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L(A) = L(B)?

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Abstract

The equivalence problem for deterministic pushdown automata is shown to be decidable. We exhibit a *complete formal system* for deducing equivalent pairs of deterministic rational boolean series on the alphabet associated with a dpda $\mathcal{M}$.

Keywords: deterministic pushdown automata; rational series; finite dimensional vector spaces; matrix semi-groups; complete formal systems.

1 Introduction

The so-called "equivalence problem for deterministic pushdown automata" (dpda for short), is the following decision problem:

INSTANCE: two dpda A, B. QUESTION: $\mathrm{L}(A) = \mathrm{L}(B)$?

where $\mathrm{L}(A)$ (resp. $\mathrm{L}(B)$) is the language recognized by A (resp. B). (This problem is often denoted by $Eq(D, D)$, where D stands for the class of all dpda). The question of whether this problem is *decidable* or not is raised in [GG66] and has received much attention since this time. Beside the fact that this question was natural from the point of view of *formal language* theory, it appeared later as Turing-equivalent with other equivalence-problems for different types of recursive *program schemes* (see [Cou90] for a survey). Some other Turing-equivalent problems on *semi-Thue systems* were also found (see [Sén94] for a survey) and formulations in terms of bisimulation equivalence of infinite *graphs* (or *processes*) have been found too (see [Cau95] for a survey).

Among a large number of papers let us only quote [Val74],[VP75], [Bee76], [Rom85],[Oya87],[Sti96] which proved decidability of $Eq(D', D')$ for subclasses D' of the full class D of dpda. (We refer the reader to the surveys ([Cou90],[Cau95], [Lis96]) for other results on problems related to $Eq(D, D)$). The work [Mei89,Mei92] is an attempt to solve the general problem. On account of its incompleteness

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(see for example the comment in [Lis96, p.219]) it does not provide a full solution; nevertheless it introduced a fundamental new idea: the notion of *linear independence* for languages.

We prove here that the equivalence problem for dpda is *decidable* (theorem 9.2).

We obtain this result by providing a *complete* formal system $\mathcal{D}_0$ for equivalence identities between *deterministic rational boolean series* (we use here a type of formal system inspired by [Cou83a] and a notion of deterministic series inspired by [HHY79]). The proof of this completeness property leans on two types of arguments:

- in section 3 we develop around the fundamental idea of [Mei89,Mei92] an *algebraic* theory of "d-spaces" and in sections 5,7 we turn these structure results into a construction of *strategies* (for the formal system $\mathcal{D}_0$).
- in section 8 we analyze the *infinite trees* generated by some strategies associated with $\mathcal{D}_0$. This analysis has similarities with the analysis of derivations of *pairs* of configurations which are customary in the works using Valiant's methods of alternate or parallel stacking (see for example [Rom85,Oya87]).

This version is essentially an overview of the full proof given in [Sén97].

2 Preliminaries

2.1 Pushdown automata

A *pushdown automaton* on the alphabet X is a 6-tuple $\mathcal{M} = < X, Z, Q, \delta, q_0, z_0 >$ where Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol and $\delta : QZ \times (X \cup \{\epsilon\}) \rightarrow \mathcal{P}_f(QZ^*)$, is the transition mapping.

Let $q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, f \in X^*$ and $a \in X \cup \{\epsilon\}$; we note $(qz\omega, af) \longmapsto_{\mathcal{M}} (q'\omega'\omega, f)$ if $q'\omega' \in \delta(qz, a)$. $\overset{*}{\longmapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{M}}$. For every $q\omega, q'\omega' \in QZ^*$ and $f \in X^*$, we note $q\omega \xrightarrow{f}_{\mathcal{M}} q'\omega'$ iff $(q\omega, f) \overset{*}{\longmapsto}_{\mathcal{M}} (q'\omega', \epsilon)$. $\mathcal{M}$ is said *deterministic* iff, for every $z \in Z, q \in Q$:

$$\text{either } \mathrm{Card}(\delta(qz, \epsilon)) = 1 \text{ and for every } x \in X, \mathrm{Card}(\delta(qz, x)) = 0, \tag{1}$$

$$\text{or } \mathrm{Card}(\delta(qz, \epsilon)) = 0 \text{ and for every } x \in X, \mathrm{Card}(\delta(qz, x)) \leq 1. \tag{2}$$

$\mathcal{M}$ is said *real-time* iff, for every $qz \in QZ$, $\mathrm{Card}(\delta(qz, \epsilon)) = 0$. A dpda $\mathcal{M}$ is said *normalized* iff, for every $qz \in QZ, x \in X$:

$$q'\omega' \in \delta(qz, x) \Rightarrow \mid \omega' \mid \leq 2, \text{ and } q'\omega' \in \delta(qz, \epsilon) \Rightarrow \mid \omega' \mid = 0 \tag{3}$$

Given some finite set $F \subseteq QZ^*$ of configurations, the *language recognized by $\mathcal{M}$ with final configurations F* is defined by $\mathrm{L}(\mathcal{M}, F) = \{w \in X^* \mid \exists c \in F, q_0 z_0 \xrightarrow{w}_{\mathcal{M}} c\}$.

2.2 Deterministic context-free grammars

Let $\mathcal{M}$ be some deterministic pushdown automaton (for sake of simplicity [2] , we suppose here that $\mathcal{M}$ is normalized). The *variable* alphabet $V_{\mathcal{M}}$ associated to $\mathcal{M}$ is defined as: $V_{\mathcal{M}} = \{[p,z,q] \mid p,q \in Q, z \in Z\}$. The *context-free* grammar $G_{\mathcal{M}}$ associated to $\mathcal{M}$ is then $G_{\mathcal{M}} =< X, V, P_{\mathcal{M}} >$ where $V = V_{\mathcal{M}}$ and $P_{\mathcal{M}}$ is the set of all the pairs of one of the following forms:

$$([p,z,q], x[p',z_1,p''][p'',z_2,q]) \text{ or } ([p,z,q], x'[p',z',q]) \text{ or } ([p,z,q], a) \tag{4}$$

where $p,q \in Q, z \in Z, x, x' \in X, a \in X \cup \{\epsilon\}, p'z_1z_2 \in \delta(pz,x), p'z' \in \delta(pz,x'), q \in \delta(pz,a)$. $G_{\mathcal{M}}$ is a *strict-deterministic* grammar. (A general theory of this class of grammars is exposed in [Har78] and used in [HHY79]). We call *mode* every element of $QZ \cup \{\epsilon\}$. For every $q \in Q, z \in Z$, qz is said *ϵ-bound* (respectively ϵ-free) iff condition (1) (resp. condition (2)) in the above definition of deterministic automata is realized. The mode ϵ is said ϵ-free. We define a mapping $\mu : V^* \to QZ \cup \{\epsilon\}$ by

$$\mu(\epsilon) = \epsilon \text{ and } \mu([p,z,q] \cdot \beta) = pz, \tag{5}$$

for every $p,q \in Q, z \in Z, \beta \in V^*$. For every $w \in V^*$ we call $\mu(w)$ the *mode* of the word w. For technical reasons (which will be made clear in section 7), we suppose that Z contains a special symbol e such that, for every $q \in Q, \delta(qe,\epsilon) = \{q\}$ and $\mathrm{im}(\delta) \subseteq \mathcal{P}_f(Q(Z - \{e\})^*)$.

2.3 Free monoids acting on semi-rings

2.3.1 Semi-ring $\mathsf{B} < W >$

Let $(\mathsf{B}, +, \cdot, 0, 1)$ where $\mathsf{B} = \{0,1\}$ denote the semi-ring of "booleans". Let W be some alphabet. By $(\mathsf{B} < W >, +, \cdot, \emptyset, \epsilon)$ we denote the semi-ring of *boolean series* over W: every boolean series $S \in \mathsf{B} < W >$ can be written in a unique way as: $S = \Sigma_{w \in W^*} S_w \cdot w$, where, for every $w \in W^*$, $S_w \in \mathsf{B}$. The *support* of S is the language

$$\mathrm{supp}(S) = \{w \in W^* \mid S_w \neq 0\}.$$

In the particular case where the semi-ring of coefficients is B (which is the only case considered in this article [3]) we sometimes identify the series S with its support. We recall that for every $S \in \mathsf{B} < W >$, S^* is the series defined by: $S^* = \sum_{0 \leq n} S^n$. Given two alphabets W, W', a map $\psi : \mathsf{B} < W > \to \mathsf{B} < W' >$ is said *σ-additive* iff it fulfills: for every denumerable family $(S_i)_{i \in \mathsf{N}}$ of elements of $\mathsf{B} < W >$, $\psi(\sum_{i \in \mathsf{N}} S_i) = \sum_{i \in \mathsf{N}} \psi(S_i)$. A map $\psi : \mathsf{B} < W > \to \mathsf{B} < W' >$ which is both a semi-ring homomorphism and a σ-additive map is usually called a *substitution.*

2.3.2 Actions of monoids

Given a semi-ring $(\mathsf{S}, +, \cdot, 0, 1)$ and a monoid $(\mathsf{M}, \cdot, 1_M)$, a map $\circ : \mathsf{S} \times \mathsf{M} \to \mathsf{S}$ is called a *right-action* of the monoid M over the semi-ring S iff, for every

[2] but without loss of generality for the equivalence problem

[3] generalizations to other types of coefficients will be done elsewhere

$S, T \in \mathsf{S}, m, m' \in \mathsf{M}$:

$$0 \circ m = 0, \quad S \circ 1_M = S, \quad (S+T) \circ m = (S \circ m) + (T \circ m) \quad \text{and} \tag{6}$$

$$S \circ (m \cdot m') = (S \circ m) \circ m' \tag{7}$$

In the particular case where $\mathsf{S} = \mathsf{B} < W >$, $\circ$ is said to be a σ-right-action if it fulfills the additional property that, for every denumerable family $(S_i)_{i \in \mathsf{N}}$ of elements of S and $m \in \mathsf{M}$:

$$(\sum_{i \in \mathsf{N}} S_i) \circ m = \sum_{i \in \mathsf{N}} (S_i \circ m). \tag{8}$$

2.3.3 The action of W^* *on* $\mathsf{B} < W >$

We recall the following classical σ-right-action $\bullet$ of the monoid W^* over the semi-ring $\mathsf{B} < W >$: for all $S, S' \in \mathsf{B} < W >, u \in W^*$

$$S \bullet u = S' \Leftrightarrow \forall w \in W^*, \;\; S'_w = S_{u \cdot w}. \tag{9}$$

(i.e. $S \bullet u$ is the *left-quotient* of S by u , or the *residual* of S by u). For every $S \in \mathsf{B} < W >$ we denote by $\mathsf{Q}(S)$ the set of residuals of S: $\mathsf{Q}(S) = \{S \bullet u \mid u \in W^*\}$. We recall that S is said *rational* iff the set $\mathsf{Q}(S)$ is *finite*. We define the *norm* of a series $S \in \mathsf{B} < W >$, denoted $\|S\|$ by: $\|S\| = \mathrm{Card}(\mathsf{Q}(S)) \in \mathsf{N} \cup \{\infty\}$.

2.3.4 The action of X^* *on* $\mathsf{B} < V >$

Let us fix now a deterministic (normalized) pda $\mathcal{M}$ and consider the associated grammar G. We define a σ-right-action $\otimes$ of the monoid $(X \cup \{e\})^*$ over the semi-ring $\mathsf{B} < V >$ by: for every $p, q \in Q, A \in Z, H \in \mathsf{B} < V >, \beta \in V^*, x \in X$

$$[p, A, q] \cdot \beta \otimes x = H \cdot \beta \text{ iff } H = \sum_{([p,A,q], x \cdot h) \in P_{\mathcal{M}}} h, \tag{10}$$

$$[p, A, q] \cdot \beta \otimes e = \beta \text{ iff } ([p, A, q], \epsilon) \in P_{\mathcal{M}}, \tag{11}$$

$$[p, A, q] \cdot \beta \otimes e = \emptyset \text{ iff } (\{[p, A, q]\} \times V^*) \cap P_{\mathcal{M}} = \emptyset, \tag{12}$$

$$\epsilon \otimes x = \emptyset, \quad \epsilon \otimes e = \emptyset. \tag{13}$$

A series $S \in \mathsf{B} < V >$ is said ϵ-*free* iff $\forall w \in V^*, S_w = 1 \Rightarrow \mu(w)$ is $\epsilon -$ free. We denote by $\mathsf{B}_\epsilon < V >$ the subset of ϵ-free series. We define the map $\rho_\epsilon : \mathsf{B} < V > \rightarrow \mathsf{B} < V >$ as the unique σ-additive map such that,

$$\rho_\epsilon(\emptyset) = \emptyset, \quad \rho_\epsilon(\epsilon) = \epsilon$$

and for every $p \in Q, z \in Z, q \in Q, \beta \in V^*$,

$$\rho_\epsilon([p, z, q] \cdot \beta) = \rho_\epsilon(([p, z, q] \otimes e) \cdot \beta) \text{ if } pz \text{ is } \epsilon - \text{bound},$$

$$\rho_\epsilon([p, z, q] \cdot \beta) = [p, z, q] \cdot \beta \text{ if } pz \text{ is } \epsilon - \text{free}.$$

The above definition is sound because, by hypothesis (3), every $[p, z, q] \otimes e$ is either the unit series ϵ or the empty series $\emptyset$. One can notice that for every $w \in V^*$, $\rho_\epsilon(w) \in V^* \cup \{\emptyset\}$. We call ρ_ϵ the ϵ-reduction map. We then define $\odot$ as the unique right-action of the monoid X^* over the semi-ring $\mathsf{B} < V >$

such that: for every $S \in \mathsf{B} < V >, x \in X$, $S \odot x = \rho_\epsilon(\rho_\epsilon(S) \otimes x)$. One can notice that if $u \neq \epsilon$, then $S \odot u$ is ϵ-free. Let us consider the unique substitution $\varphi : \mathsf{B} < V > \to \mathsf{B} < X >$ fulfilling: for every $p, q \in Q, z \in Z$, $\varphi([p, z, q]) = \{u \in X^* \mid [p, z, q] \odot u = \epsilon\}$, (in other words, φ maps every subset $L \subseteq V^*$ on the language generated by the grammar G from the set of axioms L).

Lemma 2.1 *φ is a morphism of right-actions i.e. for every $S \in \mathsf{B} < V >$, $u \in X^*$, $\varphi(S \odot u) = \varphi(S) \bullet u$.*

We denote by $\equiv$ the kernel of φ i.e.: for every $S, T \in \mathsf{B} < V >$, $S \equiv T \Leftrightarrow \varphi(S) = \varphi(T)$.

3 Series and matrices

3.1 Deterministic series and matrices

We introduce here a notion of *deterministic* series which, in the case of the alphabet V associated to a dpda $\mathcal{M}$, generalizes the classical notion of *configuration* of $\mathcal{M}$. The main advantage of this notion is that, unlike for configurations, we shall be able to define *nice algebraic operations* on these series (this is done in section 3.2). Let us consider a pair $(W, \sim)$ where W is an alphabet and $\sim$ is an equivalence relation over W. We call $(W, \sim)$ a *structured* alphabet. The two examples we have in mind are:

- the case where $W = V$, the variable alphabet associated to $\mathcal{M}$ and $[p, A, q] \sim [p', A', q']$ iff $p = p'$ and $A = A'$ (see [Har78])
- the case where $W = X$, the terminal alphabet of $\mathcal{M}$ and $x \sim y$ holds for every $x, y \in X$ (see [Har78]).

3.1.1 Definitions

Definition 3.1 *Let $S \in \mathsf{B} < W >$. S is said* left-deterministic *iff either (1) $S = \emptyset$ or (2) $S = \epsilon$ or (3) $\forall w, w' \in W^*, S_w = S_{w'} = 1 \Rightarrow \exists A, A' \in W, w_1, w_1' \in W^*, A \sim A', w = A \cdot w_1$ and $w' = A' \cdot w_1'$.*

A left-deterministic series S is said to have the type $\emptyset$ (resp. ϵ, $[A]_\sim$) if case (1) (resp. (2), (3)) occurs.

Definition 3.2 *Let $S \in \mathsf{B} < W >$. S is said* deterministic *iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.*

This notion is the straight-forward extension to the infinite case of the notion of (finite) *set of associates* defined in [HHY79].
We denote by $\mathsf{DB} < W >$ the subset of deterministic boolean series over W.
Let us denote by $\mathsf{B}_{n,m} < W >$ the set of (n, m)-matrices with entries in the semi-ring $\mathsf{B} < W >$.

Definition 3.3 *Let $m \in \mathsf{N}, S \in \mathsf{B}_{1,m} < W >$: $S = (S_1, \cdots, S_m)$. S is said* left-deterministic *iff either (1) $\forall i \in [1, m], S_i = \emptyset$ or (2) $\exists i_0 \in [1, m], S_{i_0} = \epsilon$ and $\forall i \neq i_0, S_i = \emptyset$ or (3) $\exists i_0 \in [1, m], S_{i_0} \neq \emptyset$ and $\forall w, w' \in W^*, \forall i, j \in [1, m], [(S_i)_w = (S_j)_{w'} = 1 \Rightarrow \exists A, A' \in W, w_1, w_1' \in V^*, A \sim$*

$A', w = A \cdot w_1$ *and* $w' = A' \cdot w'_1$].

A left-deterministic row-vector S is said to have the type $\emptyset$ (resp. (ϵ, i_0), $[A]_\sim$) if case (1) (resp. (2), (3)) occurs.

The right-action $\bullet$ on $\mathsf{B} < W >$ is extended componentwise to $\mathsf{B}_{n,m} < W >$: for every $S = (s_{i,j})$, $u \in W^*$, the matrix $T = S \bullet u$ is defined by $t_{i,j} = s_{i,j} \bullet u$. The ordering $\leq$ on B is also extended componentwise to $\mathsf{B}_{n,m} < W >$.

Definition 3.4 *Let* $S \in \mathsf{B}_{1,m} < W >$. S *is said* deterministic *iff, for every* $u \in W^*$, $S \bullet u$ *is left-deterministic.*

We denote by $\mathsf{DB}_{1,m} < W >$ the subset of deterministic row-vectors of dimension m over $\mathsf{B} < W >$.

Definition 3.5 *Let* $S \in \mathsf{B}_{n,m} < W >$. S *is said* deterministic *iff, for every* $i \in [1, n]$, $S_{i,*}$ *is a deterministic row-vector.*

3.1.2 Residuals

Lemma 3.1 *Let* $S \in \mathsf{DB}_{1,m} < W >, T \in \mathsf{B}_{m,1} < W >, u \in W^*$ *and* $U = S \cdot T$. *Exactly one of the following cases is true:*

(1) $\exists j, S_j \bullet u \notin \{\emptyset, \epsilon\}$; *in this case* $U \bullet u = (S \bullet u) \cdot T$.

(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \bullet u' = \epsilon$; *in this case* $U \bullet u = T_{j_0} \bullet u''$.

(3) $\forall j, S_j \bullet u = \emptyset$ *and* $\forall u' \preceq u, S_j \bullet u' \neq \epsilon$; *in this case* $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Lemma 3.2 *For every* $S \in \mathsf{DB}_{n,m} < W >, T \in \mathsf{DB}_{m,s} < W >$, $S \cdot T \in \mathsf{DB}_{n,s} < W >$.

3.1.3 W=V

Let $(W, \sim)$ be the structured alphabet $(V, \sim)$ associated with $\mathcal{M}$ and let us consider a bijective numbering of the elements of Q: $(q_1, q_2, \ldots, q_{n_Q})$. Some particular "vectorial" notions turn out to be useful:

- we define a *Q-series* to be a family $(S_q)_{q \in Q}$ such that the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ is deterministic
- we define a *Q-form* to be a family $\Phi = (\Phi_q)_{q \in Q}$ of deterministic series.

Given a Q-series S and a Q-form Φ, their Q-product $S * \Phi$ is the deterministic series defined by $S * \Phi = \sum_{q \in Q} S_q \cdot \Phi_q$. If the Q-series $(S_q)_{q \in Q}$ is identified with the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ and the Q-form $(\Phi_q)_{q \in Q}$ with the column-vector $(\Phi_{q_j})_{j \in [1, n_Q]}$, then the Q-product appears to be just the ordinary product of matrices.

Let us define here handful notations for some particular row-vectors or Q-series. Let us use the *Kronecker symbol* $\delta_{i,j}$ meaning ϵ if $i = j$ and $\emptyset$ if $i \neq j$. For every $1 \leq i \leq n$, we define the row-vector ϵ_i^n as: $\epsilon_i^n = (\epsilon_{i,j}^n)_{1 \leq j \leq n}$ where $\forall j, \epsilon_{i,j}^n = \delta_{i,j}$. We call *unit row-vector* any vector of the form ϵ_i^n.

For every $\omega \in Z^*, p, q \in Q$, $[p\omega q]$ is the deterministic series defined inductively by:

$$[p\epsilon q] = \emptyset \text{ if } p \neq q, [p\epsilon q] = \epsilon \text{ if } p = q,$$
$$[p\omega q] = \sum_{r \in Q} [pAr] \cdot [r\omega' q] \text{ if } \omega = A \cdot \omega' \text{ for some } A \in Z, \omega' \in Z^*.$$

By $[p\omega]$ we denote the Q-series: $[p\omega] = ([p\omega q])_{q\in Q}$. (In particular $[q_i] = \epsilon_i^{n_Q}$).

By $[\omega]$ we denote the Q-matrix: $[\omega] = ([p\omega q])_{p\in Q, q\in Q}$. Here also we identify $[\omega]$ with the matrix $([p_i \omega q_j])_{1\leq i\leq n_Q, 1\leq j\leq n_Q} \in \mathsf{DB}_{n_Q,n_Q} < V >$. The next lemma relates the right-action $\odot$ with the right-action $\bullet$.

Lemma 3.3 *Let $S \in \mathsf{DB} < V >, u \in X^*$. One of the three following cases must occur:* (1) $S \odot u = \emptyset$, *or* (2) $S \odot u = \epsilon$, *or* (3) $\exists u_1, u_2 \in X^*, v_1 \in V^*, q \in Q, A \in Z, \Phi$ *$Q-$form such that* $u = u_1 \cdot u_2, \rho_\epsilon(S) \odot u_1 = S \bullet v_1 = [qA] * \Phi$ *and* $S \odot u = ([qA] \odot u_2) * \Phi$.

Corollary 3.4 *Let $S \in \mathsf{DB} < V >, u \in X^*$. Then $S \odot u \in \mathsf{DB} < V >$.*

The particular letters $[p, e, q]$ for $p, q \in Q$ play a special role in sections 7 and 8: we use them as *marks* in the series (somehow like the ceilings of [Val74]). We define below a map ρ_e which removes the marks in the series. Let us define $\rho_e : \mathsf{DB} < V > \longrightarrow \mathsf{B} < V >$ as the unique substitution such that:

$$\rho_e([p, e, q]) = \epsilon \quad \text{if } p = q, \quad \rho_e([p, e, q]) = \emptyset \quad \text{if } p \neq q.$$

Lemma 3.5 *For every $S \in \mathsf{DB} < V >$, $\rho_e(S) \in \mathsf{DB} < V >$ and $\|\rho_e(S)\| \leq \|S\|$.*

3.1.4 Rational series, norm

Let us generalize the definition of *rationality* of series in $\mathsf{B} < W >$ to matrices. Given $M \in \mathsf{B}_{n,m} < W >$ we denote by $\mathsf{Q}(M)$ the set of *residuals* of M: $\mathsf{Q}(M) = \{M \bullet u \mid u \in W^*\}$. Similarly, we denote by $\mathsf{Q}_r(M)$ the set of *row-residuals* of M: $\mathsf{Q}_r(M) = \bigcup_{1\leq i\leq n} \mathsf{Q}(M_{i,*})$. M is said *rational* iff the set $\mathsf{Q}(M)$ is finite. One can check that it is equivalent to the property that every coefficient $M_{i,j}$ is rational, or to the property that $\mathsf{Q}_r(M)$ is finite. We denote by $\mathsf{DRB}_{n,m} < W >$ the set of deterministic, rational matrices over $\mathsf{B} < W >$. For every $M \in \mathsf{DRB}_{n,m} < W >$, we define the norm of M as:

$$\|M\| = \mathrm{Card}(\mathsf{Q}_r(M)).$$

Lemma 3.6 *Let $A \in \mathsf{DB}_{n,m} < W >, B \in \mathsf{DB}_{m,s} < W >$. Then $\|A \cdot B\| \leq \|A\| + \|B\|$.*

3.2 Deterministic spaces

We adapt here the key-idea of [Mei89,Mei92] to series.

3.2.1 Definitions

Let $(W, \sim)$ be some structured alphabet and let us consider the set $E = \mathsf{DRB} < W >$. A series $U = \sum_{i=1}^n \gamma_i \cdot U_i$ where $\vec{\gamma} \in \mathsf{DRB}_{1,n} < W >$, $U_i \in \mathsf{DRB} < W >$ is called a *linear combination* of the U_i's. We call *deterministic space* of rational

series (d-space for short) any subset V of E which is closed under finite linear combinations. Given any set $\mathcal{G} = \{U_i | i \in I\}$, one can check that the set V of all (finite) linear combinations of elements of $\mathcal{G}$ is a d-space (by lemma 3.2) and that it is the smallest d-space containing $\mathcal{G}$. Therefore we call V the d-space *generated* by $\mathcal{G}$ and we call $\mathcal{G}$ a *generating set* of V (we note $\mathsf{V} = \mathsf{V}(\{U_i | i \in I\})$). (Similar definitions can be given for *families* of series).
We let now $W = V$. Following an analogy with classical linear algebra, we develop now a notion corresponding to a kind of *linear independence* of the images by φ of the given series . Let us extend the equivalence relation $\equiv$ to d-spaces by: for every d-spaces $\mathsf{V}_1, \mathsf{V}_2$, $\mathsf{V}_1 \equiv \mathsf{V}_2 \Leftrightarrow \forall i, j \in \{1, 2\}, \forall S \in \mathsf{V}_i, \exists S' \in \mathsf{V}_j, S \equiv S'$.

Lemma 3.7 *Let* $S_1, \ldots, S_j, \ldots, S_m \in \mathsf{DRB} < V >$. *The following are equivalent*

(i) $\exists \vec{\alpha}, \vec{\beta} \in \mathsf{DRB}_{1,m} < V >, \vec{\alpha} \not\equiv \vec{\beta}$, *such that* $\sum_{1 \leq j \leq m} \alpha_j \cdot S_j \equiv \sum_{1 \leq j \leq m} \beta_j \cdot S_j$,

(ii) $\exists j_0 \in [1, m], \exists \vec{\gamma} \in \mathsf{DRB}_{1,m} < V >, \vec{\gamma} \not\equiv \epsilon^m_{j_0}$, *such that* $S_{j_0} \equiv \sum_{1 \leq j \leq m} \gamma_j \cdot S_j$,

(iii) $\exists j_0 \in [1, m], \exists \vec{\gamma}' \in \mathsf{DRB}_{1,m} < V >, \gamma'_{j_0} \equiv \emptyset$, *such that* $S_{j_0} \equiv \sum_{1 \leq j \leq m} \gamma'_j \cdot S_j$,

(iv) $\exists j_0 \in [1, m]$, *such that* $\mathsf{V}((S_j)_{1 \leq j \leq m}) \equiv \mathsf{V}((S_j)_{1 \leq j \leq m, j \neq j_0})$.

The equivalence between (i),(ii) and (iii) was first proved in [Mei89,Mei92], in the case where the S_j's are configurations $q_j\omega$, with the same ω.

3.3 Height, defect and linearity

We define here notions of height and defect (for deterministic rational series) and a subsequent notion of (d, d')-linearity which will play a crucial role in section 8. We then relate these "size notions" with the notion of norm.

3.3.1 Definitions

Let $S \in \mathsf{DRB} < V >$. We call *linear decomposition of* S any pair $([p\omega], \Phi)$ where $p \in Q, \omega \in Z^*, \Phi \in \mathsf{DB}_{Q,1} < V >$ such that

$$S = [p\omega] * \Phi.$$

We denote by $\mathcal{D}(S)$ the set of all linear decompositions of S. We define the *right-defect* of S (rd(S) in short) by

$$\mathrm{rd}(S) = \min\{\|\Phi\| \mid \exists p \in Q, \omega \in Z^*, ([p\omega], \Phi) \in \mathcal{D}(S)\}.$$

One can easily see that the right-defect of S is finite and not greater than $\|S\|$. We call *minimal* decomposition of S the decomposition $([p_0\omega_0], \Phi_0)$ which makes the triple $(p, |\omega|, \|\Phi\|)$ minimal for the right-to-left lexicographic ordering in $Q \times \mathsf{N} \times \mathsf{N}$. We then define the *linear-height* of S (noted $|S|$) as the integer $|S| = |\omega_0|$ (and it is clear that $\mathrm{rd}(S) = \|\Phi_0\|$).
The height and right-defect of a Q-form can be defined similarly. We say that S is *marked* iff S contains some occurrence of some letter $[p, e, q] \in V$ (we assumed the existence of such a "dummy " letter e in Z in section 2.2).

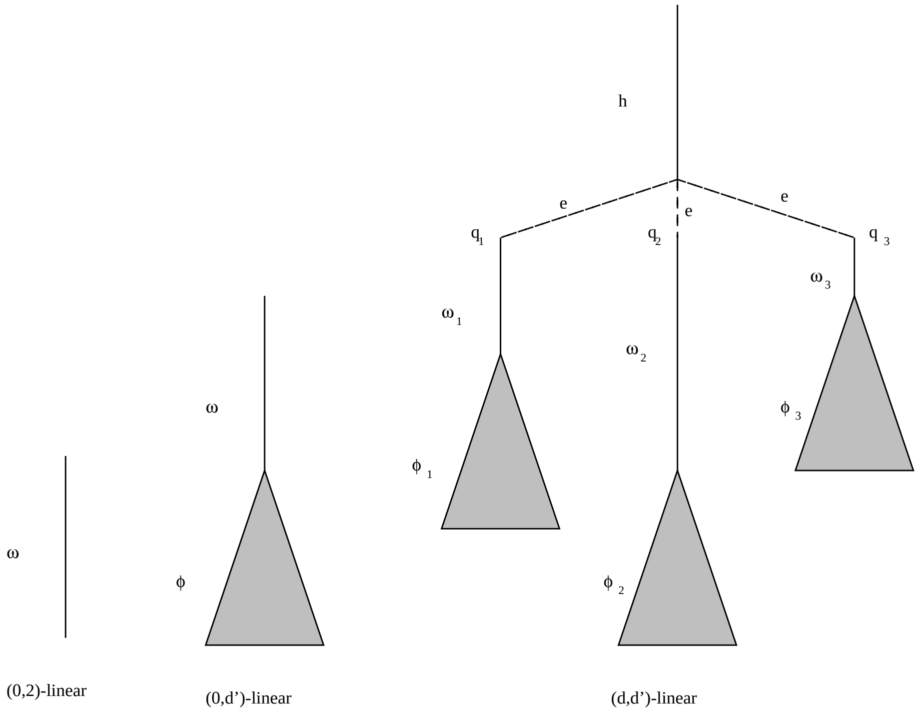

Fig. 1. (d, d')-linearity.

Definition 3.6 *Let* $S \in \mathsf{DRB} < V >$ *and* $d' \in \mathsf{N}$.

(i) *S is said $(0, d')$-linear iff* $\mathrm{rd}(S) \leq d'$ *and S is not marked*

(ii) *if $d \in \mathsf{N} - \{0\}$, S is said (d, d')-linear iff S has a decomposition $S = \sum_{q \in Q}[phq] \cdot [qeq] \cdot \Phi_q$ where every Φ_q is a $(0, d')$-linear series and $|h| \leq d$.*

In case (ii) we call the series Φ_q the d'-linear components of S. It should be clear that the d'-linear components are independent of the value of d (for d large enough) and that the set of d'-linear components is uniquely defined. We denote by $\mathsf{DBlin}^{d'} < V >$ (resp. $\mathsf{DRBlin}^{d'} < V >$) the set of series in $\mathsf{DB} < V >$ (resp. $\mathsf{DRB} < V >$) which are $(0, d')$-linear.

Examples:

No series can be $(0 - 0)$-linear. S is $(0, 1)$-linear iff $S = \emptyset$. S is $(0, 2)$-linear iff it has the form $S = \sum_{q \in Q'}[p\omega q]$ for some $p \in Q, Q' \subseteq Q, \omega \in (Z - \{e\})^*$.
Hence one can view the $(0, d)$-linear series as series which have a structure " not too far" from the *linear* structure of the configurations of the initial dpda $\mathcal{M}$.

(We illustrate on figure 1 the above definitions).

3.4 Height and norm

Lemma 3.8 *Let* $S \in \mathsf{DRB} < V >, x \in X, d, d' \in \mathsf{N}$.

(i) $\mathrm{rd}(S \odot x) \leq \mathrm{rd}(S)$

(ii) *S is (d, d')-linear $\Rightarrow$ $S \odot x$ is $(d + 1, d')$-linear*

(iii) $\|S \odot x\| \leq \|S\| + |Q|$.

Here $|Q|$ denotes the cardinality of the set Q.

Lemma 3.9 *Let* $\omega \in Z^+, A', A \in Z, p \in Q$, $\Phi \in \mathsf{DB}_{Q,1} < V >$. *If* $\|[A] * \Phi\| > \|\Phi\|$, *then* $\|[\omega A] * \Phi\| = |Q| \cdot |\omega| + \|[A] * \Phi\|$ *and* $\|[pA'\omega A] * \Phi\| = 1 + |Q| \cdot |\omega| + \|[A] * \Phi\|$.

The following lemma serves as a crucial technical argument in section 8.

Lemma 3.10 *Let* $U = [ph] * \Phi, H = U \odot u$ *where* $p \in Q, h \in Z^*, |h| \geq 1, \Phi$ *is a* $Q - form$, $|\Phi| \geq 1, u \in X^*, |u| \leq k$. *Let us suppose that* $\|H\| \geq 1 + k \cdot |Q| + \|\Phi\|$. *Then*
$H = ([ph] \odot u) * \Phi$ *where* $[ph] \odot u = [q\omega]$ *for some* $q \in Q, |\omega| \geq k$.

Intuitive meaning: If U is a deterministic rational series admitting a decomposition over the Q-form Φ and, $U \odot u$ has a sufficiently large norm (compared to $\|\Phi\|$ and $|u|$), then the action of u on U cannot have "touched" the form Φ.

4 Deduction systems

4.1 General deduction systems

We follow here the general philosophy of [HHY79,Cou83a]. Let us call *deduction system* any triple $\mathcal{D} =< \mathcal{A}, H, \models\!\!\!- >$ where $\mathcal{A}$ is a denumerable set called the *set of assertions*, H, the *cost function* is a mapping $\mathcal{A} \to \mathsf{N} \cup \{\infty\}$ and $|\!\!-\!\!-$, the *deduction relation* is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$; $\mathcal{A}$ is given with a fixed bijection with N (an "encoding" or "Gödel numbering") so that the notions of recursive subset, recursively enumerable subset, recursive function, ... over $\mathcal{A}, \mathcal{P}_f(A)$,... are defined, up to this fixed bijection ; we assume that $\mathcal{D}$ satisfies the following axioms:
(A 1) $|\!\!-\!\!-$ is recursively enumerable
(A 2) $\forall (P, A) \in |\!\!-\!\!-$, (min $\{H(p), p \in P\} < H(A)$) or ($H(A) = \infty$). (We let $\min(\emptyset) = \infty$).

In the sequel we use the notation $P |\!\!-\!\!- A$ for $(P, A) \in |\!\!-\!\!-$. We call *proof* in the system $\mathcal{D}$, *relative to the set of hypotheses* $\mathcal{H} \subseteq \mathcal{A}$, any subset $P \subseteq \mathcal{A}$ fulfilling : $\forall p \in P, (\exists Q \subseteq P, Q |\!\!-\!\!- p)$ or $(p \in \mathcal{H})$. We call P a *proof* iff $\forall p \in P, (\exists Q \subseteq P, Q |\!\!-\!\!- p)$ (i.e. iff P is a proof relative to $\emptyset$).
Let us define the total map $\chi : \mathcal{A} \to \{0, 1\}$ and the partial map $\overline{\chi} : \mathcal{A} \to \{0, 1\}$ by :
$\chi(A) = 1$ if $H(A) = \infty, \chi(A) = 0$ if $H(A) < \infty$, $\overline{\chi}(A) = 1$ if $H(A) = \infty, \overline{\chi}$ is undefined if $H(A) < \infty$. (χ is the 'truth-value function", $\overline{\chi}$ is the "1-value function").

Lemma 4.1 *Let* P *be a proof relative to* $\mathcal{H} \subseteq H^{-1}(\infty)$ *and* $A \in P$. *Then* $\chi(A) = 1$.

In other words : if an assertion is provable from true hypotheses, then it is true. The deduction system $\mathcal{D}$ will be said *complete* iff, conversely, $\forall A \in \mathcal{A}, \chi(A) = 1 \Longrightarrow$ there exists some *finite* proof P such that $A \in P$. (In other words, $\mathcal{D}$ is complete iff every true assertion is "finitely" provable).

Lemma 4.2 : *If $\mathcal{D}$ is complete, $\overline{\chi}$ is a recursive partial map.*

In order to define deduction relations from more elementary ones, we set the following definitions. Let $\vdash\!\!- \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$. For every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set: $P \overset{[0]}{\vdash\!\!-} Q$ iff $P \supseteq Q$; $P \overset{[1]}{\vdash\!\!-} Q$ iff $\forall q \in Q, \exists R \subseteq P, R \vdash\!\!- q$; $P \overset{<0>}{\vdash\!\!-} Q$ iff $P \overset{[0]}{\vdash\!\!-} Q$; $P \overset{<1>}{\vdash\!\!-} Q$ iff $\forall q \in Q, (\exists R \subseteq P, R \vdash\!\!- q)$ or $(q \in P)$; $P \overset{<n+1>}{\vdash\!\!-} Q$ iff $\exists R \in \mathcal{P}_f(A), P \overset{<1>}{\vdash\!\!-} R$ and $R \overset{<n>}{\vdash\!\!-} Q$ (for every $n \geq 0$).; $\overset{<*>}{\vdash\!\!-} = \bigcup_{n \geq 0} \overset{<n>}{\vdash\!\!-}$.
Given $\vdash\!\!-_1, \vdash\!\!-_2 \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(\mathcal{A})$, for every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set : $P(\vdash\!\!-_1 \circ \vdash\!\!-_2)Q$ iff $\exists R \subseteq \mathcal{A}, (P \vdash\!\!-_1 R) \wedge (R \vdash\!\!-_2 Q)$.

4.2 System $\mathcal{D}_0$

Let us define here a particular deduction system $\mathcal{D}_0$ "Taylored for the equivalence problem for dpda's".

Given a fixed dpda $\mathcal{M}$ over the terminal alphabet X, we consider the variable alphabet V associated to $\mathcal{M}$ (see section 3.1) and the set $\mathsf{DRB} < V >$ (the set of Deterministic Rational Boolean series over V^*). The set of assertions is defined by : $\mathcal{A} = \mathsf{N} \times \mathsf{DRB} < V > \times \mathsf{DRB} < V >$ i.e. an assertion is here a *weighted equation* over $\mathsf{DRB} < V >$.
The "cost-function" $H : \mathcal{A} \rightarrow \mathsf{N} \cup \{\infty\}$ is defined by : $H(n, S, S') = n + 2 \cdot \mathrm{Div}(S, S')$, where $\mathrm{Div}(S, S')$, the *divergence* between S and S', is defined by : $\mathrm{Div}(S, S') = \min\{|\, u \,|\mid u \in \triangle(\varphi(S), \varphi(S'))\}$. (We recall $\min(\emptyset) = \infty$).
Let us notice that here : $\chi(n, S, S') = 1 \Longleftrightarrow S \equiv S'$.

We define a binary relation $|\!\!\vdash\!\!- \subset \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

$(R0)$	$\{(p, S, T)\}$	$\vert\!\vdash\!\!-\ (p + 1, S, T)$	
$(R1)$	$\{(p, S, T)\}$	$\vert\!\vdash\!\!-\ (p, T, S)$	
$(R2)$	$\{(p, S, S'), (p, S', S'')\}$	$\vert\!\vdash\!\!-\ (p, S, S'')$	
$(R3)$	$\emptyset$	$\vert\!\vdash\!\!-\ (0, S, S)$	
$(R'3)$	$\emptyset$	$\vert\!\vdash\!\!-\ (0, S, T)$	(for $T \in \{\emptyset, \epsilon\}, S \equiv T$)
$(R4)$	$\{(p + 1, S \odot x, T \odot x) \mid x \in X\}$	$\vert\!\vdash\!\!-\ (p, S, T)$	(for $S \not\equiv \epsilon \wedge T \not\equiv \epsilon$)
$(R5)$	$\{(p, S, S')\}$	$\vert\!\vdash\!\!-\ (p + 2, S \odot x, S' \odot x)$	(for $x \in X$)
$(R6)$	$\{(p, S \cdot T' + T, T')\}$	$\vert\!\vdash\!\!-\ (p, S^* \cdot T, T')$	(for $S \not\equiv \varepsilon$)
$(R7)$	$\{(p, S, S')\}$	$\vert\!\vdash\!\!-\ (p, S + T, S' + T)$	

$$(R8)\ \ \{(p,S,S')\}\ \ |\!\!\vdash\!\!-\ \ (p,S\cdot T,S'\cdot T)$$

$$(R9)\ \ \{(p,T,T')\}\ \ |\!\!\vdash\!\!-\ \ (p,U\cdot T,U\cdot T')$$

where $p \in \mathsf{N}, S, S', T, T' \in \mathsf{DRB} < V >, U \in \mathsf{RB} < V >$. (By set of "all" these pairs we mean, all the pairs which fulfill both properties "to belong to $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$" and "to have one of these 11 possible forms" ; but of course, for example, not all the triples $(p, S+T, S'+T)$ belong to $\mathcal{A}$ because $\mathsf{DRB} < V >$ is not closed under sum).

Lemma 4.3 *: Let $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$ such that $P\ |\!\!\vdash\!\!-\ A$. Then $min\{H(p) \mid p \in P\} \leq H(A)$.*

Let us define $\vdash\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$, $P \vdash\!\!- A \Longleftrightarrow P\ \overset{<*>}{|\!\!\vdash\!\!-} \circ \overset{[1]}{|\!\!\vdash\!\!-}_{0,3,4} \circ\ \overset{<*>}{|\!\!\vdash\!\!-}\ \{A\}$, where $|\!\!\vdash\!\!-_{0,3,4}$ is the relation defined by R_0, R_3, R_3', R_4 only.
We let $\mathcal{D}_0 =< \mathcal{A}, H, \vdash\!\!- >$.

Lemma 4.4 *: $\mathcal{D}_0$ is a deduction system.*

The key-statement of this work is that $\mathcal{D}_0$ is complete (theorem 9.1). We prove this completeness result by exhibiting a "strategy" $\mathcal{S}$ which, for every true assertion (n, S, S'), constructs a finite $\mathcal{D}_0$-proof of this assertion. Notice that, by lemma 4.2 we don't need to prove that $\mathcal{S}$ is computable in any sense to establish that $\overline{\chi}$ is partial-recursive.

4.3 Strategies

Let $\mathcal{D} =< \mathcal{A}, H, \vdash\!\!- >$ be a deduction system. We call a *strategy* for $\mathcal{D}$ any partial map $\mathcal{S} : \mathcal{A}^+ \rightarrow \mathcal{A}^*$ such that :
(S1) if $\mathcal{S}(A_1A_2\cdots A_n) = B_1 \cdots B_m$ then $\exists Q \subseteq \{A_i \mid 1 \leq i \leq n-1\}$ such that

$$\{B_j \mid 1 \leq j \leq m\} \cup Q \vdash\!\!- A_n,$$

(S2) if $\mathcal{S}(A_1A_2\cdots A_n) = B_1 \cdots B_m$ then

$$\min\{H(A_i) \mid 1 \leq i \leq n\} = \infty \Longrightarrow \min\{H(B_j) \mid 1 \leq j \leq m\} = \infty.$$

Given a strategy $\mathcal{S}$, we define $\mathcal{T}(\mathcal{S}, A)$, the proof-tree associated to the strategy $\mathcal{S}$ and the assertion A as the unique tree t such that :
$\varepsilon \in dom(t)$, $t(\varepsilon) = A$, and, for every path $x_0x_1, \cdots x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \leq i \leq n-1$) if x_{n-1} has m sons $x_{n-1}\cdot 1, \cdots . x_{n-1}\cdot m \in \mathrm{dom}(t)$ with labels $t(x_{n-1}\cdot j) = B_j$ (for $1 \leq j \leq m$) then

$$\begin{array}{l}(\forall i \in [1, n-1], A_i \neq A_n \text{ and } \mathcal{S}(A_1\cdots A_n) = B_1\cdots B_m) \text{ or}\\ (\exists i \in [1, n-1], A_i = A_n \text{ and } m = 0) \text{ or}\\ \quad (A_1\cdots A_n \notin \mathrm{dom}(\mathcal{S}) \text{ and } m = 0).\end{array} \tag{14}$$

(Notice that "x_{n-1} is a leaf" $\Leftrightarrow m = 0 \Leftrightarrow (\mathcal{S}(A_1\cdots A_n) = \varepsilon$ or $\exists i \in [\,1\,,\,n-1\,], A_i = A_n$ or $A_1\cdots A_n \notin dom(\mathcal{S})))$.
Let us say that $\mathcal{S}$ *terminates* iff, $\forall A \in \chi^{-1}(1), \mathcal{T}(\mathcal{S}, A)$ is finite; $\mathcal{S}$ is said *closed*

iff, $\forall W \in \mathcal{A}^+, W \in (\chi^{-1}(1))^+ \Longrightarrow W \in dom(\mathcal{S})$ (i.e. $\mathcal{S}$ is defined on every non-empty sequence of true assertions).

Lemma 4.5 *If $\mathcal{S}$ is a strategy for the deduction-system $\mathcal{D}$ then, for every true assertion A*
(1) the set of labels of $\mathcal{T}(\mathcal{S}, A)$ is a $\mathcal{D}$-proof, relative to the set of labels of the leaves of $\mathcal{T}(\mathcal{S}, A)$.
(2) every label of a leaf is true.

Lemma 4.6 *: If $\mathcal{S}$ is a closed strategy for $\mathcal{D}$, then, for every true assertion A, the set of labels of $\mathcal{T}(\mathcal{S}, A)$ is a $\mathcal{D}$-proof.*

Lemma 4.7 *: If $\mathcal{D}$ admits some terminating, closed strategy then $\mathcal{D}$ is complete.*

5 Triangulations

Let $S_1, S_2, \cdots, S_d$ be a family of deterministic series over the structured alphabet V (we recall V is the alphabet associated with some dpda $\mathcal{M}$ as defined in section 2.2).
Let us consider a sequence $\mathcal{S}$ of n "weighted" linear equations :

$$(\mathcal{E}_i) : p_i, \sum_{j=1}^{d} \alpha_{i,j} S_j \ , \ \sum_{j=1}^{d} \beta_{i,j} S_j$$

where $p_i \in \mathsf{N}$, and $A = (\alpha_{i,j}), B = (\beta_{i,j})$ are deterministic rational matrices of dimension (n, d), with indices $m \leq i \leq m+n-1, 1 \leq j \leq d$. For any weighted equation, $\mathcal{E} = (p, S, S')$, we recall the "cost" of this equation is : $H(\mathcal{E}) = p + 2 \cdot \mathrm{Div}(\varphi(S), \varphi(S'))$.
We associate to such a system another system of equations, $\mathrm{INV}(\mathcal{S})$, which "translates the equations of $\mathcal{S}$ into equations over $(\alpha_{i,j}, \beta_{i,j})$ only". This function INV is in some sense an "elaborated version" of the *inverse* systems defined in [Mei89,Mei92]. The general idea of the construction of INV consists in iterating the transformation used in the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in lemma 3.7, i.e. the classical idea of *triangulating* a system of linear equations. Of course we must deal with the weights and relate the construction with the deduction system $\mathcal{D}_0$. Let us assume here that

$$\forall j \in [1, d], S_j \not\equiv \emptyset. \tag{15}$$

For every $S \in \mathsf{B} < X >$ (resp. $S' \in \mathsf{B}_{1,d} < X >$), we define $\nu(S) = \min\{|u|, u \in \mathrm{supp}(S)\}$ (resp. $\nu(S') = \min\{|u|, u \in \cup_{1 \leq j \leq d} \mathrm{supp}(S'_j)\}$). Let us define $\mathrm{INV}(\mathcal{S})$, $\mathrm{W}(\mathcal{S}) \in \mathsf{N} \cup \{\perp\}, \mathrm{D}(\mathcal{S}) \in \mathsf{N}$ by induction on n. $\mathrm{W}(\mathcal{S})$ is the *weight* of $\mathcal{S}$. $\mathrm{D}(\mathcal{S})$ is the *weak codimension* of $\mathcal{S}$.
Case 1 : $\varphi(\alpha_{m,*}) = \varphi(\beta_{m,*})$ [4].

$$\mathrm{INV}(\mathcal{S}) = ((W(\mathcal{S}), \alpha_{m,j}, \beta_{m,j}))_{1 \leq j \leq d}, \ \mathrm{W}(\mathcal{S}) = p_m - 1, \ \mathrm{D}(\mathcal{S}) = 0.$$

[4] Given any matrix $\gamma \in \mathsf{DB}_{I,J} < V >$ and $i \in I$, by $\gamma_{i,*}$ we mean the row vector $(\gamma_{i,j})_{j \in J}$.

Case 2 : $\varphi(\alpha_{m,*}) \neq \varphi(\beta_{m,*}), n \geq 2, p_{m+1} - p_m \geq 2 \cdot \nu(\triangle(\varphi(\alpha_{m,*}), \varphi(\beta_{m,*}))) + 1$.
Let $u = \min\triangle(\varphi(\alpha_{m,*}), \varphi(\beta_{m,*}))$. Suppose $u \in \triangle(\varphi(\alpha_{m,j_0}), \varphi(\beta_{m,j_0}))$.
Subcase 1 : $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u = \emptyset$.
Let us consider the equation $(p_m + 2 \cdot |u|, S_{j_0}, \sum_{j=1}^{d}(\beta_{m,j} \odot u)S_j)$ and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by :

$$(\mathcal{E}'_i) : p_i, \sum_{j \neq j_0} (\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j} \odot u))S_j \ , \ \sum_{j \neq j_0} (\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j} \odot u))S_j$$

where the above equation is seen as as an equation between two linear combinations of the S_i's where the j_0-th coefficient is $\emptyset$ on both sides. We then define :

$$\mathrm{INV}(\mathcal{S}) = \mathrm{INV}(\mathcal{S}'), \mathrm{W}(\mathcal{S}) = \mathrm{W}(\mathcal{S}'), \mathrm{D}(\mathcal{S}) = \mathrm{D}(\mathcal{S}') + 1.$$

Subcase 2 : $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u \neq \emptyset$.

Let us consider the w-equation $(p_m + 2 \cdot |u|, S_{j_0} \ , \ \sum_{j=1}^{d}(\beta_{m,j_0} \odot u)^*(\beta_{m,j} \odot u)S_j)$ and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by :

$$(\mathcal{E}'_i) : p_i, \sum_{j \neq j_0} [(\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j_0} \odot u)^*(\beta_{m,j} \odot u)]S_j \ , \ \sum_{j \neq j_0} [(\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j_0} \odot u)^*(\beta_{m,j} \odot u)]S_j.$$

(Here also the above equation is seen as an equation between two linear combinations of the S_j's, $1 \leq j \leq d$). We then define:

$$\mathrm{INV}(\mathcal{S}) = \mathrm{INV}(\mathcal{S}'), \mathrm{W}(\mathcal{S}) = \mathrm{W}(\mathcal{S}'), \mathrm{D}(\mathcal{S}) = \mathrm{D}(\mathcal{S}') + 1.$$

Subcase 3 : $\alpha_{m,j_0} \odot u = \emptyset, \beta_{m,j_0} \odot u = \varepsilon$
(analogous to subcase 1).
Subcase 4 : $\alpha_{m,j_0} \odot u \neq \emptyset, \beta_{m,j_0} \odot u = \varepsilon$
(analogous to subcase 2).
Case 3 : $\varphi(\alpha_{m,*}) \neq \varphi(\beta_{m,*}), n = 1$.
We then define:

$$\mathrm{INV}(\mathcal{S}) = \perp, \mathrm{W}(\mathcal{S}) = \perp, \mathrm{D}(\mathcal{S}) = 0,$$

where $\perp$ is a special symbol which can be understood as meaning "undefined".
Case 4 : $\varphi(\alpha_{m,*}) \neq \varphi(\beta_{m,*}), n \geq 2, p_{m+1} - p_m \leq 2 \cdot \nu(\triangle(\varphi(\alpha_{m,*}), (\varphi(\beta_{m,*})))$.
We then define:

$$\mathrm{INV}(\mathcal{S}) = \perp, \mathrm{W}(\mathcal{S}) = \perp, \mathrm{D}(\mathcal{S}) = 0,$$

Lemma 5.1 *: Let $\mathcal{S}$ be a system of linear equations. If $\mathrm{INV}(\mathcal{S}) \neq \perp$ then, $\forall \mathcal{E} \in \mathrm{INV}(\mathcal{S}), \mathcal{E}$ is a weighted linear equation with deterministic coefficients.*

Lemma 5.2 *: Let $\mathcal{S}$ be a system of linear equations. If $\mathrm{INV}(\mathcal{S}) \neq \perp$ then $\mathrm{INV}(\mathcal{S}) = (\bar{\mathcal{E}}_j)_{1 \leq j \leq d}$ fulfills:*

(i) $\{\bar{\mathcal{E}}_j \mid 1 \leq j \leq d\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + \mathrm{D}(\mathcal{S}) - 1\} \mid\!\!- \mathcal{E}_{m+\mathrm{D}(\mathcal{S})}$

(ii) $\min\{H(\mathcal{E}_i) \mid m \leq i \leq m+\mathrm{D}(\mathcal{S})\} = \infty \Longrightarrow \min\{H(\bar{\mathcal{E}}_j) \mid 1 \leq j \leq d\} = \infty$.

Lemma 5.3 *: Let $\mathcal{S}$ be a system of linear equations satisfying the hypothesis of case 2. Then, $\forall i \in [m+1, m+n-1]$,*
$\| \alpha'_{i,*} \| \leq \| \alpha_{i,*} \| + \| \beta_{m,*} \| + | Q || u |, \| \beta'_{i,*} \| \leq \| \beta_{i,*} \| + \| \beta_{m,*} \| + | Q || u |$
.

Let us consider the function F defined by :

$$F(n) = max\{\nu(\varphi(A)\triangle\varphi(B)) \mid A, B \in \mathsf{DRB}_{1,d} < V >, \| A \| \leq n, \| B \| \leq n, \varphi(A) \neq \varphi(B)\}.$$

For every integer parameters $K_1, K_2, K_3, K_4 \in \mathsf{N} - \{0\}$, we define integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ by :

$$\delta_m = 0, \ell_m = 0, L_m = K_2, s_m = K_3 \cdot K_2 + K_4, S_m = 0, \Sigma_m = 0, \tag{16}$$

and for every $m \leq i \leq m+n-2$,

$$\begin{aligned} &\delta_{i+1} = 2 \cdot F(s_i + \Sigma_i) + 1, && \ell_{i+1} = 5 \cdot \delta_{i+1} + 14, \\ &L_{i+1} = K_1 \cdot (L_i + \ell_{i+1}) + K_2, && s_{i+1} = K_3 \cdot L_{i+1} + K_4, \\ &S_{i+1} = s_i + \Sigma_i + \mid Q \mid F(s_i + \Sigma_i), && \Sigma_{i+1} = \Sigma_i + S_{i+1}. \end{aligned} \tag{17}$$

For every weighted, deterministic rational linear equation $\mathcal{E} = (p, \sum_{j=1}^{d} \alpha_j S_j \ , \ \sum_{j=1}^{d} \beta_j S_j)$, we define

$$||| \mathcal{E} ||| = max\{\| \vec{\alpha} \|, \| \vec{\beta} \|\}.$$

Lemma 5.4 *Let $\mathcal{S} = (\mathcal{E}_i)_{m \leq i \leq m+d-1}$ be a system of d weighted linear equations such that $H(\mathcal{E}_i) = \infty$ (for every i) and:*

(1) $\forall i \in [m, m+d-1], ||| \mathcal{E}_i ||| \leq s_i$

(2) $\forall i \in [m, m+d-2], \mathrm{W}(\mathcal{E}_{i+1}) - \mathrm{W}(\mathcal{E}_i) \geq \delta_{i+1}$.

Then $\mathrm{INV}(\mathcal{S}) \neq \perp, \mathrm{D}(\mathcal{S}) \leq d-1, \forall \mathcal{E} \in \mathrm{INV}(\mathcal{S}), ||| \mathcal{E} ||| \leq \Sigma_{m+\mathrm{D}(\mathcal{S})} + s_{m+\mathrm{D}(\mathcal{S})}$.

All the lemmas can be easily extended to the more general case where assumption (15) is replaced by the weaker assumption that:

$$\exists j \in [1, d], S_j \not\equiv \emptyset. \tag{18}$$

6 Constants

Let us define some integers which are *constant* in the sense that they depend on the dpda $\mathcal{M}$ and an initial true assertion $A_0 = (\Pi_0, S_0^-, S_0^+)$ only.

$k_0 = \max\{\nu([pAq]) \mid p, q \in Q, A \in Z, [pAq] \not\equiv \emptyset\}$, $k_1 = \max\{2k_0 + 1, 3\}$,

$D_1 = 4k_0 + 3$, $\quad k_2 = (D_1 + 4) \cdot k_1 + k_0 + 1$,

$K_1 = k_1 + 1$, $\quad K_2 = (D_1 + 1) \cdot k_1^2 + (D_2 + 1) \cdot k_1 + 1$,

$K_3 = k_0|Q|$, $\quad K_4 = k_0 \cdot |Q|^2 + (k_2 + k_1 + 7) \cdot |Q| + 3$,

$d_0 = 2 \cdot |Q| \cdot (\mathrm{Card}(X^{\leq k_1}) + 1)$.

We consider now the integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ defined by the relations (16,17) of section 5 where the parameters $K_1, \ldots, K_4$ are chosen to be the above constants and $m = 1, n = d_0, d = d_0$.

$$D_2 = \max\{\Sigma_{d_0} + s_{d_0}, \|S_0^-\|, \|S_0^+\|\}, \qquad N_0 = 1 + (k_2+2)|Q| + D_2,$$

$$C_2 = \mathrm{Card}\{U \in \mathsf{DRB} < V >, \|U\| \leq D_2\},$$

$$K_6 = 5 \cdot [(C_2 \cdot |Q||Z|^{k_2+D_2+3})^{|Q|} \cdot |Q||Z|^{D_1}]^2, \quad K_5 = (K_6+1) \cdot k_0 \cdot |Q|.$$

7 Strategies for $\mathcal{D}_0$

Let us define strategies for the particular system $\mathcal{D}_0$.
We define first auxiliary strategies $T_{cut}, T_\emptyset, T_\varepsilon, T_A, T_B, T_C$ and then derive some closed strategies from them. Let us fix here some total ordering on $X : x_1 < x_2 < \cdots < x_\alpha$ and also some total ordering $\leq$ of type ω on $\mathcal{A}$ (inherited from the usual well-ordering of N by the fixed encoding). From these orderings one can construct in the usual way an ordering of type ω on the sets $X^*, \mathcal{A}^*$ and $\mathsf{N}^* \times (\mathsf{DRB} < V >)^*$.

Let us adapt the usual notion of *stacking derivation* to derivations of series. For every $u \in X^*$ we define the binary relation $\uparrow (u)$ over $\mathsf{DB} < V >$ by: for every $S, S' \in \mathsf{DB} < V >, S \uparrow (u) S' \Leftrightarrow \exists A \in Z, \omega \in Z^+, p, q \in Q, \Psi \in \mathsf{DB}_{Q,1} < V >$ such that

$$S = [pA] * \Psi, [pA] \odot u = [q\omega], S' = [q\omega] * \Psi.$$

A sequence of deterministic series $S_0, S_1, \ldots, S_n$ is a *derivation* iff there exist $x_1, \ldots, x_n \in X$ such that $S_0 \odot x_1 = S_1, \ldots, S_{n-1} \odot x_n = S_n$. If $u = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ we call $S_0, S_1, \ldots, S_n$ the derivation *associated* with (S, u). A derivation $S_0, S_1, \ldots, S_n$ is said to be *stacking* iff it is the derivation associated to a pair (S, u) such that $S = S_0$ and $S_0 \uparrow (u) S_n$.

T_{cut}: $T_{cut}(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $\exists i \in [1, n-1], \exists S, T,$

$$A_i = (p_i, S, T), A_n = (p_n, S, T), p_i < p_n \text{ and } m = 0$$

$T_\emptyset$: $T_\emptyset(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ iff $\exists S, T, A_n = (p, S, T), p \geq 0, S \equiv T \equiv \emptyset$ and $m = 0$

T_ε: $T_\varepsilon(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $A_n = (p, S, T), p \geq 0, S \equiv T \equiv \varepsilon$ and $m = 0$

T_A: $T_A(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

$$A_n = (p, S, T), m = \mid X \mid, B_1 = (p+1, S \odot x_1, T \odot x_1), \cdots, B_m = (p+1, S \odot x_m, T \odot x_m),$$

where $S \not\equiv \varepsilon, T \not\equiv \varepsilon$

T_B^+: $T_B^+(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $n \geq k_1 + 1, A_{n-k_1} = (\pi, \overline{U}, U')$, (where $\overline{U}$ is unmarked)

$$U' = \sum_{q \in Q} [\overline{p} A q] \cdot V_q \quad (\text{ for some } \overline{p} \in Q)$$

$A_i = (\pi + k_1 + i - n, U_i, U_i')$ for $n - k_1 \leq i \leq n$, $(U_i)_{n-k_1 \leq i \leq n}$ is a derivation, $(U_i')_{n-k_1 \leq i \leq n}$ is a "stacking derivation" (see the above definition),

$$U_n' = \sum_{q \in Q} [p\tau q] \cdot V_q, \quad \text{for some } p \in Q, \tau \in Z^+,$$

$m = 1, B_1 = (\pi + k_1 - 1, V, V'), V = U_n, V' = \sum_{q \in Q'} [p\tau q] \cdot [qeq] \cdot (\overline{U} \odot u_q)$, where $Q' = \{q \in Q \mid [\overline{p}Aq] \not\equiv \emptyset\}, \forall q \in Q', u_q = \min(\varphi([\overline{p}Aq]))$.

T_B^-: T_B^- is defined in the same way as T_B^- by exchanging the left series (S^-) and right series (S^+) in every assertion (p, S^-, S^+).

T_C: $T_C(A_1 \cdots A_n) = B_1 \cdots B_m$ iff there exists $d \in [1, d_0], D \in [\,0\,,\ d-1\,], S_1, S_2, \cdots, S_d \in \mathsf{DRB} < V >, 1 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_{D+1} = n$, such that,

(C1) every equation $(\mathcal{E}_i) = (p_{\kappa_i}, S^-_{p_{\kappa_i}}, S^+_{p_{\kappa_i}})$, for $1 \leq i \leq D+1$, is a weighted equation over $S_1, S_2, \cdots, S_d$,

(C2) $\mathcal{S} = (\mathcal{E}_i)_{1 \leq i \leq D+1}$ is such that, $\mathrm{INV}(\mathcal{S}) \neq \perp, \mathrm{D}(\mathcal{S}) = D$ and $\|| \mathrm{INV}(\mathcal{S}) \|| \leq \Sigma_{d_0} + s_{d_0}$,

(C3) $(\kappa_1, \kappa_2, \cdots, \kappa_{D+1}, S_1, \cdots, S_d) \in \mathsf{N}^* \times (\mathsf{DRB} < V >)^*$ is the minimal vector satisfying conditions (C1,C2) for the given sequence $(A_1 \cdots A_n)$ and

(C4) $B_1 \cdots B_m = \rho_e(\mathrm{INV}(\mathcal{S}))$ (where ρ_e is the obvious extension of ρ_e to pairs of series and then to sequences of weighted equations; in other words the result of T_C is $\mathrm{INV}(\mathcal{S})$ where the marks have been removed).

Lemma 7.1 *: $T_{cut}, T_\emptyset, T_\varepsilon, T_A, T_B, T_C$ are $\mathcal{D}_0$-strategies.*

Let us define the strategy $\mathcal{S}_{AB}$ by : for every $W = A_1 A_2 \cdots A_n$,

(0) if $W \in \mathrm{dom}(T_{cut})$, then $\mathcal{S}_{AB}(W) = T_{cut}(W)$ (1) elsif $W \in \mathrm{dom}(T_\emptyset)$, then $\mathcal{S}_{AB}(W) = T_\emptyset(W)$

(2) elsif $W \in \mathrm{dom}(T_\varepsilon)$, then $\mathcal{S}_{AB}(W) = T_\varepsilon(W)$ (4) elsif $W \in \mathrm{dom}(T_B^+)$, then $\mathcal{S}_{AB}(W) = T_B^+(W)$

(5) elsif $W \in \mathrm{dom}(T_B^-)$, then $\mathcal{S}_{AB}(W) = T_B^-(W)$ (6) elsif $W \in \mathrm{dom}(T_A)$, then $\mathcal{S}_{AB}(W) = T_A(W)$

(7) else $\mathcal{S}_{AB}(W)$ is undefined.

The strategy $\mathcal{S}_{ABC}$ is obtained by inserting "(3) elsif $W \in \mathrm{dom}(T_C)$, then $\mathcal{S}_{ABC}(W) = T_C(W)$" in the above list of cases.

Lemma 7.2 *$\mathcal{S}_{ABC}, \mathcal{S}_{AB}$ are closed.*

8 Tree analysis

This section is devoted to the analysis of the proof-trees τ produced by the strategy $\mathcal{S}_{AB}$ defined in section 7. The main results are lemma 8.8 and 8.9 whose combination asserts that if some path (from a node x to a node y) of τ is such that its origin has a "small norm" and its length is "large enough", then the transformation T_C is defined at some ancestor of y. This key technical result will ensure termination of the strategy $\mathcal{S}_{ABC}$ (see section 9).

The next lemma just shows that the *weight* and the *depth* of a given node are closely related.

Lemma 8.1 *: Let τ be a proof-tree associated to the strategy $\mathcal{S}_{AB}$. Let $x, x' \in dom(\tau), x \leq x'$. Then $\mid W(x') - W(x) \mid \leq |x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3$.*

We fix throughout the remaining of this section a tree $\tau = \mathcal{T}(\mathcal{S}_{AB}, (\pi_0, U_0^-, U_0^+))$ (i.e. τ is the proof tree associated to the assertion $(\pi_0, U_0^-, U_0^+))$ by the strategy $\mathcal{S}_{AB}$). We suppose that

$$\mathrm{rd}(U_0^-) \leq D_2, \quad \mathrm{rd}(U_0^+) \leq D_2, \quad U_0^-, U_0^+ \text{ are both unmarked .} \tag{19}$$

We recall that, formally, τ is a map $\mathrm{dom}(\tau) \to \mathsf{N} \times \mathsf{DRB} < V > \times \mathsf{DRB} < V >$ such that $\mathrm{dom}(\tau) \subseteq \{1, \dots, |X|\}^*$ is closed under prefix and under "left-brother" (i.e. $w \cdot (i+1) \in \mathrm{dom}(\tau) \Rightarrow w \cdot i \in \mathrm{dom}(\tau)$). We denote by $pr_{2,3} : \mathsf{N} \times \mathsf{DRB} < V > \times \mathsf{DRB} < V > \to \mathsf{DRB} < V > \times \mathsf{DRB} < V >$ the projection $(\pi, U, U') \mapsto (U, U')$. By τ_s we denote the tree obtained from τ by forgetting the weights: $\tau_s = \tau \circ pr_{2,3}$. We establish some linearity properties of the labels of τ.

Lemma 8.2 *For every label (π, U^-, U^+) of τ,*

(1) U^α is (D_1, D_2)-linear (for $\alpha \in \{-,+\}$)

(2) if U^α is unmarked, then U^α is $(0, D_2)$-linear

(3) $\exists \alpha \in \{-,+\}, U^\alpha$ is unmarked.

We show now that the tree τ is somewhat "smooth" in the sense that its labels cannot be varying in a too chaotic way along a given branch. We shall establish that every "sufficiently long" branch must contain a "reasonnably short factor" (a "N-stacking sequence") where at least d labels (U, U') are belonging to the same d-space V of dimension $d \leq d_0$ with coordinates not greater than s_{d_0} (over some fixed generating family of cardinality d). Let us define now this notion of stacking sequence which is , roughly speaking, an extension to sequences of *pairs* (U_i, U_i') appearing in τ_s, of the notion of *stacking derivation.*
For every $U, U' \in \mathsf{DRB} < V >$ we set:

$$|||U||| = \max\{\|\tilde{U}\|, \;\; \tilde{U} \in \mathsf{Q}(U) \text{ and } \tilde{U} \text{ is unmarked }\}, \;\; |||(U, U')||| = \max\{|||U|||, |||U'|||\}.$$

Let $x \in \mathrm{dom}(\tau)$. We define now a kind of norm on nodes which in some sense "erases" the short-term variations of $|||(*,*)|||$: $\mathrm{N}(x) = \max\{|||\tau_s(x(j))|||, j \in \mathsf{N}, |x| - k_1 \leq j \leq |x|\}$.

Lemma 8.3 *Let $(x_i)_{i \in I}$ (where I is a beginning section of N) be some branch of τ.*

(1) The function $N : I \to \mathsf{N}$ is $(k_0 \cdot |Q|)$-up-Lipschitz.

(2) The restriction $N : J \to \mathsf{N}$ to any interval $J \subseteq I$ such that neither T_B^-, nor T_B^+ occur in J, is $|Q|$-up-Lipschitz.

Let $\sigma = (x_i)_{i \in I}$ be a path in τ, where $I \subseteq \mathsf{N}$ is a non-empty interval and $i_0 = \min\{I\}$. We call σ a *N-stacking sequence* iff:

$$\forall i \in I, \mathrm{N}(x_i) \geq \mathrm{N}(x_{i_0}) \text{ and } \mathrm{N}(x_{i_0}) \geq N_0. \tag{20}$$

From now on and until lemma 8.9 , we fix a N-stacking sequence $\sigma = (x_i)_{i \in I}$. We call $\mathrm{Card}(I) - 1$ the *length* of σ (denoted $|\sigma|$) and we suppose $|\sigma| \geq k_1$. Let us use the simplified notations $\mathrm{N}(i)$ for $\mathrm{N}(x_i)$ and let us note $\tau_s(x_i) = (U_i^-, U_i^+)$.

Lemma 8.4 $\forall i \in \mathsf{N}$, *if* $i+k_1 \in I$, *then* $\exists j \in [i, i+k_1], |||\ (U_j^-, U_j^+)\ ||| \geq \mathrm{N}(i_0)$.

Proof: This follows from the fact that $\mathrm{N}(i+k_1) \geq \mathrm{N}(i_0)$. □
We let

- $i_1 = \min\{i \in [i_0 - k_1, i_0] \cap \mathsf{N},\ |||\ (U_i^-, U_i^+)\ ||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|\}$, if no T_B^α occurs in $[i_0 - k_1, i_0]$
- and i_1 is the unique element of $[i_0 - k_1, i_0] \cap \mathsf{N}$ where some T_B^α occurs otherwise.

For every $\alpha \in \{-, +\}$, we define a Q-series $[p^\alpha \omega^\alpha]$ and a Q-forms Φ^α as follows.
case 1: $U_{i_1}^\alpha$ is unmarked and $\|U_{i_1}^\alpha\| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.
As $\mathrm{rd}(U_{i_1}^\alpha) \leq D_2$, $U_{i_1}^\alpha$ has a minimal decomposition: $U_{i_1}^\alpha = [p^\alpha \omega_1^\alpha] * \Phi_1^\alpha$ with $\|\Phi_1^\alpha\| \leq D_2$. Using the inequality $\|U_{i_1}^\alpha\| \geq \mathrm{N}(i_0) - (k_2+1)|Q| \geq D_2$, we conclude that ω_1^α admits a decomposition $\omega_1^\alpha = \omega^\alpha \omega_2^\alpha$ such that

$$U_{i_1}^\alpha = [p^\alpha \omega^\alpha] * \Phi^\alpha, \quad \Phi^\alpha = [\omega_2^\alpha] * \Phi_1^\alpha, \tag{21}$$

with

$$\mathrm{N}(i_0) - (k_2+1)|Q| \leq \|\Phi^\alpha\| < \mathrm{N}(i_0) - k_2|Q|, \quad \|\Phi^\alpha\| \geq D_2 + |Q|, \quad |\Phi^\alpha| \geq 1. \tag{22}$$

case 2: $U_{i_1}^\alpha$ is marked and $|||\ U_{i_1}^\alpha\ ||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.
Hence $U_{i_1}^\alpha$ can be written as $U_{i_1}^\alpha = \Sigma_{q \in Q'} [p_1^\alpha \mu_1^\alpha q] \cdot [qeq] \cdot (\overline{U_{i_1}^\alpha} \odot u_q)$ for some $p_1^\alpha \in Q, Q' \subseteq Q, 1 \leq |\mu_1^\alpha| \leq D_1, |u_q| \leq k_0, \overline{U_{i_1}^\alpha} = U_j^{-\alpha}, j < i_1$.
Let $q \in Q'$ such that $|||\ U_{i_1}^\alpha\ ||| = \|\overline{U_{i_1}^\alpha} \odot u_q\|$. We have:

$$\|\overline{U_{i_1}^\alpha}\| \geq \|\overline{U_{i_1}^\alpha} \odot u_q\| - |u_q||Q| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q| - k_0|Q| = \mathrm{N}(i_0) - k_2|Q|.$$

From this inequality we conclude, as in case 1, that $\overline{U_{i_1}^\alpha}$ has a decomposition:

$$\overline{U_{i_1}^\alpha} = [p^\alpha \omega^\alpha] * \Phi^\alpha, \tag{23}$$

with

$$\mathrm{N}(i_0) - (k_2+1)|Q| \leq \|\Phi^\alpha\| < \mathrm{N}(i_0) - k_2|Q|, \quad \|\Phi^\alpha\| \geq D_2 + |Q|, \quad |\Phi^\alpha| \geq 1. \tag{24}$$

case 3: $|||\ U_{i_1}^\alpha\ ||| < \mathrm{N}(i_0) - (k_2 - k_0)|Q|$. In this case , $[p^\alpha \omega^\alpha], \Phi^\alpha$ are both undefined.

Let us remark that, by the inequality $|||\ (U_{i_1}^-, U_{i_1}^+)\ ||| \geq \mathrm{N}(i_0) - (k_2 - k_1)|Q|$, there exists at least one $\alpha \in \{-, +\}$ such that $[p^\alpha \omega^\alpha], \Phi^\alpha$ are both defined.
Let us define now the following families of series and d-spaces of series:

$$W^\alpha = \mathsf{V}((\Phi_q^\alpha)_{q \in Q}), \quad V^\alpha = \mathsf{V}((\Phi_q^\alpha \odot u)_{q \in Q, u \in X^*, 0 \leq |u| \leq k_0}), \tag{25}$$

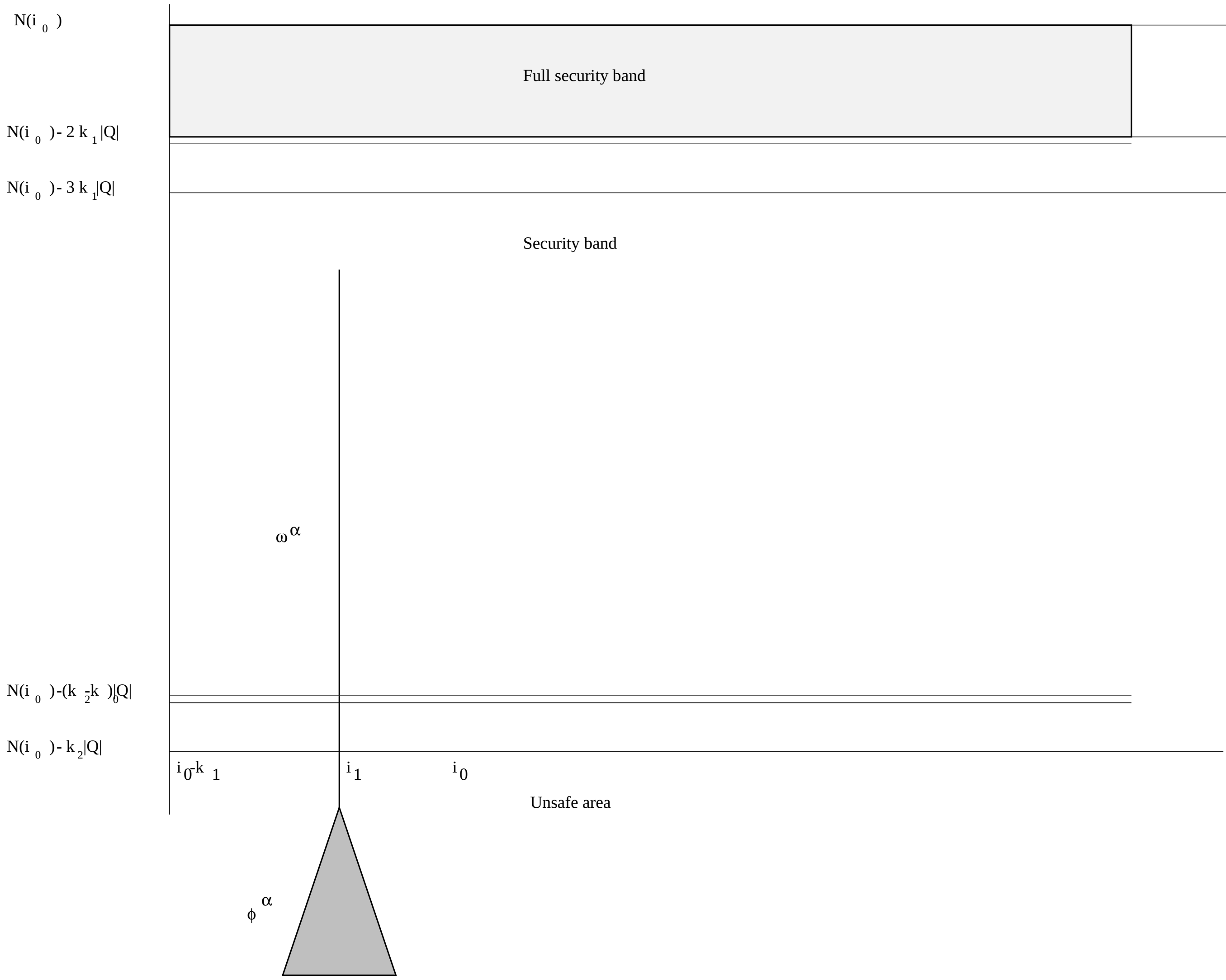

Fig. 2. the generating family.

for every $\alpha \in \{-,+\}$;

$$\bar{V} = V^- + V^+, \quad \tilde{W} = W^- \cup W^+, \quad \tilde{V} = V^- \cup V^+, \tag{26}$$

where , for every d-spaces V_1, V_2, $V_1 + V_2$ denotes the smallest d-space containing $V_1 \cup V_2$. We illustrate on figure 2 the above definitions. The band $\{U \in \mathsf{DRB} < V > | \mathrm{N}(i_0) - 2k_1|Q| \leq ||| \ U \ |||\}$ is a *full security band* in the sense that every U_i^α in this band *belongs to* $\bar{V}$ (lemma 8.5). The band $\{U \in \mathsf{DRB} < V > | \ \mathrm{N}(i_0) - (k_2 - k_0)|Q| \leq ||| \ U \ ||| < \mathrm{N}(i_0)\}$ is a *security band* in the sense that if $[U_j^\alpha$ is in this band , $j < i$, U_i^α belongs to the full security band, and $U_j^\alpha \longrightarrow U_i^\alpha]$, then U_j^α *belongs to* $\bar{V}$ (property $\mathcal{P}(\alpha, i, j)$ established in the proof of lemma 8.5).

Lemma 8.5 *Let* $i \geq i_1, \alpha \in \{-,+\}$ *such that* $||| \ U_i^\alpha \ ||| \geq \mathrm{N}(i_0) - 2k_1|Q|$. *Then* $U_i^\alpha \in \bar{V}$.

Sketch of proof: Let us consider the following property $\mathcal{P}(\alpha, i, j)$:
$\{j \leq i$ and $||| \ U_i^\alpha \ ||| \geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q|$ and no T_B^α occurs in $[j+1, i]\} \Longrightarrow$
$\{($ if U_j^α is unmarked then $U_j^\alpha \in \tilde{W})$ and

(if U_j^α is marked then, there exists some $\alpha' \in \{-,+\}$ such that ,
every linear component of U_j^α is in $(W^{\alpha'} \cap \mathsf{DRBlin}^{D_2} < V >) \odot X^{<1,k_0>})\}$.
(Here we denote by $X^{<m,m'>}$ the set $\{u \in X^*, m \leq |u| \leq m'\}$). We prove by lexicographic induction on the pair (i,j) that: $\forall i \geq i_1, \forall j \geq i_1, [\forall \alpha \in \{-,+\}, \mathcal{P}(\alpha,i,j)]$. □
Let us define a new space of series by: $\mathcal{G}_1 = (\theta_j)_{1 \leq j \leq d}$ where

$$\{\theta_j | 1 \leq j \leq d\} = \bigcup_{\alpha \in \{-,+\}} \{\Phi_q^\alpha \odot u | \ q \in Q, u \in X^{\leq k_1}\} \cup \{\rho_\epsilon(\Phi_q^\alpha), \ \ q \in Q\},$$

(one can notice that $d \leq d_0$), and

$$\bar{V_1} = \mathsf{V}(\mathcal{G}_1).$$

Lemma 8.6 *Let $L > 0$ such that $(i_0 + K_1 \cdot L + K_2) + k_1 \in I$. There exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ such that, $U_i^- \in \bar{V_1}, U_i^+ \in \bar{V_1}$.*

Let us give now a stronger version of lemma 8.6 where we analyze the *size of the coefficients* of the linear combinations whose *existence* only is proved in lemma 8.6.

Lemma 8.7 *Let $L > 0$ such that $(i_0 + K_1 \cdot L + K_2) + k_1 \in I$. There exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ such that, for every $\alpha \in \{-,+\}$, there exists a deterministic rational vector $(\beta_{i,j}^\alpha)_{1 \leq j \leq d}$ fulfilling*

(1) $U_i^\alpha = \sum_{j=1}^d \beta_{i,j}^\alpha \theta_j$
(2) $\|\beta_{i,*}^\alpha\| \leq K_3 \cdot (i - i_0) + K_4$.

By *iteration* of lemma 8.7 we obtain the following lemma.

Lemma 8.8 *Suppose that $i_0 + L_{d_0} + k_1 \in I$. Then there exists $i_0 \leq \kappa_1 < \kappa_2 < \ldots < \kappa_d$ and deterministic rational vectors $(\beta_{i,j}^\alpha)_{1 \leq j \leq d}$ (for every $i \in [1,d]$) such that*

(0)$W(\kappa_1) \geq 1$,
(1)$\forall i, \forall \alpha, U_{\kappa_i}^\alpha = \sum_{j=1}^d \beta_{i,j}^\alpha \theta_j \in \overline{V_1}$,
(2)$\forall i, \forall \alpha, \|\beta_{i,*}^\alpha\| \leq s_i$,
(3)$\forall i, W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}$.

where the sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \sigma_i)$ are those defined by relations (16, 17) in section 5, $K_1, \ldots, K_4$ are the constants defined in section 6, and $m = 1, n = d$.

Lemma 8.9 *Let $(x_i)_{i \in I}$ be a path in τ (we suppose $I \subseteq \mathsf{N}$ is a non-empty interval). Let $L > 0$. One of the following cases is true:*

(0) $N(i_0) \geq N_0$, *where* $i_0 = \min(I)$,

(1) $|I| \leq K_5 \cdot L + K_6$,

(2) $(x_i)_{i \in I}$ *contains a* N*-stacking sequence of length* $\geq L$.

9 Completeness of $\mathcal{D}_0$

We show that, up to some slight details, $\mathcal{S}_{ABC}$ is terminating.

Lemma 9.1 *: Let A_0 be some true assertion which is supposed unmarked. Then the tree $\mathcal{T}(\mathcal{S}_{ABC}, A_0)$ is finite.*

Proof: Suppose $A_0 = (\Pi_0, S_0^-, S_0^+)$ is true, unmarked and $t = \mathcal{T}(\mathcal{S}_{ABC}, A_0)$ is infinite.

By Koenig's lemma, t contains an infinite branch whose (infinite) labelling word is $A_0 A_1 \cdots A_n \cdots$. Lemma 5.4 applied to $m = 1$ and $d \leq d_0$ and combined with lemma 3.5 shows that the equations produced by T_C have size $\leq D_2$, hence that the number of possible results of T_C is *finite*. Hence T_C occurs only a finite number of times on this branch (otherwise T_{cut} would occur on this branch, which is impossible on an infinite branch). Let n_0 be the last point where T_C occurs or $n_0 = 0$ if T_C does not occur on this branch. $(A_{n_0+i})_{i \geq 0}$ is a branch of the tree $t' = \mathcal{T}(\mathcal{S}_{AB}, A_{n_0})$. As every result of T_C is a weighted linear equation which has size bounded by D_2 (by the above arguments) and which is unmarked (by point (C4) in the definition of T_C), if $n_0 \neq 0$ then t' fulfills the hypothesis (19) assumed in section 8.
If $n_0 = 0$, by definition of D_2 (see section 6) $rd(S_0^-) \leq D_2, rd(S_0^+) \leq D_2$ and by hypothesis S_0^-, S_0^+ are unmarked. Hence, in this case too, t' fulfills the hypothesis (19) assumed in section 8. Let us now apply the results of section 8.
By lemma 8.9, the branch $(A_{n_0} + i)_{i \geq 0}$ must contain an infinite N-stacking sequence. Let us remark that, as $T_\emptyset$ does not occur (otherwise the branch would be finite) every equation (π, U^-, U^+) labelling this branch is such that $U^- \not\equiv \emptyset, U^+ \not\equiv \emptyset$. By lemma 8.8 such an N-stacking sequence contains a subsequence $(A_{\kappa_1}, A_{\kappa_2}, \cdots, A_{\kappa_i}, \cdots, A_{\kappa_d})$ with $d \leq d_0$, fulfilling hypotheses (1,2) of lemma 5.4, and by the above remark it fulfills hypothesis (18) of section 5 too. Hence some finite prefix of $(A_{n_0+i})_{i \geq 0}$ belongs to dom(T_C). The priority ordering given in the definition of $\mathcal{S}_{ABC}$ then implies that either $T_{cut}, T_\emptyset, T_\varepsilon$ or T_C occurs at some $n_0 + i + 1$. But the three first cases cannot occur on an infinite branch and the fourth one contradicts the maximality of n_0. □

Theorem 9.1 *The system $\mathcal{D}_0$ is complete.*

Proof: By lemma 7.1 $\mathcal{S}_{ABC}$ is a strategy for $\mathcal{D}_0$, by lemma7.2 $\mathcal{S}_{ABC}$ is closed and by lemma 9.1 $\mathcal{S}_{ABC}$ is terminating on every unmarked true assertion. By a slight variant of lemma 4.7, every unmarked true assertion has a $\mathcal{D}_0$-proof. But for every $A \in \mathcal{A}$, there exists a finite $\mathcal{D}_0$-proof of $(0, A, \rho_e(A))$. It follows that every true assertion A has a $\mathcal{D}_0$-proof. □

Theorem 9.2 *The equivalence problem for deterministic pushdown automata is decidable.*

Proof: Let $\mathcal{M}$ be some dpda. The equivalence relation $\equiv$ on $\mathsf{DRB} < V >$ (where V is the structured alphabet associated to the given $\mathcal{M}$) has a recursively enumerable complement (this is well-known). By theorem 9.1 and lemma 4.2 $\equiv$ is recursively enumerable too. Hence $\equiv$ is recursive. In addition, the system $\mathcal{D}_0$ associated with $\mathcal{M}$ is computable from $\mathcal{M}$, hence the theorem follows. □

10 Elimination

Let us sketch here some simplifications of the system $\mathcal{D}_0$ which can be obtained via the *elimination* of rule $R5$. Let us call $\mathcal{D}_1$ the deduction system obtained by removing rule $R5$ from $\mathcal{D}_0$ and by replacing rule $R'3$ by the more restricted rule:

$$(R''3)\ \emptyset \mid\!\vdash\!\!- (0,[qzr],\epsilon)(\text{for } q,r \in Q, z \in Z, [qzr] \equiv \epsilon).$$

Let us call $\mathcal{C}$ the deduction system obtained by removing rule $R4$ from $\mathcal{D}_1$.

Lemma 10.1 *Let $A_0 \in \mathcal{A}$, A_0 unmarked and $t = \mathcal{T}(\mathcal{S}_{ABC}, A_0)$. Then, $\forall(\pi, S, S') \in \mathrm{im}(t)$ either $S \equiv S' \equiv \emptyset$ or $S \equiv S' \equiv \epsilon$ or*
$\forall x \in X, \mathrm{im}(t) \overset{<*>}{\mid\!\vdash\!\!-}_{\mathcal{C}} (\pi+1, S \odot x, S' \odot x)$.

(This property of $\mathrm{im}(t)$ is analogous to the property "to be self-proving " considered in [Cou83b, p.162]).

Lemma 10.2
(1)$\forall S \in \mathsf{DRB} < V >$ *if $S \equiv S' = \emptyset$ or $S \equiv S' = \epsilon$ then*

$$\{(0,[pzq],\emptyset) \mid p,q \in Q, z \in Z, [pzq] \equiv \emptyset\} \overset{<*>}{\mid\!\vdash\!\!-}_{\mathcal{C}} (0,S,S').$$

(2)$\forall p,q \in Q, z \in Z$ *such that $[pzq] \equiv \emptyset$ there exists a finite $\mathcal{D}_1$-proof $P_{[pzq]}$ of the assertion $(0,[pzq],\emptyset)$.*

By the two above lemma one can see that, $\mathrm{im}(t) \cup \bigcup_{\substack{p,q\in Q, z\in Z\\ [pzq]\equiv\emptyset}} P_{[pzq]}$ is a $\mathcal{D}_1$-proof of A_0. The case of a marked assertion A_0 can be easily deduced from the unmarked case. Hence we get the

Theorem 10.1 *The system $\mathcal{D}_1$ is complete.*

Let us define now a new formal system $\mathcal{D}_2$.

$$\mathcal{A}_2 = \mathsf{DRB} < V > \times \mathsf{DRB} < V > .$$

The "cost-function" $H_2 : \mathcal{A} \to \mathsf{N} \cup \{\infty\}$ is defined by :

$$H_2(S,S') = \mathrm{Div}(S,S').$$

The binary relation $\mid\!\vdash\!\!-_{\mathcal{D}_2} \subset \mathcal{P}_f(\mathcal{A}_2) \times \mathcal{A}_2$, the *elementary deduction relation*, is the set of all the pairs having one of the following forms:

$$
\begin{array}{llll}
(R21) & \{(S,T)\} & \mid\!\vdash\!\!-_{\mathcal{D}_2} (T,S) & \\
(R22) & \{(S,S'),(S',S'')\} & \mid\!\vdash\!\!-_{\mathcal{D}_2} (S,S'') & \\
(R23) & \emptyset & \mid\!\vdash\!\!-_{\mathcal{D}_2} (S,S) & \\
(R'23) & \emptyset & \mid\!\vdash\!\!-_{\mathcal{D}_2} ([qzr],\epsilon) & (\text{for } q,r \in Q, z \in Z, [qzr] \equiv \epsilon) \\
(R24) & \{(S \odot x, T \odot x) \mid x \in X\} & \mid\!\vdash\!\!-_{\mathcal{D}_2} (S,T) & (\text{for } S \not\equiv \epsilon \wedge T \not\equiv \epsilon) \\
(R25) & \{(S \cdot T' + T, T')\} & \mid\!\vdash\!\!-_{\mathcal{D}_2} (S^* \cdot T, T') & (\text{for } S \not\equiv \varepsilon) \\
(R26) & \{(S,S')\} & \mid\!\vdash\!\!-_{\mathcal{D}_2} (S+T, S'+T) & \\
(R27) & \{(S,S')\} & \mid\!\vdash\!\!-_{\mathcal{D}_2} (S \cdot T, S' \cdot T) & \\
(R28) & \{(T,T')\} & \mid\!\vdash\!\!-_{\mathcal{D}_2} (U \cdot T, U \cdot T') &
\end{array}
$$

where $S, S', T, T' \in \mathsf{DRB} < V >, U \in \mathsf{RB} < V >$.

Let us define $\vdash\!\!-_{\mathcal{D}_2}$ by : for every $P \in \mathcal{P}_f(\mathcal{A}_2), A \in \mathcal{A}_2$, $P \vdash\!\!- A \Longleftrightarrow P \overset{<*>}{\mid\!\vdash\!\!-} \circ \overset{[1]}{\mid\!\vdash\!\!-}_{3,4} \circ \overset{<*>}{\mid\!\vdash\!\!-} \{A\}$, where $\mid\!\vdash\!\!-_{3,4}$ is the relation defined by $R23, R'23, R24$ only.

We let $\mathcal{D}_2 =< \mathcal{A}_2, H_2, \vdash\!\!-_{\mathcal{D}_2} >$.

Lemma 10.3 *: $\mathcal{D}_2$ is a deduction system.*

Theorem 10.2 *The system $\mathcal{D}_2$ is complete.*

Sketch of proof: This can be easily deduced from lemma 10.1,lemma 10.2 and the fact that the image (by erasing the weights) of every proof $P_{[pzq]}$ (for $[pzq] \equiv \emptyset$), is also a $\mathcal{D}_2$-proof of $([pzq], \emptyset)$. □

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References

[Bee76] C. Beeri. An improvement on Valiant's decision procedure for equivalence of deterministic finite-turn pushdown automata. *TCS 3*, pages 305–320, 1976.

[Cau95] D. Caucal. Bisimulation of context-free grammars and of pushdown automata. In *Modal Logic and process algebra*, pages 85–106. CSLI Lecture Notes,vol. 53, Stanford CA, 1995.

[Cou83a] B. Courcelle. An axiomatic approach to the Korenjac-Hopcroft algorithms. *Math. Systems theory*, pages 191–231, 1983.

[Cou83b] B. Courcelle. Fundamental properties of infinite trees. *Theoretical Computer Science 25*, pages 95–169, 1983.

[Cou90] B. Courcelle. Recursive applicative program schemes. In *Handbook of Theoretical Computer Science, edited by J. Van Leeuwen*, pages 461–490. Elsevier, 1990.

[GG66] S. Ginsburg and S. Greibach. Deterministic context-free languages. *Information and Control*, pages 620–648, 1966.

[Har78] M.A. Harrison. *Introduction to Formal Language Theory.* Addison-Wesley, Reading, Mass., 1978.

[HHY79] M.A. Harrison, I.M. Havel, and A. Yehudai. On equivalence of grammars through transformation trees. *TCS 9*, pages 173–205, 1979.

[Lis96] L.P. Lisovik. Hard sets methods and semilinear reservoir method with applications. In *Proceedings 23rd ICALP*, pages 229–231. Springer, LNCS 1099, 1996.

[Mei89] Y.V. Meitus. The equivalence problem for real-time strict deterministic pushdown automata. *Kibernetika 5 (in russian, english translation in Cybernetics and Systems analysis)*, pages 14–25, 1989.

[Mei92] Y.V. Meitus. Decidability of the equivalence problem for deterministic pushdown automata. *Kibernetika 5 (in russian, english translation in Cybernetics and Systems analysis)*, pages 20–45, 1992.

[Oya87] M. Oyamaguchi. The equivalence problem for real-time d.p.d.a's. *J. assoc. Comput. Mach. 34*, pages 731–760, 1987.

[Rom85] V.Yu. Romanovskii. Equivalence problem for real-time deterministic pushdown automata. *Kibernetika no 2*, pages 13–23, 1985.

[Sén94] G. Sénizergues. Formal languages and word-rewriting. In *Term Rewriting, Advanced Course*, pages 75–94. Springer, LNCS 909, edited by H. Comon and J.P.Jouannaud, 1994.

[Sén97] G. Sénizergues. L(A) = L(B)? Technical report, LaBRI, Université Bordeaux I, report nr1161-97, can be accessed at URL, http://www.labri.u-bordeaux.fr/, 1997.

[Sti96] C. Stirling. Decidability of bisimulation equivalence for normed pushdown processes. In *Proceedings CONCUR 96*, pages 217–232. Springer-Verlag, LNCS 1119, 1996.

[Val74] L.G. Valiant. The equivalence problem for deterministic finite-turn pushdown automata. *Information and Control 25*, pages 123–133, 1974.

[VP75] L.G. Valiant and M.S. Paterson. Deterministic one-counter automata. *Journal of Computer and System Sciences 10*, pages 340–350, 1975.