On the termination-problem for one-rule semi-Thue systems

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Abstract. We solve the u-termination and the termination problems for the one-rule semi-Thue systems S of the form $0^p 1^q \longrightarrow v$, $(p, q \in \mathbb{N} - \{0\}, v \in \{0,1\}^*)$. We obtain a structure theorem about a monoid that we call the *termination-monoid* of S. As a consequence, for every fixed system S of the above form, the termination-problem has a *linear* time-complexity.

Keywords: semi-Thue systems; termination; finite automata; rational monoid; automatic structure;

1 Introduction

The semi-Thue systems are among the simplest type of rewriting systems that occur in the literature: they can be seen as the special case of term rewriting systems where function symbols are of arity 1 only, which implies also that they are linear (with only one variable). Despite this apparent simplicity it has been shown recently that the *termination* and *u-termination* problems (see §2.4 for precise definitions) are both *undecidable* for 3 rules semi-Thue systems ([MS96]). In the case of semi-Thue systems with only *one* rule, these two algorithmic problems are still open. Partial results are obtained in [Kur90, Ges91, DH93, McN94, ZG95].

We show here that for the one rule semi-Thue systems of the form $0^p 1^q \longrightarrow v, (p, q \in \mathbb{N} - \{0\}, v \in \{0,1\}^*)$, these two problems are decidable, thus generalizing the results of [ZG95].

The general idea of our solution is that the termination (resp. u-termination) problem for a system $u \longrightarrow v$ can be reduced to the study of either the set of irreducible words $L_+(S) = \{\beta \in Irr(S) \mid \rho(\beta u) = \infty\}$ or its analogue on the right $R_+(S) = \{\gamma \in Irr(S) \mid \rho(u\gamma) = \infty\}$ (provided u has no self-overlap). We show that for semi-Thue systems of the above form, it is always true that $L_+(S)$ (or $R_+(S)$) is *rational* (Theorem 3.10).

We describe in two steps a way to *construct effectively* a finite automaton $\mathcal{A}_S$ recognizing $L_+(S)$ (or $R_+(S)$):

- we first show that a word w is the initial word of some infinite derivation iff w is the initial word of a *looping* derivation (see §4); this is sufficient to solve the termination problem (theorem 4.7),
- we then show that an integer parameter k_0, which is in some sense the key-parameter allowing to construct $\mathcal{A}_S$, is in fact always smaller than $2 \cdot p$; this second step turns out to be rather tricky, it consists in the detailed inspection of 11 different cases (see §5); it follows that $\mathcal{A}_S$ is constructible and that the u-termination problem is decidable (theorem 5.4).

At last, we study the structure of a monoid $\bar{M}$ expressing in some sense the termination problem for S (therefore, we call it the *termination*-monoid): $\bar{M}$ is the Rees-quotient of $M = X^*/\overset{*}{\longleftrightarrow}_S$ by the image in M of the ideal $I_+ = \{w \in \{0,1\}^* \mid w \overset{\infty}{\longrightarrow}_S\}$. It turns out that when S is not u-terminating, $\bar{M}$ is a *rational* monoid (in the sense of [Sak87, PS90]). This implies that, for every given S, the termination problem is solvable in *linear* time (theorem 6.1 and corollary 6.5).

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2 Preliminaries

2.1 Words

Given an alphabet X, by $(X^*, \cdot, \epsilon)$ we denote the *free monoid* generated by X (where $\cdot$ is the concatenation-product and ϵ is the empty word). Given a word u, by $|u|$ we denote the length of u. X^+ denotes the subset $X^* - \{\epsilon\}$. As $X^*$is embedded in $F(X)$ (the *free group* generated by X) we sometimes use the notation u^{-1} for the *inverse* in $F(X)$ of a word u. A prefix (resp. strict prefix)of a word $u \in X^*$ is any word v such that there exists $w \in X^*$ (resp. $w \in X^+$) fulfilling $u = v \cdot w$. The notions of suffix (resp. strict suffix) are defined similarly. Given $u \in X^*$ and $K \in \mathbb{N}$, we denote by $u^{(K)}$ the suffix of length K of u: either $|u| \leq K$ and $u^{(K)} = u$ or $|u| > K$ and $u = wu^{(K)}$ with $|u^{(K)}| = K$. By $u^{<p}$ we denote the set of all the powers of u with an exponent strictly less than p: $\{u^i \mid 0 \leq i < p\}$. As we deal with termination problems, it will be convenient to consider the monoid X^*_∞ defined by: $X^*_\infty = X^* \cup \{\infty\}$ (∞ is just one new symbol); the product in the monoid X^* is extended to X^*_∞ by the formula: $\forall w \in X^*_\infty, w \cdot \infty = \infty \cdot w = \infty$. One can check that $(X^*_\infty, \cdot, \epsilon)$ is a monoid too.

2.2 Semi-Thue systems

We recall a semi-Thue system over X is a subset $S \subseteq X^* \times X^*$. The one-step derivation generated by S, denoted $\longrightarrow_S$, is the binary relation defined by: $\forall f, g \in X^*, f \longrightarrow_S g$ iff there exists $(u, v) \in S$ and $\alpha, \beta \in X^*$ such that $f = \alpha u \beta, g = \alpha v \beta$. The relations $\xrightarrow{*}_S$ (the derivation generated by S), $\xleftrightarrow{*}_S$(the *congruence* generated by S), ..., are then deduced from $\longrightarrow_S$ as usual. We denote by $\mathrm{Irr}(S)$ the set of *irreducible* words: $\mathrm{Irr}(S) = X^* - \bigcup_{(u,v)\in S} X^* u X^*$. The relation ${}_l\!\longrightarrow_S$, the one-step *leftmost derivation* generated by S, is classical. Let us mention that, in the particular case where S consists of only one rule $u \longrightarrow v$, it can be simply defined by: $f \;{}_l\!\longrightarrow_S g \iff \exists r, s \in X^*$ such that $f = rus, g = rvs$ and every decomposition $f = r'us'$ is such that $|r| \leq |r'|$. ${}_l\!\xrightarrow{*}_S$ is the leftmost-derivation generated by S. We note $w \xrightarrow{\infty}_S$ iff there is an infinite derivation starting on w.

We refer the reader to [BO93, DJ91] for more information on rewriting systems.

2.3 Automata

A n-tape *generalized finite automaton* (abbreviated n-g.f.a. in the sequel) is a 5-tuple:

$$\mathcal{A} = < X, Q, \delta, q_0, F >$$

where X is a finite alphabet, the input-alphabet; Q is a finite set, the set of states; q_0 is a distinguished state , the initial state; $F \subseteq Q$ is a set of distinguished states , the final states; δ, the set of transitions, is a subset of $Q \times (X^*)^n \times Q$.

The *one-step move relation* $\longrightarrow_{\mathcal{A}}$ is the binary relation on $\mathrm{Config}(\mathcal{A})$ defined by: $(q, \mathbf{u}) \longrightarrow_{\mathcal{A}} (q', \mathbf{u}') \iff \exists \mathbf{a} \in (X^*)^n$ such that $(q, \mathbf{a}, q') \in \delta$ and $\mathbf{u} = \mathbf{a} \cdot \mathbf{u}'$.
The language recognized by $\mathcal{A}$ is: $\mathrm{L}(\mathcal{A}) = \{\mathbf{u} \in (X^*)^n \mid \exists q \in F, (q_0, \mathbf{u}) \xrightarrow{*}_{\mathcal{A}} (q, (\epsilon)^n)\}$. For every $\mathbf{u} \in (X^*)^n$, the binary relation $\xrightarrow{\mathbf{u}}_{\mathcal{A}}$ over Q is defined by: $q \xrightarrow{\mathbf{u}}_{\mathcal{A}} q' \iff (q, \mathbf{u}) \xrightarrow{*}_{\mathcal{A}} (q', (\epsilon)^n)$.

One can easily see that the class of languages of the form $\mathrm{L}(\mathcal{A})$ for some 1-g.f.a. $\mathcal{A}$ is exactly the class of *rational languages* and the class of binary relations of the form $\mathrm{L}(\mathcal{A})$ for some 2-g.f.a. $\mathcal{A}$ is exactly the class of *rational relations* (see [Ber79, chap.III-IV]).

2.4 Termination problems

We focus here on the two following algorithmic problems.

termination problem **:** the termination-problem for the alphabet X and the semi-Thue system $S \subseteq X^* \times X^*$ is the following
instance: $w \in X^*$
question : does every derivation (mod S) starting on w have finite length ?
(when the answer is "yes", we say that S *terminates* on w).

uniform termination problem **:** the uniform-termination problem for a class $\mathcal{S}$ of semi-Thue systems is the following

instance: an alphabet X and a finite semi-Thue system $S \subseteq X^* \times X^*$ which belongs to $\mathcal{S}$.
question : does every derivation (mod S) starting from a word in X^* have finite length ?
(when the answer is "yes" we say that S is *uniformly-terminating*, sometimes abbreviated as u-terminating).

For more information about termination problems, the reader should refer to [Der87, DJ91].

3 The ideal I_+

Let $S : u \to v$ be a one-rule semi-Thue system over the alphabet $X = \{0, 1\}$. We suppose u has no self-overlap i.e. : $\forall u_1, u_2 \in \{0, 1\}^*, u = u_1 u_2 u_1 \Longrightarrow u_1 = \varepsilon$. We define a map $\rho : X^*_\infty \longrightarrow \mathrm{Irr}(S) \bigcup \{\infty\}$ by:

$$w \mapsto \begin{cases} w' & \text{if } w \xrightarrow{*}_S w' \in \mathrm{Irr}(S) \\ \infty & \text{if } w \xrightarrow{\infty}_S \text{ or } w = \infty \end{cases}$$

As u is overlap-free, ρ is well-defined.

Lemma 3.1 *Let $w \in \{0, 1\}^*$. The following properties are equivalent*

1. $\rho(w) = \infty$,
2. $\exists \beta \in \mathrm{Irr}(S), \exists \gamma \in \{0, 1\}^*, w \,{}_l\!\xrightarrow{*}_S \beta u \gamma$ *and* $\rho(\beta u) = \infty$.

Proof: It is clear that (2) $\Longrightarrow$ (1). Let us suppose that (2) is false. Let us consider the decomposition :

$$w = x_1 \ldots x_i \ldots x_p \text{ where } x_i \in \{0, 1\}. \quad (1)$$

Let us consider the sequence of elements of X^*_∞ :

$$w_0 = \epsilon\ ,\ z_0 = w\ ,\ w_{i+1} = \rho(w_i x_{i+1}) \text{ and } z_{i+1} = (x_{i+1})^{-1} z_i \text{ for } 0 \leq i < p.$$

Using decomposition (1) and hypothesis $\neg$(2), one can show by induction on i that, for every $i \in [0, p]$: $w_i \neq \infty$ and $w \,{}_l\!\xrightarrow{*}_S w_i z_i$. It follows that $\rho(w) = w_p z_p = w_p \neq \infty$. □

Let us define

$L_+(S) = \{\beta \in Irr(S) \mid \rho(\beta u) = \infty\}, L_-(S) = \{\beta \in Irr(S) \mid \rho(\beta u) \neq \infty\}$,
$R_+(S) = \{\gamma \in Irr(S) \mid \rho(u\gamma) = \infty\}, R_-(S) = \{\gamma \in Irr(S) \mid \rho(u\gamma) \neq \infty\}$,
$I_+(S) = \{w \in \{0, 1\}^* \mid \rho(w) = \infty\}, I_-(S) = \{w \in \{0, 1\}^* \mid \rho(w) \neq \infty\}$.

Corollary 3.2 *:*
1- One of the sets $L_\varepsilon(S), R_\varepsilon(S), I_\varepsilon(S) (\varepsilon \in \{+, -\})$ is recursive iff all these sets are recursive
2- S is u-terminating $\Longleftrightarrow L_+(S) = \emptyset \Longleftrightarrow R_+(S) = \emptyset$.

Let us say that a word $w_1 \in X^*$ *crosses* (from left to right) a word $w_2 \in X^+$ iff there exist a leftmost derivation :

$$w_1 \cdot w_2 \,{}_l\!\xrightarrow{*}_S w_3$$

which uses the rightmost letter of w_2.

We say that a word $w \in Irr(S)$ is a *left-separator* iff the following situation is impossible :

$$\exists w_1, w_2, w_3 \in \{0, 1\}^*, ww_2 \in \mathrm{Irr}(S), w_1 \cdot ww_2 \xrightarrow{*}_S w_3,$$

where this derivation uses the rightmost letter of w. A notion of *right-separator* can be defined similarly.

In all the remaining we suppose $u = 0^p 1^q$ with $p > 0, q > 0$.

Lemma 3.3 *Let $S = 0^p1^q \to v'0$ where $p > 0, q > 0, v' \in \{0,1\}^*$. Let $n \in \mathbb{N} - \{0\}$. One (and only one) of the following cases is true.*

C1: $\forall q' < q, 0^p1^{q'}$ *does not cross* 1^n *and* $\rho(0^p1^{q'}1^n) < \infty$,
C2: $\forall q' < q, 0^p1^{q'}$ *does not cross* 1^n *and* $\rho(0^p1^{q'}1^n) = \infty$,
C3:

$$\exists q' < q \begin{cases} 0^p1^{q'} \text{ crosses } 1^n, \\ \rho(0^p1^{q'}1^n) \neq \infty, \\ \forall q" < q, q" \neq q' \Longrightarrow 0^p1^{q"} \text{ does not cross } 1^n \quad \text{and } \rho(0^p1^{q"}1^n) \neq \infty. \end{cases}$$

C4:

$$\exists q' < q \begin{cases} 0^p1^{q'} \text{ crosses } 1^n, \\ \rho(0^p1^{q'}1^n) = \infty, \\ \forall q" < q', 0^p1^{q"} \text{ does not cross } 1^n \quad \text{and } \rho(0^p1^{q"}1^n) \neq \infty, \\ \forall q" \in]q', q[, 0^p1^{q"} \text{ does not cross } 1^n \quad \text{and } \rho(0^p1^{q"}1^n) = \infty. \end{cases}$$

Remark. In any case, 0^p1^q has *at most one* strict prefix which crosses 1^n.

Lemma 3.4 *Let $S = 0^p1^q \to v'0$ where $p > 0, q > 0, v' \in \{0,1\}^*1\{0,1\}^*$. Let $n \in \mathbb{N} - \{0\}$ realizing case C3. Let $q" < q$ such that $0^p1^{q"}$ does not cross 1^n. Then $\rho(0^p1^{q"}1^n0)$ is a left-separator.*

Sketch of proof: Suppose $0^p1^{q'}$ crosses 1^n and $\rho(0^p1^{q'}1^n) \neq \infty$. It shows that

$$n = q - q' \tag{2}$$
$$\text{or } v'0 = v"0^p \text{ for some } v" \in (0+1)^*. \tag{3}$$

If (2) is true : $\rho(0^p1^{q"}1^n) \in (0+1)^*011^{<q-1}$ and every element of $011^{<q-1}0$ is a left-separator. Hence $\rho(0^p1^{q"}1^n0)$ is a left-separator.
If (3) is true : $\rho(0^p1^{q"}1^n) \in (0+1)^*0^p11^{<q-1}$. Hence $\rho(0^p1^{q"}1^n0)$ is a left-separator. □

Lemma 3.5 *Let $S = 0^p1^q \to 1^rv'0^s$ where $p, q, r, s > 0, v' \in (0^+1^+)^*$, one of the blocks of 0 in $1^rv'0^s$ is not divided by p. Let $w \in 1^rv'0^s1^tv'(0+1)^*$ for some integer $t \geq 0$ such that $v'1^t \neq \epsilon$. Then $\rho(w) \in 1^r0(0+1)^* \cup \{\infty\}$.*

Let us define the sets $\mathrm{M}(S), M_+(S), M_-(S)$ by:

$$\mathrm{M}(S) = \{f \in X^* \mid \exists n \in \mathbb{N}, f = f_n \cdot f_{n-1} \cdots f_r \cdots f_0, \forall i \in [0,n] f_i \text{ is a strict prefix of } u,$$
$$f_0v \in u(0+1)^* \text{ and } \forall i \in [1,n] f_i\rho(f_{i-1}\cdots f_0u) \in u(0+1)^*\} \cup \{\epsilon\}.$$

$$M_+(S) = M(S) \cap L_+(S), \quad M_-(S) = M(S) \cap L_-(S).$$

Lemma 3.6 $L_+(S) = ((0+1)^*M_+(S)) \cap \mathrm{Irr}(S)$

Hence $L_\varepsilon(S)$ is rational iff $M_\varepsilon(S)$ is rational (for every $\varepsilon \in \{+,-\}$).

Lemma 3.7 *Let $S = 0^p1^q \to 1^rv'0^s$ where $p, q, r, s > 0, v' \in (0^+1^+)^*$, one of the blocks of 0 in $1^rv'0^s$ is not divided by p. Then $M_+(S), M_-(S)$ are rational.*

Sketch of proof: By lemma 3.3, the integer $n = r$ must fulfill one of cases C1-C4. Let us give a description of a generalized f.a. $\mathcal{A}_S$ recognizing $M_+(S)$ in each of these cases.
C1 : $\forall q' < q, 0^p1^{q'}$ does not cross 1^r and $\rho(0^p1^{q'}1^r) < \infty$. Hence $r > q$ and $s < p$. So that $M_+(S) = \emptyset$. The g.f.a. $\mathcal{A}_S$ is trivial.
C2 : $\forall q' < q, 0^p1^{q'}$ does not cross 1^r and $\rho(0^p1^{q'}1^r) = \infty$. Hence $M_+(S) = 0^p1^{<q}$. The g.f.a. $\mathcal{A}_S$ is straightforward.
C3 : $\exists! q' < q, 0^p1^{q'}$ crosses 1^r, and $\rho(0^p1^{q'}1^r) \neq \infty$.

Let us note $\alpha = 0^p1^{q'}$. Let us define : $Q_i = (0^p1^q)1^i (i = 0$ or $r \dot{-} q \leq i < r)$.
For every $i \in \{0\} \cup [r \dot{-} q, r[$ we define :

$$k_i = \inf\{k \mid \rho(\alpha^k Q_i) = \infty\} \tag{4}$$

(notice that this definition implies that $[k_i = \infty \Longleftrightarrow \forall k, \rho(\alpha^k Q_i) < \infty]$ and also

$$k_o \geq k_{r \dot{-} q} \geq \cdots \geq k_{r-1} \geq 0).$$

Let us define : $P_{i,j} = \rho(\alpha^j Q_i)$ for $i \in \{0\} \cup [r \dot{-} q, r[, 0 \leq j \leq k_i$.
The g.f.a $\mathcal{A}_S$ has the set of states :

$\mathcal{Q}_S = \{P_{i,j} \mid i \in \{0\} \cup [r \dot{-} q, r[, 0 \leq j \leq k_i\} \cup \{\infty\}$,
(notice that all the states $P_{i,j}$ with $j = k_i < \infty$ are equal to ∞),
its set of transitions δ is:

$$\delta = \{(P_{i,j+1}, \alpha, P_{i,j}, \mid 0 \leq j \leq k_i - 1 < \infty\} \tag{5}$$
$$\cup \{(P_{i,0}, \alpha, P_{i,0}) \mid k_i = \infty\} \tag{6}$$
$$\cup \{(P_{\ell,0}, 0^p1^{q"}, P_{i,j}) \mid q \dot{-} r \leq q" < q, q" \neq q', \ell = r - q + q"\}, \tag{7}$$

its initial state is: ∞,
its terminal state is: $P_{0,0}$.

One can prove by induction on $\mid f \mid$ that, for all $f \in \{0,1\}^*, i \in \{0\} \cup [r \dot{-} q, r[, 0 \leq j \leq k_i$:

$$f \in \mathrm{M}(S) \Longrightarrow \exists P \in \mathcal{Q}_S, P \xrightarrow{f}_{\mathcal{A}_S} P_{0,0}, \tag{8}$$
$$i = 0 \text{ and } P_{i,j} \xrightarrow{f}_{\mathcal{A}_S} P_{i,0} \Longrightarrow f \in \mathrm{M}(S) \text{ and } \rho(fQ_0) = P_{i,j}, \tag{9}$$
$$i \neq 0, k_i = \infty \text{ and } P_{i,0} \xrightarrow{f}_{\mathcal{A}_S} P_{0,0} \Longrightarrow f \in \mathrm{M}(S) \text{ and } \exists k \in \mathbb{N}, \rho(fQ_0) \in P_{i,k}0(0+1)^*, \tag{10}$$
$$i \neq 0, k_i \neq \infty \text{ and } P_{i,j} \xrightarrow{f}_{\mathcal{A}_S} P_{0,0} \Longrightarrow f \in \mathrm{M}(S) \text{ and } \rho(fQ_0) \in P_{i,j}0(0+1)^*. \tag{11}$$

The key-arguments are that, for $i \neq 0$, if $\rho(fQ_0) = P_{i,j}0w$ then
1- $\rho(\alpha f Q_0) = P_{i,j+1}0w$, because, by lemma 3.4, $P_{i,0}0$ is a left-separator;
2- for $q \dot{-} r \leq q" < q, q" \neq q', \ell = r - q + q"$, if $\rho(fQ_0) = P_{i,j}0w$, then

$$\rho(0^p1^{q''}fQ_0) = \rho(0^p1^{q''}P_{i,j}0w) = \rho(0^p1^{q''}P_{i,j}0)w \quad (\text{ because } P_{i,0}0 \text{ is a left-separator})$$
$$\in \rho(0^p1^{q''}1^r0(0+1)^*)w \text{ (using lemma 3.5) } = \rho(0^p1^{q''}1^r)0w' \text{ (for some } w' \in (0+1)^*),$$
(because $\rho(0^p1^{q''}1^r)0 = P_{\ell,0}0, q' \neq q'' \Longrightarrow \ell \neq 0$, hence $P_{\ell,0}0$ is either a left-separator or $= \infty$)
$$= P_{\ell,0}0w'.$$

By equivalences (8-11),

$$\mathrm{L}(\mathcal{A}_S) = M_+(S).$$

C4 : $\exists! q' < q, 0^p1^{q'}$ crosses 1^r, and $\rho(0^p1^{q'}1^r) = \infty$
The same g.f.a $\mathcal{A}_S$ as in C3 recognizes $L_+(S)$. Let us notice that in this case :

$$(i \geq r - q' \Longrightarrow P_{i,0} = \infty, k_i = 0) \text{ and } (r \dot{-} q \leq i < r - q' \Longrightarrow P_{i,0} \neq \infty, k_i = 1).$$

(see lemma 3.3, C4). □

Lemma 3.8 *Let $S = 0^p1^q \rightarrow 0^t1^rv'$ where $p, q, r, t > 0, p > t, q \leq r, v' \in 0(0+1)^* \cup \{\epsilon\}$. Then $\mathrm{M}_+(S)$ is rational.*

Sketch of proof: Given $w \in 0^t1(0+1)^* \cap \mathrm{Irr}(S)$, there exists at most one strict prefix α of 0^p1^q such that αw is reducible. Hence $M_+(S) = \emptyset$ or $M_+(S) = \{\alpha^{k_0}\}$ for some integer k_0.
□

Lemma 3.9 *Let $S = 0^p1^q \to 1^r v' 0^s$, where $p,q,r,s > 0, v' \in (0^+1^+)^*$, every block of 0 in $1^r v' 0^s$ is divided by p and every block of 1 in $1^r v' 0^s$ is divided by q. Then $M_+(S)$ is rational.*

Sketch of proof: If $v' \neq \varepsilon$ then 0^p1^q is factor of v' and $L_+(S) = \mathrm{Irr}(S)$.
If $v' = \varepsilon : S = 0^p1^q \to (1^q)^\lambda (0^p)^\mu$. Then

- if $\lambda \geq 2$ and $\mu \geq 2$, then $\rho(0^p u) = \infty$. One can construct a g.f.a. $\mathcal{A}_S$ recognizing $M_+(S)$ as in lemma 3.7, C2.
- if $\lambda = 1$, then: $\forall n \in \mathbb{N}, \forall i < r = q, \quad \rho((0^p)^n u 1^i) = 1^q 0^{\mu \cdot p \cdot (n+1)} 1^i < \infty$.
  As in lemma 3.7, C1, we have: $M_+(S) = \emptyset$.
- if $\mu = 1$, by the above argument $R_+(S) = \emptyset$, hence by corollary 3.2, point 2, $L_+(S) = \emptyset$, so that $M_+(S) = \emptyset$.

□

Theorem 3.10 *Let $S = 0^p1^q \to v \quad (p > 0, q > 0, v \in \{0,1\}^*)$. Then either $L_+(S), L_-(S)$ are rational or $R_+(S), R_-(S)$ are rational.*

Proof: If ($v \in 1(0+1)^*0$ and v has at least one block of 0 not divided by p) or $v \in 0(0+1)^*$, then S fulfills the hypothesis of one of the lemma 3.7,3.8,3.9, hence $L_\varepsilon(S)$ are rational. In every other case where $v \notin 0^* + 1^*$ by a symmetry argument $R_\varepsilon(S)$ are rational. In the remaining case where $v \in 0^* + 1^*$, $L_+(S) = R_+(S) = \emptyset$. □

4 Infinite derivations

Lemma 4.1 *Let S fulfilling the hypothesis of lemma 3.7, case C2. Then*
$\forall q' < q, \exists q'' < q, \exists \beta, \gamma \in \{0,1\}^*$ such that $0^p1^{q'}Q_0 \xrightarrow{+}_S \beta(0^p1^{q''}Q_0)\gamma$.

We skip the proof which is analogous to (but simpler than) the proof of lemma 4.3 below.

Lemma 4.2 *Let S fulfilling the hypothesis of lemma 3.7, case C2, and $w \in \{0,1\}^*$. $\rho(w) = \infty \iff \exists q' < q, \exists \beta, \gamma, \beta', \gamma' \in \{0,1\}^*$, $w \xrightarrow{*}_S \beta'(0^p1^{q'}Q_0)\gamma'$ and $0^p1^{q'}Q_0 \xrightarrow{+}_S \beta(0^p1^{q'}Q_0)\gamma$.*

Proof: The non-trivial direction of the equivalence is ($\Longrightarrow$). Let $w \in \{0,1\}^*$, $\rho(w) = \infty$. By lemma 3.1, there exist $\beta'_0 \in \mathrm{Irr}(S), \gamma'_0 \in \{0,1\}^*, w \,{}_l\!\xrightarrow{*}_S \beta'_0 u \gamma'_0$ and $\rho(\beta'_0 u) = \infty$. As $\beta'_0 v$ must be reducible,

$$\beta'_0 = \beta'_1 0^p 1^{q_1} \qquad \text{for some } \beta'_1 \in \mathrm{Irr}(S), \ \ 0 \leq q_1 < q.$$

By lemma 4.1, there exist infinite sequences $(q_k)_{k\geq 1}, (\beta_k)_{k \geq 1}, (\gamma_k)_{k\geq 1}$ such that:

$$0^p1^{q_k}Q_0 \xrightarrow{+}_S \beta_k(0^p1^{q_{k+1}}Q_0)\gamma_k, \qquad \text{for all } k \geq 1.$$

There must exist $k_0 \geq 1, d \geq 1$ such that $q_{k_0+d} = q_{k_0}$. Let us define:

$$q' = q_{k_0} \ , \ \beta = \beta_{k_0}\beta_{k_0+1}\ldots\beta_{k_0+d-1} \ , \ \gamma = \gamma_{k_0+d-1}\ldots\gamma_{k_0+1}\gamma_{k_0} \ , \ \beta' = \beta'_1\beta_1\ldots\beta_{k_0-1} \ , \ \gamma' = \gamma_{k_0-1}\ldots\gamma_1\gamma'_0.$$

We have

$$w \xrightarrow{*}_S \beta'(0^p1^{q'}Q_0)\gamma' \text{ and } 0^p1^{q'}Q_0 \xrightarrow{+}_S \beta(0^p1^{q'}Q_0)\gamma$$

as required. □

Lemma 4.3 *Let S fulfilling the hypothesis of lemma 3.7, case C3 or C4. Let α, Q_i, k_i be as defined in the proof of lemma 3.7. Then*
$\forall i \in \{0\} \cup [r-q, r[$, if $k_i \neq \infty$ then $\exists i' \in \{0\} \cup [r-q, r[, \exists \beta, \gamma \in \{0,1\}^*$ such that $\alpha^{k_i}Q_i \xrightarrow{+}_S \beta(\alpha^{k_{i'}}Q_{i'})\gamma$.

Proof: Suppose $k_i \neq \infty$. $\rho(\alpha^{k_i} Q_i) = \infty$, hence, by lemma 3.1, $\exists \beta_1 \in \mathrm{Irr}(S), \exists \gamma_1 \in \{0,1\}^*$,

$$\alpha^{k_i} Q_{i\ l} \xrightarrow{*}_S \beta_1 u \gamma_1 \text{ and } \rho(\beta_1 u) = \infty. \tag{12}$$

By lemma 3.6, $\beta_1 = \beta_2 \beta_3$ with $\beta_3 \in M_+(S)$. Let us consider an accepting computation of $\mathcal{A}_S$ over the word β_3. It must have the form:

$$\infty = P_{i',k_{i'}} \xrightarrow{\alpha^{k_{i'}}}_{\mathcal{A}_S} P_{i',0} \xrightarrow{0^p 1^{q''}}_{\mathcal{A}_S} P_{i'',j} \xrightarrow{\beta_4}_{\mathcal{A}_S} P_{0,0} \tag{13}$$

where $i', i'' \in \{0\} \cup [r-q, r[, 0 \neq q'' < q, 0 \neq j < k_{i''}, i' = r-q+q''$ (see definition 7) and $\beta_3 = \alpha^{k_{i'}} 0^p 1^{q''} \beta_4$.
By (12) (13),

$$\begin{aligned} \alpha^{k_i} Q_i \xrightarrow{*}_S \beta_1 Q_0 \gamma_1 &\xrightarrow{*}_S \beta_2 \alpha^{k_{i'}} (0^p 1^{q''} \rho(\beta_4 Q_0)) \gamma_1 \\ &= \beta_2 \alpha^{k_{i'}} (0^p 1^{q''} 1^r 0 \gamma_2) \gamma_1 \text{ (for some } \gamma_2 \in (0+1)^*, \text{ by lemma 3.5)} \\ &= \beta_2 \alpha^{k_{i'}} Q_{i'} 0 \gamma_2 \gamma_1. \end{aligned}$$

Letting $\beta = \beta_2$ and $\gamma = 0\gamma_2\gamma_1$, the lemma is proved. □

Lemma 4.4 *Let S fulfilling the hypothesis of lemma 3.7, case C3 or C4 and $w \in \{0,1\}^*$.*
$\rho(w) = \infty \iff \exists i \in \{0\} \cup [r-q, r[, \exists \beta, \gamma, \beta', \gamma' \in \{0,1\}^*, w \xrightarrow{*}_S \beta' \alpha^{k_i} Q_i \gamma'$ and $\alpha^{k_i} Q_i \xrightarrow{+}_S \beta(\alpha^{k_i} Q_i)\gamma$.

Proof: The proof is analogous to that of lemma 4.2: it leans on lemma 3.1, lemma 4.3 and finiteness of $\{0\} \cup [r-q, r[$. □

Lemma 4.5 *Let $S = 0^p 1^q \to 0^t 1^r v'$, fulfilling the hypothesis of lemma 3.8. Let α, k_0 be as defined in the proof of lemma 3.8 and let us suppose that $k_0 \neq \infty$. Then*
1- $\exists \beta_S, \gamma_S \in \{0,1\}^*$ such that $\alpha^{k_0} Q_0 \xrightarrow{+}_S \beta_S(\alpha^{k_0} Q_0)\gamma_S$.
2- $\forall w \in \{0,1\}^*, \rho(w) = \infty \iff \exists \beta', \gamma' \in \{0,1\}^*, w \xrightarrow{*}_S \beta'(\alpha^{k_0} Q_0)\gamma'$.

Proof: Analogous to the proof of lemma 4.2. □

Lemma 4.6 *Let $S = 0^p 1^q \to (1^q)^\lambda (0^p)^\mu$ where $\lambda \geq 2$ and $\mu \geq 2$. Then*
1- $\exists \beta_S, \gamma_S \in \{0,1\}^*$ such that $0^p Q_0 \xrightarrow{+}_S \beta_S(0^p Q_0)\gamma_S$.
2- $\forall w \in \{0,1\}^*, \rho(w) = \infty \iff \exists \beta', \gamma' \in \{0,1\}^*, w \xrightarrow{*}_S \beta'(0^p Q_0)\gamma'$.

Proof: Point (1) is straightforward. Point (2) is a direct consequence of lemma 3.1. □

Theorem 4.7 *Given any system $S = 0^p 1^q \to v$ (where $v \in \{0,1\}^*$), and any word $w \in \{0,1\}^*$, one can decide whether S terminates on w or not.*

Proof: The termination problem for a pair (S, w) where S is any finite semi-Thue system over $\{0,1\}$ and w any word over $\{0,1\}$ is semi-decidable. By the above lemmas 4.2, 4.4, 4.5, 4.6, for systems S of the form $0^p 1^q \to v$ the non-termination of S on w is also semi-decidable. This proves the theorem. □

5 Computation of the structure of I_+

Our proofs of lemma 3.7, 3.8 and 3.9 consist in describing (abstractly) a finite automaton recognizing the set $\mathrm{M}_+(S)$. This automaton is *constructible* from two data:

- the case that the system S is fulfilling (i.e. cases C1-C4 of lemma 3.7, the hypothesis of lemma 3.8 or the hypothesis of lemma 3.9),
- a finite set of integers $k_0, \ldots, k_{r-1}$ in cases C1-C4 of lemma 3.7 , a single integer k_0 in the case of lemma 3.8.

We will show now that these cases and integers are effectively computable from the system S.

Lemma 5.1 *Let S fulfill the hypothesis of lemma 3.7. The property "w_1 crosses (from left to right) w_2" is decidable.*

Proof: Let u' such that $0^p1^q = u'1$. Let $v = 1^r w_0 0^s$ such that at least one block of 0 in v is not divided by p. For every $w_1, w_2 \in \{0,1\}^*$, w_1 crosses w_2 iff $w_2 = w_2'1$ (for some w_2') and

$$w_1 \cdot w_2' \;{}_l\!\xrightarrow{*}_S \beta \cdot u' (\text{ for some } \beta \in \{0,1\}^*) \text{ and } \beta \cdot u' \in \mathrm{Irr}(S). \tag{14}$$

One can then decide this condition (14) by the following algorithm:
1- decide whether $\rho(w_1 \cdot w_2') = \infty$ or not (by theorem 4.7)
2- if $\rho(w_1 \cdot w_2') = \infty$ then w_1 does not cross w_2
3- if $\rho(w_1 \cdot w_2') \neq \infty$ then compute $\rho(w_1 \cdot w_2')$
4- if $\rho(w_1 \cdot w_2')$ does not admit the suffix u' then w_1 does not cross w_2 else w_1 crosses w_2. □

Lemma 5.2 *Let S fulfill the hypothesis of lemma 3.7. One can decide which of cases C1-C4 is fulfilled.*

Proof: Clearly, it is sufficient to decide for every $q' < q$ whether $0^p1^{q'}$ crosses 1^r or not, and whether $\rho(0^p1^{q'+r}) = \infty$ or not. These properties can be decided by lemma 5.1 and theorem 4.7. □

Lemma 5.3 *Let S fulfill the hypothesis of lemma 3.7, case C3 or C4. Then $k_0 \leq k_{r-1}+1$. In particular, $k_0 = \infty \Longleftrightarrow k_{r-1} = \infty$.*

Proof: Let α be the unique strict prefix of 0^p1^q which crosses (from left to right)1^r. Suppose that $\alpha^k 0^p 1^q 1^{r-1} \xrightarrow{\infty}_S$. Then, by confluence of S:

$$\alpha^k 1^r v' 0^s 1^{r-1} \xrightarrow{\infty}_S \tag{15}$$

Let us consider the unique integer λ such that $r - q \leq \lambda \cdot q < r$. The $(r-1) - (\lambda \cdot q)$ last symbols of 1^{r-1} cannot be used in this derivation. Hence $\alpha^k 1^r v' 0^s 1^{\lambda \cdot q} \xrightarrow{\infty}_S$, which implies that

$$\alpha^k 0^p 1^q 1^{\lambda \cdot q} \xrightarrow{\infty}_S . \tag{16}$$

Let us consider the integer q' such that $\alpha = 0^p1^{q'}$. Both integers $\lambda \cdot q$ and $q' + r - q$ are divided by q and belong to $[r-q, r[$. Hence

$$\lambda \cdot q = q' + r - q. \tag{17}$$

Hence

$$\begin{aligned} \alpha^{k+1}0^p1^q &\longrightarrow_S \alpha^k(\alpha 1^r v' 0^s) = \alpha^k(0^p1^{q'+r}v'0^s) = \alpha^k(0^p1^q1^{\lambda \cdot q}v'0^s) \text{ by equality (17)} \\ &\xrightarrow{\infty}_S \text{ (by derivation (16)).} \end{aligned}$$

We have proved that $k_0 \leq k_{r-1} + 1$. □
By the definition of the k_i and lemma 5.3 we know that

$$k_{r-1} + 1 \geq k_0 \geq k_{\dot{r-q}} \geq \ldots \geq k_{r-1} \geq 0. \tag{18}$$

Owing to lemma 5.2, the fact that S fulfills case Ci of lemma 3.7 or the hypothesis of lemma j, is decidable.

In cases C1-C4 of lemma 3.7, by the inequalities (18), from the knowledge that $k_0 = \infty$ or any finite upper bound on k_0, one can compute the exact values of all the k_i's and then the automaton $\mathcal{A}_S$.

In the case of lemma 3.8, from the knowledge that $k_0 = \infty$ or any finite upper bound on k_0, one can compute the exact value of k_0, hence an automaton recognizing $\mathrm{M}_+(S)$.

In the case of lemma 3.9, the computation of a finite automaton is straightforward.

Let us consider some one rule semi-Thue system: $\mathrm{S} = 0^p1^q \longrightarrow v$, where $p, q > 0, v \in (0+1)^*$ and u is not factor of v. Let us denote by S' the system $\mathrm{S}' = 0^q1^p \longrightarrow v'$, where v' is obtained from the mirror image of v by exchanging letters 0 and 1.
The remaining of this section is a discussion which distinguishes 11 main cases D1,...,D8,D'2,D'5,D'6. In every case Di we give the aforementioned bounds and in every case D'i we show that S' fulfills the corresponding case Di. The remaining case where u is factor of v is trivial (then $\mathrm{L}_+(S) = \mathrm{Irr}(S)$).

(We omit here the detailed treatment of cases D6,D7 and D8, see the annex of [Sén96] for more complete calculations; anyway D6 is rather similar to D5 and D7,D8 are easy).

5.1 Case D1: $v = 1^r w_1 1^{\overline{q}} w_2 0^{\overline{p}} w_3 0^s$

with $r, s > 0$, $w_1 \in (0^+1^+)^* \cdot 0^+$, $w_2 \in (0^+1^+)^*$, $w_3 \in 1^+ \cdot (0^+1^+)^*$, $p \not| \overline{p}$, $q \not| \overline{q}$.
(That is: v contains an *internal* block of 1 not divided by q, followed by an *internal* block of 0 not divided by p). The hypothesis of lemma 3.7 is fulfilled. Let $\alpha = 0^p 1^{q'}$ and $q' + r = \lambda \cdot q$ $(\lambda \geq 1)$.

$$\begin{aligned}\alpha 0^p 1^q &\longrightarrow_S 0^p 1^{q'+r} w_1 1^{\overline{q}} w_2 0^{\overline{p}} w_3 0^s \longrightarrow_S 1^r w_1 1^{\overline{q}} w_2 (0^{\overline{p}} w_3 0^s 1^{(\lambda-1)\cdot q)} w_1 1^{\overline{q}} w_2) 0^{\overline{p}} w_3 0^s \\ &\xrightarrow{*}_S 1^r w_1 1^{\overline{q}} w_2 \rho(0^{\overline{p}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2) 0^{\overline{p}} w_3 0^s.\end{aligned}$$

where

$$\rho(0^{\overline{p}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2) \in (\mathrm{Irr}(S) \cap 0(0+1)^*1) \cup \{\infty\}. \tag{19}$$

As well, for every $k \geq 1$

$$\alpha^k 0^p 1^q \xrightarrow{*}_S 1^r w_1 1^{\overline{q}} w_2 \rho(0^{\overline{p}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2)^k 0^{\overline{p}} w_3 0^s.$$

and by (19),

$$1^r w_1 1^{\overline{q}} w_2 \rho(0^{\overline{p}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2)^k 0^{\overline{p}} w_3 0^s \in \mathrm{Irr}(S) \cup \{\infty\}.$$

It follows that either $k_0 = 1$ (if $\rho(0^{\overline{p}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2) = \infty$)
or $k_0 = \infty$ (if $\rho(0^{\overline{p}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2) \neq \infty$).
Conclusion: either $k_0 = 1$ or $k_0 = \infty$.

5.2 Case D2: $v = 1^r w_1 1^{\overline{q}} w_2 0^s$

with $r, s > 0$, $w_1 \in (0^+1^+)^* \cdot 0^+$, $w_2 \in (0^+1^+)^*$, $q \not| \overline{q}$, $p \not| s$.
(That is: v contains an *internal* block of 1 not divided by q and the *right-external* block of 0 is not divided by p). Let $\alpha = 0^p 1^{q'}$ and $q' + r = \lambda \cdot q$ $(\lambda \geq 1)$. Then, for every $k \geq 1$,

$$\rho(\alpha^k 0^p 1^q) = 1^r w_1 1^{\overline{q}} w_2 \rho(0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2)^k 0^s.$$

It follows that either $k_0 = 1$ (if $\rho(0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2) = \infty$) or $k_0 = \infty$ (if $\rho(0^s 1^{(\lambda-1)\cdot q} w_1 1^{\overline{q}} w_2) \neq \infty$).
Conclusion: either $k_0 = 1$ or $k_0 = \infty$.

5.3 Case D'2: $v = 1^r w_1 0^{\overline{p}} w_2 0^s$

with $r, s > 0$, $w_1 \in (0^+1^+)^*$, $w_2 \in (1^+0^+)^* \cdot 1^+$, $q \not| r$, $p \not| \overline{p}$. Then S' fulfills case D2.

5.4 Case D3: $v = 1^r w_1 0^{\overline{p}} w_2 1^{\overline{q}} w_3 0^s$

with $r, s > 0$, $w_1 \in (0^+1^+)^*$, $w_2 \in (1^+0^+)^*$, $w_3 \in (0^+1^+)^*$, $p \not| \overline{p}$, $p \mid s$, $q \not| \overline{q}$, $q \mid r$.
(That is: v contains an *internal* block of 0 not divided by p, followed by an *internal* block of 1 not divided by q). We suppose (without loss of generality) that $1^{\overline{q}}$ is the *leftmost* block of 1 in v which is not divided by q and that $0^{\overline{p}}$ is the *rightmost* block of 0 in v which is not divided by p. Let $\alpha = 0^p$ and $r = \lambda \cdot q$ $(\lambda \geq 1)$.

$$\alpha 0^p 1^q \xrightarrow{2}_S 1^r w_1 0^{\overline{p}} w_2 1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}} w_3 0^s \xrightarrow{*}_S 1^r w_1 0^{\overline{p}} w_2 \rho(1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}}) w_3 0^s.$$

Owing to the divisibility of r and s, there exist $\mu_1, \mu_2 \in \mathbb{Z}$ and $\gamma_1 \in (0^+1^+)^*$, $\gamma_2 \in 1^+(0^+1^+)^*$ such that:

$$\rho(1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}}) = 1^{\overline{q}+\mu_1 q} \gamma_1 0^{\overline{p}+\mu_2 p} \gamma_2 \text{ or } \rho(1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}}) = \infty. \tag{20}$$

It follows that

$$\rho(\alpha 0^p 1^q) = \rho(1^r w_1 0^{\overline{p}} w_2 1^{\overline{q}+\mu_1 q}) \gamma_1 0^{\overline{p}+\mu_2 p} \gamma_2 w_3 0^s \text{ or } \rho(\alpha 0^p 1^q) = \infty. \tag{21}$$

Similarly

$$\alpha^2 0^p 1^q \longrightarrow_S 1^r w_1 0^{\overline{p}} w_2 (1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}+\mu_1 q}) \gamma_1 0^{\overline{p}+\mu_2 p} \gamma_2 w_3 0^s \tag{22}$$

$$\xrightarrow{*}_S 1^r w_1 0^{\overline{p}} w_2 \rho(1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}+\mu_1 q}) \gamma_1 0^{\overline{p}+\mu_2 p} \gamma_2 w_3 0^s. \tag{23}$$

There exist $\mu_3 \in \mathbb{Z}$ and $\gamma_3 \in 1^+(0^+1^+)^*$ such that:

$$\rho(1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}+\mu_1 q}) = 1^{\overline{q}+\mu_1 q}\gamma_1 0^{\overline{p}+\mu_3 p}\gamma_3 \text{ or } \rho(1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}+\mu_1 q}) = \infty. \tag{24}$$

It follows that, if $\rho(1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}}), \rho(1^{\overline{q}} w_3 0^s 1^{(\lambda-1)\cdot q} w_1 0^{\overline{p}} w_2 1^{\overline{q}+\mu_1 q})$ are both finite, then

$$\rho(\alpha^2 0^p 1^q) = \rho(1^r w_1 0^{\overline{p}} w_2 1^{\overline{q}+\mu_1 q})(\gamma_1 0^{\overline{p}+\mu_3 p}\gamma_3)(\gamma_1 0^{\overline{p}+\mu_2 p}\gamma_2) w_3 0^s, \tag{25}$$

and more generally, for every $k \geq 2$,

$$\rho(\alpha^k 0^p 1^q) = \rho(1^r w_1 0^{\overline{p}} w_2 1^{\overline{q}+\mu_1 q})(\gamma_1 0^{\overline{p}+\mu_3 p}\gamma_3)^{k-1}(\gamma_1 0^{\overline{p}+\mu_2 p}\gamma_2) w_3 0^s. \tag{26}$$

Conclusion: $k_0 \leq 2$ or $k_0 = \infty$.

5.5 Case D4: $v = 1^r 0^s$

with $r, s > 0, p \not| s$ and $q \not| r$.
Let $r = \lambda q + r'(0 < r' < q), s = \mu p + s'(0 < s' < p), \alpha = 0^p 1^{q-r'}$. For every $k \geq 0$,

$$\alpha^k 0^p 1^q \xrightarrow{*}_S 1^r 0^s (1^{\lambda\cdot q} 0^s)^k. \tag{27}$$

D4.1: $\lambda = 0$
Then, for every $k \geq 0$, $\rho(\alpha^k 0^p 1^q) = 1^r 0^{(k+1)\cdot s}$. It follows that $k_0 = \infty$.

D4.2: $\mu = 0$
Then, S' fulfills case D4.1, hence $k_0 = \infty$.

D4.3: $\lambda = 1, \mu \geq 2$
Let $k \geq 1$. By (27) and the special value of λ we have

$$\alpha^k 0^p 1^q \xrightarrow{*}_S 1^r 0^s (1^q 0^s)^k = 1^r (0^s 1^q)^k 0^s. \tag{28}$$

Using the fact that $\mu \geq 1$ we get

$$\begin{aligned}
(0^s 1^q)^k &\xrightarrow{*}_S (0^{s-p} 1^r 0^s)^k \xrightarrow{*}_S (0^{s'}(1^r 0^s)^\mu)^k && (29)\\
&= (0^{s'} 1^r (0^s 1^r)^{\mu-1} 0^s)^k && (30)\\
&= 0^{s'} 1^r (0^s 1^r)^{\mu-1} [0^{s+s'} 1^r (0^s 1^r)^{\mu-1}]^{k-1} 0^s && (31)\\
&= 0^{s'} 1^r (0^s 1^r)^{\mu-1} [0^{s'} (0^s 1^r)^\mu]^{k-1} 0^s. && (32)
\end{aligned}$$

One can check that $0^s 1^r \xrightarrow{*}_S 0^{s'} 1^r (0^s 1^{r'})^\mu \in \mathrm{Irr}(S)$.

Plugging (32) into (28) and using the derivation just above we obtain

$$\alpha^k 0^p 1^q \xrightarrow{*}_S 1^r 0^{s'} 1^r (0^{s'} 1^r (0^s 1^{r'})^\mu)^{\mu-1} [0^{s'} (0^{s'} 1^r (0^s 1^{r'})^\mu)^\mu]^{k-1} 0^s 0^s. \tag{33}$$

If $2 \cdot s' < p$, the right-handside of (33) is irreducible , hence $k_0 = \infty$.
If $2 \cdot s' > p$, $\rho(0^{2\cdot s'} 1^r) = 0^{2\cdot s'-p} 1^r 0^s 1^{r'}$ so that

$$\rho(\alpha^k 0^p 1^q) = 1^r 0^{s'} 1^r (0^{s'} 1^r (0^s 1^{r'})^\mu)^{\mu-1} [\rho(0^{2\cdot s'} 1^r)(0^s 1^{r'})^\mu (0^{s'} 1^r (0^s 1^{r'})^\mu)^{\mu-1}]^{k-1} 0^s 0^s,$$

hence $k_0 = \infty$. Let us suppose now that $2 \cdot s' = p$. We can rewrite equation (33) as

$$\alpha^k 0^p 1^q \xrightarrow{*}_S \gamma_1 1 0^s 1^{r'} [0^{2\cdot s'} 1^r 0 \gamma_2 1 0^s 1^{r'}]^{k-1} 0^{2\cdot s} \text{ where } \gamma_1 1, 0\gamma_2 1 \in \mathrm{Irr}(S). \tag{34}$$

We then compute:

$$0^{2\cdot s'} 1^r \xrightarrow{*}_S 1^r 0^s 1^{r'}, \tag{35}$$

$$0^s 1^{r+r'} \longrightarrow_S 0^{s-p} 1^r 0^s 1^{2.r'} \longrightarrow_S 0^{s-2p}(1^r 0^s) 1^{r'} 0^s 1^{2.r'} \xrightarrow{*}_S 0^{s'}(1^r 0^s)(1^{r'} 0^s)^{\mu-1} 1^{2.r'},$$

which implies that

$$\rho(0^s 1^{r+r'}) = 0^{s'} 1\gamma_3, \text{ where } 1\gamma_3 \in \mathrm{Irr}(S). \tag{36}$$

Using (34),(35) and (36) we obtain that for $k \geq 2$

$$\begin{aligned}
\alpha^k 0^p 1^q &\xrightarrow{*}_S \gamma_1 1 0^s 1^{r'} [0^{2\cdot s'} 1^r 0 \gamma_2 1 0^s 1^{r'}]^{k-1} 0^{2\cdot s} \\
&\xrightarrow{*}_S \gamma_1 1 0^s 1^{r'} [1^r 0^s 1^{r'} 0 \gamma_2 1 0^s 1^{r'}]^{k-1} 0^{2\cdot s} \\
&= \gamma_1 1 (0^s 1^{r+r'}) [0^s 1^{r'} 0 \gamma_2 1 (0^s 1^{r+r'})]^{k-2} 0^s 1^{r'} 0 \gamma_2 1 0^s 1^{r'} 0^{2\cdot s} \\
&\xrightarrow{*}_S \gamma_1 1 (0^{s'} 1 \gamma_3) [0^s 1^{r'} 0 \gamma_2 1 (0^{s'} 1 \gamma_3)]^{k-2} 0^s 1^{r'} 0 \gamma_2 1 0^s 1^{r'} 0^{2\cdot s} \in \mathrm{Irr}(S),
\end{aligned}$$

hence $k_0 = \infty$.

D4.4: $\lambda = \mu = 1$

This subcase is similar to D4.3. Now equation (33) reduces to

$$\alpha^k 0^p 1^q \xrightarrow{*}_S 1^r 0^{s'} 1^r [0^{2s'} 1^r 0^s 1^{r'}]^{k-1} 0^{2s}. \tag{37}$$

If $2 \cdot s' \neq p$, $\rho(\alpha^k 0^p 1^q) = 1^r 0^{s'} 1^r [\rho(0^{2s'} 1^r 0^s 1^{r'})]^{k-1} 0^{2s}$, hence $k_0 = \infty$.
If $2 \cdot s' = p$, assuming $k \geq 2$ and using (37),(35),(36) we get

$$\rho(\alpha^k 0^p 1^q) = 1^r 0^{s'} 1^{2r} [0^s 1^{r'} 0^{s'} 1 \gamma_3]^{k-2} 0^s 1^{r'} 0^s 1^{r'} 0^{2s}.$$

Hence $k_0 = \infty$.

D4.5: $\lambda \geq 2, \mu = 1$ Then S' fulfills case D4.3, hence $k_0 = \infty$.

D4.6: $\lambda \geq 2, \mu \geq 2$

Let $\alpha = 0^p$. Let us consider the derivation

$$\alpha 0^p 1^q \longrightarrow_S \alpha 1^r 0^s \longrightarrow_S 1^r 0^s 1^{r-q} 0^s = 1^r 0^{s-2p} (\alpha 0^p 1^q) 1^{r-2q} 0^s \xrightarrow{\infty}_S .$$

Hence S fulfills the hypothesis of lemma 3.7, case C2.
Conclusion: $k_0 = 1$ or $k_0 = \infty$.

5.6 Case D5: $v = 1^r w_0 0^s$

with $r, s > 0, w_0 \in (0^+ 1^+)^*$, at least one block of 0 in $w_0 0^s$ is not divided by p and every block of 1 is divided by q.
Let us reduce ourselves to the case where $q = 1$ (the value of k_0 in the general case is equal to the value of k_0 for the system obtained by replacing every factor 1^q in u or v by the letter 1): the system under consideration is then $0^p 1 \longrightarrow 1^r w_0 0^s$, where at least one block of 0 in $w_0 0^s$ is not divided by p.
As we exclude the case where u is a factor of v, it follows that every internal block of 0 in v must belong to $0^{<p}$. Let $\alpha = 0^p$.

D5.1: $p > s$

if $r \geq 2$: for every $k \geq 0$

$$\rho(0^{p\cdot k} 0^p 1) = 1^r w_0 0^s (1^{r-1} w_0 0^s)^k.$$

hence $k_0 = \infty$.
if $r = 1$: let us show by induction on $n \geq 1$ that,

$$\forall w \in (10^{<p})^{n-1} 10^*, \rho(0^p w) \in (10^{<p})^* 10^* \tag{38}$$

Let $n = 1$ and $w = 10^m$ for some $m \geq 0$. Then

$$\rho(0^p w) = \rho(0^p 10^m) = 1^r w_0 0^{s+m} \in (10^{<p})^* 10^*.$$

Let $n \geq 1$ and $w = 10^{m_1}10^{m_2}\dots10^{m_n}10^{m_{n+1}}$ where $m_i < p$ (for $i \leq n$) and $m_{n+1} \geq 0$. Then

$$\begin{aligned}\rho(0^p w) &= \rho(0^p 10^{m_1}10^{m_2}\dots10^{m_n}10^{m_{n+1}})\\ &= \rho(1^r w_0 0^{s+m_1}10^{m_2}\dots10^{m_n}10^{m_{n+1}})\\ &= 1^r w_0 0^{s+m_1}10^{m_2}\dots10^{m_n}10^{m_{n+1}}(\text{ if } s+m_1 < p) \text{ or}\\ &= 1^r w_0 0^{s+m_1-p}\rho(0^p 10^{m_2}\dots10^{m_n}10^{m_{n+1}})(\text{ if } p \leq s+m_1 < 2\cdot p)\\ &\in (10^{<p})^* 10^* (\text{ by induction hypothesis }).\end{aligned}$$

Using (38), one can then prove by induction on $k \geq 0$ that

$$\forall k \geq 0, \rho(0^{p\cdot k}0^p 1) \in (10^{<p})^* 10^*. \tag{39}$$

It follows from (39) that $k_0 = \infty$.

D5.2: $p = s$ Let us consider the unique morphism $\varphi : \{0,1\}^* \to \{0,1\}^*$ such that

$$\varphi(0) = 0, \quad \varphi(1) = 1^r w_0.$$

Then , for every $k \geq 0$

$$\rho(\alpha^k 0^p 1) = \varphi^k(1^r w_0)0^{p\cdot(k+1)}.$$

Hence $k_0 = \infty$.

D5.3: $p < s$
if $|v|_1 = 1$: that is $v = 10^s$.
Then, for every $k \geq 0$

$$\rho(\alpha^k 0^p 1) = 10^{s\cdot(k+1)},$$

which shows that $k_0 = \infty$.
if $|v|_1 \geq 2$: let us consider some integer k such that

$$k \cdot s \geq (k+1)\cdot p. \tag{40}$$

Using the fact that $p < s$, one can prove by induction on l that, for every $l \geq 0$,

$$\alpha^l 1 \xrightarrow{*}_S w_l 0^{l\cdot s}, \text{ for some } w_l \in \{0,1\}^*.$$

We obtain that

$$\alpha^k(0^p 1) \longrightarrow_S (\alpha^k 1)1^{r-1}w_0 0^s \xrightarrow{*}_S w_k 0^{k\cdot s}1^{r-1}w_0 0^s,$$

and by (40), $w_k 0^{k\cdot s}1^{r-1}w_0 0^s = \gamma 0^{(k+1)\cdot p}1\delta$ for some $\gamma, \delta \in \{0,1\}^*$. Hence

$$\alpha^k(0^p 1) \xrightarrow{*}_S \gamma\alpha^k(0^p 1)\delta,$$

which shows that for every k fulfilling (40), $\rho(\alpha^k(0^p 1)) = \infty$. Hence $k_0 \leq \lceil p/(s-p)\rceil$.
Conclusion: $k_0 \leq \lceil p/(s-p)\rceil$ or $k_0 = \infty$.
Case D'5: $v = 1^r w_0 0^s$, where $r, s > 0$, at least one block of 1 is not divided by q and every block of 0 is divided by p. Then S' fulfills case D5.
Case D6: $v = 0^t 1^r w_0 0^s$, with $r, s, t > 0$, $w_0 \in (0^+1^+)^*$. Then $k_0 \leq \lceil p/(s-h)\rceil$ or $k_0 = \infty$.
Case D'6: $v = 1^t 0^r w_0 1^s$, with $r, s, t > 0$, $w_0 \in (1^+0^+)^*$. Then S' fulfills case D6.
Case D7: $v = 0^t 1^r w_0 1^s$, with $r, s, t > 0$, $w_0 \in (0^+1^+)^*0^+$. Then $k_0 = 1$ or $k_0 = \infty$.
Case D8: $v = 0^t 1^s$, with $s, t \geq 0$. Then $\mathrm{L}_+(S) = \emptyset$.

By the above discussion the proof of the following theorem is achieved.

Theorem 5.4 *Given any system $S = 0^p 1^q \to v$ (where $p > 0, q > 0, v \in \{0,1\}^*$)*
(1) *one can construct either a finite automaton recognizing $\mathrm{L}_+(S)$ (in cases Di, $1 \leq i \leq 8$), or a finite automaton recognizing $\mathrm{R}_+(S)$ (in cases D'2,D'5,D'6).*
(2)*one can decide whether S is u-terminating or not.*

6 The termination monoid

Let us consider a system $S = u \longrightarrow v$, such that u has no self-overlap. We recall that X is the alphabet $\{0,1\}$. We introduce the alphabet $\bar{X} = \{0,1,\infty\}$ and the following monoids:

$$M = X^*/\overset{*}{\longleftrightarrow}_S,\ \bar{M} = M/\bar{I_+}$$

where $\bar{I_+}$ is the image of the ideal I_+ of X^* in the quotient M by the canonical projection, and $M/\bar{I_+}$ is the so-called Rees-quotient of M by the ideal $\bar{I_+}$. By definition, $\bar{M}$ admits the following presentation over $\bar{X}$:

$$u = v; x \cdot \infty = \infty(\forall x \in \bar{X}); \infty \cdot x = \infty(\forall x \in \bar{X}); w = \infty(\forall w \in I_+). \tag{41}$$

Let us denote by $\Pi : M \longrightarrow \bar{M}$ the canonical projection induced by the Rees-quotient. As S is confluent, I_+ is saturated by relation $\overset{*}{\longleftrightarrow}_S$. It follows that the restriction of Π on $M - \bar{I_+}$ is injective. We denote by $\bar{S} \subseteq \bar{X}^* \times \bar{X}^*$ the set of all pairs of words shown in (41). By the above remark, the subset $\bar{W} = \mathrm{Irr}(S) \cup \{\infty\} \subseteq \bar{X}^*$ is a system of representatives $\pmod{\overset{*}{\longleftrightarrow}_{\bar{S}}}$. Let us define a map $\bar{\rho} : \bar{X}^* \longrightarrow \bar{W}$ by:

$$\forall w \in \bar{X}^*, w \overset{*}{\longleftrightarrow}_{\bar{S}} \bar{\rho}(w) \text{ and } \bar{\rho}(w) \in \bar{W}.$$

Theorem 6.1 *Let $S = 0^p1^q \longrightarrow v$, where $p > 0, q > 0$ and $v \in (0+1)^*$. If S is not u-terminating then $\bar{\rho}$ is a rational function.*

In other words, $\bar{M}$ is a *rational* monoid in the sense of [Sak87, PS90].
The fact that

$$\rho = \bar{\rho} \cap X_\infty^* \times \bar{X}^*, \bar{\rho} = \rho \cup \{(w,\infty) \mid w \in \bar{X}^* \cdot \infty \cdot \bar{X}^*\}$$

shows that $\bar{\rho}$ is rational iff ρ is rational. Hence we are reduced to prove that ρ is rational. By symmetry we can also suppose that v is not in $(1^+0^+)^*1$. For any word $w \in X^*$ we define the map $\rho_w : \bar{W} \longrightarrow \bar{W}$ by

$$\forall w' \in \bar{W},\ \ \rho_w(w') = \rho(w' \cdot w).$$

Lemma 6.2 *ρ_u is rational.*

Proof:

$$\begin{aligned}\rho_u = &\{(\beta w_1, \beta w_2) \mid w_1 \in \mathrm{M}_-(S), \beta w_1 \in \mathrm{Irr}(S), \rho(w_1 u) = w_2\}\\ &\cup \{(\beta w_1, \infty) \mid w_1 \in \mathrm{M}_+(S), \beta w_1 \in \mathrm{Irr}(S)\} \cup \{(\infty,\infty)\},\end{aligned}$$

hence it suffices to prove that $\rho_u \cap (\mathrm{M}_-(S) \times \bar{W})$ is rational. If $v \in 0(0+1)^*$, $\mathrm{M}_-(S)$ is finite hence $\rho_u \cap (\mathrm{M}_-(S) \times \bar{W})$ is trivially rational. Let us suppose now that $v = 1^r v' 0^s$ where $r > 0, s > 0, v' \in (0^+1^+)^*$. If S fulfills the hypothesis of lemma 3.7 case C2 or the hypothesis of lemma 3.9, then $\mathrm{M}_-(S)$ is finite too and the same argument applies. It remains to treate the case where S fulfills the hypothesis of lemma 3.7 case C3 or C4 (C1 is impossible because S is supposed to be non u-terminating). Let us suppose case C3 is fulfilled and let $\alpha = 0^p1^{q'}$ be the unique strict prefix of u which crosses 1^r. Let $w \in \mathrm{M}_-(S)$, it must have the form

$$w = \alpha^{k_1}\alpha'_1\alpha^{k_2}\alpha'_2 \ldots \alpha^{k_m}\alpha'_m\alpha^{k_{m+1}} \tag{42}$$

where $m \geq 0, \forall i \in [1, m+1], 0 \leq k_i < k_0$ and $\forall i \in [1,m], \alpha'_i = 0^p1^{q'_i}$ for some $q'_i \neq q'$. Then

$$\rho_u(w) = \rho(\alpha^{k_1}\alpha'_1 1^r)[1^{-r}\rho(\alpha^{k_2}\alpha'_2 1^r)] \ldots [1^{-r}\rho(\alpha^{k_m}\alpha'_m 1^r)][1^{-r}\rho(\alpha^{k_{m+1}}u)], \tag{43}$$

because every $\rho(\alpha^{k_i}\alpha'_i 1^r)$ begins by 1^r0 (by lemma 3.5) and every $\rho(\alpha'_i 1^r 0)$ is a left-separator (by lemma 3.4). Formula (42) (43) show that ρ_u is rational (it is, indeed, a right-to-left *subsequential function* (see [Ber79, p.100] for a definition)). Case C4 is very similar and could be treated in the same way. □

Lemma 6.3 *$\rho_0, \rho_1, \rho_\infty$ are rational functions.*

Proof:

$$\begin{aligned}&\rho_0 = \{(w, w0) \mid w \in \mathrm{Irr}(S)\} \cup \{(\infty,\infty)\},\ \ \rho_\infty = \{(w,\infty) \mid w \in \bar{W}\},\\ &\rho_1 = \{(w, w1) \mid w1 \in \mathrm{Irr}(S)\} \cup \{(w, \rho_u(w1u^{-1}) \mid w1 \in \mathrm{Irr}(S) \cdot u\} \cup \{(\infty,\infty)\}.\end{aligned}$$

Considering these formula, ρ_0, ρ_∞ are trivially rational and ρ_1 is rational by lemma 6.2. □

Remark. Lemma 6.3 shows that $\bar{M}$ has an *automatic* structure over the set of generators $\{0, 1, \infty\}$, with set of representatives $\bar{W}$ (in a sense generalizing in a straightforward way the definition of [ECH$^+$92]).

Lemma 6.4 *Let S be a system fulfilling hypothesis of lemma 3.7 case C3 or C4 and non u-terminating. Then there exists an integer K such that $\forall w \in \mathrm{M}_-(S), |\rho(wu)| < K$ or $(\rho(wu))^{(K)}$ is a right-separator .*

Sketch of proof: Let us suppose C3 is fulfilled, let k_0 be the integer associated to S (by hypothesis $k_0 \neq \infty$) and $\alpha = 0^p 1^{q'}$ the word associated to S. Then for every $\alpha_1 = 0^p 1^{q_1}, \alpha_2 = 0^p 1^{q_2}$ with $0 \leq q_i < q, q_i \neq q'$ and every $k_1 < k_0, k_2 < k_0$,

$$\alpha_1 \alpha^{k_1} \alpha_2 \alpha^{k_2} u \xrightarrow{*}_S 1^r v' 0^s (1^{r-q+q_1} v' 0^s)(1^{r-q} v' 0^s)^{k_1} (1^{r-q+q_2} v' 0^s)(1^{r-q} v' 0^s)^{k_2}.$$

By lemma 3.4 $\rho(1^r v' 0^s 1^{r-q+q_1})0$ is a left-separator. The leftmost block of 0, among all the blocks of 0 in $1v'0^s(1^{r-q}v'0^s)^{k_1} 1^{r-q+q_2}$ which are not divided by p, together with the symbol 1 immediately on its left, is a right-separator. It follows that for every $\beta \in (0+1)^*$,

$$\rho(\beta \alpha_1 \alpha^{k_1} \alpha_2 \alpha^{k_2} u) = [\rho(\beta 1^r v' 0^s 1^{r-q+q_1} 0)(10)^{-1}] \rho(1v'0^s(1^{r-q} v' 0^s)^{k_1} (1^{r-q+q_2} v' 0^s)(1^{r-q} v' 0^s)^{k_2}).$$

where $\rho(1v'0^s(1^{r-q}v'0^s)^{k_1}(1^{r-q+q_2}v'0^s)(1^{r-q}v'0^s)^{k_2})$ is a right-separator. Let us set

$$K_0 = 2 \cdot k_0 \cdot \mid u \mid \text{ and } K = \max\{\mid \rho(wu) \mid \ \mid \ w \in \mathrm{M}_-(S), \mid w \mid \leq K_0\} + 1.$$

One can check that every word $w \in \mathrm{M}_-(S)$ of length $\geq K_0$ has the form $\beta \alpha_1 \alpha^{k_1} \alpha_2 \alpha^{k_2}$ considered above, that $|\alpha_1 \alpha^{k_1} \alpha_2 \alpha^{k_2}| \leq K_0$ and consequently

$$\rho(1v'0^s(1^{r-q}v'0^s)^{k_1}(1^{r-q+q_2}v'0^s)(1^{r-q}v'0^s)^{k_2})| \leq K.$$

□

Outline of proof of theorem 6.1

Let us notice that, if $w = x_1 x_2 \ldots x_i \ldots x_n$ (where $x_i \in X$) then,

$$\bar{\rho}(w) = \rho_{x_n}(\ldots(\rho_{x_i}(\ldots \rho_{x_2}(\rho_{x_1}(\epsilon))\ldots)\ldots). \tag{44}$$

1- Under the hypothesis of lemma 3.7 case C2, or lemma 3.8 or lemma 3.9, $\mathrm{M}_-(S)$ is finite. Let us set $K = \max\{|w|, w \in \mathrm{M}_-(S)\}$. The i-th step of the calculation (44) consists in computing $\rho(\rho(x_1 x_2 \ldots x_{i-1}) x_i)$ and does only use the suffix $\rho(x_1 x_2 \ldots x_{i-1})^{(K)}$. Hence a finite left-to-right subsequential transducer, which keeps in its memory this suffix of length K of the computed part of the argument, can compute $\bar{\rho}$.
2- Under the hypothesis of lemma 3.7 case C3 or C4, the above argument is no more valid. Nevertheless, with the help of lemma 6.2 and lemma 6.4, one can define a finite 2-automaton which recognizes $\bar{\rho}$ (but it is no more right-subsequential as is ρ_u, nor left-subsequential as was $\bar{\rho}$ in part 1 of the proof).□

Corollary 6.5 *Let $S = 0^p 1^q \longrightarrow v$, where $p > 0, q > 0$ and $v \in (0+1)^*$, be a fixed system. Then the termination problem for S can be solved in linear time.*

Proof: If S is u-terminating then the termination problem for S is trivial. Let us suppose now that S is not u-terminating. For every $w \in \{0,1\}^*$, S terminates on w iff $\bar{\rho}(w) \neq \infty$. By theorem 6.1 $\bar{\rho}$ is a rational function, hence it can be computed in linear time (see theorem 5.2 of [Ber79]). □

7 Examples

Let us apply our results and constructions on some particular systems.

Example 1 $S = 0011 \longrightarrow 1111000$. It fulfills case D5 of the discussion of §5, hence $k_0 \leq 2$ or $k_0 = \infty$. We see that $\alpha = 00$. Hence we compute

$$\rho(\alpha 0011) = 1^4 01^4 0^6, (\alpha)^2 0011 \xrightarrow{*}_S 1^4 01^8 [(\alpha)^2 0011] 0^6$$

$$Q_0 = 0011, k_0 = 2; Q_1 = 00111, k_1 = 2; Q_2 = 001111, k_2 = 1; Q_3 = 0011111, k_3 = 1.$$

The automaton $\mathcal{A}_S$ (which recognizes $\mathrm{M}_+(S)$) follows. We show on figure 1 the "trim" version of $\mathcal{A}_S$. Notice that $\infty = P_{0,2} = P_{3,1}$ and that $P_{2,0}$ is *not* co-accessible, hence it does not appear on this schema.

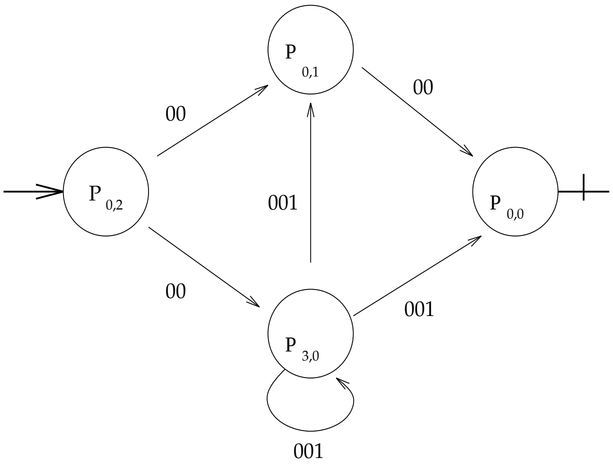

Fig. 1. the g.f.a.$\mathcal{A}_S$.

Example 2 $S = 0^2 1^2 \longrightarrow 1^3 0^5 1 0 1^4 0^3$. It fulfills case D1 of the discussion of §5, hence $k_0 \leq 1$ or $k_0 = \infty$. We see that $\alpha = 001$. Hence we compute $\rho(\alpha 0011) = 1^3(0^5 1 0 1^4 0 1^3 0^5 1 0 1^4 0^3) 0^5 1 0 1^4 0^3$.
We conclude that $k_0 = \infty$, i.e. $\mathrm{M}_+(S) = \emptyset$ and S is u-terminating.

Example 3 $S = 0001 \longrightarrow 01011000$. It fulfills case D6.2.2 of the discussion of §5, hence $k_0 \leq \lceil p + h/(s-h) \rceil = 5$ or $k_0 = \infty$. We see that $\alpha = 00$. Hence we compute

$$\rho(\alpha 0001) = (01011)0(01011)^2 0^6, [(\alpha)^2 0001] \xrightarrow{*}_S (01011)0(01011)^3 0(01011)^2 [(\alpha)^2 0001] 1 (01011) 0^6$$

We conclude that $k_0 = 2$. Hence $\mathrm{M}_+(S) = \{0^4\}$.
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Annex

Let us give here the detailed calculations of cases D6,D7,D8.

Case D6: $v = 0^t 1^r w_0 0^s$

with $r, s, t > 0$, $w_0 \in (0^+1^+)^*$.
If $p \leq t$ and $q \leq r$, then u is factor of v , (trivial case of non-u-termination).
If $q > r$, then $\mathrm{M}(S) = \{\epsilon\}$, hence $\mathrm{M}_+(S) = \emptyset$.
We assume now that: $p > t$ and $q \leq r$. Let $p = t + h$ where $h > 0$ and $\alpha = 0^h$.
The system S has the form:

$$S = 0^t 0^h 1^q \longrightarrow 0^t 1^r w_0 0^s.$$

Let us introduce the system

$$\bar{S} = 0^h 1^q \longrightarrow 1^r w_0 0^s.$$

It is clear that $\xrightarrow{*}_S \subseteq \xrightarrow{*}_{\bar{S}}$.

D6.1: one block of 1 in v is not divided by q
For every $k \geq 1$ we have

$$\alpha^k 0^p 1^q \longrightarrow_S 0^{h \cdot k} 0^t 1^r w_0 0^s \xrightarrow{*}_S 0^t 1^r w_0 0^s (1^{r-q} w_0 0^s)^k = 0^t 1^r w_0 (0^s 1^{r-q} w_0)^k 0^s. \tag{45}$$

Let us suppose that $k_0 \geq 2$. Then $\rho(\alpha 0^p 1^q) < \infty$, which by the above formula implies that $\rho(0^t 1^r w_0)$, $\rho(0^s 1^{r-q} w_0)$ are both finite. It is clear that $\rho(0^t 1^r w_0) \in (0+1)^* 1$ and, as $1^r w_0$ contains one block of 1 which is not divided by q, $\rho(0^s 1^{r-q} w_0) \in 0(0+1)^* 1$. We deduce from these remarks and formula (45) that

$$\rho(\alpha^k 0^p 1^q) = \rho(0^t 1^r w_0)(\rho(0^s 1^{r-q} w_0))^k 0^s.$$

Hence $k_0 = \infty$. Hence in this subcase $k_0 = 1$ or $k_0 = \infty$.

D6.2: every block of 1 in v is divided by q
Hence we are reduced to the case where $q = 1$.

D6.2.1: $h \geq s$ or ($h < s$ and $\mid v \mid_1 = 1$) By the results of cases D5.1, D5.2, $h \geq s$ implies that $\bar{S}$ is u-terminating, hence that S is u-terminating too. By the result of case D5.3, subcase where ($h < s$ and $\mid v \mid_1 = 1$), $\bar{S}$ is u-terminating, hence S is u-terminating too, i.e. $k_0 = \infty$.

D6.2.2: $h < s$, $\mid v \mid_1 \geq 2$ and every two consecutive symbols 1 are separated by at least $p - s$ symbols 0
Let us consider some integer k such that

$$k \cdot s \geq (k+1) \cdot h + p. \tag{46}$$

By calculations analogous to those of case D5.3, subcase where $\mid v \mid_1 \geq 2$, we obtain that there exist $\gamma, \delta \in \{0,1\}^*$ such that

$$\alpha^k (0^p 1) \xrightarrow{*}_S \gamma \alpha^k (0^p 1) \delta.$$

Hence, for every k fulfilling (46), $\rho(\alpha^k(0^p 1)) = \infty$. Hence, $k_0 \leq \lceil (p+h)/(s-h) \rceil$.

D6.2.3: $h < s < p$ and v has a factor in $10^{<p-s}1$ For every $k \geq 0$

$$\alpha^k (0^p 1) \xrightarrow{*}_s 0^t 1^r w_0 (0^s 1^{r-1} w_0)^k 0^s \tag{47}$$

Let us suppose that $k_0 \geq 2$. Formula (47), for $k = 1$ shows that, under this hypothesis $\rho(0^s 1^{r-1} w_0) < \infty$. Moreover, as $1^r w_0$ has a factor in $10^{<p-s}1$, either ($r \geq 2$ and $\rho(0^s 1^{r-1} w_0) = 0^s 1^{r-1} w_0$) or ($r = 1$ and $0^s 1^{r-1} w_0$ is still irreducible or contains a factor in $10^{<p-s}1$). In both situations $\rho(0^s 1^{r-1} w_0) \in 0(0+1)^* 1$. Using (47) we obtain that for every $k \geq 0$

$$\rho(\alpha^k (0^p 1)) = 0^t 1^r w_0 (\rho(0^s 1^{r-1} w_0))^k 0^s.$$

Hence $k_0 = 1$ or $k_0 = \infty$.
Conclusion: $k_0 \leq \lceil (p+h)/(s-h) \rceil$ or $k_0 = \infty$.

Case D7: $\boldsymbol{v = 0^t 1^r w_0 1^s}$

with $r, s, t > 0$, $w_0 \in (0^+1^+)^*0^+$.
By the same arguments as in case D6, we are reduced to the case where: $p > t$ and $q \leq r$. Let $p = t + h$ where $h > 0$ and $\alpha = 0^h$.
For every $k \geq 1$ we have

$$\alpha^k 0^p 1^q \longrightarrow_S 0^{h \cdot k} 0^t 1^r w_0 1^s \xrightarrow{*}_S 0^t 1^r w_0 1^s (1^{r-q} w_0 1^s)^k = 0^t 1^r (w_0 1^s 1^{r-q})^k w_0 1^s \tag{48}$$
$$\xrightarrow{*}_S 0^t 1^r (\rho(w_0 1^s 1^{r-q}))^k w_0 1^s. \tag{49}$$

Let us suppose that $k_0 \geq 2$, which is equivalent to: $\rho(\alpha 0^p 1^q) < \infty$. Then, by formula (49), $\rho(w_0 1^s 1^{r-q})$ is finite too, but the rule of S cannot modify the first (resp. last) letter of a word. Hence $\rho(w_0 1^s 1^{r-q}) \in 0(0+1)^*1$. Hence the word $0^t 1^r (\rho(w_0 1^s 1^{r-q}))^k w_0 1^s$ is irreducible and by (49), $k_0 = \infty$.
Conclusion: $k_0 = 1$ or $k_0 = \infty$.

Case D8: $\boldsymbol{v = 0^t 1^s}$

with $s, t \geq 0$. It is clear that, as we suppose u is not a factor of v, ($p > t$ or $q > s$), which implies that $\mathrm{L}_+(S) = \emptyset$.

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