbb{S}_k)$.
The equations (72,73) can then be rewritten as

$$D(n) \cdot u_i(n+1) = R_i(n, u_1(n), \ldots, u_j(n), \ldots u_p(n)) \tag{76}$$

for $1 \le i \le p, n \in \mathbb{N}, R_i \in \mathcal{D}(\mathbb{S}_k)[X_1, \ldots, X_p]$, where the R_i are homogeneous of degree d, and

$$u_i(0) = c_i, \tag{77}$$

for $1 \le i \le p$.
Let us define the sequence $F(n)$ by:

$$F(n+1) = D(n) \cdot F(n)^d$$

for all $n \ge 0$, and

$$F(0) = 1.$$

One can check that:

$$F(n+1)\cdot u_i(n+1) = R_i(n, F(n)u_1(n), \dots, F(n)u_j(n), \dots F(n)u_p(n)) \quad (78)$$

for $1 \le i \le p, n \in \mathbb{N}$ and

$$F(0)u_i(0) = c_i. \quad (79)$$

Using step 1 of this proof, we know that both sequences $(F(n)\cdot u_1(n))_{n\geq 0}$ and $(F(n))_{n\geq 0}$ belong to $\mathcal{D}(\mathbb{S}_k)$. It follows that $u_1 \in \mathcal{F}(\mathbb{S}_k)$.

Step 3: General case.

Let $d \geq 0$ be the maximum degree of all polynomials $P_1, \dots, P_p$. Let us introduce a new undeterminate Z and consider the polynomials $Q_i(n, X_1, X_2, \dots, X_p, Z)$ which are homogeneous of degree d and such that

$$Q_i(n, X_1, X_2, \dots, X_p, 1) = P_i(n, X_1, X_2, \dots, X_p).$$

We also introduce the constant sequence

$$u_{p+1}(n) = 1$$

for every $n \geq 0$. One can check that the sequences $u_1(n), u_2(n), \dots, u_p(n), u_{p+1}(n)$ are fulfilling the conditions

$$u_i(n+1) = Q_i(n, u_1(n), \dots, u_j(n), \dots u_p(n), u_{p+1}(n)) \quad (80)$$

for $1 \le i \le p+1, n \in \mathbb{N}$ and

$$u_i(0) = c_i \text{ if } 1 \le i \le p, \;\; u_{p+1}(0) = 1. \quad (81)$$

By step 2 of this proof, we can conclude that $u_1 \in \mathcal{F}(\mathbb{S}_k)$. $\Box$

Theorem 29. *Let $u \in \mathcal{F}(\mathbb{S}_{k+1})$ and $v \in \mathcal{F}(\mathbb{S}_k)$ for some integer $k \geq 3$. Then the convolution-product $u \times v$ belongs to $\mathcal{F}(\mathbb{S}_{k+1})$.*

Proof:

Let $a^+, a^-, b^+, b^- \in \mathbb{S}_{k+1}, c^+, c^-, d^+, d^- \in \mathbb{S}_k$ such that

$$u = \frac{a^+ - a^-}{b^+ - b^-}, v = \frac{c^+ - c^-}{d^+ - d^-}.$$

We introduce the auxiliary sequences:

$$b = b^+ - b^-, \;\; d = d^+ - d^-; \;\; B(n) = \Pi_{\ell=0}^n b(\ell); \;\; D(n) = \Pi_{\ell=0}^n d(\ell),$$

and

$$\bar{B}(m,n) = \Pi_{\substack{\ell=0 \\ \ell\neq m}}^n b(\ell); \;\; \bar{D}(m,n) = \Pi_{\substack{\ell=0 \\ \ell\neq m}}^n d(\ell).$$

One can check that, for every $n \in \mathbb{N}$:

$$u(n) = \frac{1}{B(n)\cdot D(n)} \cdot (\sum_{m=0}^{n} a(m)\bar{B}(m,n)\cdot c(n-m)\bar{D}(n-m,n)). \quad (82)$$

The product $B(n) \cdot D(n)$ can be decomposed as

$$B(n) \cdot D(n) = (B^+(n)D^+(n) + B^-(n)D^-(n)) - (B^+(n)D^-(n) + B^-(n)D^+(n)) \tag{83}$$

where the B^ϵ fulfill the equations

$$\begin{aligned} B_2^\epsilon(n,n) &= b^\epsilon(n) \\ B_2^+(\ell,n) &= b^+(\ell)B_2^+(\ell+1,n) + b^-(\ell)B_2^-(\ell+1,n) \\ B_2^-(\ell,n) &= b^+(\ell)B_2^-(\ell+1,n) + b^-(\ell)B_2^+(\ell+1,n) \end{aligned}$$

for $0 \le \ell < n$, and $B^\epsilon(n) = B_2^\epsilon(0,n)$.
It follows from lemma 10 that $B_2^\epsilon \in \mathbb{S}_{k+1}^{(2)}$ and from lemma 12 that $B^\epsilon \in \mathbb{S}_{k+1}$.
Similarly, $D^\epsilon \in \mathbb{S}_k$.
The double sequence $\bar{B}(m,n)$ can be defined through the triple sequence $\bar{B}_3^\epsilon(\ell,m,n)$:

$$\bar{B}_3(\ell,m,n) = \Pi^n_{\substack{\ell'=\ell \\ \ell' \neq m}} b(\ell')$$

via the formula

$$\bar{B}(m,n) = \bar{B}_3(0,m,n).$$

We can decompose $\bar{B}_3(\ell,m,n)$ as

$$\bar{B}_3(\ell,m,n) = \bar{B}_3^+(\ell,m,n) - \bar{B}_3^-(\ell,m,n)$$

where the $\bar{B}_3^\epsilon$ are fulfilling the equations:

$$\begin{aligned} &\bar{B}_3^+(n,n,n) = 1 \\ &\bar{B}_3^-(n,n,n) = 0 \\ &\bar{B}_3^\epsilon(\ell,m,n) = \bar{B}_3^\epsilon(\ell+1,m,n) \text{ if } \ell = m, \ell < n, \epsilon \in \{+,-\} \\ &\bar{B}_3^\epsilon(n,m,n) = b^\epsilon(n) \text{ if } 0 \le m < n \\ &\bar{B}_3^+(\ell,m,n) = b^+(\ell)\bar{B}_3^+(\ell+1,m,n) + b^-(\ell)\bar{B}_3^-(\ell+1,m,n) \text{ if } \ell \neq m, \ell < n \\ &\bar{B}_3^-(\ell,m,n) = b^-(\ell)\bar{B}_3^+(\ell+1,m,n) + b^+(\ell)\bar{B}_3^-(\ell+1,m,n) \text{ if } \ell \neq m, \ell < n \end{aligned}$$

It follows from lemma 11 that $\bar{B}^\epsilon \in \mathbb{S}_{k+1}^{(2)}$. Similarly, $\bar{D}^\epsilon \in \mathbb{S}_k^{(2)}$.
Using now lemma 13 (simple sequences viewed as double), lemma 14 (closure under ordinary product) and lemma 15 (closure under pseudo-convolution), we obtain that the numerator of the right-hand-side of equation (82) belongs to $\mathcal{D}(\mathbb{S}_{k+1})$. By theorem 22 the set $\mathbb{S}_{k+1}$ is closed under ordinary product and under sum. The decomposition (83) thus shows that the denominator of the right-hand-side of equation (82) belongs to $\mathcal{D}(\mathbb{S}_{k+1})$. Finally, $u \in \mathcal{F}(\mathbb{S}_{k+1})$. $\square$

Comparison with other classes The set of *rational* sequences of rational numbers is a subset of $\mathcal{F}(\mathbb{S}_3)$: such sequences are defined by recurrences of the form (72,73)

with polynomials P_i of degree 1, with constant rational coefficients; hence, by theorem 28, they belong to $\mathcal{F}(\mathbb{S}_3)$.
The set of the so-called P-*recurrent* sequences of rationals is a subset of $\mathcal{F}(\mathbb{S}_3)$: by theorem 5 polynomial functions (with coefficients in $\mathbb{N}$) belong to $\mathbb{S}_2$, hence polynomial functions (with coefficients in $\mathbb{Q}$) belong to $\mathcal{F}(\mathbb{S}_2)$, and by theorem 28 solutions of equations with polynomial rational coefficients belong to $\mathcal{F}(\mathbb{S}_3)$. Let us recall that, the set of P-recurrent sequences is closed under Hadamard-product ([Sta80, theorem 2.10]) and also under convolution-product ([Sta80, theorem 2.3]). This last property is not known for the set $\mathcal{F}(\mathbb{S}_3)$. Theorem 29 can be seen as a "weak" closure property in this respect.

8 Perspectives

We think that the classes of sequences (or formal power series) introduced here deserve further study.
1- Other closure properties for $\cup_{k\geq 2}\mathbb{S}_k$ should be investigated. For example we strongly believe that it is closed under the operation $\bullet$ (the series-substitution) and also under recurrent equations with coefficients and also *exponents* in this class. Whether these closure properties would transfer to $\cup_{k\geq 2}\Sigma\mathbb{S}_k$ and whether the closure under operation $\bullet$ would also transfer to $\cup_{k\geq 2}\mathcal{F}(\mathbb{S}_k)$ is an interesting question.
2- Comparison with other classes of sequences defined by means of automata would be interesting too: we think $\cup_{k\geq 2}\mathbb{S}_k$ is included in the class of "residually ultimately periodic" sequences studied in [CT00] (work in preparation). Comparisons with the various classes considered in [AS03] should be done too. A combination of these different methods can be hoped.
3- The notion of *multiple* sequences of level k just sketched in §5 should be studied in full generality.
4- Corollary 5 might be used in two directions:
- we could derive from some undecidability problem for sequences of numbers an undecidability result for deterministic automata of level $k \geq 2$,
- any progress toward an equivalence algorithm for automata of level $k \geq 2$ could lead to new techniques allowing to solve equality problems for sequences of numbers.

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