L(A) = L(B)?

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Abstract. The equivalence problem for deterministic pushdown automata is shown to be decidable. We exhibit a *complete formal system* for deducing equivalent pairs of deterministic rational series on the alphabet associated with a dpda $\mathcal{M}$.

Keywords: deterministic pushdown automata; rational series; finite dimensional vector spaces; matrix semi-groups; complete formal systems.

1 Introduction

The so-called "equivalence problem for deterministic pushdown automata" (dpda for short), is the following decision problem:

INSTANCE: two dpda A, B.
QUESTION: $\mathrm{L}(A) = \mathrm{L}(B)$?

where $\mathrm{L}(A)$ (resp. $\mathrm{L}(B)$) is the language recognized by A (resp. B). (This problem is often denoted by $Eq(D, D)$, where D stands for the class of all dpda). The question of whether this problem is *decidable* or not is raised in [GG66] and has received much attention since this time. Beside the fact that this question was natural from the point of view of *formal language* theory, it appeared later as Turing-equivalent with other equivalence-problems for different types of recursive *program schemes* (see [Cou90] for a survey). Some other Turing-equivalent problems on *semi-Thue systems* were also found (see [Sén94] for a survey) and formulations in terms of bisimulation equivalence of infinite *graphs* (or *processes*) have been found too (see [Cau95] for a survey).

Among a large number of papers let us only quote [Val74, VP75, Bee76, Rom85, Oya87, Sti96] which proved decidability of $Eq(D', D')$ for subclasses D' of the full class D of dpda. (We refer the reader to the surveys ([Lis96, Cou90, Cau95]) for other results on problems related to $Eq(D, D)$). The work [Mei89, Mei92] is an attempt to solve the general problem. On account of its incompleteness (see for example the comment in [Lis96, p.219]) it does not provide a full solution; nevertheless it introduced a fundamental new idea: the notion of *linear independance* for languages.

We prove here that the equivalence problem for dpda is *decidable* (theorem 9.3).

We obtain this result by providing a *complete* formal system $\mathcal{D}_0$ for equivalence identities between *deterministic rational series* (we use here a type of formal system inspired by [Cou83] and a notion of deterministic series inspired by [HHY79]). The proof of this completeness property leans on two types of arguments:

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– in section 3 we develop around the fundamental idea of [Mei89, Mei92] an *algebraic* theory of "d-spaces" and in sections 5,7 we turn these structure results into a construction of *strategies* (for the formal system $\mathcal{D}_0$).
– in section 8 we analyze the *infinite trees* generated by some strategies associated with $\mathcal{D}_0$. This analysis has similarities with the analysis of derivations of *pairs* of configurations which are customary in the works using Valiant's methods of alternate or parallel stacking (see for example [Rom85, Oya87]).

Table of Contents

2 Preliminaries

2.1 Pushdown automata

A *pushdown automaton* on the alphabet X is a 6-tuple $\mathcal{M} = < X, Z, Q, \delta, q_0, z_0 >$ where Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol and δ, the transition function, is a mapping $\delta : QZ \times (X \cup \{\epsilon\}) \rightarrow \mathcal{P}_f(QZ^*)$.

Let $q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, f \in X^*$ and $a \in X \cup \{\epsilon\}$; we note $(qz\omega, af) \longmapsto_{\mathcal{M}} (q'\omega'\omega, f)$ if $q'\omega' \in \delta(qz, a)$. $\overset{*}{\longmapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{M}}$.

For every $q\omega, q'\omega' \in QZ^*$ and $f \in X^*$, we note $q\omega \xrightarrow{f}_{\mathcal{M}} q'\omega'$ iff $(q\omega, f) \overset{*}{\longmapsto}_{\mathcal{M}} (q'\omega', \epsilon)$.

$\mathcal{M}$ is said *deterministic* iff, for every $z \in Z, q \in Q$:

$$\text{either } \mathrm{Card}(\delta(zq, \epsilon)) = 1 \text{ and for every } x \in X, \mathrm{Card}(\delta(zq, x)) = 0, \qquad (1)$$

$$\text{or } \mathrm{Card}(\delta(zq, \epsilon)) = 0 \text{ and for every } x \in X, \mathrm{Card}(\delta(zq, x)) \leq 1. \qquad (2)$$

$\mathcal{M}$ is said *real-time* iff, for every $qz \in QZ$, $\mathrm{Card}(\delta(qz, \epsilon)) = 0$. A dpda $\mathcal{M}$ is said *normalized* iff, for every $qz \in QZ, x \in X$:

$$q'\omega' \in \delta(qz, x) \Rightarrow \mid \omega' \mid \leq 2, \qquad (3)$$

$$q'\omega' \in \delta(qz, \epsilon) \Rightarrow \mid \omega' \mid = 0 \qquad (4)$$

Given some finite set $F \subseteq QZ^*$ of configurations, the *language recognized by $\mathcal{M}$ with final configurations F* is

$$\mathrm{L}(\mathcal{M}, F) = \{w \in X^* \mid \exists c \in F, q_0 z_0 \xrightarrow{w}_{\mathcal{M}} c\}.$$

It is a "folklore" result that , given a dpda $\mathcal{M}$ and a finite set of configurations F, one can effectively compute another dpda $\mathcal{M}'$ which is normalized and a finite set of configurations F' such that:

$$\mathrm{L}(\mathcal{M}, F) = \mathrm{L}(\mathcal{M}', F').$$

2.2 Deterministic context-free grammars

Let $\mathcal{M}$ be some deterministic pushdown automaton (for sake of simplicity, we suppose here that $\mathcal{M}$ is normalized). The *variable* alphabet $V_{\mathcal{M}}$ associated to $\mathcal{M}$ is defined as:

$$V_{\mathcal{M}} = \{[p, z, q] \mid p, q \in Q, z \in Z\}$$

The *context-free* grammar $G_{\mathcal{M}}$ associated to $\mathcal{M}$ is then

$$G_{\mathcal{M}} =< X, V, P >$$

where
$V = V_{\mathcal{M}}$
P is the set of all the pairs of one of the following forms:

$$([p, z, q], x[p', z_1, p''][p'', z_2, q]) \tag{5}$$

where $p, q, p', p'' \in Q, x \in X, p'z_1z_2 \in \delta(pz, x)$

$$([p, z, q], x[p', z', q]) \tag{6}$$

where $p, q, p' \in Q, x \in X, p'z' \in \delta(pz, x)$

$$([p, z, q], a) \tag{7}$$

where $p, q, \in Q, a \in X \cup \{\epsilon\}, q \in \delta(pz, a)$. $G_{\mathcal{M}}$ is a *strict-deterministic* grammar. (A general theory of this class of grammars is exposed in [Har78] and used in [HHY79]).

We call *mode* every element of $QZ \cup \{\epsilon\}$. For every $q \in Q, z \in Z$, qz is said ϵ-*bound* (respectively ϵ-free) iff condition (1) (resp. condition (2)) in the above definition of deterministic automata is realized. The mode ϵ is said ϵ-free. We define a mapping $\mu : V^* \to QZ \cup \{\epsilon\}$ by

$$\mu(\epsilon) = \epsilon \text{ and } \mu([p, z, q] \cdot \beta) = pz,$$

for every $p, q \in Q, z \in Z, \beta \in V^*$. For every $w \in V^*$ we call $\mu(w)$ the *mode* of the word w.

For technical reasons (which will be made clear in section 7), we suppose that Z contains a special symbol e such that, for every $q \in Q$, $\delta(qe, \epsilon) = \{q\}$ and $\mathrm{im}(\delta) \subseteq \mathcal{P}_f(Q(Z - \{e\})^*)$. Equivalently,

$$\forall q \in Q, ([q, e, q], \epsilon) \in P, \tag{8}$$

and

$$\forall (v, w) \in P, w \in (V - \{[q, e, q] | q \in Q\})^*. \tag{9}$$

2.3 Free monoids acting on semi-rings

Semi-ring $\mathsf{B} < W >$ Let $(\mathsf{B}, +, \cdot, 0, 1)$ where $\mathsf{B} = \{0, 1\}$ denote the semi-ring of "booleans". Let W be some alphabet. By $(\mathsf{B} < W >, +, \cdot, \emptyset, \epsilon)$ we denote the semi-ring of *boolean series* over W:
the set $\mathsf{B} < W >$ is defined as B^{W^*}; the sum and product are defined as usual; each word $w \in W^*$ can be identified with the element of B^{W^*} mapping the word w on 1 and every other word $w' \neq w$ on 0; every boolean series $S \in \mathsf{B} < W >$ can then be written in a unique way as:

$$S = \Sigma_{w \in W^*} S_w \cdot w,$$

where, for every $w \in W^*$, $S_w \in \mathsf{B}$.
The *support* of S is the language

$$\mathrm{supp}(S) = \{w \in W^* \mid S_w \neq 0\}.$$

In the particular case where the semi-ring of coefficients is B (which is the only case considered in this article) we sometimes identify the series S with its support. The usual ordering $\leq$ on B extends to $\mathsf{B} < W >$ by:

$$S \leq S' \text{ iff } \forall w \in W^*, S_w \leq S'_w.$$

A distance $\mathrm{d} : \mathsf{B} < W > \times \mathsf{B} < W > \longrightarrow \mathbb{R}_+$ is defined by: for every $S, S' \in \mathsf{B} < W >$

$$\mathrm{d}(S, S') = 2^{-\min\{|w| \mid S_w \neq S'_w\}}$$

where it is understood that $min(\emptyset) = \infty$ and $2^{-\infty} = 0$.

The operations $+$ and $\cdot$ are *continuous* for the topology $\mathcal{T}_{\mathrm{d}}$ associated to the distance d. Due to the fact that $\mathcal{T}_{\mathrm{d}}$ is defined by a distance, the continuity property of the operations $+, \cdot$ are equivalent to the following: for every converging sequences $(S^{(i)})_{i \in \mathbb{N}}$, $(T^{(i)})_{i \in \mathbb{N}}$, if $S = \lim(S^{(i)})$ and $T = \lim(T^{(i)})$ then

$$S + T = \lim(S^{(i)} + T^{(i)}) \text{ and } S \cdot T = \lim(S^{(i)} \cdot T^{(i)}). \tag{10}$$

We call *topological* semi-ring (t-semi-ring in short) any semi-ring $(\mathsf{S}, +, \cdot, 0, 1)$ endowed with a topology $\mathcal{T}$ such that the operations $\cdot, +$ are both continuous. (Hence $(\mathsf{B} < W >, +, \cdot, \emptyset, \epsilon, \mathcal{T}_{\mathrm{d}})$ is a topological semi-ring). We recall that for every $S \in \mathsf{B} < W >$, S^* is the series defined by:

$$S^* = \sum_{0 \leq n} S^n = \lim(\sum_{0 \leq n \leq i} S^n). \tag{11}$$

One can notice that, if $\epsilon \not\leq S$ then

$$\lim S^n = \emptyset \tag{12}$$

(In fact the topological structure of $\mathsf{B} < W >$ is not essential in this work; we just use it in axiom (D6) of d-universes, in section 3.2. One could alternatively use a similar axiom dealing with the star operation).

Given two alphabets W, W', a map $\psi : \mathsf{B} < W > \rightarrow \mathsf{B} < W' >$ is said *σ-additive* iff it fulfills: for every denumerable family $(S_i)_{i \in \mathbb{N}}$ of elements of $\mathsf{B} < W >$,

$$\psi(\sum_{i \in \mathbb{N}} S_i) = \sum_{i \in \mathbb{N}} \psi(S_i). \tag{13}$$

A map $\psi : \mathsf{B} < W > \rightarrow \mathsf{B} < W' >$ which is both a semi-ring homomorphism and a σ-additive map is usually called a *substitution*.

Actions of monoids Given a t-semi-ring $(\mathsf{S}, +, \cdot, 0, 1, \mathcal{T})$ and a monoid $(M, \cdot, 1_M)$, a map $\circ : \mathsf{S} \times M \rightarrow \mathsf{S}$ is called a *topological right-action* of the monoid M over the t-semi-ring S iff, for every $S, T \in \mathsf{S}, m, m' \in M$:

$$0 \circ m = 0 \tag{14}$$

$$S \circ 1_M = S \tag{15}$$

$$(S + T) \circ m = (S \circ m) + (T \circ m) \tag{16}$$

$$S \circ (m \cdot m') = (S \circ m) \circ m' \tag{17}$$

$$\text{the mapping } U \mapsto U \circ m \text{ is continuous.} \tag{18}$$

When $\circ$ fulfills only properties (14,15,16,17) it is just called a *right-action* of the monoid M over the semi-ring S.

The action of W^* on $\mathsf{B} < W >$ We recall the following classical topological right-action $\bullet$ of the monoid W^* over the t-semi-ring $\mathsf{B} < W >$: for all $S, S' \in \mathsf{B} < W >, u \in W^*$

$$S \bullet u = S' \Leftrightarrow \forall w \in W^*, (S'_w = 1 \text{ iff } S_{u \cdot w} = 1).$$

(i.e. $S \bullet u$ is the *left-quotient* of S by u , or the *residual* of S by u).

For every $S \in \mathsf{B} < W >$ we denote by $\mathsf{Q}(S)$ the set of residuals of S:

$$\mathsf{Q}(S) = \{S \bullet u \mid u \in W^*\}.$$

We recall that S is said *rational* iff the set $\mathsf{Q}(S)$ is *finite*. We define the *norm* of a series $S \in \mathsf{B} < W >$, denoted $||S||$ by:

$$||S|| = \mathrm{Card}(\mathsf{Q}(S)) \in \mathbb{N} \cup \{\infty\}.$$

The action of X^* on $\mathsf{B} < V >$ Let us fix now a deterministic (normalized) pda $\mathcal{M}$ and consider the associated grammar G. We define a topological right-action $\otimes$ of the monoid $(X \cup \{e\})^*$ over the t-semi-ring $\mathsf{B} < V >$ by: for every $p, q \in Q, A \in Z, H \in V^*, \beta \in V^*, x \in X, a \in X \cup \{\epsilon\}$

$$[p, A, q] \cdot \beta \otimes x = H \cdot \beta \text{ iff } ([p, A, q], x \cdot H) \in P \tag{19}$$

$$[p, A, q] \cdot \beta \otimes e = H \cdot \beta \text{ iff } ([p, A, q], H) \in P \tag{20}$$

$$\epsilon \otimes x = \emptyset \tag{21}$$

$$\epsilon \otimes e = \emptyset \tag{22}$$

A series $S \in \mathsf{B} < V >$ is said *ϵ-free* iff

$$\forall w \in V^*, S_w = 1 \Rightarrow \mu(w) \text{ is } \epsilon - \text{free}.$$

We denote by $\mathsf{B}_\epsilon < V >$ the subset of ϵ-free series. We define the map $\rho_\epsilon : \mathsf{B} < V > \rightarrow \mathsf{B} < V >$ as the unique σ-additive mapping such that, for every $p \in Q, z \in Z, q \in Q, \beta \in V^*$,

$$\rho_\epsilon([p, z, q] \cdot \beta) = \rho_\epsilon(([p, z, q] \otimes e) \cdot \beta) \text{ if } pz \text{ is } \epsilon - \text{bound},$$

$$\rho_\epsilon([p, z, q] \cdot \beta) = [p, z, q] \cdot \beta \text{ if } pz \text{ is } \epsilon - \text{free, and}$$

$$\rho_\epsilon(\epsilon) = \epsilon.$$

The above definition is sound because, by hypothesis (4), every $[p, z, q] \otimes e$ is either the unit series ϵ or the empty series $\emptyset$. One can notice that for every $w \in V^*$, $\rho_\epsilon(w) \in V^* \cup \{\emptyset\}$. We call ρ_ϵ the ϵ-reduction map. We then define $\odot$ as the unique right-action of the monoid X^* over the semi-ring $\mathsf{B} < V >$ such that: for every $S \in \mathsf{B} < V >, x \in X$

$$S \odot x = \rho_\epsilon(\rho_\epsilon(S) \otimes x).$$

One can notice that if $u \neq \epsilon$, then $S \odot u$ is ϵ-free. Let us remark that $\otimes$ is continuous but $\odot$ is not continuous in general. Let us consider the unique substitution $\varphi : \mathsf{B} < V > \rightarrow \mathsf{B} < X >$ fulfilling: for every $p, q \in Q, z \in Z$,

$$\varphi([p, z, q]) = \{u \in X^* \mid [p, z, q] \odot u = \epsilon\},$$

(in other words, φ maps every subset $L \subseteq V^*$ on the language generated by the grammar G from the set of axioms L).

Lemma 2.1 *φ is a morphism of right-actions i.e. for every $S \in \mathsf{B} < V >, u \in X^*$, $\varphi(S \odot u) = \varphi(S) \bullet u$.*

Proof: Obvious and well-known. □

We denote by $\equiv$ the kernel of φ i.e.: for every $S, T \in \mathsf{B} < V >$,

$$S \equiv T \Leftrightarrow \varphi(S) = \varphi(T).$$

For every integer n, we denote by $\equiv_n$ the following approximation of $\equiv$:

$$S \equiv_n T \Leftrightarrow \varphi(S) \cap X^{\leq n} = \varphi(T) \cap X^{\leq n},$$

and we denote by $\ll_n$ the following preoder:

$$S \ll_n T \Leftrightarrow \varphi(S) \cap X^{\leq n} \subseteq \varphi(T) \cap X^{\leq n}.$$

3 Series and languages

3.1 Deterministic series and matrices

We introduce here a notion of *deterministic* series which, in the case of the alphabet V associated to a dpda $\mathcal{M}$, generalizes the classical notion of *configuration* of $\mathcal{M}$. The main advantage of this notion is that, unlike for configurations, we shall be able to define *nice algebraic operations* on these series (this is done in section 3.2). Let us consider a pair $(W, \sim)$ where W is an alphabet and $\sim$ is an equivalence relation over W. We call $(W, \sim)$ a *structured* alphabet. The two examples we have in mind are:

- the case where $W = V$, the variable alphabet associated to $\mathcal{M}$ and $[p, A, q] \sim [p', A', q']$ iff $p = p'$ and $A = A'$ (see [Har78])
- the case where $W = X$, the terminal alphabet of $\mathcal{M}$ and $x \sim y$ holds for every $x, y \in X$ (see [Har78]).

Definitions

Definition 3.1 *Let $S \in \mathsf{B} < W >$. S is said* left-deterministic *iff either*

(1) $S = \emptyset$ *or*
(2) $S = \epsilon$ *or*
(3) $\forall w, w' \in W^*$,

$$S_w = S_{w'} = 1 \Rightarrow \exists A, A' \in W, w_1, w_1' \in W^*, A \sim A', w = A \cdot w_1 \text{ and } w' = A' \cdot w_1'.$$

A left-deterministic series S is said to have the type $\emptyset$ (resp. ϵ, $[A]_\sim$) if case (1) (resp. (2), (3)) occurs.

Definition 3.2 *Let $S \in \mathsf{B} < W >$. S is said* deterministic *iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.*

This notion is the straighforward extension to the infinite case of the notion of (finite) *set of associates* defined in [HHY79].
We denote by $\mathsf{DB} < W >$ the subset of deterministic boolean series over W. Let us denote by $\mathsf{B}_{n,m} < W >$ the set of (n, m)-matrices with entries in the semi-ring $\mathsf{B} < W >$.

Definition 3.3 *Let $m \in \mathbb{N}, S \in \mathsf{B}_{1,m} < W >$: $S = (S_1, \cdots, S_m)$. S is said* left-deterministic *iff either*

(1) $\forall i \in [1, m], S_i = \emptyset$ *or*
(2) $\exists i_0 \in [1, m], S_{i_0} = \epsilon$ *and* $\forall i \neq i_0, S_i = \emptyset$ *or*
(3) $\forall w, w' \in W^*, \forall i, j \in [1, m], (S_i)_w = (S_j)_{w'} = 1 \Rightarrow \exists A, A' \in W, w_1, w_1' \in V^*$, $A \sim A', w = A \cdot w_1$ *and* $w' = A' \cdot w_1'$.

A left-deterministic row-vector S is said to have the type $\emptyset$ (resp. (ϵ, i_0), $[A]_\sim$) if case (1) (resp. (2), (3)) occurs.

The right-action $\bullet$ on $\mathsf{B} < W >$ is extended componentwise to $\mathsf{B}_{n,m} < W >$: for every $S = (s_{i,j})$, $u \in W^*$, the matrix $T = S \bullet u$ is defined by

$$t_{i,j} = s_{i,j} \bullet u.$$

The ordering $\leq$ on B is also extended componentwise to $\mathsf{B}_{n,m} < W >$.

Definition 3.4 *Let $S \in \mathsf{B}_{1,m} < W >$. S is said* deterministic *iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.*

We denote by $\mathsf{DB}_{1,m} < W >$ the subset of deterministic row-vectors of dimension m over $\mathsf{B} < W >$.

Definition 3.5 *Let $S \in \mathsf{B}_{n,m} < W >$. S is said* deterministic *iff, for every $i \in [1, n]$, $S_{i,.}$ is a deterministic row-vector.*

Let us notice first some easy facts about deterministic matrices.

Fact 3.6 *Let $S \in \mathsf{DB} < W >$. For every $T \in \mathsf{B} < W >, u \in W^*$*

(1) $T \leq S \Rightarrow T \in \mathsf{DB} < W >$
(2) $T = S \bullet u \Rightarrow T \in \mathsf{DB} < W >$

Residuals

Lemma 3.7 *Let $S \in \mathsf{DB} < W >, T \in \mathsf{B} < W >, u \in W^*$. If $S \bullet u \neq \emptyset$ then $(S \cdot T) \bullet u = (S \bullet u) \cdot T$.*

Proof: Let $S \in \mathsf{DB} < W >, T \in \mathsf{B} < W >, u \in W^*$, such that $S \bullet u \neq \emptyset$. Let $u', u'' \in W^*$ such that $u = u' \cdot u''$ and let $w \in \mathrm{supp}(S)$. If $w \bullet u' = \epsilon$ then $S \bullet u' = \epsilon$ (because $S \bullet u'$ is left-deterministic), hence $S \bullet u = \epsilon \bullet u'' = \emptyset$, which would contradict the hypothesis. It follows that

$$\forall u' \prec u, \forall w \in \mathrm{supp}(S), w \bullet u' \neq \epsilon.$$

Hence for every $w_1 \in \mathrm{supp}(S), w_2 \in \mathrm{supp}(T)$

$$(w_1 \cdot w_2) \bullet u \neq \emptyset \Rightarrow (w_1 \cdot w_2) \bullet u = (w_1 \bullet u) \cdot w_2.$$

This proves that $(S \cdot T) \bullet u = (S \bullet u) \cdot T$. $\square$

Lemma 3.8 *Let $S \in \mathsf{DB} < W >, T \in \mathsf{B} < W >, u \in W^*$ and $U = S \cdot T$. Exactly one of the following cases is true:*

(1) $S \bullet u \neq \emptyset$;
in this case $U \bullet u = (S \bullet u) \cdot T$.

(2) $S \bullet u = \emptyset, \exists u', u'', u = u' \cdot u'', S \bullet u' = \epsilon$;
in this case $U \bullet u = T \bullet u''$.
(3) $S \bullet u = \emptyset, \forall u' \preceq u, S \bullet u' \neq \epsilon$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Proof: Clearly, one of the hypotheses (1-3) must occur. Let us examine each of these cases.
In case (1), by lemma 3.7, $U \bullet u = (S \bullet u) \cdot T$.
In case (2), $U \bullet u = (U \bullet u') \bullet u''$ and by case (1), $U \bullet u' = (S \bullet u') \cdot T$. It follows that $U \bullet u = T \bullet u''$.
In case (3), if $S = \emptyset$, the conclusion of the lemma is clearly true. Let us suppose now that $S \neq \emptyset$ and let $u' \prec u$ be the maximum prefix of u such that $S \bullet u' \neq \emptyset$. Then, there exist some $A \in W, u'' \in W^*$such that $u = u' \cdot A \cdot u''$ and there exist some $B_1, \cdots, B_q \in W, S_1, \cdots, S_q \in \mathsf{B} < W >$ such that $S \bullet u' = \sum_{1 \leq i \leq q} B_q \cdot S_q$ and $B_1 \sim \cdots \sim B_i \sim \cdots \sim B_q$(because $S \bullet u'$ is left-deterministic). By maximality of u', A does not belong to $\{B_1, \cdots, B_q\}$, hence

$$S \bullet u = ((\sum_{1 \leq i \leq q} B_q \cdot S_q) \bullet A) \bullet u'' = \emptyset \bullet u'' = \emptyset.$$

□

Lemma 3.9 *Let* $S \in \mathsf{DB}_{1,m} < W >, T \in \mathsf{B}_{m,1} < W >, u \in W^*$ *and* $U = S \cdot T$. *Exactly one of the following cases is true:*

(1) $\exists j, S_j \bullet u \neq \emptyset$ *and* $\forall j (S_j \bullet u \neq \emptyset$ *or* $\forall u' \prec u, S_j \bullet u \neq \epsilon)$;
in this case $U \bullet u = (S \bullet u) \cdot T$.
(2) $\exists j_0, S_{j_0} \bullet u = \emptyset, \exists u', u'', u = u' \cdot u'', S_{j_0} \bullet u' = \epsilon$;
in this case $U \bullet u = T_{j_0} \bullet u''$.
(3) $\forall j, \forall u' \preceq u, S_j \bullet u' = \emptyset$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Proof: Let us note $S = (S_j)_{1 \leq j \leq m}, T = (T_j)_{1 \leq j \leq m}$. Clearly, one of the hypotheses (1-3) must occur. Let us examine each of these cases.
In case (1), every 3-tuple (S_j, T_j, u) fulfills case (1) or (3) of lemma 3.8, hence $(S_j \cdot T_j) \bullet u = (S_j \bullet u) \cdot T_j$. Hence

$$U \bullet u = \sum_{1 \leq j \leq m} (S_j \cdot T_j) \bullet u = \sum_{1 \leq j \leq m} (S_j \bullet u) \cdot T_j = (S \bullet u) \cdot T.$$

In case (2), $S \bullet u'$ must be left-deterministic of type (ϵ, j_0) , hence $\forall j \neq j_0, S_j \bullet u' = \emptyset$. It follows that

$$U \bullet u = T_{j_0} \bullet u''.$$

In case (3), every 3-tuple (S_j, T_j, u) fulfills case (3) of lemma 3.8, hence $(S_j \cdot T_j) \bullet u = \emptyset = (S_j \bullet u) \cdot T_j$. It follows that

$$U \bullet u = \emptyset = (S \bullet u) \cdot T.$$

□

Lemma 3.10 *For every $S \in \mathsf{DB}_{n,m} < W >, T \in \mathsf{DB}_{m,s} < W >, S \cdot T \in \mathsf{DB}_{n,s} < W >$.*

Proof: As the notion of deterministic matrix is defined row by row, it is sufficient to prove this lemma in the particular case where $n = 1$. Using the same notations as in the proof of lemma 3.9, we have:

$$U_k = \sum_{1 \le j \le m} S_j \cdot T_{j,k} \text{ for every } k \in [1, s]. \tag{23}$$

Let $u \in W^*$. Let us show that $U \bullet u$ is left-deterministic. Let us notice that the hypothesis of the 3 cases considered in lemma 3.9 depends on the vector S and the word u only (and not on the integer k). Let us examine every one of these cases.
In case (1)or (3),

$$U_k \bullet u = \sum_{1 \le j \le m} (S_j \bullet u) \cdot T_{j,k} \text{ for every } k \in [1, s],$$

which proves that $U \bullet u = (S \bullet u) \cdot T$. In case (2),

$$U_k \bullet u = T_{j_0,k} \bullet u'' \text{ for every } k \in [1, s],$$

which proves that $U \bullet u = T \bullet u''$. In both cases, $U_k \bullet u$ is left-deterministic. $\square$

W=V Let $(W, \sim)$ be the structured alphabet $(V, \sim)$ associated with $\mathcal{M}$ and let us consider a bijective numbering of the elements of Q: $(q_1, q_2, \ldots, q_{n_Q})$. Some particular "vectorial" notions turn out to be useful:

- we define a *Q-series* to be a family $(S_q)_{q \in Q}$ such that the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ is deterministic
- we define a *Q-form* to be a family $\Phi = (\Phi_q)_{q \in Q}$ of deterministic series.

Given a Q-series S and a Q-form Φ, their Q-product $S * \Phi$ is the deterministic series defined by

$$S * \Phi = \sum_{q \in Q} S_q \cdot \Phi_q.$$

Given the above ordering of the elements of Q, one can identify the Q-series $(S_q)_{q \in Q}$ with the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ and the Q-form $(\Phi_q)_{q \in Q}$ with the column-vector $(\Phi_{q_j})_{j \in [1,n_Q]}$ The Q-product appears then to be just the ordinary product of matrices.

Let us define here handful notations for some particular row-vectors or Q-series. Let us use the *Kronecker symbol* $\delta_{i,j}$ meaning ϵ if $i = j$ and $\emptyset$ if $i \ne j$. For every $1 \le n, 1 \le i \le n$, we define the row-vector ϵ_i^n as:

$$\epsilon_i^n = (\epsilon_{i,j}^n)_{1 \le j \le n} \text{ where } \forall j, \epsilon_{i,j}^n = \delta_{i,j}.$$

We call *unit row-vector* any vector of the form ϵ_i^n.
For every $\omega \in Z^*, p, q \in Q$, $[p\omega q]$ is the deterministic series defined inductively by:

$$[p\epsilon q] = \emptyset \text{ if } p \neq q, [p\epsilon q] = \epsilon \text{ if } p = q,$$

$$[p\omega q] = \sum_{r \in Q} [pAr] \cdot [r\omega' q] \text{ if } \omega = A \cdot \omega' \text{ for some } A \in Z, \omega' \in Z^*.$$

By $[p\omega]$ we denote the Q-series:

$$[p\omega] = ([p\omega q])_{q \in Q}.$$

(In particular $[q_i] = \epsilon_i^{n_Q}$). These Q-series represent faithfully the configurations of $\mathcal{M}$ in the sense that, for every ϵ-free configurations $q\omega, q'\omega'$ and word $f \in X^*$,

$$q\omega \xrightarrow{f}_{\mathcal{M}} q'\omega' \text{ iff } [q\omega] \odot f = [q'\omega']. \tag{24}$$

By $[\omega]$ we denote the Q-matrix:

$$[\omega] = (\Omega_{p,q})_{p \in Q, q \in Q} \text{ such that for every } p, q, \Omega_{p,q} = [p\omega q].$$

Here also we identify $[\omega]$ with the matrix $([p_i \omega q_j])_{1 \leq i \leq n_Q, 1 \leq j \leq n_Q} \in \mathsf{DB}_{n_Q, n_Q} < V >$. The next lemmas relate the mapping ρ_ϵ and right-action $\odot$ with the right-action $\bullet$.

Lemma 3.11 *Let* $S \in \mathsf{DB} < V >$.

1. *there exists* $v \in V^*$ *such that* $\rho_\epsilon(S) = S \bullet v$.
2. $\rho_\epsilon(S) \equiv S$.

Proof: If for every $w \in \mathrm{supp}(S), \rho_\epsilon(w) = \emptyset$, then $\rho_\epsilon(S) = \emptyset$, which is a residual of S, hence point (1) of the lemma is true. Moreover, in this case every $w \in \mathrm{supp}(S)$ contains a letter $[p, z, q]$ which is ϵ-bound and such that $[p, z, q] \otimes e = \emptyset$, hence $S \equiv \emptyset = \rho_\epsilon(S)$, which establishes point (2) of the lemma.
Let us suppose now that there exists some $w_0 \in \mathrm{supp}(S)$ such that $\rho_\epsilon(w_0) = w' \in V^*$. Then $w_0 = [p_1, z_1, q_1] \cdots [p_n, z_n, q_n] \cdot w'$, where $n \geq 0$ anf for every $i \in [1, n]$, $[p_i, z_i, q_i] \otimes e = \epsilon$. Let us set $v = [p_1, z_1, q_1] \cdots [p_n, z_n, q_n]$. We consider the set of words: $D(v) = \{v' \cdot [p_{j+1}, z_{j+1}, q'_{j+1}], 0 \leq j \leq n-1, q'_{j+1} \in Q, q'_{j+1} \neq q_{j+1}\}$, and we set $S' = \sum_{w \in D(v)} w \cdot (S \bullet w)$. It is clear that

$$S = v \cdot (S \bullet v) + S'. \tag{25}$$

Moreover one can check that, for every $w \in D(v)$, $\rho_\epsilon(w) = \emptyset$ (because the letters $[p_{j+1}, z_{j+1}, q'_{j+1}]$ fulfill $\rho_\epsilon([p_{j+1}, z_{j+1}, q'_{j+1}]) = \emptyset$). Hence $\rho_\epsilon(S') = \emptyset$. As S is deterministic, $S \bullet u$ must be left-deterministic of the same type as w', hence $S \bullet u$ is ϵ-free. Using now(25) we obtain:

$$\rho_\epsilon(S) = S \bullet v + \emptyset = S \bullet v.$$

Point (1) is then proved. Applying φ to the two members of equation (25) and using point (1) we obtain that

$$\varphi(S) = \varphi(v \cdot \rho_\epsilon(S)) = \varphi(v) \cdot \varphi(\rho_\epsilon(S)).$$

But, by the hypothesis on the letters $[p_i, z_i, q_i]$, $\varphi(v) = \epsilon$. It follows that $\varphi(S) = \varphi(\rho_\epsilon(S))$ i.e. point (2) is true. □

Corollary 3.12 $\forall S \in \mathsf{DB} < V >, \rho_\epsilon(S) \in \mathsf{DB} < V >$.

Lemma 3.13 *Let $S \in \mathsf{DB} < V >, u \in X^*$. One of the three following cases must occur:*

(1) $S \odot u = \emptyset$,
(2) $S \odot u = \epsilon$,
(3) $\exists u_1, u_2 \in X^*, v_1 \in V^*, q \in Q, A \in Z, \Phi$ *$Q-$form such that*

$$u = u_1 \cdot u_2, S \odot u_1 = S \bullet v_1 = [qA] * \Phi \text{ and } S \odot u = ([qA] \odot u_2) * \Phi.$$

Proof: Easy. □

Corollary 3.14 $\forall S \in \mathsf{DB} < V >, u \in X^*, S \odot u \in \mathsf{DB} < V >$.
$\forall S \in \mathsf{DRB} < V >, u \in X^*, S \odot u \in \mathsf{DRB} < V >$.

Proof: Let us consider case (3) of the above lemma. Due to the form of the rules generating the right-action $\otimes$ (see section 2.3), $[qA] \odot u_2$ is of the form $[q'\omega']$ for some $q' \in Q, \omega' \in Z^*$. Hence $S \odot u$ is the Q-product of a Q-series by a Q-form, which is a deterministic series by lemma 3.10. □
The particular letters $[p, e, q]$ for $p, q \in Q$ play a special role in sections 7 and section 8: we use them as *marks* in the series (somehow like the ceilings of [Val74]). We define below a map ρ_e which removes the marks in the series. Let us define $\rho_e : \mathsf{DB} < V > \rightarrow \mathsf{B} < V >$ as the unique substitution such that:

$$\rho_e([p, e, q]) = \epsilon \quad \text{if } p = q, \quad \rho_e([p, e, q]) = \emptyset \quad \text{if } p \neq q.$$

We note $\bar{V}_e = \{[p, e, q] \,|\, p, q, \in Q\}, V_e = V - \bar{V}_e$. A deterministic series $S \in \mathsf{DB} < V >$ is said e-free iff its type is $(\emptyset)$ or (ϵ) or$([pA])$, with $A \neq e$.

Lemma 3.15 *For every $S \in \mathsf{DB} < V >$*

(1) $\rho_e(S) \in \mathsf{DB} < V >$
(2) $\|\rho_e(S)\| \leq \|S\|$.

Sketch of proof: We establish first that, for every $u \in V_e^*, \exists u' \in V^*$ such that

$$\rho_e(S) \bullet u = \rho_e(S \bullet u') \text{ and } S \bullet u' \text{ is } e\text{-free}. \tag{26}$$

Let us prove (26) by induction on $|u|$.
$|u|$**=0**: Either $S \in \{\emptyset, \epsilon\}$ and (26) is clearly true, or, using the determinism of S, one can show that there exists a maximal integer n such that:

$$\exists (p_i)_{1 \leq i \leq n} \in Q^n, S \bullet ([p_1, e, p_1] \cdots [p_n, e, p_n]) \neq \emptyset.$$

Then $u' = [p_1, e, p_1] \cdots [p_n, e, p_n]$ satisfies (26).
$|u|$**=m+1**: $u = u_1 \cdot v_1$ where $u_1 \in V_e^*, |u_1| = m, v_1 \in V_e$.
By induction hypothesis there exists $u_1' \in V^*$ such that

$$\rho_e(S) \bullet u_1 = \rho_e(S \bullet u_1') \text{ and } S \bullet u_1' \text{ is } e\text{-free}.$$

If $S \bullet u_1' \in \{\emptyset, \epsilon\}$, then $u' = u_1' v_1$ satisfies (26). Otherwise let $([pA])$ be the type of $S \bullet u_1'$ $(p \in Q, A \neq e)$. We then have

$$\rho_e(S) \bullet u_1 = \rho_e(S \bullet u_1') \tag{27}$$

$$S \bullet u_1' = [pA] * \Phi \tag{28}$$

for some Q-form Φ. If $v_1 = [p, A, q_1]$ (for some $q_1 \in Q$), then let us consider the series Φ_{q_1}: by induction hypothesis, there exists some $w_1' \in V^*$ such that

$$\rho_e(\Phi_{q_1}) \bullet \epsilon = \rho_e(\Phi_{q_1} \bullet w_1') \text{ and } \Phi_{q_1} \bullet w_1' \text{ is } e\text{-free}. \tag{29}$$

Combining equations (27,28,29) we see that $u' = u_1' \cdot [p, A, q_1] \cdot w_1'$ fulfills:

$$\begin{aligned}
\rho_e(S) \bullet (u_1 \cdot v_1) &= (\rho_e(S) \bullet u_1) \bullet v_1 \\
&= \rho_e(S \bullet u_1') \bullet [p, A, q_1] \\
&= \rho_e([pA] * \Phi) \bullet [p, A, q_1] \\
&= \rho_e(\Phi_{q_1}) \\
&= \rho_e(\Phi_{q_1} \bullet w_1')
\end{aligned}$$

where $S \bullet (u_1' \cdot [p, A, q_1] \cdot w_1') = \Phi_{q_1} \bullet w_1'$ and $\Phi_{q_1} \bullet w_1'$ is e-free. Hence (26) is fulfilled by our choice of u'.
Let us prove the lemma now. By (26) every residual $\rho_e(S) \bullet u$ is left-deterministic of the same type as $S \bullet u'$. Hence $\rho_e(S)$ is deterministic. Moreover formula (26) shows that $\|\rho_e(S)\| \leq \|S\|$. □

3.2 Deterministic spaces

We elaborate here on the key-idea of [Mei89, Mei92].

Definitions Let $(W, \sim)$ be some structured alphabet. We call *deterministic universe* over $\mathsf{DB} < W >$ (d-universe over $\mathsf{DB} < W >$, in short), any 3-tuple $(E, \circ_l, \circ_r)$ such that, E is a set, $\circ_l$ is a map from $\bigcup \mathsf{DB}_{1,m} < W > \times E^m$ to E and $\circ_r$ is a map from $E \times W^*$ to E , such that, their componentwise extensions:

$$\circ_l : \bigcup \mathsf{DB}_{n,m} < W > \times E^m \to E^n \text{ and } \circ_r : E^n \times W^* \to E^n$$

are fulfilling

(D1) $\forall M \in \mathsf{DB}_{n,m} < W >, M' \in \mathsf{DB}_{m,s} < W >, \mathbf{e} \in E^s, (M \cdot M') \circ_l \mathbf{e} = M \circ_l (M' \circ_l \mathbf{e})$
(D2) $\forall j \in [1, m], \forall \mathbf{e} \in E^m, \epsilon_j^m \circ_l \mathbf{e} = e_j$
(D3) $\forall u, u' \in W^*, \forall \mathbf{e} \in E^m, \mathbf{e} \circ_r (u \cdot u') = (\mathbf{e} \circ_r u) \circ_r u'$
(D4) $\forall e \in E, e \circ_r \epsilon = e$
(D5) $\forall S \in \mathsf{DB}_{1,n} < W >, \mathbf{e} \in E^n, u \in W^*$, one of the two following cases occurs:
  (C1) $\forall u' \preceq u$, $S \bullet u'$ is not a unit and then $(S \circ_l \mathbf{e}) \circ_r u = (S \bullet u) \circ_l \mathbf{e}$
  (C2) $\exists u', u'', u' \cdot u'' = u$, $S \bullet u' = \epsilon_j^n$ and then $(S \circ_l \mathbf{e}) \circ_r u = (S \bullet u') \circ_l (\mathbf{e} \circ_r u'')$.
(D6) for every converging sequence $(S^{(i)})_{i \in \mathbb{N}}$ of elements of $\mathsf{DB}_{1,n} < W >$ and every $\mathbf{e}, \mathbf{e}' \in E^n$, $(\forall i \in \mathbb{N}, S^{(i)} \circ_l \mathbf{e} = \mathbf{e}') \Rightarrow \lim(S^{(i)}) \circ_l \mathbf{e} = \mathbf{e}'$.

An element $e' = S \circ_l \mathbf{e}$ where $S \in \mathsf{DB}_{1,n} < W >$, $\mathbf{e} = (e_1, \ldots, e_i, \ldots, e_n)$ is called a *linear combination* of the e_i's.

Remark 3.16 *The lefthand-side of the equation in (D1) makes sense because, by lemma 3.10,* $(M \cdot M')$ *is a deterministic matrix too.*

A pair $(E', \circ_l')$ such that E' is a subset of E, E' is closed under left-action $\circ_l$ and $\circ_l'$ is the restriction of $\circ_l$ to E', is called a *deterministic space* (d-space in short) in the universe $(E, \circ_l, \circ_r)$. More generally, a pair $(E', \circ_l')$ is called a deterministic space over $\mathsf{DB} < W >$ iff there exists some d-universe over $\mathsf{DB} < W >$, $(E, \circ_l, \circ_r)$, such that $(E', \circ_l')$ is a d-space in $(E, \circ_l, \circ_r)$. Given a family of elements of E, $(U_i)_{i \in I}$, we denote by $\mathsf{V}((U_i)_{i \in I})$ the smallest d-space in E containing $\{U_i \mid i \in I\}$. We call it the d-space generated by $(U_i)_{i \in I}$. One can check that $\mathsf{V}((U_i)_{i \in I})$ is simply the set of all the (finite) linear combinations of elements of $\{U_i \mid i \in I\}$.

For every integers n, let us consider the triple $(E_n, \circ_l, \circ_r)$ where $E_n = \mathsf{DB}_{1,n} < W >$, $\circ_l$ is the matrix product and $\circ_r$ is the residual action $\bullet$. Axiom (D1) is fulfilled by E_n (it expresses just the associativity of the matrix product), axioms (D2-4) are easily verified, axiom (D5) is fulfilled as a corollary of lemma 3.9 and axiom (D6) is fulfilled because of the continuity of operations $+$ and $\cdot$ in the t-semi-ring $\mathsf{B} < W >$ (see section 2.3). Hence $(E_n, \circ_l, \circ_r)$ is a particular d-universe (and also d-space) over $\mathsf{DB} < W >$.

Rational series, norm Let us generalize the definition of *rationality* of series in $\mathsf{B} < W >$ to matrices. Given $M \in \mathsf{B}_{n,m} < W >$ we denote by $\mathsf{Q}(M)$ the set of *residuals* of M:

$$\mathsf{Q}(M) = \{M \bullet u \mid u \in W^*\}.$$

Similarly, we denote by $\mathsf{Q}_r(M)$ the set of *row-residuals* of M:

$$\mathsf{Q}_r(M) = \bigcup_{1 \leq i \leq n} \mathsf{Q}(M_{i,*}).$$

M is said *rational* iff the set $\mathsf{Q}(M)$ is finite. One can check that it is equivalent to the property that every coefficient $M_{i,j}$ is rational, or to the property that $\mathsf{Q}_r(M)$ is finite. We denote by $\mathsf{RB}_{n,m} < W >$ (resp. $\mathsf{DRB}_{n,m} < W >$) the set of rational (resp. deterministic, rational) matrices over $\mathsf{B} < W >$. For every $M \in \mathsf{RB}_{n,m} < W >$, we define the norm of M as:

$$||M|| = \mathrm{Card}(\mathsf{Q}_r(M)).$$

Lemma 3.17 *Let* $A \in \mathsf{DB}_{n,m} < W >, B \in \mathsf{DB}_{m,s} < W >$. *Then* $||A \cdot B|| \leq ||A|| + ||B||$.

Proof: Let $A = (a_{i,k}), B = (b_{k,j}), C = A \cdot B, C = (c_{i,j})$. Let $1 \leq i \leq n, H \in \mathsf{Q}(C_{i,*})$. Let $u \in W^*$ such that

$$H = ((\sum_{1 \leq k \leq m} a_{i,k} \cdot b_{k,j}) \bullet u)_{1 \leq j \leq s} = (\sum_{1 \leq k \leq m} (a_{i,k} \cdot b_{k,j}) \bullet u)_{1 \leq j \leq s}.$$

We apply lemma 3.9 to $S = A_{i,*}$ and $T_j = B_{*,j}$ for every $1 \leq j \leq s$. But the case fulfilled by (S, T_j) is independant of j.
If case (1)or (3) of lemma 3.9 is realized then

$$\begin{aligned} H &= (\sum_{1 \leq k \leq m} (a_{i,k} \bullet u) \cdot b_{k,j}))_{1 \leq j \leq s} \\ &= A_{i,*} \cdot B. \end{aligned}$$

If case (2) of lemma 3.9 is realized then

$$H = (b_{k_0,j} \bullet u'')_{1 \leq j \leq s} = B_{k_0,*} \bullet u''$$

The number of residuals H obtained by case (1) is less or equal than $||A||$ and the number obtained by case (2) is less or equal than $||B||$. This proves the inequality. □

Linear independance Following an analogy with classical linear algebra, we develop here a notion of *linear independance* in a d-universe E. The analogue of the field of coefficients here is the set of coefficients $\mathsf{DB} < W >$. This set is a monoid only, and there exists no inverse element for a non-null element S (except for ϵ). Nevertheless, for every non-null $S \in \mathsf{DB} < W >$, there exists $s \in W^*$ such that $S \bullet s = \epsilon$. By virtue of axiom (D5), the right-action of s (by $\circ_r$) will provide a kind of *inverse* to the left-action of S (by $\circ_l$). This idea will lead to an analogue, in the context of d-universes over $\mathsf{DB} < W >$, of the classical notions of linear independance and basis for vector spaces over a field

F.
We need first two technical lemmas about two new operations on row-vectors. Given $A, B \in \mathsf{DB}_{1,m} < W >$ and $1 \leq j_0 \leq m$ we define the vector $C = A \Box_{j_0} B$ as follows:
if $A = (a_1, \ldots, a_j, \ldots, a_m), B = (b_1, \ldots, b_j, \ldots, b_m)$ then $C = (c_1, \ldots, c_j, \ldots, c_m)$ where

$$c_j = a_j + a_{j_0} \cdot b_j \text{ if } j \neq j_0 \ , \ \ c_j = \emptyset \text{ if } j = j_0.$$

Lemma 3.18 *Let $A, B \in \mathsf{DB}_{1,m} < W >$ and $1 \leq j_0 \leq m$.*
Then $A \Box_{j_0} B \in \mathsf{DB}_{1,m} < W >$ and $\|A \Box_{j_0} B\| \leq \|A\| + \|B\|$.

Proof:
Let $C = A \Box_{j_0} B$.
1 Let us prove first that if A, B are both left-deterministic, then C is left-deterministic too.
If A is left-deterministic of type $[pz]$, then C is left-deterministic of the same type.
If A is left-deterministic of type (ϵ, j_1) with $j_1 \neq j_0$, then $C = A$, hence C is left-deterministic.
If A is left-deterministic of type (ϵ, j_0), then $C \leq B$, hence C is left-deterministic.
If A is left-deterministic of type $(\emptyset)$, then $C = \emptyset$, hence C is left-deterministic.
2 Let us examine a residual $C \bullet u$, for some $u \in W^*$. Lemma 3.8 applies on $S = a_{j_0}$ and $T = b_j$ for every $j \neq j_0$. But the case of the lemma fulfilled by (S, T_j, u) depends on (S, u) only.
Suppose $a_{j_0} \bullet u \neq \emptyset$ (case 1); in this case

$$C \bullet u = (A \bullet u) \Box_{j_0} B \tag{30}$$

Suppose $a_{j_0} \bullet u = \emptyset, \exists u', u'', u = u' \cdot u'', a_{j_0} \bullet u' = \epsilon$ (case 2); in this case

$$C \bullet u = < (B \bullet u'') | \emptyset^m_{j_0} > \tag{31}$$

where $\emptyset^m_{j_0}$ is the row vector $\epsilon^m_{j_0}$ in which $\emptyset$ and ϵ have been exchanged and $< *, * >$ is the "scalar product" defined by $< S, T >= \sum_{j=1}^m S_j \cdot T_j$.
Suppose $a_{j_0} \bullet u = \emptyset, \forall u' \preceq u, a_{j_0} \bullet u' \neq \epsilon$ (case 3); in this case, equation (30) is true again. When equation (30) is true, $C \bullet u$ is left-deterministic by part (1) of this proof, and when equation (31) is true, $C \bullet u$ is left-deterministic because B is assumed deterministic. We have proved that $C \in \mathsf{DB}_{1,m} < W >$.
3 The number of residuals of the form (30) is bounded above by $\|A\|$ and the number of residuals of the form (31) is bounded above by $\|B\|$. Hence $\|C\| \leq \|A\| + \|B\|$. $\Box$
Given $A \in \mathsf{DB}_{1,m} < W >$ and $1 \leq j_0 \leq m$ we define the vector $A' = \Box^*_{j_0}(A)$ as follows:
if $A = (a_1, \ldots, a_j, \ldots, a_m)$ then $A' = (a'_1, \ldots, a'_j, \ldots, a'_m)$ where

$$a'_j = a^*_{j_0} \cdot a_j \text{ if } j \neq j_0 \ , \ \ a'_j = \emptyset \text{ if } j = j_0.$$

Lemma 3.19 *Let $A \in \mathsf{DB}_{1,m} < W >$ and $1 \leq j_0 \leq m$.*
Then $\Box^*_{j_0}(A) \in \mathsf{DB}_{1,m} < W >$ and $\|\Box^*_{j_0}(A)\| \leq \|A\|$.

Proof: Let us examine a residual $A' \bullet u$, for some $u \in W^*$. Let $u' = \max\{v \preceq u \mid v \in a_{j_0}^*\}$. Let $u'' \in W^*$ such that $u = u' \cdot u''$. One can check that for every $S, T \in \mathsf{B} < W >$

$$(S \cdot T) \bullet u = (S \bullet u) \cdot T + \sum_{\substack{u = u_1 \cdot u_2, \\ \epsilon \in S \bullet u_1}} a_j \bullet u_2.$$

Applying this formula to $S = a_{j_0}^*$ and $T = a_j$, with $j \neq j_0$ we obtain

$$a'_j \bullet u = (a_{j_0}^* \bullet u) \cdot a_j + \sum_{\substack{u = u_1 \cdot u_2, \\ \epsilon \in a_{j_0}^* \bullet u_1}} a_j \bullet u_2. \tag{32}$$

As a_{j_0} is deterministic, one can check that

$$a_{j_0}^* \bullet u = (a_{j_0} \bullet u'') \cdot a_{j_0}^*.$$

As A is deterministic, if u_2 has some prefix u'_2 in a_{j_0}, then $a_j \bullet u'_2 = \emptyset$ so that $a_j \bullet u_2 = \emptyset$. Hence

$$\sum_{\substack{u = u_1 \cdot u_2, \\ \epsilon \in a_{j_0}^* \bullet u_1}} a_j \bullet u_2 = a_j \bullet u''.$$

Plugging the two last equations into (32) we obtain

$$a'_j \bullet u = (a_{j_0} \bullet u'') \cdot a_j^* + a_j \bullet u''(\text{ for } j \neq j_0), \quad \text{and } a'_j \bullet u = \emptyset(\text{ for } j = j_0)$$

which can be rewritten as

$$A' \bullet u = (A \bullet u'') \Box_{j_0} A' \tag{33}$$

Let us show that A' is left-deterministic. If A is left-deterministic of type $[pz]$, then A' is left-deterministic of the same type.
If A is left-deterministic of type (ϵ, j_1) with $j_1 \neq j_0$, then $A' = A$ (notice that $\emptyset^* = \epsilon$), hence A' is left-deterministic.
If A is left-deterministic of type (ϵ, j_0) or $(\emptyset)$, then $A' = \emptyset$, hence A' is left-deterministic.
By part (1) of the proof of lemma 3.18, the fact that $A \bullet u''$ and A' are both left-deterministic implies that $(A \bullet u'') \Box_{j_0} A'$ is left-deterministic too. By formula (33), $A' \bullet u$ is left-deterministic. We have proved that $A' \in \mathsf{DB}_{1,m} < W >$.
Moreover, by formula (33), $\mathrm{Card}(\mathsf{Q}(A')) \leq \mathrm{Card}(\mathsf{Q}(A))$, i.e. $\|A'\| \leq \|A\|$. $\Box$
From a purely technical point of view, the two next lemmas (3.20 and 3.21) are *not used* in the sequel. We nevertheles believe they are fundamental for the general understanding of d-spaces. The proofs of these lemmas (especially the parts (1) $\Rightarrow$ (2) $\Rightarrow$ (3)) develop in a pure algebraic form (hence in a form which is more easily understandable), some ideas which will be used in section 5 but in a more complex form. In this further section these ideas will be combined with considerations on *weights* of linear equations and on *deduction systems* over weighted linear equations.

Lemma 3.20 (fundamental lemma) *Let $U_1, \ldots, U_j, \ldots, U_m$ be elements of a d-universe E over $\mathsf{DB} < W >$. The following are equivalent*

1. $\exists \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathsf{DB}_{1,m} < W >, \boldsymbol{\alpha} \neq \boldsymbol{\beta}$, *such that*

   $$(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m) = (\beta_1, \ldots, \beta_j, \ldots, \beta_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m)$$

2. $\exists j_0 \in [1, m], \exists \boldsymbol{\gamma} \in \mathsf{DB}_{1,m} < W >, \boldsymbol{\gamma} \neq \epsilon^m_{j_0}$, *such that*

   $$U_{j_0} = (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m)$$

3. $\exists j_0 \in [1, m], \exists \boldsymbol{\gamma} \in \mathsf{DB}_{1,m} < W >, \gamma_{j_0} = \emptyset$, *such that*

   $$U_{j_0} = (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m)$$

4. $\exists j_0 \in [1, m]$, *such that*

   $$\mathsf{V}((U_j)_{1 \leq j \leq m}) = \mathsf{V}((U_j)_{1 \leq j \leq m, j \neq j_0}).$$

.

The equivalence between (1),(2) and (3) was first proved in [Mei89, Mei92], in the case where $E = \mathsf{DB} < X >$ and the U_j's are the languages recognized by some configurations $q_j \omega$, with the same ω. This is the key-idea around which we have developped the notion of d-spaces.

Proof: (1) $\Rightarrow$ (2)**:**

Let us suppose (1) and let us consider $u = \min(\bigcup_{1 \leq j \leq m} \alpha_j \Delta \beta_j)$. Without loss of generality we can suppose that $u \in \alpha_{j_0} - \beta_{j_0}$.

$$((\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m)) \circ_r u = ((\beta_1, \ldots, \beta_j, \ldots, \beta_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m)) \circ_r u. \tag{34}$$

As $\alpha_{j_0} \bullet u = \epsilon$ and $\boldsymbol{\alpha}$ is deterministic, $\boldsymbol{\alpha} \bullet u$ must be left-deterministic of type (ϵ, j_0),i.e. $\boldsymbol{\alpha} \bullet u = \epsilon^m_{j_0}$. Using successively axiom (D5, case (C2)), axiom (D4) and axiom (D2) we obtain

$$\begin{aligned}((\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m)) \circ_r u &= \epsilon^m_{j_0} \circ_l ((U_1, \ldots, U_j, \ldots, U_m) \circ_r \epsilon) \\ &= \epsilon^m_{j_0} \circ_l (U_1, \ldots, U_j, \ldots, U_m) \\ &= U_{j_0}.\end{aligned} \tag{35}$$

Let us examine now the righthand-side of equality (34). Let $u' \prec u$. By minimality of u, $\boldsymbol{\beta} \bullet u'$ is a unit iff $\boldsymbol{\alpha} \bullet u'$ is a unit. But if $\boldsymbol{\alpha} \bullet u'$ is a unit, then $\boldsymbol{\alpha} \bullet u = \emptyset$, which is false. Hence $\boldsymbol{\beta} \bullet u'$ is not a unit. By hypothesis, $\boldsymbol{\beta} \bullet u$ is not a unit either. Hence, $\forall u' \preceq u$, $\boldsymbol{\beta} \bullet u'$ is not a unit . By axiom (D5, case C1)

$$((\beta_1, \ldots, \beta_j, \ldots, \beta_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m)) \circ_r u = ((\beta_1, \ldots, \beta_j, \ldots, \beta_m) \bullet u) \circ_l (U_1, \ldots, U_j, \ldots, U_m). \tag{36}$$

Let us set $\boldsymbol{\gamma} = \boldsymbol{\beta} \bullet u$. $\boldsymbol{\gamma} \neq \epsilon^m_{j_0}$ and by equalities (34),(35), (36),

$$U_{j_0} = (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \circ_l (U_1, \ldots, U_j, \ldots, U_m).$$

(2) $\Rightarrow$ (3):
Let us suppose (2). Let us consider the matrix $\Gamma \in \mathsf{DB}_{m,m} < W >$ defined by:

$$\Gamma_{j,*} = \delta_{j,*} \text{ (for every } j \neq j_0) \ , \ \Gamma_{j,*} = \gamma \ \text{ (for } j = j_0).$$

Equality (2) can be rewritten in terms of $\mathbf{U} = (U_1, \ldots, U_j, \ldots, U_m)$ as: $\mathbf{U} = \Gamma \circ_l \mathbf{U}$. It follows that ,

$$\forall p \in \mathbb{N}, \mathbf{U} = \Gamma^p \circ_l \mathbf{U}. \tag{37}$$

One can check that

$$(\Gamma^p)_{i,j} = \emptyset \text{ for } i \neq j_0 \ , \ (\Gamma^p)_{i,j} = \sum_{0 \leq k \leq p} \gamma_{j_0}^k \cdot \gamma_j \text{ for } i = j_0, j \neq j_0$$
$$(\Gamma^p)_{i,j} = \gamma_{j_0}^p \text{ for } i = j = j_0$$

As $\gamma_{j_0} \neq \epsilon$ and is deterministic, by assertion (12), $\lim(\gamma_{j_0}^p) = \emptyset$. By the definition of the operation $*$ and by continuity of $\cdot$, $\lim(\sum_{0 \leq k \leq p} \gamma_{j_0}^k \cdot \gamma_j) = \gamma_{j_0}^* \cdot \gamma_j$, for every $j \neq j_0$. It follows that $\lim \Gamma^p = \Gamma'$ where

$$\Gamma'_{j,*} = \delta_{j,*} \text{ (for every } j \neq j_0) \ , \ \Gamma'_{j,*} = \square_{j_0}^*(\gamma) \ \text{ (for } j = j_0).$$

Using axiom (D6) and assertion (37) we obtain

$$\mathbf{U} = \Gamma' \circ_l \mathbf{U},$$

and the above value of Γ' shows that

$$U_{j_0} = \square_{j_0}^*(\boldsymbol{\gamma}) \circ_l (U_1, \ldots, U_j, \ldots, U_m)$$

As the component j_0 of $\square_{j_0}^*(\boldsymbol{\gamma})$ is empty, point (3) is proved.
(3) $\Rightarrow$ (4):
Let us suppose (3). Let us denote by $\hat{U}$ the vector $(U_1, \ldots, U_{j_0-1}, U_{j_0+1} \ldots, U_m) \in E^{m-1}$. Let us consider the matrix $\hat{\Gamma} \in \mathsf{DB}_{m,m-1} < W >$ defined as the matrix Γ where the j_0-th column has been removed. Every vector $\mathbf{e} = \boldsymbol{\alpha} \circ_l \mathbf{U}$ for some $\boldsymbol{\alpha} \in \mathsf{DB}_{1,m} < W >$ can be expressed as $\mathbf{e} = (\boldsymbol{\alpha} \cdot \hat{\Gamma}) \circ_l \hat{U}$. (One can also notice that $\mathbf{e} = (\boldsymbol{\alpha} \square_{j_0} \boldsymbol{\gamma}) \circ_l \hat{U}$).
Hence

$$\mathsf{V}((U_j)_{1 \leq j \leq m}) \subseteq \mathsf{V}((U_j)_{1 \leq j \leq m, j \neq j_0}).$$

As the reverse inclusion is trivially true, (4) is proved.
(4) $\Rightarrow$ (1):
Let us suppose (4) is true for some integer j_0. The element U_{j_0} has clearly two different representations as linear combination of the U_j's. Hence (1) is true. □

A family $(U_i)_{i \in I}$ (in a d-universe E) is said to be *linearly dependant* iff one of cases (1-4) occurs for some finite subfamily $(U_{i_j})_{1 \leq j \leq m}$ defined by some injection $j \mapsto i_j$. It is said *linearly independant* or *free* iff it is *not* linearly dependant. A family $(U_i)_{i \in I}$ (in a d-space V) is a *basis* of V iff it is both a generating family (i.e. $\mathsf{V} = \mathsf{V}((U_i)_{i \in I})$) and a free family.

Let us apply the above lemma to the d-universe $E = \mathsf{DB} < X >$ (over $\mathsf{DB} < X >$) and then, "lift" (by means of φ) the result to the d-space $\mathsf{DB} < V >$ (over $\mathsf{DB} < V >$) endowed with the congruence $\equiv$ and the right-action $\odot$. Let us extend the equivalence relation $\equiv$ to d-spaces in the universe $\mathsf{DB} < V >$ by: if $\mathsf{V}_1, \mathsf{V}_2$ are d-spaces ,

$$\mathsf{V}_1 \equiv \mathsf{V}_2 \Leftrightarrow \forall i,j \in \{1,2\}, \forall S \in \mathsf{V}_i, \exists S' \in \mathsf{V}_j, S \equiv S'.$$

Lemma 3.21 *Let $S_1, \ldots, S_j, \ldots, S_m \in \mathsf{DB} < V >$. The following are equivalent*

1. $\exists \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathsf{DB}_{1,m} < V >, \boldsymbol{\alpha} \not\equiv \boldsymbol{\beta}$, *such that*

   $$(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m) \circ_l (S_1, \ldots, S_j, \ldots, S_m) \equiv (\beta_1, \ldots, \beta_j, \ldots, \beta_m) \circ_l (S_1, \ldots, S_j, \ldots, S_m)$$

2. $\exists j_0 \in [1,m], \exists \boldsymbol{\gamma} \in \mathsf{DB}_{1,m} < V >, \boldsymbol{\gamma} \not\equiv \epsilon^m_{j_0}$, *such that*

   $$S_{j_0} \equiv (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \circ_l (S_1, \ldots, S_j, \ldots, S_m)$$

3. $\exists j_0 \in [1,m], \exists \boldsymbol{\gamma}' \in \mathsf{DB}_{1,m} < V >, \gamma'_{j_0} \equiv \emptyset$, *such that*

   $$S_{j_0} \equiv (\gamma'_1, \ldots, \gamma'_j, \ldots, \gamma'_m) \circ_l (S_1, \ldots, S_j, \ldots, S_m)$$

4. $\exists j_0 \in [1,m]$, *such that*

   $$\mathsf{V}((S_j)_{1 \le j \le m}) \equiv \mathsf{V}((S_j)_{1 \le j \le m, j \ne j_0}).$$

.

Proof: Let us extend componentwise the map φ, i.e.

$$\forall A \in \mathsf{DB}_{n,m} < V >, (\varphi(A))_{i,j} = \varphi(A_{i,j}) \text{ for } 1 \le i \le n, 1 \le j \le m.$$

We already know that $\varphi : \mathsf{B} < V > \to \mathsf{B} < X >$ is a semi-ring morphism, a morphism of right-actions (w.r. to $\odot$ and $\bullet$) and is σ-additive. φ is also a d-space morphism in the sense that: for every $\boldsymbol{\alpha} \in \mathsf{DB}_{1,m} < V >, S \in \mathsf{DB}_{m,1} < V >$

$$\varphi(\boldsymbol{\alpha} \circ_l S) = \varphi(\boldsymbol{\alpha}) \circ_l \varphi(S).$$

Let us follow step by step the proof of lemma 3.20 and lift it .

(1) $\Rightarrow$ (2):

Let $u = \min(\bigcup_{1 \le j \le m} \varphi(\alpha_j) \Delta \varphi(\beta_j))$. Let $\boldsymbol{\gamma} = \boldsymbol{\beta} \odot u$.

$$S_{j_0} \equiv (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_m) \circ_l (S_1, \ldots, S_j, \ldots, S_m).$$

(2) $\Rightarrow$ (3):

By the σ-additivity of φ, $\varphi(\gamma^*_{j_0}) = (\varphi(\gamma_{j_0}))^*$. Hence

$$S_{j_0} \equiv \square^*_{j_0}(\gamma) \circ_l (S_1, \ldots, S_j, \ldots, S_m).$$

(3) $\Rightarrow$ (4):

Let us denote by $\hat{S}$ the vector $(S_1, \ldots, S_{j_0-1}, S_{j_0+1} \ldots, S_m) \in \mathsf{DB}_{m-1,1} < V >$.

If $T = \boldsymbol{\alpha} \circ_l S$ then $T \equiv (\boldsymbol{\alpha} \Box_{j_0} \boldsymbol{\gamma}) \circ_l \hat{S}$.
(4) $\Rightarrow$ (1):
Let us suppose (4) is true for some integer j_0. The element S_{j_0} is clearly equivalent (mod $\equiv$) to two linear combinations of the S_j's with non-equivalent vectors of coefficients. Hence (1) is true. $\Box$

Let us call *d-submonoid* of $\mathsf{DB} < W >$ any $K \subseteq \mathsf{DB} < W >$ such that,

(1)$\epsilon \in K$,
(2) for every $A \in \mathsf{DB}_{1,m} < W > \cap \mathsf{M}_{1,m}(K)$, $B \in \mathsf{DB}_{m,1} < W > \cap \mathsf{M}_{m,1}(K)$, $j_0 \in [1, m]$, $A \cdot B \in K$ and $\Box^*_{j_0}(A) \in \mathsf{M}_{1,m}(K)$.

The notions of universe and space over $\mathsf{DB} < W >$ can be adapted to K in a straightforward way and both lemma 3.20, lemma 3.21 remain true in this context. One can check that $K = \mathsf{DRB} < W >$ fulfills conditions (1)(2) above. This set of coefficients K will be used in the sequel.

Dimension The results of this paragraph are not used in the sequel and can be skipped by the reader interested in the main decidability result only. We present them for the algebraic pleasure (and also because they might be useful for future works).

Lemma 3.22 *Let $(U_i)_{i \in I}$ be a finite family generating a d-space V. Then there is a subfamily $(U_{i_j})_{1 \leq j \leq m}$ which is a basis of V.*

Proof: Immediate from lemma 3.20. $\Box$

Corollary 3.23 (existence of a basis for spaces of finite type) *Let V be some d-space admitting a finite generating family. Then V has a basis.*

Example 3.24 $\mathsf{DB}_{1,2} < W >$ *admits the basis $(\epsilon^2_1, \epsilon^2_2)$.*

Remark 3.25 *We do not know if the above statement does hold in general, i.e. for spaces which are not of finite type.*

Lemma 3.26 (dimension invariance) *Let $(U_i)_{1 \leq i \leq n}$,$(U'_j)_{1 \leq j \leq m}$ be two bases of some d-space V. Then $n = m$.*

Proof: Let us suppose that $(U_i)_{1 \leq i \leq n}$,$(U'_j)_{1 \leq j \leq m}$ are two basis of some d-space V. Then there are two matrices $A \in \mathsf{DB}_{n,m} < W >$, $B \in \mathsf{DB}_{m,n} < W >$ such that:

$$\mathbf{U} = A \circ_l \mathbf{U}' \text{ and } \mathbf{U}' = B \circ_l \mathbf{U}.$$

It follows that $\mathbf{U} = (A \cdot B) \circ_l \mathbf{U}$, which implies, by the freeness of $\mathbf{U}$ that $A \cdot B = \mathrm{I}_n$ (the unit matrix of order n). Let us denote by ψ the semi-ring homomorphism $\mathsf{B} < W > \longrightarrow \mathsf{B}$ mapping every non-emtyset on 1 and the empty set on 0. We denote by the same symbol $\psi : \mathsf{DB}_{n,m} < W > \longrightarrow \mathsf{M}_{n,m}(\mathsf{B})$ the componentwise extensions of the above map. We then have:

$$\psi(A) \cdot \psi(B) = \mathrm{I}_n . \tag{38}$$

Let us consider the binary relations R_A, R_B defined by $\psi(A)$ (resp. $\psi(B)$):

$$(i,j) \in R_A \Leftrightarrow \psi(A)_{i,j} = 1 \text{ and } (j,i) \in R_B \Leftrightarrow \psi(B)_{j,i} = 1.$$

Identity (38) means that $R_B \circ R_A = \mathrm{Id}_{[1,n]}$. This is possible only if $m \leq n$. Similarly we must have $n \leq m$, so that $n = m$. □
This proves, in particular, that , independently of the chosen d-universe E over $\mathsf{DB} < W >$, every finitely generated d-space (in E) must be isomorphic to some $\mathsf{DB}_{1,n} < W >$ (by lemma 3.23) where n is unique (by lemma 3.26).

3.3 Height, defect and linearity

We define here notions of height and defect (for deterministic rational series) and a subsequent notion of (d,d')-linearity which will play a crucial role in section 8. We then relate these "size notions" with the notion of norm.

Definitions Let $S \in \mathsf{DRB} < V >$. We call *linear decomposition of* S any pair $([p\omega], \Phi)$ where $p \in Q, \omega \in Z^*, \Phi \in \mathsf{DB}_{Q,1} < V >$ such that

$$S = [p\omega] * \Phi.$$

We denote by $\mathcal{D}(S)$ the set of all linear decompositions of S. We define the *right-defect* of S ($\mathrm{rd}(S)$ for short) by

$$\mathrm{rd}(S) = \min\{||\Phi|| \mid \exists p \in Q, \omega \in Z^*, ([p\omega], \Phi) \in \mathcal{D}(S)\}.$$

One can easily see that the right-defect of S is finite and not greater than $||S||$. We call *minimal* decomposition of S the decomposition $([p_0\omega_0], \Phi_0)$ which makes the pair $(|\omega|, ||\Phi||)$ minimal for the right-to-left lexicographic ordering in $\mathbb{N} \times \mathbb{N}$. We then define the *linear-height* of S (noted $|S|$) as the integer $|S| = |\omega_0|$ (and it is clear that $\mathrm{rd}(S) = ||\Phi_0||$).

The height and right-defect of a Q-form are defined similarly.
Let $\Phi \in \mathsf{DRB}_{Q,1} < V >$. We call *linear decomposition of* S any pair $([\omega], \Psi)$ where $\omega \in Z^*, \Psi \in \mathsf{DB}_{Q,1} < V >$ such that

$$\Phi = [\omega] * \Psi.$$

$\mathcal{D}(\Phi)$ denotes the set of all linear decompositions of Φ. The *minimal* decomposition of Φ is the $([\omega_0], \Psi_0)$ which makes the pair $(|\omega|, ||\Psi||)$ minimal for the right-to-left lexicographic ordering in $\mathbb{N} \times \mathbb{N}$. The integers $|\Phi|, \mathrm{rd}(\Phi)$ are then defined by:

$$|\Phi| = |\omega_0|, \mathrm{rd}(\Phi) = ||\Psi_0||.$$

We say that S is *marked* iff S contains some occurence of some letter $[qeq] \in V$ (we assumed the existence of such a "dummy " letter e in Z in section 2.2).

Definition 3.27 *Let $S \in \mathsf{DRB} < V >$ and $d, d' \in \mathbb{N}, d \geq 1$.*

1. *S is said $(0, d')$-linear iff $\mathrm{rd}(S) \leq d'$ and S is not marked*
2. *S is said (d, d')-linear iff S has a decomposition $S = \sum_{q \in Q}[phq] \cdot [qeq] \cdot \Phi_q$ where every Φ_q is a $(0, d')$-linear series and $|h| \leq d$.*

In case 2 we call the series Φ_q the d'-linear components of S. It should be clear that the d'-linear components are independent of the value of d (for d large enough) and that the set of d'-linear components is uniquely defined (it is empty if S is not (d, d')-linear for any d, otherwise it is the set of all maximal unmarked residuals of S). We denote by $\mathsf{DBlin}^{d'} < V >$ (resp. $\mathsf{DRBlin}^{d'} < V >$) the set of series in $\mathsf{DB} < V >$ (resp. $\mathsf{DRB} < V >$) which are $(0, d')$-linear.
Examples:
No series can be $(0-0)$-linear. S is $(0, 1)$-linear iff $S = \emptyset$. S is $(0, 2)$-linear iff it has the form $S = [p\omega q]$ for some $p, q \in Q, \omega \in (Z - \{e\})^*$.
Hence one can view the $(0, d)$-linear series as series which have a structure " not too far" from the *linear* structure of the configurations of the initial dpda $\mathcal{M}$.
(We illustrate on figure 1 the above definitions).

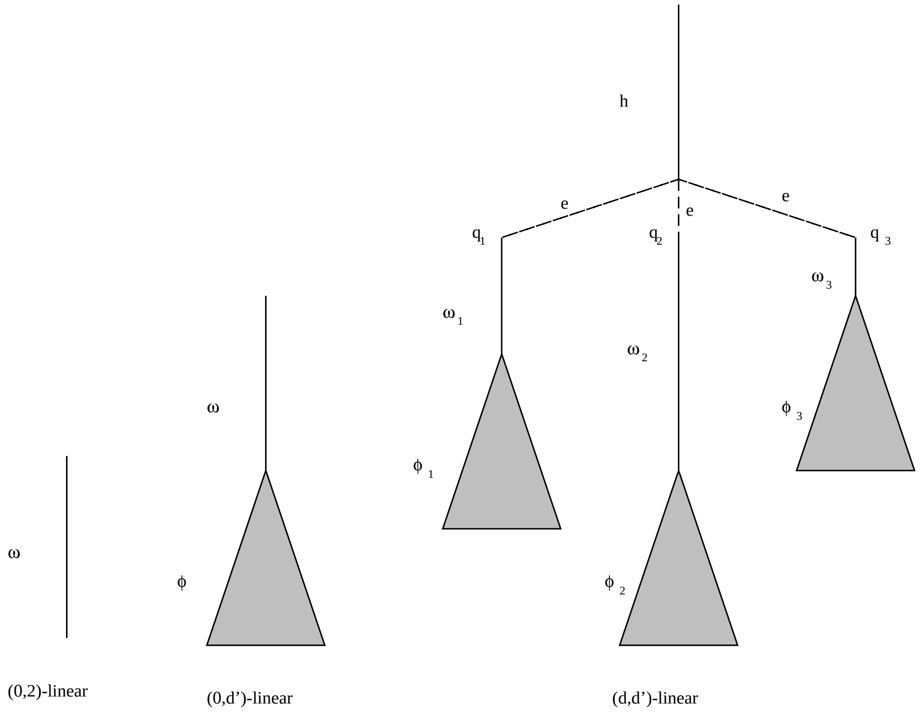

Fig. 1. (d, d')-linearity.

3.4 Height and norm

Lemma 3.28 *Let* $S \in \mathsf{DRB} < V >, x \in X, d, d' \in \mathbb{N}$.

1. $\mathrm{rd}(S \odot x) \leq \mathrm{rd}(S)$
2. S *is* (d, d')*-linear* $\Rightarrow S \odot x$ *is* $(d+1, d')$*-linear*
3. $\|S \odot x\| \leq \|S\| + |Q|$.

Here $|Q|$ denotes the cardinality of the set Q.
Sketch of proof: Point (1) follows from lemmas 3.9 and 3.13. Points (2),(3) follow from the hypothesis that the dpda $\mathcal{M}$ is normalized. □

Lemma 3.29 *Let* $B, A \in Z$, $\Phi \in \mathsf{DB}_{Q,1}, V >$. *If* $\|[A] * \Phi\| > \|\Phi\|$ *then,* $\forall q \in Q, [qBA] * \Phi \notin \mathsf{Q}_r([A] * \Phi)$.

Proof: Suppose that $\exists q \in Q$, such that $[qBA] * \Phi \in \mathsf{Q}_r([A] * \Phi)$. Let us fix some $q \in Q, u_q \in V^*, q' \in Q$, fulfilling

$$[qBA] * \Phi = ([q'A] * \Phi) \bullet u_q.$$

case 1: $u_q = \epsilon$.
Hence $q = q', B = A, [qB] * [A] * \Phi = [qB] * \Phi$. It follows that $[A] * \Phi = \Phi$ and finally $\|[A] * \Phi\| = \|\Phi\|$.
case 2: $u_q \neq \epsilon, \Phi \neq \emptyset_Q$.
Then $u_q = [q'Ar]v_q$ for some $r \in Q, v_q \in V^*$.

$$[qBA] * \Phi = (([q'A] * \Phi) \bullet [q'Ar]) \bullet v_q = \Phi_r \bullet v_q.$$

It follows that , $\forall p \in Q$,

$$[pA] * \Phi = \Phi_r \bullet v_q \bullet [qBp] \in \mathsf{Q}_r(\Phi),$$

hence $\|[A] * \Phi\| = \|\Phi\|$.
case 3: $u_q \neq \epsilon, \Phi = \emptyset_Q$.
then $\|[A] * \Phi\| = \|\Phi\| = 0$.
In all cases the lemma is proved by contraposition. □

Lemma 3.30 *Let* $\omega \in Z^+, A', A \in Z, p \in Q$, $\Phi \in \mathsf{DB}_{Q,1} < V >$. *If* $\|[A] * \Phi\| > \|\Phi\|$, *then*

(1) $\|[\omega A] * \Phi\| = |Q| \cdot |\omega| + \|[A] * \Phi\|$
(2) $\|[pA'\omega A] * \Phi\| = 1 + |Q| \cdot |\omega| + \|[A] * \Phi\|$

Proof: Let us prove point (1) by induction on $n = |\omega|$.
$n = 0$:
formula (1) is obvious in this case.
$n = 1$: $\omega = B$.
$||[\omega A] * \Phi|| = ||[B] * ([A] * \Phi)||$. By lemma 3.4,

$$\mathsf{Q}_r([BA] * \Phi) = \{[qBA] * \Phi \mid q \in Q\} \dot\cup \mathsf{Q}_r([A] * \Phi),$$

(where $\dot\cup$ denotes a *disjoint* union) hence $||[BA] * \Phi|| = |Q| + ||[A] * \Phi||$.
$n = m + 1, m \geq 1$: $\omega = CB\omega'$ for some $C, B \in Z, \omega' \in Z^*$.
$||[\omega A] * \Phi|| = ||[CB] * ([\omega' A] * \Phi)||$. As a consequence of the induction hypothesis, $||[B] * [\omega' A] * \Phi|| > ||[\omega' A] * \Phi)||$, hence, by lemma 3.4

$$||[\omega A] * \Phi|| = |Q| + ||[B\omega' A] * \Phi)||.$$

By induction hypothesis, $||[B\omega' A] * \Phi)|| = m \cdot |Q| + ||[A] * \Phi||$, hence

$$||[\omega A] * \Phi|| = (m + 1) \cdot |Q| + ||[A] * \Phi||.$$

Let us prove now point (2).
By the above induction, $||[A'\omega A]*\Phi|| = |Q|+||[\omega A]*\Phi||$, i.e. $\forall q \in Q, [qA'\omega A]*\Phi \notin \mathsf{Q}_r([\omega A] * \Phi)$. Hence

$$\mathsf{Q}_r([pA'\omega A] * \Phi) = \{[pA'\omega A] * \Phi\} \dot\cup \mathsf{Q}_r([\omega A] * \Phi).$$

It follows that

$$||[pA'\omega A] * \Phi|| = 1 + ||[\omega A] * \Phi|| = 1 + |\omega| \cdot |Q| + ||[A] * \Phi||.$$

□

Lemma 3.31 *Let* $\omega \in Z^*, B \in Z, p \in Q$, $\Phi \in \mathsf{DB}_{Q,1} < V >$ *. Then* $||[pB\omega] * \Phi|| \leq 1 + |Q| \cdot |\omega| + ||\Phi||$.

Proof: One can notice that $||[pB\omega]|| = 1 + |Q| \cdot |\omega|$. Hence , by lemma 3.17,

$$||[pB\omega] * \Phi|| \leq ||[pB\omega]|| + ||\Phi|| = 1 + |Q| \cdot |\omega| + ||\Phi||.$$

□

Lemma 3.32 *Let* $S \in \mathsf{DRB} < V >$*. Then* $1 + |Q| \cdot (|S| - 2) + \mathrm{rd}(S) \leq ||S|| \leq 1 + |Q| \cdot |S| + \mathrm{rd}(S)$.

Proof: The upper-bound on $||S||$ follows from lemma 3.31 . The lower bound follows from lemma 3.30 in the case where $|S| \geq 2$ and is clear for $|S| \leq 1$.
□
The following lemma serves as a crucial technical argument in section 8.

Lemma 3.33 *Let* $U = [ph] * \Phi, H = U \odot u$ *where* $p \in Q, h \in Z^*, |h| \geq 1, \Phi$ *is a* $Q-$ *form* , $|\Phi| \geq 1, u \in X^*, |u| \leq k$*. Let us suppose that* $||H|| \geq 1 + (k-1)|Q| + ||\Phi||$*. Then*
$H = ([ph] \odot u) * \Phi$ *where* $[ph] \odot u = [q\omega]$ *for some* $q \in Q, |\omega| \geq k$*.*

Intuitive meaning: If U is a deterministic rational series admitting a decomposition over the Q-form Φ and, $U \odot u$, where $|u| \leq k$, has a sufficiently large norm (compared to $||\Phi||$ and $|u|$), then the action of u on U cannot have "touched" the form Φ.
Proof: By lemma 3.9 combined with lemma 3.13, one of the following 3 cases occurs.
case 1: $\forall r \in Q, [phr] \odot u \neq \emptyset$.
Hence, $\exists \omega \in Z^*, \exists q \in Q, [ph] \odot u = [q\omega], H = [q\omega] * \Phi$ and, by lemma 3.30,

$$||H|| = 1 + |Q|(|\omega| - 1) + ||\Phi||.$$

The hypothesis about $||H||$ is then equivalent to $|\omega| \geq k$.
case 2: $\exists q \in Q, \exists u', u'' \in X^*, u = u'u'', [phq] \odot u' = \epsilon$.
Hence $H = \Phi_q \odot u''(|u''| \leq k)$. By lemma 3.13, $\exists u''_1, u''_2 \in X^*, u'' = u''_1, u''_2, \exists r \in Q, A \in Z, \Phi'$ $Q-$form such that

$$\Phi_q \odot u''_1 = [rA] * \Phi' \in \mathsf{Q}_r(\Phi) \text{ and } \Phi_q \odot u'' = ([rA] \odot u''_2) * \Phi'.$$

As $|h| \geq 1, |u'| \geq 1$, hence $|u''_2| \leq |u''| \leq k-1$. It follows that

$$\begin{aligned} ||H|| = ||\Phi_q \odot u''|| &\leq ||[rA] * \Phi'|| + (k-1)|Q| \\ &\leq ||\Phi|| + (k-1)|Q| \\ &< 1 + (k-1)|Q| + ||\Phi||, \end{aligned}$$

contradicting the hypothesis about $||H||$. This case is impossible.
case 3: $H = \emptyset$.
This contradicts also the hypothesis. □

Lemma 3.34 *Let* $D \geq 0$*. Let* $\Phi = (\Phi_q)_{q \in Q}$ *be a* Q *-form and let* $S \in \mathsf{V}((\Phi_q)_{q\in Q})$ *such that:*

(1) $||\Phi|| \geq D + |Q|, |\Phi| \geq 1$
(2) $\mathrm{rd}(S) \leq D$*.*

Then, $\exists \omega \in Z^*, \exists p \in Q, S = [p\omega] * \Phi$*.*

Proof: Let S fulfilling hypotheses (1),(2). S can be written

$$S = \sum_{q \in Q} \alpha_q \Phi_q = [p_0\omega_0] * \Psi,$$

where $([p_0\omega_0], \Psi)$ is the minimal decomposition of S. Let us suppose that

$$\exists q \in Q, \exists u \in \alpha_q, \exists u', u'' \in V^*, u = u' \cdot u'' \text{ and } S \bullet u' = \Psi_q. \tag{39}$$

Then $\Psi_q \bullet u'' = \Phi_q$, hence $\mathsf{Q}_r(\Psi) \supseteq \mathsf{Q}_r(\Phi_q)$. As $|\Phi| \geq 1$, we must have $\Phi = [A] * \Phi'$ for some $A \in Z$, hence

$$\mathrm{Card}(\mathsf{Q}(\Phi_q)) = \mathrm{Card}(\mathsf{Q}_r(\Phi)) - |Q| + 1. \tag{40}$$

It follows that

$$||\Psi|| \geq ||\Phi_q|| = ||\Phi|| - |Q| + 1 \geq D + 1.$$

But this contradicts the fact that $\mathrm{rd}(S) \leq D$. We have established that (39) is impossible. In other words, case (2) of lemma 3.9 cannot occur in the action of a word $u \in \alpha_q$ on the linear combination $\sum_{q \in Q} \alpha_q \Phi_q$. Hence $\forall q \in Q, \forall u \in \alpha_q, \exists q_u \in Q, \omega_u \in Z^+$,

$$S \bullet u = ([p_0 \omega_0] \bullet u) * \Psi = [q_u \omega_u] * \Psi. \tag{41}$$

But for such q, u we have both following equalities:

$$||S \bullet u|| = ||\Phi_q|| = ||\Phi|| - |Q| + 1 \tag{42}$$

by equation (40);

$$||S \bullet u|| = 1 + |Q|(|\omega_u| - 1) + ||\Psi|| \tag{43}$$

because , ω_0, ω_u have the same rightmost letter A, the minimality of decomposition $([p_0\omega_0], \Psi)$ implies that $||[A] * \Psi|| > ||\Psi||$, and lemma 3.30 point (2) gives the equality. These two equations (42),(43) show that there exists some unique integer $1 \leq n \leq |\omega|$, such that

$$\forall q \in Q, \forall u \in \alpha_q, |\omega_u| = n.$$

But all the words ω_u are suffixes of ω_0, hence there exists some unique words $\omega_0', \omega_0'', \omega_0 = \omega_0' \cdot \omega_0''$ such that

$$\forall q \in Q, \forall u \in \alpha_q, S \bullet u = [q\omega_0''] * \Psi = \Phi_q.$$

It follows that $[\omega_0''] * \Psi = \Phi$ and $S = [p_o \omega_0'] * \Phi$. □

4 Deduction systems

4.1 General deduction systems

We follow here the general philosophy of [HHY79, Cou83]. Let us call *deduction system* any triple $\mathcal{D} =< \mathcal{A}, H, \models\!\!-\; >$ where $\mathcal{A}$ is a denumerable set called the *set of assertions*, H, the *cost function* is a mapping $\mathcal{A} \to \mathbb{N} \cup \{\infty\}$ and $\models\!\!-$, the *deduction relation* is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$; $\mathcal{A}$ is given with a fixed bijection with $\mathbb{N}$ (an "encoding" or "Gödel numbering") so that the notions of recursive subset, recursively enumerable subset, recursive function, ... over $\mathcal{A}, \mathcal{P}_f(A)$,... are defined, up to this fixed bijection ; we assume that $\mathcal{D}$ satisfies the following axioms:

(A 1) $\models\!\!-$ is recursively enumerable

(A 2) $\forall (P, A) \in \;\vdash\!\!-$, $(\min \{H(p), p \in P\} < H(A))$ or $(H(A) = \infty)$. (We let $\min(\emptyset) = \infty$).

In the sequel we use the notation $P \vdash\!\!- A$ for $(P, A) \in \;\vdash\!\!-$. We call *proof* in the system $\mathcal{D}$, *relative to the set of hypotheses* $\mathcal{H} \subseteq \mathcal{A}$, any subset $P \subseteq \mathcal{A}$ fulfilling :

$$\forall p \in P, (\exists Q \subseteq P, Q \vdash\!\!- p) \text{ or } (p \in \mathcal{H}).$$

We call P a *proof* iff

$$\forall p \in P, (\exists Q \subseteq P, Q \vdash\!\!- p)$$

(i.e. iff P is a proof relative to $\emptyset$).
Let us define the total map $\chi : \mathcal{A} \to \{0, 1\}$ and the partial map $\overline{\chi} : \mathcal{A} \to \{0, 1\}$ by :

$$\chi(A) = 1 \text{ if } H(A) = \infty, \chi(A) = 0 \text{ if } H(A) < \infty,$$

$$\overline{\chi}(A) = 1 \text{ if } H(A) = \infty, \overline{\chi} \text{ is undefined if } H(A) < \infty.$$

(χ is the 'truth-value function", $\overline{\chi}$ is the "1-value function").

Lemma 4.1 *Let P be a proof relative to $\mathcal{H} \subseteq H^{-1}(\infty)$ and $A \in P$. Then $\chi(A) = 1$.*

In other words : if an assertion is provable from true hypotheses, then it is true.
Proof: Let P be a proof. We prove by induction on n that,

$$\mathcal{P}(n) : \forall p \in P, H(p) \geq n.$$

It is clear that, $\forall p \in P, H(p) \geq 0$. Suppose that $\mathcal{P}(n)$ is true. Let $p \in P - \mathcal{H} : \exists Q \subseteq P, Q \vdash\!\!- p$. By induction hypothesis, $\forall q \in Q, H(q) \geq n$ and by (A2),$H(p) \geq n + 1$. It follows that : $\forall p \in P - \mathcal{H}, H(p) = \infty$. But by hypothesis, $\forall p \in \mathcal{H}, H(p) = \infty$. □

The deduction system $\mathcal{D}$ will be said *complete* iff, conversely, $\forall A \in \mathcal{A}, \chi(A) = 1 \Longrightarrow$ there exists some *finite* proof P such that $A \in P$. (In other words, $\mathcal{D}$ is complete iff every true assertion is "finitely" provable).

Lemma 4.2 *: If $\mathcal{D}$ is complete, $\overline{\chi}$ is a recursive partial map.*

Proof: Let $i \mapsto P_i$ be some recursive function whose domain is $\mathbb{N}$ and whose image is $\mathcal{P}_f(\mathcal{A})$. Let $h : (\mathcal{P}_f(\mathcal{A}) \times \mathcal{A} \times \mathbb{N}) \to \{0, 1\}$ be a total recursive function such that :

$$P \vdash\!\!- A \text{ iff } \exists n \in \mathbb{N}, h(P, A, n) = 1$$

(such an h exists, because the r.e. sets are the projections of the recursive sets, see [Rog67]).

The following (informal) semi-algorithm computes $\overline{\chi}$ on the assertion A :

```
1. i := 0 ; n := 0 ; s := 0;
```

2. P := P_i;
3. b := $\min_{p \in P}\{\max_{Q \subseteq P}\{h(Q, p, n)\}\}$;
4. c:= $(A \in P)$
5. **if** $(b \wedge c)$ **then** $(\overline{\chi}(A) = 1$; **stop**);
6. **if** $i = 0$ **then** $(s := s + 1$; $n := s)$
   else $(i := i - 1$; $n := n + 1)$;
7. **goto** 2 ;

$\square$

In order to define deduction relations from more elementary ones, we set the following definitions.

Let $\vdash\!\!- \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$. For every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set :

- $P \overset{[0]}{\vdash\!\!-} Q$ iff $P \supseteq Q$
- $P \overset{[1]}{\vdash\!\!-} Q$ iff $\forall q \in Q, \exists R \subseteq P, R \vdash\!\!- q$
- $P \overset{<0>}{\vdash\!\!-} Q$ iff $P \overset{[0]}{\vdash\!\!-} Q$
- $P \overset{<1>}{\vdash\!\!-} Q$ iff $\forall q \in Q, (\exists R \subseteq P, R \vdash\!\!- q)$ or $(q \in P)$
- $P \overset{<n+1>}{\vdash\!\!-} Q$ iff $\exists R \in \mathcal{P}_f(A), P \overset{<1>}{\vdash\!\!-} R$ and $R \overset{<n>}{\vdash\!\!-} Q$ (for every $n \geq 0$).
- $\overset{<*>}{\vdash\!\!-} = \bigcup_{n \geq 0} \overset{<n>}{\vdash\!\!-}$.

Given $\vdash\!\!-_1, \vdash\!\!-_2 \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(\mathcal{A})$, for every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set :

$$P(\vdash\!\!-_1 \circ \vdash\!\!-_2)Q \text{ iff } \exists R \subseteq \mathcal{A}, (P \vdash\!\!-_1 R) \wedge (R \vdash\!\!-_2 Q).$$

4.2 System $\mathcal{D}_0$

Let us define here a particular deduction system $\mathcal{D}_0$ "Taylored for the equivalence problem for dpda's".

Given a fixed dpda $\mathcal{M}$ over the terminal alphabet X, we consider the variable alphabet V associated to $\mathcal{M}$ (see section 3.1) and the set $\mathsf{DRB} < V >$ (the set of Deterministic Rational Boolean series over V^*). The set of assertions is defined by :

$$\mathcal{A} = \mathbb{N} \times \mathsf{DRB} < V > \times \mathsf{DRB} < V >$$

i.e. an assertion is here a *weighted equation* over $\mathsf{DRB} < V >$.

The "cost-function" $H : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is defined by :

$$H(n, S, S') = n + 2 \cdot \mathrm{Div}(S, S'),$$

where $\mathrm{Div}(S, S')$, the *divergence* between S and S', is defined by :

$$\mathrm{Div}(S, S') = \min\{| u || u \in \triangle(\varphi(S), \varphi(S'))\}.$$

(We recall $\min(\emptyset) = \infty$).

Let us notice that here :

$$\chi(n, S, S') = 1 \Longleftrightarrow S \equiv S'.$$

We define a binary relation $|\!\!\vdash\!\!- \subset \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

$(R0)$ $\{(p, S, T)\}$	$\vert\!\!\vdash\!\!-$ $(p + 1, S, T)$	
$(R1)$ $\{(p, S, T)\}$	$\vert\!\!\vdash\!\!-$ (p, T, S)	
$(R2)$ $\{(p, S, S'), (p, S', S'')\}$	$\vert\!\!\vdash\!\!-$ (p, S, S'')	
$(R3)$ $\emptyset$	$\vert\!\!\vdash\!\!-$ $(0, S, S)$	
$(R'3)$ $\emptyset$	$\vert\!\!\vdash\!\!-$ $(0, [qzr], \epsilon)$	(for $q, r \in Q, z \in Z, [qzr] \equiv \epsilon$)
$(R4)$ $\{(p + 1, S \odot x, T \odot x) \mid x \in X\}$	$\vert\!\!\vdash\!\!-$ (p, S, T)	(for $S \not\equiv \epsilon \wedge T \not\equiv \epsilon$)
$(R5)$ $\{(p, S, S')\}$	$\vert\!\!\vdash\!\!-$ $(p + 2, S \odot x, S' \odot x)$	(for $x \in X$)
$(R6)$ $\{(p, S \cdot T' + T, T')\}$	$\vert\!\!\vdash\!\!-$ $(p, S^* \cdot T, T')$	(for $S \not\equiv \varepsilon$)
$(R'6)$ $\{(p, S^* \cdot T, T')\}$	$\vert\!\!\vdash\!\!-$ $(p, S \cdot T' + T, T')$	
$(R7)$ $\{(p, S, S')\}$	$\vert\!\!\vdash\!\!-$ $(p, S + T, S' + T)$	
$(R8)$ $\{(p, S, S')\}$	$\vert\!\!\vdash\!\!-$ $(p, S \cdot T, S' \cdot T)$	
$(R9)$ $\{(p, S, S')\}$	$\vert\!\!\vdash\!\!-$ $(p, U \cdot S, U \cdot S')$	

where $p \in \mathbb{N}, S, S', T, T' \in \mathsf{DRB} < V >, U \in \mathsf{RB} < V >$. (By set of "all" these pairs we mean, all the pairs which fulfill both properties "to belong to $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$" and "to have one of these 13 possible forms" ; but of course, for example, not all the triples $(p, S + T, S' + T)$ belong to $\mathcal{A}$ because $\mathsf{DRB} < V >$ is not closed under sum; as well not all the triples $(p, U \cdot S, U \cdot S')$ belong to $\mathcal{A}$ because U may be non-deterministic and in general $U \cdot S$ may be non-deterministic too).

Remark 4.3 *We do not claim that this system is minimal. For example, it is not clear whether (R0) can be deduced from (R5) (R4), because of the condition on ϵ in (R4).*

Lemma 4.4 *: Let $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$ such that $P \,|\!\!\vdash\!\!-\, A$. Then $min\{H(p) \mid p \in P\} \leq H(A)$.*

Proof: Let us check this property for every type of rule.
R0: $p + 2Div(S, T) \leq p + 1 + 2Div(S, T)$.
R1: $p + 2Div(S, T) = p + 2Div(T, S)$.
R2: as the weight p is the same in all the considered equations, we are reduced to prove that :

$\forall n \in \mathbb{N}, S \equiv_n S' \wedge S' \equiv_n S" \Longrightarrow S \equiv_n S"$ (obvious)
R3, R'3: $\infty = \mathrm{Div}(S, S)$.
R4: Let $S, T \in \mathsf{DRB} < V >, S \not\equiv \varepsilon, T \not\equiv \varepsilon$.If $Div(S,T) = \infty$ the required inequality is true. If $Div(S,T) = n \in \mathbb{N}$, let us consider some $u \in \triangle(\varphi(S), \varphi(T)), \mid u \mid = n$. We can suppose, for example, that $S \odot u = \varepsilon, T \odot u \neq \varepsilon$. As $S \not\equiv \varepsilon, \exists x_0 \in X, \exists v \in X^*, u = x_0 \cdot v$. Hence $(S \odot x_0) \odot v = \varepsilon, (T \odot x_0) \odot v \neq \varepsilon, Div(S \odot x_0, T \odot x_0) \leq \mid v \mid$. Hence we have

$$\begin{aligned}\min\{H(p+1, S \odot x, T \odot x) \mid x \in X)\} &\leq H(p+1, S \odot x_0, T \odot x_0)\\ &= (p+1) + 2 \mid v \mid\\ &\leq (p+1) + 2 \cdot \mathrm{Div}(S,T) - 2\\ &< h(p, S, T).\end{aligned}$$

R5: Let us suppose $H(p+2 \mid x \mid, S \odot x, S' \odot x) \neq \infty$. Let $\mathrm{Div}(S \odot x, S' \odot x) = n, v \in \triangle(\varphi(S \odot x), \varphi(S' \odot x))$. As $xv \in \triangle(\varphi(S), \varphi(S')$, $\mathrm{Div}(S, S') \leq n+1$. Hence

$$H(p, S, S') \leq p+2(n+1) = (p+2)+2 \cdot \mathrm{Div}(S \odot x, S' \odot x) = H(p+2, S \odot x, S' \odot x).$$

R6: We are reduced to prove that, for every $n \geq 0, S, T, T' \in \mathsf{DRB} < V >, S \not\equiv \varepsilon$,

$$S \cdot T' + T \equiv_n T' \Longrightarrow S^* \cdot T \equiv_n T'. \tag{44}$$

Let us suppose $S \cdot T' + T \equiv_n T'$.
• Clearly $S^* \cdot T = T' \ll_n T'$.
• Suppose $S^k T \ll_n T$.
Let $u \in \varphi(S^{k+1}T) = \varphi(S) \cdot \varphi(S^k T), \mid u \mid \leq n$. Then $u = v \cdot w$ with $v \in \varphi(S), w \in \varphi(S^k T), \mid w \mid \leq n$. By the above hypothesis, $w \in \varphi(T')$, hence $u \in \varphi(S \cdot T')$ as $S \cdot T' \ll_n T', u \in \varphi(T')$.
We have proved by induction on k that, for every $k \geq 0$, $S^k T \ll_n T'$, hence $S^* T \ll_n T'$.
Let $u \in \varphi(T')$. If $u \in \varphi(T)$, then $u \in \varphi(S^* T)$.
Otherwise, $u \in \varphi(S.T')$ i.e. $u = v \cdot w$ for some $v \in \varphi(S), w \in \varphi(T')$. As $\varepsilon \neq S$ and S is deterministic, $\varepsilon \notin \varphi(S)$, hence $v \neq \varepsilon, \mid w \mid < \mid u \mid$. We can then prove by induction on $\mid u \mid$ that :

$$u \in \varphi(T'), \mid u \mid \geq n \Longrightarrow u \in \varphi(S^* T),$$

i.e $T' \ll_n S^* T$. This ends the proof of implication (44).
R'6: Suppose $S^* T \equiv_n T'$
$ST' + T \equiv_n S(S^* T) + T$ (by R8, see below),
$S(S^* T) + T = (SS^* + \varepsilon)T = S^* T$ (by semi-ring properties and definition of *),
hence $ST' = T \equiv_n S^* T$.
R7: $S \equiv_n S' \Longrightarrow S + T \equiv_n S' + T$ (obvious)
R8: $S \equiv_n \Longrightarrow S.T \equiv_n S'.T$ (clear because, every prefix of a word of length $\leq n$ has length $\leq n$).
R9: $S \equiv_n S' \Longrightarrow U.S \equiv_n U.S'$ (analogous to R8). □

Let us define $\mid\!\!-\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$,

$$P \mid\!\!-\!\!- A \Longleftrightarrow P \overset{<*>}{|\!\!\mid\!\!-\!\!-} \circ \overset{[1]}{|\!\!\mid\!\!-\!\!-}{}_{0,3,4} \circ \overset{<*>}{|\!\!\mid\!\!-\!\!-} \{A\}.$$

where $|\!\!\mid\!\!-\!\!-_{0,3,4}$ is the relation defined by R_0, R_3, R_3', R_4 only. We let

$$\mathcal{D}_0 =< \mathcal{A}, H, \mid\!\!-\!\!- > .$$

Lemma 4.5 *: $\mathcal{D}_0$ is a deduction system.*

Proof: It should be clear, from the well-known decidability properties of finite automata, that $|\!\!\mid\!\!-\!\!-$ is recursively enumerable. Using lemma 4.4, one can show by induction on n that :

$$P \overset{<n>}{|\!\!\mid\!\!-\!\!-} Q \Longrightarrow \forall q \in Q, \min\{H(A) \mid A \in P\} \leq H(q).$$

The proof of lemma 4.4 also reveals that :

$$P \,|\!\!\mid\!\!-\!\!-_{\{0,3,4\}} q \Longrightarrow (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

It follows that, for every $m, n \geq 0$:

$$P \overset{<n>}{|\!\!\mid\!\!-\!\!-} Q \overset{[1]}{|\!\!\mid\!\!-\!\!-}{}_{0,3,4} R \overset{<m>}{|\!\!\mid\!\!-\!\!-} q \Longrightarrow (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

Both axioms $(A1), (A2)$ are fulfilled. $\square$

The key-statement of this work is that $\mathcal{D}_0$ is complete (theorem 9.2). We prove this completeness result by exhibiting a "strategy" $\mathcal{S}$ which, for every true assertion (n, S, S'), constructs a finite $\mathcal{D}_0$-proof of this assertion. By lemma 4.2 we don't need to prove that $\mathcal{S}$ is computable in any sense to establish that $\overline{\chi}$ is partial-recursive.

4.3 Strategies

Let $\mathcal{D} =< \mathcal{A}, H, \mid\!\!-\!\!- >$ be a deduction system. We call a *strategy* for $\mathcal{D}$ any partial map $\mathcal{S} : \mathcal{A}^+ \rightarrow \mathcal{A}^*$ such that :

(S1) if $\mathcal{S}(A_1A_2\cdots A_n) = B_1 \cdots B_m$ then $\exists Q \subseteq \{A_i \mid 1 \leq i \leq n-1\}$ such that

$$\{B_j \mid 1 \leq j \leq m\} \cup Q \mid\!\!-\!\!- A_n,$$

(S2) if $\mathcal{S}(A_1A_2\cdots A_n) = B_1 \cdots B_m$ then

$$\min\{H(A_i) \mid 1 \leq i \leq n\} = \infty \Longrightarrow \min\{H(B_j) \mid 1 \leq j \leq m\} = \infty.$$

Remark 4.6 *Axiom (A1) on systems is similar to the "monotonicity" condition of [HHY79] or axiom (2.4.2') of [Cou83].*
Axiom (S2) on strategies is similar to the "validity" condition of [HHY79] or property (2.4.1') of [Cou83]. Notice that (S2) is imposed on the strategy $\mathcal{S}$ only, but not on the inverse of the deduction relation ($\vdash\!\!-$ $)^{-1}$. The trick is that ($\vdash\!\!-$ $)^{-1}$ is not valid in general (see R2,R"6) but is computable while $\mathcal{S}$ is valid but is not required to be computable *in general. In the case of $\mathcal{D}_0$, the strategy $\mathcal{S}_{ABC}$ defined below (section 7) turns out to be computable but we shall not be in position to show this computability property before knowing that $\mathcal{D}_0$ is complete (i.e. theorem 9.2).*

Given a strategy $\mathcal{S}$, we define $\mathcal{T}(\mathcal{S}, A)$, the proof-tree associated to the strategy $\mathcal{S}$ and the assertion A as the unique tree t such that :

$$\varepsilon \in dom(t), \quad t(\varepsilon) = A,$$

and, for every path $x_0 x_1, \cdots x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \leq i \leq n-1$) if x_{n-1} has m sons $x_{n-1} \cdot 1, \cdots . x_{n-1} \cdot m \in \mathrm{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \leq j \leq m$) then

$$(\mathcal{S}(A_1 \cdots A_n) = B_1 \cdots B_m) \text{ or } (m = 0 \text{ and } \exists i \in [1, n-1], A_i = A_n) \text{ or } (m = 0 \text{ and } A_1 \cdots A_n \notin \mathrm{dom}(\mathcal{S})). \tag{45}$$

Notice that x_{n-1} is a leaf (i.e. $m = 0$) iff:

$$(\exists i \in [1, n-1], A_i = A_n) \text{ or } (\mathcal{S}(A_1 \cdots A_n) = \varepsilon) \text{ or } (A_1 \cdots A_n \notin dom(\mathcal{S})). \tag{46}$$

Let us say that $\mathcal{S}$ *terminates* iff, $\forall A \in \chi^{-1}(1), \mathcal{T}(\mathcal{S}, A)$ is finite; $\mathcal{S}$ is said *closed* iff, $\forall W \in \mathcal{A}^+, W \in (\chi^{-1}(1))^+ \implies W \in dom(\mathcal{S})$ (i.e. $\mathcal{S}$ is defined on every non-empty sequence of true assertions). For every tree t let us define:

$$\mathcal{L}(t) = \{t(x) | \forall y \in \mathrm{dom}(t), x \preceq y \Rightarrow x = y\}, \quad \mathcal{I}(t) = \{t(x) | \exists y \in \mathrm{dom}(t), x \prec y\}.$$

(Here $\mathcal{L}$ stands for "leaves" and $\mathcal{I}$ stands for "internal nodes").

Lemma 4.7 *If $\mathcal{S}$ is a strategy for the deduction-system $\mathcal{D}$ then, for every true assertion A*
(1) the set of labels of $\mathcal{T}(\mathcal{S}, A)$ is a $\mathcal{D}$-proof, relative to the set $\mathcal{L}(\mathcal{T}(\mathcal{S}, A)) - \mathcal{I}(\mathcal{T}(\mathcal{S}, A))$.
(2) every label of a leaf is true.

Proof: Let us suppose that $H(A) = \infty$. Let $t = \mathcal{T}(\mathcal{S}, A), P = \mathrm{im}(t)$ (the set of labels of t), $\mathcal{H} = \mathcal{L}(\mathcal{T}(\mathcal{S}, A)) - \mathcal{I}(\mathcal{T}(\mathcal{S}, A))$.
Using (S2), one can prove by induction on the depth of $x \in \mathrm{dom}(t)$ that, $H(t(x)) = \infty$. Point (2) is then proved. Let x be an internal node of t, with sons $x \cdot 1, x \cdot 2, \cdots, x \cdot m$ ($m \geq 0$), and with ancestors $y_1, y_2, \cdots, y_{n-1}, y_n = x$ ($n \geq 1$), such that

$$t(y_1) \cdots t(y_n) = A_1 \cdots A_n, \quad t(x_1)t(x_2) \cdots t(x_m) = B_1 \cdots B_m.$$

By definition of $\mathcal{T}(\mathcal{S}, A)$,

$$\mathcal{S}(A_1 \cdots A_n) = B_1 \cdots B_m$$

and by condition (S1):

$$\exists Q \subseteq \{A_i \mid i \leq n-1\}, \text{ such that } \{B_j \mid 1 \leq j \leq m\} \bigcup Q \mid\!\!\!- A_n.$$

It follows that for every $p \notin \mathcal{H}, \exists R \subseteq P, R \mid\!\!\!- p$, hence

$$\forall p \in P, (\exists R \subseteq P, R \mid\!\!\!- p) \text{ or } p \in \mathcal{H}.$$

Point (1) is proved. □

Lemma 4.8 *: If $\mathcal{S}$ is a closed strategy for $\mathcal{D}$, then, for every true assertion A, the set of labels of $\mathcal{T}(\mathcal{S}, A)$ is a $\mathcal{D}$-proof.*

Proof : Let us suppose that $H(A) = \infty$. Let $t = \mathcal{T}(\mathcal{S}, A)$ and let $P, \mathcal{H}$ be defined as above. By lemma 4.7, P is a $\mathcal{D}$-proof relative to $\mathcal{H}$. By lemma 4.7 point (2) and lemma 4.1, every label of a node of t is true. By the definition of a closed strategy, if $p \in \mathcal{H}$ and x is a leaf of t such that $p = t(x)$ then, the only possible true assertion in clause (46) is "$\mathcal{S}(A_1 \cdots A_n) = \varepsilon$", which implies that

$$\exists Q \subseteq P, Q \mid\!\!\!- t(x).$$

Lemma 4.7 point (1) and this fact show that P is a proof.

Lemma 4.9 *: If $\mathcal{D}$ admits some terminating, closed strategy then $\mathcal{D}$ is complete.*

Proof: Clear from lemma 4.8. □

Remark 4.10 *By the same arguments, if $\mathcal{D}$ admits some closed (but not necessarily terminating) strategy then $\mathcal{D}$ is ∞-complete in the sense that every true assertion has a $\mathcal{D}$-proof. This might be helpful only in cases where the proof-trees and the associated proofs are* regular *in a reasonable sense. This point of view will not be developped here. The comparison algorithms based on Valiant's methods of alternate-staking or parallel-stacking, might be seen in this way (this idea is due to B. Courcelle, thanks to him and to M. Oyamaguchi for discussions on this subject).*

5 Triangulations

Let $S_1, S_2, \cdots, S_d$ be a family of deterministic series over the structured alphabet V (We recall V is the alphabet associated with some dpda $\mathcal{M}$ as defined in section 2.2).
Let us consider a sequence $\mathcal{S}$ of n "weighted" linear equations :

$$(\mathcal{E}_i) : p_i, \sum_{j=1}^{d} \alpha_{i,j} S_j \ , \ \sum_{j=1}^{d} \beta_{i,j} S_j$$

where $p_i \in \mathbb{N} - \{0\}$, and $A = (\alpha_{i,j}), B = (\beta_{i,j})$ are deterministic rational matrices of dimension (n, d), with indices $m \leq i \leq m + n - 1, 1 \leq j \leq d$.

For any weighted equation, $\mathcal{E} = (p, S, S')$, we recall the "cost" of this equation is : $H(\mathcal{E}) = p + 2 \cdot \mathrm{Div}(\varphi(S), \varphi(S'))$.

We associate to such a system another system of equations, $\mathrm{INV}(\mathcal{S})$, which "translates the equations of $\mathcal{S}$ into equations over $(\alpha_{i,j}, \beta_{i,j})$ only". This function INV is in some sense an "elaborated version" of the *inverse* systems defined in [Mei89, Mei92].
The general idea of the construction of INV consists in iterating the transformation used in the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in lemma 3.21, i.e. the classical idea of *triangulating* a system of linear equations. Of course we must deal with the weights and relate the construction with the deduction system $\mathcal{D}_0$.

5.1 Restricted systems

We assume here that

$$\forall j \in [1, d], S_j \not\equiv \emptyset. \tag{47}$$

A system $\mathcal{S}$ fulfilling hypothesis (47) will be called a *restricted* system of weighted linear equations.
In all the following we use the valuation map, $\nu : \mathsf{B}_{1,d} < X > \longrightarrow \mathbb{N} \cup \{\infty\}$ defined by: for every $\gamma \in \mathsf{B}_{1,d} < X >$

$$\nu(\gamma) = \min\{|u|, u \in \cup_{1 \leq j \leq d} \gamma_j\}.$$

Let us define $\mathrm{INV}(\mathcal{S})$, $\mathrm{W}(\mathcal{S}) \in \mathbb{N} \cup \{\perp\}$, $\mathrm{D}(\mathcal{S}) \in \mathbb{N}$ by induction on n. $\mathrm{W}(\mathcal{S})$ is the *weight* of $\mathcal{S}$. $\mathrm{D}(\mathcal{S})$ is the *weak codimension* of $\mathcal{S}$.
Case 1 : $\varphi(\alpha_{m,*}) = \varphi(\beta_{m,*})$ or $n = 1$

$$\mathrm{INV}(\mathcal{S}) = ((W(\mathcal{S}), \alpha_{m,j}, \beta_{m,j}))_{1 \leq j \leq d}, \ \mathrm{W}(\mathcal{S}) = p_m - 1, \ \mathrm{D}(\mathcal{S}) = 0.$$

Case 2 : $\varphi(\alpha_{m,*}) \neq \varphi(\beta_{m,*}), n \geq 2, p_{m+1} - p_m \geq 2 \cdot \nu(\triangle(\varphi(\alpha_{m,*}), \varphi(\beta_{m,*}))) + 1$
Let $u = \min\triangle(\varphi(\alpha_{m,*}), \varphi(\beta_{m,*}))$. Suppose $u \in \triangle(\varphi(\alpha_{m,j_0}), \varphi(\beta_{m,j_0}))$.

Subcase 1 : $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u = \emptyset$.
Let us consider the equation

$$p_m, S_{j_0}, \sum_{j=1}^{d} (\beta_{m,j} \odot u) S_j$$

and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by :

$$(\mathcal{E}'_i) : p_i, \sum_{j \neq j_0} (\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j} \odot u)) S_j \ , \ \sum_{j \neq j_0} (\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j} \odot u)) S_j$$

where the above equation is seen as as an equation between two linear combinations of the S_i's where the j_0-th coefficient is $\emptyset$ on both sides. We then define :

$$\mathrm{INV}(\mathcal{S}) = \mathrm{INV}(\mathcal{S}'), \mathrm{W}(\mathcal{S}) = \mathrm{W}(\mathcal{S}'), \mathrm{D}(\mathcal{S}) = \mathrm{D}(\mathcal{S}') + 1.$$

Subcase 2 : $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u \neq \emptyset$.

Let us consider the w-equation

$$p_m, S_{j_0} \ , \ \sum_{j=1}^{d} (\beta_{m,j_0} \odot u)^* (\beta_{m,j} \odot u) S_j$$

and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by :

$$(\mathcal{E}'_i) : p_i, \sum_{j \neq j_0} [(\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j_0} \odot u)^* (\beta_{m,j} \odot u)] S_j \ , \ \sum_{j \neq j_0} [(\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j_0} \odot u)^* (\beta_{m,j} \odot u)] S_j.$$

(Here also the above equation is seen as an equation between two linear combinations of the S_i's, $1 \leq i \leq d$).
Subcase 3 : $\alpha_{m,j_0} \odot u = \emptyset, \beta_{m,j_0} \odot u = \varepsilon$
(analogous to subcase 1).
Subcase 4 : $\alpha_{m,j_0} \odot u \neq \emptyset, \beta_{m,j_0} \odot u = \varepsilon$
(analogous to subcase 2).
Case 3 : $\varphi(\alpha_{m,*}) \neq \varphi(\beta_{m,*}), n \geq 2, p_{m+1} - p_m \leq 2 \cdot \nu(\triangle(\varphi(\alpha_{m,*}), (\varphi(\beta_{m,*})))$.
We then define:

$$INV(\mathcal{S}) = \perp, W(\mathcal{S}) = \perp, D(\mathcal{S}) = 0,$$

where $\perp$ is a special symbol which can be understood as meaning "undefined".

In all cases we call $D(\mathcal{S})$ the *weak co-dimension* of $\mathcal{S}$. One can check that $D(\mathcal{S}) \leq d - \dim_{\mathsf{DB}<\mathrm{X}>}(\mathsf{V}\{\varphi(S_1), \varphi(S_2), \cdots, \varphi(S_d)\})$, which we call the *strong co-dimension* of $\mathcal{S}$.

Lemma 5.1 *: Let $\mathcal{S}$ be a restricted system of linear equations. If $INV(\mathcal{S}) \neq \perp$ then, $\forall \mathcal{E} \in INV(\mathcal{S}), \mathcal{E}$ is a weighted linear equation with deterministic coefficients.*

Proof: Follows from lemmas 3.18,3.19 and the formula defining $\mathcal{S}'$ from $\mathcal{S}$. $\Box$

Lemma 5.2 *: Let $\mathcal{S}$ be a restricted system of linear equations. If $INV(\mathcal{S}) \neq \perp$ then $INV(\mathcal{S}) = (\bar{\mathcal{E}}_j)_{1 \leq j \leq d}$ fulfills:*

1. $\{\bar{\mathcal{E}}_j \mid 1 \leq j \leq d\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + D(\mathcal{S}) - 1\} \vdash\!\!- \mathcal{E}_{m+D(\mathcal{S})}$
   If, in addition, $n \geq d$ then :
2. $\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + D(\mathcal{S})\} = \infty \Longrightarrow \min\{H(\bar{\mathcal{E}}_j) \mid 1 \leq j \leq d\} = \infty$.

Proof: Let us prove by induction on D($\mathcal{S}$) statement (1) of the lemma.
if D($\mathcal{S}$) $= 0$: as INV($\mathcal{S}$) $\neq \perp$, $\mathcal{S}$ must fulfill the hypothesis of case 1.

$$\mathcal{E}_m = (p_m, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j) = \mathcal{E}_{m+D(\mathcal{S})}$$

$$\mathrm{INV}(\mathcal{S}) = ((p_m - 1, \alpha_{m,j}, \beta_{m,j}))_{1 \leq j \leq d}.$$

Using rules (R3,R7,R8) we obtain :

$$INV(\mathcal{S}) \overset{<2d>}{|\!\vdash\!\!-} (p_m - 1, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d}, \beta_{m,j} S_j) \ |\!\vdash\!\!-_0 \ \mathcal{E}_m.$$

if $D(\mathcal{S}) = m + 1, n \geq 0$: $\mathcal{S}$ must fulfill case 2.
• Suppose **case 2, subcase 1** occurs.
Let us note $\mathcal{E}'_m = (p_m + 2 \mid u \mid, S_{j_0}, \sum_{j=1}^{d} (\beta_{m,j} \odot u) S_j)$. By induction hypothesis :

$$INV(\mathcal{S}') \cup \{\mathcal{E}'_i \mid m + 1 \leq i \leq m + 1 + D(\mathcal{S}') - 1\} \vdash\!\!- \mathcal{E}'_{m+1+D(\mathcal{S}')}$$

which is equivalent to

$$INV(\mathcal{S}) \cup \{\mathcal{E}'_i \mid m + 1 \leq i \leq m + D(\mathcal{S}) - 1\} \vdash\!\!- \mathcal{E}_{m+D(\mathcal{S})}.$$

It remains to show that, $\forall i \in [m + 1, m + D(\mathcal{S}) - 1]$:

$$\{\mathcal{E}_k \mid m \leq k \leq m + D(\mathcal{S}) - 1\} \overset{<*>}{|\!\vdash\!\!-} \mathcal{E}'_i. \tag{48}$$

Using (R6) we get:

$$\mathcal{E}_m \overset{<*>}{|\!\vdash\!\!-} \mathcal{E}'_m.$$

Using (R3) we get

$$\{\mathcal{E}_i, \mathcal{E}'_m\} \overset{<*>}{|\!\vdash\!\!-} (\max\{p_i, p_m{+}2 \mid u \mid\}, \sum_j (\alpha_{i,j}{+}\alpha_{i,j_0}(\beta_{m,j}{\odot}u)) S_j, \sum_j (\beta_{i,j}{+}\beta_{i,j_0}(\beta_{m,j}{\odot}u)) S_j)$$

but the hypothesis of case 2 implies that $\max\{p_{m+1}, p_m{+}2 \mid u \mid\} = p_{m+1}$ and the fact that INV($\mathcal{S}'$) is defined implies that $\forall i \in [m + 1, m + \mathrm{D}(\mathcal{S}) - 1], p_i \geq p_{m+1}$, hence, $\forall i, \max\{p_i, p_m + 2 \mid u \mid\} = p_i$ and the right-hand side of the above

deduction is exactly $\mathcal{E}'_i$. We conclude that statement (1) of the lemma is true.
• Suppose now that **case 2, subcase 2** occurs.

$$\mathcal{E}'_m = (p_m + 2 \mid u \mid, S_{j_0}, \sum_{\substack{j=1 \\ j \neq j_0}}^{d} (\beta_{m,j_0} \odot u)^* (\beta_{m,j} \odot u) S_j).$$

Using (R5) and then (R6) , (this is possible because $\epsilon \notin \varphi(\beta_{m,j_0} \odot u)$), we get:

$$\mathcal{E}_m \overset{<*>}{\mid\!\vdash\!-} \mathcal{E}'_m,$$

and by same arguments as in previous subcase we also have

$$\{\mathcal{E}_i, \mathcal{E}'_m\} \overset{<*>}{\mid\!\vdash\!-} \mathcal{E}'_i, \quad \text{for } m+1 \leq i \leq m + \mathrm{D}(\mathcal{S}) - 1.$$

As in previous subcase we conclude that (48) is true, hence that statement (1) of the lemma is true. The two other subcases can obviously be treated in the same way.
Let us prove statement (2) of the lemma.
Let $\mathrm{supp}(\mathcal{S}) = \{j \in [1,d] \mid \sum_{i=m}^{m+n-1} \alpha_{i,j} + \beta_{i,j} \neq \emptyset\}$ and $\delta(\mathcal{S}) = \mathrm{Card}(\mathrm{supp}(\mathcal{S}))$.
We prove by induction on $\mathrm{D}(\mathcal{S})$ the statement: if $n \geq \delta(\mathcal{S})$ then ,

$$\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + D(\mathcal{S})\} = \infty \Longrightarrow \min\{H(\bar{\mathcal{E}}_j) \mid 1 \leq j \leq d\} = \infty. \quad (49)$$

if $\mathrm{D}(\mathcal{S}) = 0$: as $INV(\mathcal{S}) \neq \perp$, case 1 must occur.

- If $\varphi(\alpha_{m,*}) = \varphi(\beta_{m,*}), \min\{H(\bar{\mathcal{E}}_j) \mid 1 \leq j \leq d\} = \infty$, hence the statement is true.
- If $\varphi(\alpha_{m,*}) \neq \varphi(\beta_{m,*})$ and $n = 1$, let us suppose that $1 = n \geq \delta(\mathcal{S})$.
  - If $\delta(\mathcal{S}) = 1, \mathrm{supp}(\mathcal{S}) = \{j_0\}$, for some $j_0 \in [1,d]$. The implication $[(\alpha_{m,j_0} S_{j_0} \equiv \beta_{m,j_0} S_{j_0}) \Longrightarrow \alpha_{m,j_0} \equiv \beta_{m,j_0}]$ holds (see for example [HHY79]). Hence

    $$\min\{H(\mathcal{E}_i) \mid m \leq i \leq m+D(\mathcal{S})\} = \infty \Longrightarrow \min\{H(\bar{\mathcal{E}}_j) \mid 1 \leq j \leq d\} = \infty.$$

  - If $\delta(\mathcal{S}) = 0$, $\forall j \in [1,d], \bar{\mathcal{E}}_j = (\mathrm{W}(\mathcal{S}), \emptyset, \emptyset)$ and $\min\{H(\bar{\mathcal{E}}_j) \mid 1 \leq j \leq d\} = \infty$.

if $\mathrm{D}(\mathcal{S}) = p+1, p \geq 0$: as $D(\mathcal{S}) \geq 1$ and $INV(\mathcal{S}) \neq \perp$, case 2 must occur.
By the treatment of statement (1) we have established that :

$$\mathcal{E}_m \overset{<*>}{\mid\!\vdash\!-} \mathcal{E}'_m, \{\mathcal{E}_i, \mathcal{E}'_m\} \overset{<*>}{\mid\!\vdash\!-} \mathcal{E}'_i(\text{ for } m+1 \leq i \leq m + D(\mathcal{S}) - 1),$$

hence

$$\{\mathcal{E}_i \mid m \leq i \leq m + D(\mathcal{S}) - 1\} \overset{<*>}{\mid\!\vdash\!-} \{\mathcal{E}'_i \mid m+1 \leq i \leq m+1+D(\mathcal{S}') - 1\}$$

which proves that

$$\min\{H(\mathcal{E}_i) \mid m \leq i \leq m+D(\mathcal{S})\} \leq \min\{H(\mathcal{E}'_i) \mid m+1 \leq i \leq m+1+D(\mathcal{S}')-1\} \tag{50}$$

Let us notice that, $j_0 \notin \mathrm{supp}(\mathcal{S}')$ while $j_0 \in \mathrm{supp}(\mathcal{S})$ and $[1,d] - \mathrm{supp}(\mathcal{S}) \subseteq [1,d] - \mathrm{supp}(\mathcal{S}')$ hence $\delta(\mathcal{S}') \leq \delta(\mathcal{S}) - 1$. It follows that $(n-1) \geq \delta(\mathcal{S}'), \mathcal{S}'$ has $(n-1)$ equations and $D(\mathcal{S}') = D(\mathcal{S}) - 1$. By induction hypothesis:

$$\min\{H(\mathcal{E}'_i) \mid m+1 \leq i \leq m+1+D(\mathcal{S}')-1\} = \infty \Longrightarrow \min\{H(\bar{\mathcal{E}'}_j) \mid 1 \leq j \leq d\} = \infty. \tag{51}$$

As $INV(\mathcal{S}) = INV(\mathcal{S}')$, (50,51) imply statement (2) of the lemma. □

Lemma 5.3 *: Let $\mathcal{S}$ be a restricted system of linear equations satisfying the hypothesis of case 2. Then, $\forall i \in [m+1, m+n-1]$,*
$\| \alpha'_{i,*} \| \leq \| \alpha_{i,*} \| + \| \beta_{m,*} \| + \mid Q \mid\mid u \mid, \| \beta'_{i,*} \| \leq \| \beta_{i,*} \| + \| \beta_{m,*} \| + \mid Q \mid\mid u \mid .$

Proof: Follows from lemmas 3.18,3.19 and the formula defining $\mathcal{S}'$ from $\mathcal{S}$. □

Let us consider the function F defined by :

$$F(n) = max\{\nu(\varphi(A) \triangle \varphi(B)) \mid A, B \in \mathsf{DRB}_{1,d} < V >, \| A \| \leq n, \| B \| \leq n, \varphi(A) \neq \varphi(B)\}.$$

For every integer parameters $K_1, K_2, K_3, K_4 \in \mathbb{N} - \{0\}$, we define integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ by :

$$\delta_m = 0, \ell_m = 0, L_m = K_2, s_m = K_3 \cdot K_2 + K_4, S_m = 0, \Sigma_m = 0, \tag{52}$$

$$\begin{cases} \delta_{i+1} &= 2 \cdot F(s_i + \Sigma_i) + 1 \\ \ell_{i+1} &= 2 \cdot \delta_{i+1} + 3 \\ L_{i+1} &= K_1 \cdot (L_i + \ell_{i+1}) + K_2 \\ s_{i+1} &= K_3 \cdot L_{i+1} + K_4 \\ S_{i+1} &= s_i + \Sigma_i + \mid Q \mid F(s_i + \Sigma_i) \\ \Sigma_{i+1} &= \Sigma_i + S_{i+1} \end{cases} \tag{53}$$

for $m \leq i \leq m+n-2$.

These sequences are intended to have the following meanings when K_1, K_2, K_3, K_4 are chosen to be the constants defined in section 6 and the equations $(\mathcal{E}_i)$ are labelling nodes of a N-stacking sequence (see section 8.4):

$\delta_{i+1} \leq$ increase of weight between $\mathcal{E}_i, \mathcal{E}_{i+1}$
$L_{i+1} \geq$ increase of depth between $\mathcal{E}_m, \mathcal{E}_{i+1}$
$\ell_{i+1} \geq$ increase of depth between $\mathcal{E}_i, \mathcal{E}_{i+1}$
$s_{i+1} \geq$ size of the coefficients of $\mathcal{E}_{i+1}$
$S_{i+1} \geq$ size of the coefficients of $\mathcal{E}_{i+1}^{(i+1-m)}$ (these systems are inroduced below in the proof of lemma 5.4)
$\Sigma_{i+1} \geq$ increase of the coefficients between $\mathcal{E}_k^{(i-m)}, \mathcal{E}_k^{(i+1-m)}$(for $k \geq i+1$).

For every weighted, deterministic rational linear equation $\mathcal{E} = (p, \sum_{j=1}^{d} \alpha_j S_j \ , \ \sum_{j=1}^{d} \beta_j S_j)$, we define

$$||| \mathcal{E} |||= max\{\| \alpha \|, \| \beta \|\}.$$

Lemma 5.4 *Let $\mathcal{S} = (\mathcal{E}_i)_{m \leq i \leq m+d-1}$ be a restricted system of d weighted equations such that :*

(1) $\forall i \in [m, m+d-1], ||| \mathcal{E}_i ||| \leq s_i$
(2) $\forall i \in [m, m+d-2], \mathrm{W}(\mathcal{E}_{i+1}) - \mathrm{W}(\mathcal{E}_i) \geq \delta_{i+1}$.

Then $\mathrm{INV}(\mathcal{S}) \neq \perp, \mathrm{D}(\mathcal{S}) \leq d-1, \forall \mathcal{E} \in \mathrm{INV}(\mathcal{S}), ||| \mathcal{E} ||| \leq \Sigma_{m+\mathrm{D}(\mathcal{S})} + s_{m+\mathrm{D}(\mathcal{S})}$.

Proof: Let us define a sequence of systems $\mathcal{S}^{(i-m)} = (\mathcal{E}_k^{(i-m)})_{m \leq i \leq k \leq m+\mathrm{D}(\mathcal{S})}$, by induction :

- $\mathcal{E}_k^{(0)} = \mathcal{E}_k$ for $m \leq k \leq m + \mathrm{D}(\mathcal{S})$
- if case 1 or case 3 is realized, $\mathrm{D}(\mathcal{S}) = 0$, hence $\mathcal{S}^{(i-m)}$ is well-defined for $m \leq i \leq m + \mathrm{D}(\mathcal{S})$
- if case 2 is realized then we set : $\forall i \geq m+1, \mathcal{E}_k^{(i-m)} = (\mathcal{E'}_k)^{(i-m)}$, for $m+1 \leq k \leq m+\mathrm{D}(\mathcal{S}) = m+1+D(\mathcal{S}')$.

Let us prove by induction on i that, $\forall k \in [i, m+\mathrm{D}(\mathcal{S})]$:

$$||| \mathcal{E}_k^{(i-m)} ||| \leq s_k + \Sigma_i.$$

$i = m$: in this case

$$||| \mathcal{E}_k^{(i-m)} |||=||| \mathcal{E} ||| \leq s_k = s_k + \Sigma_m.$$

$i+1 \leq m + \mathrm{D}(\mathcal{S})$: in this case, by lemma 5.3,

$$||| \mathcal{E}_k^{(i+1-m)} ||| \leq ||| \mathcal{E}_k^{(i-m)} ||| + ||| \mathcal{E}_i^{(i-m)} ||| + | Q || u |$$

where $u = \min\{\varphi(\alpha_{i,*}^{(i-m)}) \triangle \varphi(\beta_{i,*}^{(i-m)})\}$.
By definition of F and the induction hypothesis :

$$| u | \leq F(||| \mathcal{E}_i^{(i-m)} |||) \leq F(s_i + \Sigma_i).$$

Hence

$$\begin{aligned} ||| \mathcal{E}_k^{(i+1-m)} ||| \leq (s_k + \Sigma_i) + (s_i + \Sigma_i) + | Q | F(s_i + \Sigma_i) &= (s_k + \Sigma_i) + S_{i+1} \\ &= s_k + \Sigma_{i+1}. \end{aligned}$$

Let us notice that $\mathrm{D}(\mathcal{S})$ is always an integer and that this proof is valid for $m \leq i \leq m + \mathrm{D}(\mathcal{S}), i \leq k \leq m + \mathrm{D}(\mathcal{S})$.
Let us prove now that $INV(\mathcal{S}) \neq \perp$. Let us consider the system $(\mathcal{E}_k^{(\mathrm{D}(\mathcal{S}))})_{m+\mathrm{D}(\mathcal{S}) \leq k \leq m+d-1}$.
If $\mathrm{D}(\mathcal{S}) = d-1$, $\mathcal{E}^{(\mathrm{D}(\mathcal{S}))}$ fulfills case 1 of the definition of INV, hence $INV(\mathcal{S}) = INV(\mathcal{S}^{(\mathrm{D}(\mathcal{S}))}) \neq \perp$.
If $\mathrm{D}(\mathcal{S}) < d-1$, by hypothesis :

$$W(\mathcal{E}_{D(\mathcal{S})+1}) - W(\mathcal{E}_{\mathrm{D}(\mathcal{S})}) \geq \delta_{\mathrm{D}(\mathcal{S})+1} = 2F(s_{m+\mathrm{D}(\mathcal{S})} + \Sigma_{m+\mathrm{D}(\mathcal{S})}) + 1.$$

If $\mathcal{E}^{(\mathrm{D}(\mathcal{S}))}_{m+\mathrm{D}(\mathcal{S})}$ fulfills case 1 of the definition of INV, then $\mathrm{INV}(\mathcal{S}) \neq \perp$. Otherwise, let $u = \min\triangle(\varphi(\alpha_{m+\mathrm{D}(\mathcal{S}),*}), \varphi(\beta_{m+\mathrm{D}(\mathcal{S}),*}))$:

$$| u | \leq F(\||\, \mathcal{E}^{(\mathrm{D}(\mathcal{S}))}_{m+D(\mathcal{S})} \,\||) \leq F(s_{m+\mathrm{D}(\mathcal{S})} + \Sigma_{m+\mathrm{D}(\mathcal{S})}).$$

Hence $p_{\mathrm{D}(\mathcal{S})+1} - p_{\mathrm{D}(\mathcal{S})} \geq 2 \mid u \mid +1$ i.e. the hypothesis of case 2 is realized. This proves that $\mathrm{D}(\mathcal{S}^{(\mathrm{D}(\mathcal{S}))}) \geq 1$ while in fact, $\mathrm{D}(\mathcal{S}^{(\mathrm{D}(\mathcal{S}))}) = 0$. This contradiction shows that this last case ($\mathrm{D}(\mathcal{S}) < d-1$ and $\mathcal{E}^{(\mathrm{D}(\mathcal{S}))}_{m+\mathrm{D}(\mathcal{S})}$ not fulfilling case 1 of definition of INV) is impossible. We have proved statement (2) of the lemma. □

5.2 General systems

We consider now the general case where assumption (47) is removed. Let us suppose that there exists some $d_1 \in [0, d]$ such that:

$$\forall j \in [1, d_1], S_j \not\equiv \emptyset, \ \ \forall j \in [d_1+1, d], S_j \equiv \emptyset \tag{54}$$

We then associate to the original system $\mathcal{S}$ a new system $\hat{\mathcal{S}}$ of n weighted linear equations:

$$(\hat{\mathcal{E}}_i) : p_i, \sum_{j=1}^{d_1} \alpha_{i,j} S_j \ , \ \sum_{j=1}^{d_1} \beta_{i,j} S_j$$

where $m \leq i \leq m+n-1$.
We then define:

$$\text{If } d_1 = 0 \text{ then } \mathrm{INV}(\mathcal{S}) = \perp,\ \mathrm{W}(\mathcal{S}) = \perp,\ \mathrm{D}(\mathcal{S}) = 0.$$

$$\text{If } d_1 \geq 1 \text{ then } \mathrm{INV}(\mathcal{S}) = \mathrm{INV}(\hat{\mathcal{S}}),\ \mathrm{W}(\mathcal{S}) = \mathrm{W}(\hat{\mathcal{S}}),\ \mathrm{D}(\mathcal{S}) = \mathrm{D}(\hat{\mathcal{S}}).$$

Let us show that lemmas 5.1,5.2,5.4 remain true in the general case.

Lemma 5.5 (preliminary lemma) *For every $i \in [m, m+n-1]$*

(1) $\hat{\mathcal{E}}_i \ \overset{<*>}{|\!\vdash\!\!-} \ \mathcal{E}_i$,
(2) $\mathcal{E}_i \ \overset{<*>}{|\!\vdash\!\!-} \ \hat{\mathcal{E}}_i$.

Proof: Follows from lemma 7.5. □

Lemma 5.6 *: Let $\mathcal{S}$ be a system of linear equations.If $INV(\mathcal{S}) \neq \perp$ then, $\forall \mathcal{E} \in INV(\mathcal{S}), \mathcal{E}$ is a weighted linear equation with deterministic coefficients.*

Proof: As $\hat{\mathcal{S}}$ is a restricted system, this follows from lemma 5.1. □

Lemma 5.7 *: Let $\mathcal{S}$ be a system of linear equations. If $INV(\mathcal{S}) \neq \perp$ then $INV(\mathcal{S}) = (\bar{\mathcal{E}}_j)_{1 \leq j \leq d}$ fulfills:*

1. $\{\bar{\mathcal{E}}_j \mid 1 \leq j \leq d\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + D(\mathcal{S}) - 1\} \models\!\!- \mathcal{E}_{m+D(\mathcal{S})}$
   If, in addition, $n \geq d$ *then :*
2. $\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + D(\mathcal{S})\} = \infty \Longrightarrow \min\{H(\bar{\mathcal{E}}_j) \mid 1 \leq j \leq d\} = \infty.$

Proof: Point (1) follows from lemma 5.2 point (1) and from lemma 5.5. Point (2) follows from lemma 5.2 point (2) and from lemma 5.5, point (2). □

Lemma 5.8 *Let* $\mathcal{S} = (\mathcal{E}_i)_{m \leq i \leq m+d-1}$ *be a system of* d *weighted equations such that :*

(0) $\exists j \in [1, d], S_j \not\equiv \emptyset,$
(1) $\forall i \in [m, m + d - 1], ||| \mathcal{E}_i ||| \leq s_i,$
(2) $\forall i \in [m, m + d - 2], \mathrm{W}(\mathcal{E}_{i+1}) - \mathrm{W}(\mathcal{E}_i) \geq \delta_{i+1}.$

Then $\mathrm{INV}(\mathcal{S}) \neq \perp, \mathrm{D}(\mathcal{S}) \leq d - 1, \forall \mathcal{E} \in \mathrm{INV}(\mathcal{S}), ||| \mathcal{E} ||| \leq \Sigma_{m+\mathrm{D}(\mathcal{S})} + s_{m+\mathrm{D}(\mathcal{S})}.$

Proof: By hypothesis (0), $\hat{\mathcal{S}}$ is defined and is a restricted system of linear equations. Moreover, for every $i \in [m, m + d - 1], ||| \hat{\mathcal{E}}_i ||| \leq ||| \mathcal{E}_i |||$. Hence $\hat{\mathcal{S}}$ fulfills the hypothesis of lemma 5.4 and the conclusion of this previous lemma applied on the system $\hat{\mathcal{S}}$ gives :

$$\mathrm{INV}(\mathcal{S}) \neq \perp, \mathrm{D}(\mathcal{S}) \leq d - 1, \forall \mathcal{E} \in \mathrm{INV}(\mathcal{S}), ||| \mathcal{E} ||| \leq \Sigma_{m+\mathrm{D}(\mathcal{S})} + s_{m+\mathrm{D}(\mathcal{S})}.$$

□

6 Constants

This short section is devoted to the definition of some integer constants. The *motivation* of each of these definitions will appear later on, in different places for the different constants. The equations below provide merely an overview of the dependancies betweens these constants and allow to check that the definitions are sound (i.e. there is no hidden loop in the dependancies).

$$k_0 = \max\{\nu([pAq]) \mid p, q \in Q, A \in Z, [pAq] \not\equiv \emptyset\}, \quad k_1 = \max\{2k_0 + 1, 3\}, \tag{55}$$

$$k_2 = 4k_1 + 2(k_1)^2 + k_0, \quad D_1 = 4k_0 + 2. \tag{56}$$

k_1 is used in the definition of strategy T_B (section7), k_2 is used in the definition of a "security band" before lemma 8.9 and D_1 appears as an upper-bound on the left-defect of series in lemma 8.4.

$$K_1 = k_1 + 1, \quad K_2 = 2(k_1)^3 + 3(k_1)^2 + k_1 + 1. \tag{57}$$

These constants K_1, K_2 appear in lemma 8.12.

$$K_3 = k_0|Q|, \quad K_4 = k_0|Q|^2 + (k_2 + 6)|Q|. \tag{58}$$

These constants K_3, K_4 appear in lemma 8.13.

$$d_0 = 2 \cdot |Q| \cdot \mathrm{Card}(X^{\leq k_1}). \tag{59}$$

d_0 appears as an upper-bound on the dimension of the d-space V_1 defined by equation (103) and used in lemma 8.12. We consider now the integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ defined by the relations (53) of section 5 where the parameters $K_1, \dots, K_4$ are chosen to be the above constants and $m = 1, n = d_0$. Equivalently, they are defined by:

$$\delta_1 = 0, \ell_1 = 0, L_1 = K_2, s_1 = K_3 \cdot K_2 + K_4, S_1 = 0, \Sigma_1 = 0, \tag{60}$$

$$\begin{cases} \delta_{i+1} &= 2 \cdot F(s_i + \Sigma_i) + 1 \\ \ell_{i+1} &= 2 \cdot \delta_{i+1} + 3 \\ L_{i+1} &= K_1 \cdot (L_i + \ell_{i+1}) + K_2 \\ s_{i+1} &= K_3 \cdot L_{i+1} + K_4 \\ S_{i+1} &= s_i + \Sigma_i + \mid Q \mid F(s_i + \Sigma_i) \\ \Sigma_{i+1} &= \Sigma_i + S_{i+1} \end{cases} \tag{61}$$

for $1 \leq i \leq d_0 - 1$.

$$D_2 = \Sigma_{d_0} + s_{d_0}, \tag{62}$$

D_2 appears in the conclusion of lemma 5.8 when we take $d = d_0$ in the hypothesis and suppose that $\mathrm{D}(\mathcal{S})$ has its maximal possible value i.e. $\mathrm{D}(\mathcal{S}) = d_0 - 1$. It is used as an upper-bound on the right-defect in the definition of the trees τ analysed in part 8 (inequation (88)).

$$N_0 = 1 + (k_2 + 1)|Q| + D_2. \tag{63}$$

N_0 appears as a lower bound for the norm in the definition of a N-stacking sequence (section 8.4, condition (92)).

$$C_2 = \mathrm{Card}\{U \in \mathsf{DRB} < V >, \|U\| \leq D_2\} \tag{64}$$

$$K_6 = 7 \cdot [(C_2 + |Z|^{k_2+2})^{|Q|} \cdot |Z|^{D_1}]^2, \quad K_5 = (K_6 + 1) \cdot k_0 \cdot |Q|. \tag{65}$$

K_5, K_6 appear in lemma 8.15.

7 Strategies for $\mathcal{D}_0$

Let us define strategies for the particular system $\mathcal{D}_0$.
We shall define first auxiliary strategies $T_{cut}, T^0_\emptyset, T_\emptyset, T_\varepsilon, T_A, T_B, T_C, T_{A^2,\emptyset}$ and then derive some closed strategies from them. Let us fix here some total ordering on $X : x_1 < x_2 < \cdots < x_\alpha$ and also some total ordering $\leq$ of type ω on $\mathcal{A}$ (inherited from the usual well-ordering of $\mathbb{N}$ by the fixed encoding). From these orderings one can construct in the usual way an ordering of type ω on the sets $X^*, \mathcal{A}^*$ and $\mathbb{N}^* \times (\mathsf{DRB} < V >)^*$.

T_{cut}:

$T_{cut}(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $\exists i \in [1, n-1], \exists S, T,$

$$A_i = (p_i, S, T), A_n = (p_n, S, T), p_i < p_n \text{ and } m = 0$$

$T^0_\emptyset$:

$T^0_\emptyset(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ iff

$$A_n = (p, \emptyset, \emptyset), p \geq 0 \text{ and } m = 0$$

$T_\emptyset$:

$T_\emptyset(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ iff $\exists S, T,$

$$A_n = (p, S, T), p \geq 0, S \equiv T \equiv \emptyset \text{ and } m = 0$$

T_ε:

$T_\varepsilon(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

$$A_n = (p, S, T), p \geq 0, S \equiv T \equiv \varepsilon \text{ and } m = 0$$

T_A:

$T_A(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

$A_n = (p, S, T), m = \mid X \mid, B_1 = (p{+}1, S \odot x_1, T \odot x_1), \cdots, B_m = (p{+}1, S \odot x_n, T \odot x_m),$

where $S \not\equiv \varepsilon, T \not\equiv \varepsilon$

T^+_B:

$T^+_B(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $n \geq k_1, A_{n-k_1} = (\pi, \overline{U}, U')$, (where $\overline{U}$ is unmarked)

$$U' = \sum_{q \in Q} [\overline{p} A q] \cdot V_q \quad (\text{ for some } (\overline{p} \in Q)$$

$A_i = (\pi + k_1 + i - n, U_i, U'_i)$ for $n - k_1 \leq i \leq n$, $(U'i)_{n-k_1 \leq i \leq n}$ is a "stacking derivation" (see definition in §8.2),

$$U'_n = \sum_{q \in Q} [p \tau q] \cdot V_q, \quad \text{for some } p \in Q, \tau \in Z^+,$$

$m = 1, B_1 = (\pi + k_1 - 1, V, V'), V = U_n,$

$$V' = \sum_{q \in Q'} [p \tau q] \cdot [q e q] \cdot (\overline{U} \odot u_q)$$

where $Q' = \{q \in Q \mid [\overline{p} A q] \not\equiv \emptyset\}, \forall q \in Q', u_q = \min(\varphi([\overline{p} A q]))$.

T^-_B:

T^-_B is defined in the same way as T^-_B by exchanging the left series (S^-) and right (S^+) series in every assertion (p, S^-, S^+).

T_C:

$T_C(A_1 \cdots A_n) = B_1 \cdots B_m$ iff there exists $d \in [1, d_0], S_1, S_2, \cdots, S_d \in \mathsf{DRB} < V >, 1 \leq \kappa_1 \leq \kappa_2 < \cdots < \kappa_d = n$, such that,

(C1) every equation $(\mathcal{E}_i) = (p_{\kappa_i} S^{-}_{p_{\kappa_i}}, S^{+}_{p_{\kappa_i}})$ is a weighted equation over $S_1, S_2, \cdots, S_d$, with $p_{\kappa_i} \geq 1$,

(C2) $\mathcal{S} = (\mathcal{E}_i)_{1 \leq i \leq d}$ fulfills the hypothesis of lemma 5.8,

(C3) $(\kappa_1, \kappa_2, \cdots, \kappa_d, S_1, \cdots, S_d) \in \mathbb{N}^* \times (\mathsf{DRB} < V >)^*$ is the minimal vector satisfying conditions (C1,C2) for the given sequence $(A_1 \cdots A_n)$ and

(C4) $B_1 \cdots B_m = \rho_e(INV(\mathcal{S}))$ (where ρ_e is the obvious extension of ρ_e to pairs of series and then to sequences of weighted equations; in other words the result of T_C is $INV(\mathcal{S})$ where the marks have been removed).

$T_{A^2,\emptyset}$:

$T_\pi(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

(e 1) $A_n = (\pi, [pzq], \emptyset)$ for some $\pi \in \mathbb{N}, p, q \in Q, z \in Z$

(e 2) $(B_1, B_2, \cdots, B_m)$ is the smallest element of $\mathcal{A}^+$ fulfilling the conditions (e3,e4,e5) below :

(e 3) $\forall x \in X, [pzq] \odot x \neq \varepsilon$,

(e 4) $\forall j \in [1, m], B_j = (\pi + 1, [p_j z_j q_j], \emptyset)$ and $\exists u \in X^2, \exists w \in V^+, w \in [pzq] \odot u, [p_j z_j q_j]$ is a factor of w and $[p_j z_j q_j] \equiv \emptyset$,

(e 5) $\forall u \in X^2$, for every word $w \in [pzq] \odot u, \exists j \in [1, m], [p_j z_j q_j]$ is a factor of w.

Lemma 7.1 *: $T_{cut}, T^0_\emptyset, T_A, T_{A^2,\emptyset}$ are $\mathcal{D}_0$ strategies.*

Proof:

T_{cut} : (S1) is true by rule R0. (S2) is trivially true.

$T^0_\emptyset$: $\emptyset \models\!\!\!- (p, \emptyset, \emptyset)$ (if $p \geq 1$), which proves (S1) and $\min\{H(B_j) \mid 1 \leq j \leq m\} = \min(\emptyset) = \infty$, which proves (S2) .

T_A : by rule (R4), $\{(B_j \mid 1 \leq j \leq m\} \mid\!\models\!\!\!-_4 A_n$, which proves (S1). Suppose $H(A_n) = \infty$ i.e. $S \equiv T$. Then, $\forall x \in X$, $S \odot x \equiv T \odot x$, so that $\min\{H(B_j) \mid 1 \leq j \leq m\} = \infty$. (S2) is proved.

$T_{A^2,\emptyset}$: By $(e5)$,

$$\forall u \in X^2, \forall w \in [pzq] \odot u, \{B_j, 1 \leq j \leq m\} \overset{<*>}{\mid\!\models\!\!\!-}_{8,9} (\pi + 1, w, \emptyset).$$

Using rule R7 we get

$$\forall u \in X^2, \{B_j, 1 \leq j \leq m\} \overset{<*>}{\mid\!\models\!\!\!-}_{7} (\pi + 1, [pzq] \odot u, \emptyset)$$

Using R4 and $(e3)$

$$\{(\pi + 1, [pzq] \odot u, \emptyset), \mid u \mid \leq 2\} \overset{<*>}{\mid\!\models\!\!\!-} (\pi - 1, [pzq], \emptyset),$$

and at last

$$\{\pi - 1, [pzq], \emptyset)\} \mid\!\!\vdash\!\!-_0 (\pi, [pzq], \emptyset)\}.$$

Hence

$$\{B_j \mid 1 \leq j \leq m\} \vdash\!\!- A_n,$$

which establishes (S1). By ($e4$),

$$\min\{H(B_j) \mid 1 \leq j \leq m\} = \infty,$$

which establishes (S2).
□

Let us consider the following strategy $\mathcal{S}_\emptyset$: for every $W = A_1, A_2, ..., A_n \in \mathcal{A}$,

(0) if T_{cut} is defined on $W, \mathcal{S}_\emptyset(W) = T_{cut}(W)$
(1) elsif $T^0_\emptyset$ is defined on $W, \mathcal{S}_\emptyset(W) = T^0_\emptyset(W)$
(2) elsif $T_{A^2,\emptyset}$ is defined on $W, \mathcal{S}_\emptyset(W) = T_{A^2,\emptyset}(W)$
(3) else $\mathcal{S}_\emptyset(W)$ is undefined.

Lemma 7.2 *: Let $p, q \in Q, z \in Z.[pzq] \equiv \emptyset$ iff $\emptyset \vdash\!\!- (0, [pzq], \emptyset)$.*

Proof: Let us suppose $[pzq] \equiv \emptyset$. Let $A = (0, [pzq], \emptyset)$. Let $t = \mathcal{T}(\mathcal{S}_\emptyset, A)$. The definitions of $T^0_\emptyset, T_\varepsilon, T_{A^2,\emptyset}$ show that the labels of t belong to $\mathbb{N} \times V \times \{\emptyset\}$. Moreover the definition of $T_{A^2,\emptyset}$ implies that the sequence of weights on any branch is strictly increasing (from the root to the leaves). The definition of T_{cut}, then implies that every branch of t is finite, hence, by Koenig's lemma, t is a finite tree.

Let us consider the label of a leaf x of t : $(\pi, [p'z'q'], \emptyset)$. Let $W = A_1 A_2 \cdots A_n$ be the word associated to the branch ending at x. Suppose that $W \notin \mathrm{dom}(T_{cut}) \cup \mathrm{dom}(T^0_\emptyset)$ and $H(A) = \infty$. Then, by lemma 4.7, $H(A_n) = \infty$. It follows that

$$\forall x \in X, [p'z'q'] \odot x \neq \varepsilon \ \ (e3) \quad \text{and } \forall u \in X^2, [p'z'q'] \odot u = \emptyset,$$

hence

$$\text{every word } w \in [p'z'q'] \odot u \text{ must contain at least one letter } [p"z"q"] \equiv \emptyset. \quad (66)$$

But (66) implies that there exists some $B_1, B_2, \cdots, B_m$ fulfilling conditions ($e2, e3, e4, e5$). Hence $W \in \mathrm{dom}(\mathcal{S}_{A^2,\emptyset})$, contradicting the fact that x is a leaf. This proves that in fact every leaf of t belongs to $\mathrm{dom}(T_{cut}) \cup \mathrm{dom}(T^0_\emptyset)$. By lemma 4.7, $\mathrm{im}(t)$ is a $\mathcal{D}_0$-proof, containing A. □

Lemma 7.3 *: T^+_B, T^-_B are $\mathcal{D}_0$-strategies.*

Proof: Let us show that T_B^+ is a $\mathcal{D}_0$-strategy.
Let $\mathcal{H} = \{(\pi, \overline{U}, U'), (\pi + k_1 - 1, V, V')\}$. Let us show that

$$\mathcal{H} \vdash\!\!- (\pi + k_1, U_n, U_n'). \tag{67}$$

Using rule (R5) we obtain:

$$\{(\pi, \overline{U}, U')\} = \{(\pi, \overline{U}, \sum_{q \in Q} [\overline{p} A q] \cdot V_q)\} \mid\!\vdash\!\!-_6 (\pi + 2 \mid u_q \mid, \overline{U} \odot u_q, V_q). \tag{68}$$

By lemma 7.2, for every q such that $[\overline{p} A q] \equiv \emptyset$,

$$\emptyset \vdash\!\!- (0, [\overline{p} A q], \emptyset) \tag{69}$$

Let us show that, for every q such that $[\overline{p} A q] = \emptyset$,

$$\emptyset \overset{<*>}{\mid\!\vdash\!\!-} (0, [p, \tau, q], \emptyset). \tag{70}$$

$[\overline{p} A q] \equiv \emptyset$ and $[\overline{p} A q] \odot u = [p \tau q]$ (for some $u \in X^{k_1}$) imply that $[p \tau q] \equiv \emptyset$. Using lemma 7.2 and rules (R0,R2,R7,R8,R9), we obtain deduction (70). From this deduction, using rules (R0,R1,R2,R7,R8) we obtain

$$\emptyset \overset{<*>}{\mid\!\vdash\!\!-} (\pi + 2k_0, U_n', \sum_{[\overline{p} A q] \not\equiv \emptyset} [p \tau q] \cdot V_q). \tag{71}$$

Using rule (R'3), for every $q \in Q$,

$$\emptyset \mid\!\vdash\!\!-_{3'} (0, [q e q], \varepsilon). \tag{72}$$

Using (68) (72) and rules (R0,R1,R2,R7,R8,R9) we obtain :

$$\{(\pi, \overline{U}, U')\} \overset{<*>}{\mid\!\vdash\!\!-} (\pi + 2k_0, \sum_{[\overline{p} A q] \not\equiv \emptyset} [p \tau q] \cdot V_q, \sum_{[\overline{p} A q] \not\equiv \emptyset} [p \tau q][q e q] \cdot (\overline{U} \odot u_q)). \tag{73}$$

By (71) (73) and (R0,R3) we get:

$$\{(\pi, \overline{U}, U')\} \overset{<*>}{\mid\!\vdash\!\!-} (\pi + 2k_0, U_n', V'). \tag{74}$$

Let us recall that $U_n = V$. Hence

$$\{(\pi + k_1 - 1, V, V'), (\pi + 2k_0, U_n', V')\} \overset{<*>}{\mid\!\vdash\!\!-}_1 (\pi + k_1 - 1, U_n, U_n'). \tag{75}$$

At last :

$$\{(\pi + k_1 - 1, U_n, U_n')\} \mid\!\vdash\!\!-_0 (\pi + k_1, U_n, U_n'). \tag{76}$$

By (74,75,76), (67) is proved. Hence T_B^+ fulfills (S1).
Let us suppose now that $\forall i \in [n - k_1, n], U_i \equiv U_i'$. Then, by (74), $U_n' \equiv V'$ and by hypothesis $V = U_n \equiv U_n'$. Hence $V \equiv V'$. This shows that T_B^+ fulfills (S2).
An analogous proof can obviously be written for T_B^-. □

Lemma 7.4 *: Let $S, T, U \in \mathsf{DB} < V >$ such that $U = S^* \cdot T$ and $U \equiv \epsilon$. Then either $S = \epsilon$ or $S \equiv \emptyset$.*

Proof: Let us consider S, T, U fulfilling the hypothesis of the lemma. By lemma 3.11 $\rho_\epsilon(U) \equiv U \equiv \epsilon$. But the type of $\rho_\epsilon(U)$ cannot be $([pz])$ for any $p \in Q, z \in Z$, hence $\rho_\epsilon(U) = \epsilon$. Let us suppose that

$$\exists u \in \mathrm{supp}(S), |u| \geq 1, u \not\equiv \emptyset. \tag{77}$$

Then $u \equiv \epsilon$, hence $u = [p_1, z_1, q_1] \cdots [p_n, z_n, q_n]$ where $n \geq 0$ and for every $i \in [1, n]$, $[p_i, z_i, q_i] \otimes e = \epsilon$. By the same arguments as in the proof of lemma 3.11 we obtain

$$U = u \cdot (U \bullet u) + U'_1 \text{ where } U'_1 \equiv \emptyset.$$

But every u^k (for $k \in \mathbb{N}$) is also a prefix of some $w \in \mathrm{supp}(U)$. Hence for every $k \in \mathbb{N}$:

$$U = u^k \cdot (U \bullet u^k) + U'_k \text{ where } U'_k \equiv \emptyset. \tag{78}$$

As $\lim_{k \to +\infty} |u^k| = +\infty$, equations (78) imply that $U = \sum_{k=1}^{\infty} U'_k$, and by σ-additivity of φ,

$$\varphi(U) = \sum_{k=1}^{\infty} \varphi(U'_k) = \emptyset.$$

This last equality contradicts the hypothesis "$U \equiv \epsilon$". Hence supposition (77) is impossible i.e. $\forall u \in \mathrm{supp}(S), |u| = 0$ or $u \equiv \emptyset$. As S is supposed deterministic, this implies that either $S = \epsilon$ or $S \equiv \emptyset$. □

Lemma 7.5 *: Let $S \in \mathsf{DRB} < V >$.*

(1)$S \equiv \emptyset$ iff $\emptyset \models\!\!- (0, S, \emptyset)$.
(2)$S \equiv \epsilon$ iff $\emptyset \models\!\!- (0, S, \epsilon)$.

Sketch of proof: Let $S \in \mathsf{DRB} < V >$. We identify S with its support and consider a trim deterministic finite automaton recognizing S:

$$\mathcal{B} =< V, Q_{\mathcal{B}}, \delta, q_-, Q_{\mathcal{B},+} >,$$

where we can suppose without loss of generality that $Q = [1, n]$ for some $n \in \mathbb{N}$, $q_- = 0$. We denote by $\tilde{\delta} : Q_{\mathcal{B}} \times V^* \to Q_{\mathcal{B}}$ the usual extension of δ to words of arbitrary length. As is done in some classical algorithms translating any finite automaton into an equivalent rational expression, for every $i, j \in [1, n], k \in [0, n]$ we define $S^k_{i,j}$ as the set of all $w \in V^*$ such that the sequence of states $(q_0, q_1, \ldots, q_\ell)$ produced by the computation of $\mathcal{B}$ on w, is such that its *interior* is inside $[1, k]$ i.e.

$$\forall m \in [1, \ell - 1], q_m \in [1, k].$$

We then define

$$\bar{S}^k_{i,j} = S^k_{i,j} - S^k_{i,j} \cdot V^+ (\text{ for } i, j \in [1, n])$$

It is clear from this definition that every $\bar{S}^k_{i,j}$ is a *prefix* set. One can then establish the following recurrence relations (for $i, j \in [1, n], k \in [0, n-1]$):

$$\bar{S}^{k+1}_{i,k+1} = \bar{S}^k_{i,k+1} \tag{79}$$
$$\bar{S}^{k+1}_{k+1,j} = (\bar{S}^k_{k+1,k+1})^* \cdot \bar{S}^k_{k+1,j} (\text{ for } j \neq k+1) \tag{80}$$
$$\bar{S}^{k+1}_{i,j} = \bar{S}^k_{i,j} + \bar{S}^k_{i,k+1} \cdot (\bar{S}^k_{k+1,k+1})^* \cdot \bar{S}^k_{k+1,j} (\text{ for } i \neq k+1, j \neq k+1). \tag{81}$$

Let us prove that

$$\forall i, j \in [1, n], \forall k \in [0, n], \bar{S}^k_{i,j} \in \mathsf{DRB} < V > . \tag{82}$$

(of course the only non-trivial property is that $\bar{S}^k_{i,j}$ is *deterministic*). As $\mathcal{B}$ is trim, for every $i \in [1, n]$ there exists $u_i \in V^*, v_i \in V^*$ such that

$$\tilde{\delta}(0, u_i) = i, \tilde{\delta}(i, v_i) \in Q_{\mathcal{B},+}.$$

Let $u \in V^*$. We then have:

$$\bar{S}^k_{i,j} \bullet u \leq S^n_{i,j} \bullet u = (S^n_{0,j} \bullet u_i) \bullet u = S^n_{0,j} \bullet (u_i \cdot u).$$

Multiplying the above inequality by v_j on the right we obtain:

$$(\bar{S}^k_{i,j} \bullet u) \cdot v_j \leq (S^n_{0,j} \bullet (u_i \cdot u)) \cdot v_j \tag{83}$$
$$\leq (S^n_{0,j} \cdot v_j) \bullet (u_i \cdot u) \tag{84}$$
$$\leq S \bullet (u_i \cdot u). \tag{85}$$

If $\bar{S}^k_{i,j} \bullet u = \emptyset$ then it is left-deterministic. If $\epsilon \in \bar{S}^k_{i,j} \bullet u$ then, as $\bar{S}^k_{i,j}$ is a prefix set, $\bar{S}^k_{i,j} \bullet u = \epsilon$, hence is left-deterministic again. Otherwise, $\bar{S}^k_{i,j} \bullet u \subseteq V^+$. In this last case, inequality (85) shows that $\bar{S}^k_{i,j} \bullet u$ is left deterministic of the same type as $S \bullet (u_i \cdot u)$. Hence property (82) is established.

For every $k \in [0, n]$, we consider the following property $\mathcal{P}(k)$: $\forall i, j \in [1, n]$,

$$(\bar{S}^k_{i,j} \equiv \emptyset \Rightarrow \emptyset \vdash\!\!- (0, \bar{S}^k_{i,j}, \emptyset)) \text{ and } (\bar{S}^k_{i,j} \equiv \epsilon \Rightarrow \emptyset \vdash\!\!- (0, \bar{S}^k_{i,j}, \epsilon)). \tag{86}$$

We prove that $\mathcal{P}(k)$ is true for every $k \in [0, n]$ by induction on k.
$k = 0$: If $\bar{S}^0_{i,j} \in \{\emptyset, \epsilon\}$, property (86) is true, by rule (R3).
If $\bar{S}^0_{i,j} = [pzq]$, for some $p, q \in Q, z \in Z$, then (86) is true either by lemma 7.2 (if $[pzq] \equiv \emptyset$) or by rule (R'3) (if $[pzq] \equiv \epsilon$).
$k \rightarrow k+1$: let us suppose first that $i \neq k+1, j \neq k+1$ and let us consider relation (81).
case 1: $\bar{S}^{k+1}_{i,j} \equiv \emptyset$
Then $\bar{S}^k_{i,j} \equiv \emptyset$ and ($\bar{S}^k_{i,k+1} \equiv \emptyset$ or $\bar{S}^k_{k+1,j} \equiv \emptyset$). By induction hypothesis these equivalences imply that:

$$\emptyset \vdash\!\!- (0, \bar{S}^k_{i,j}, \emptyset) \text{ and } [\emptyset \vdash\!\!- (0, \bar{S}^k_{i,k+1}, \emptyset) \text{ or } \emptyset \vdash\!\!- (0, \bar{S}^k_{k+1,j}, \emptyset)].$$

Using now rules (R2,R7,R8,R9) and the above deductions we obtain that

$$\emptyset \vdash\!\!- (0, \bar{S}^{k+1}_{i,j}, \emptyset).$$

case 2: $\bar{S}^{k+1}_{i,j} \equiv \epsilon$
subcase 1: $\bar{S}^{k}_{i,j} \equiv \epsilon$ and ($\bar{S}^{k}_{i,k+1} \equiv \emptyset$ or $\bar{S}^{k}_{k+1,j} \equiv \emptyset$). Then, as in previous case, we can deduce from the induction hypothesis and from rules (R2,R7,R8,R9) that:

$$\emptyset \models\!\!- (0, \bar{S}^{k+1}_{i,j}, \epsilon).$$

subcase 2: ($\bar{S}^{k}_{i,j} \equiv \epsilon$ or $\bar{S}^{k}_{i,j} \equiv \emptyset$) and $\bar{S}^{k}_{i,k+1} \cdot (\bar{S}^{k}_{k+1,k+1})^{*} \cdot \bar{S}^{k}_{k+1,j} \equiv \epsilon$.
Then $\bar{S}^{k}_{i,k+1} \equiv \epsilon$ and $U = (\bar{S}^{k}_{k+1,k+1})^{*} \cdot \bar{S}^{k}_{k+1,j} \equiv \epsilon$. Moreover, as U is a residual of a deterministic series, U is deterministic too. Applying lemma 7.4 to $U, S = \bar{S}^{k}_{k+1,k+1}, T = \bar{S}^{k}_{k+1,j}$ we obtain that:

$$\bar{S}^{k}_{k+1,k+1} = \epsilon \text{ (A1)} \quad \text{or } \bar{S}^{k}_{k+1,k+1} \equiv \emptyset \text{ (A2)} .$$

If (A1) is realized, then

$$\bar{S}^{k}_{i,k+1} \cdot (\bar{S}^{k}_{k+1,k+1})^{*} \cdot \bar{S}^{k}_{k+1,j} = \bar{S}^{k}_{i,k+1} \cdot \bar{S}^{k}_{k+1,j}.$$

As in case 1, we can deduce from the induction hypothesis and from rules (R2,R7,R8,R9) that:

$$\emptyset \models\!\!- (0, \bar{S}^{k+1}_{i,j}, \epsilon).$$

If (A2) is realized, then we obtain successively:
$\emptyset \models\!\!- (0, \bar{S}^{k}_{k+1,k+1}, \emptyset)$ (by induction hypothesis
$\emptyset \models\!\!- (0, \bar{S}^{k}_{k+1,k+1} \cdot \bar{S}^{k}_{k+1,j}, \emptyset)$ (using(R8))
$\emptyset \models\!\!- (0, \bar{S}^{k}_{k+1,k+1} \cdot \bar{S}^{k}_{k+1,j} + \bar{S}^{k}_{k+1,j}, \bar{S}^{k}_{k+1,j})$ (using (R7))

$$\emptyset \models\!\!- (0, (\bar{S}^{k}_{k+1,k+1})^{*} \cdot \bar{S}^{k}_{k+1,j}, \bar{S}^{k}_{k+1,j}) \tag{87}$$

(using (R6), because $\bar{S}^{k}_{k+1,k+1} \not\equiv \epsilon$)
Using now the induction hypothesis and deduction (87) we obtain that:

$$\emptyset \models\!\!- (0, \bar{S}^{k}_{i,k+1} \cdot (\bar{S}^{k}_{k+1,k+1})^{*} \cdot \bar{S}^{k}_{k+1,j}, \epsilon)$$

i.e. $\emptyset \models\!\!- (0, \bar{S}^{k+1}_{i,j}, \epsilon)$. When the assumption that $i \neq k+1, j \neq k+1$ is removed, we can draw a similar proof with the help of relations (79) or (80) instead of (81). (End of the induction).
As S is deterministic, it is prefix, hence

$$S = \sum_{j \in Q_{\mathcal{B},+}} \bar{S}^{n}_{0,j}.$$

Using $\mathcal{P}(n)$ and rules (R2,R7), one can see that points (1) and (2) of the lemma are true. □

Lemma 7.6 *: $T_\emptyset, T_\varepsilon$ are $\mathcal{D}_0$-strategies.*

Proof: Let us suppose $T_\emptyset(A_1 \cdots A_n) = \emptyset$. Hence $A_n = (\pi, S, T)$ where $S \equiv T \equiv \emptyset$. By lemma 7.5 and rules (R1,R2),

$$\emptyset \models\!\!\!- (\pi, S, T) = A_n,$$

hence (S1) is proved.

$$\min\{H(B_j) \mid 1 \leq j \leq m\} = \min(\emptyset) = \infty,$$

hence (S2) is proved. The same arguments apply as well on T_ϵ. □

Lemma 7.7 *Let* $S \in \mathsf{DRB} < V >$. *Then* $\emptyset \models\!\!\!- (0, S, \rho_e(S)$.

Sketch of proof: Let us notice that $\emptyset \models\!\!\!- (0, S, \rho_e(S)$ is true for $S \in \{\emptyset, \epsilon\}$ (by rule R3) and also for $S = [p, e, q]$ for some $p, q \in Q$ (either by rule R'3 if $p = q$ or by lemma 7.5 if $p \neq q$). We can then extend the property to every deterministic rational series S by induction on k using the formulas (79,80,81) of the proof of lemma 7.5. □

Lemma 7.8 *Let* (p, S, S') *be a weighted equation , i.e.* $p \in \mathbb{N}, S, S' \in \mathsf{DRB} < V >$. *Then* $\{(p, S, S')\} \overset{<*>}{||\!\!\!-} \{(p, \rho_e(S), \rho_e(S'))\}$ *and* $\{(p, \rho_e(S), \rho_e(S'))\} \overset{<*>}{||\!\!\!-} \{(p, S, S')\}$.

Proof: This can be easily deduced from lemma 7.7. □

Lemma 7.9 T_C *is a* $\mathcal{D}_0$*-strategy.*

Proof: By lemma 5.7, point (1), combined with lemma 7.8, (S1) is proved. By lemma 5.7, point (2), combined with lemma 7.8, (S2) is proved. □
Let us define the strategy $\mathcal{S}_{AB}$ by : for every $W = A_1 A_2 \cdots A_n$,

(0) if $W \in \mathrm{dom}(T_{cut})$, then $\mathcal{S}_{AB}(W) = T_{cut}(W)$
(1) elsif $W \in \mathrm{dom}(T_\emptyset)$, then $\mathcal{S}_{AB}(W) = T_\emptyset(W)$
(2) elsif $W \in \mathrm{dom}(T_\varepsilon)$, then $\mathcal{S}_{AB}(W) = T_\varepsilon(W)$
(4) elsif $W \in \mathrm{dom}(T_B^+)$, then $\mathcal{S}_{AB}(W) = T_B^+(W)$
(5) elsif $W \in \mathrm{dom}(T_B^-)$, then $\mathcal{S}_{AB}(W) = T_B^-(W)$
(6) elsif $W \in \mathrm{dom}(T_A)$, then $\mathcal{S}_{AB}(W) = T_A(W)$
(7) else $\mathcal{S}_{AB}(W)$ is undefined.

The strategy $\mathcal{S}_{ABC}$ is obtained by inserting "(3) elsif $W \in \mathrm{dom}(T_C)$, then $\mathcal{S}_{ABC}(W) = T_C(W)$" in the above list of cases.

Lemma 7.10 $\mathcal{S}_{ABC}, \mathcal{S}_{AB}$ *are closed.*

Proof: Given any true assertion $A_n = (\pi, S, T)$ and any word $W = A_1 \cdots A_n$, at least one of T_ε, T_A is defined on W. □

8 Tree analysis

This section is devoted to the analysis of the proof-trees τ produced by the strategy $\mathcal{S}_{AB}$ defined in section 7. The main results are lemma 8.14 and 8.15 whose combination asserts that if some path (from a node x to a node y) of τ is such that its origin has a "small norm" and its length is "large enough", then the transformation T_C is defined at some ancestor of y. This key technical result will ensure termination of the strategy $\mathcal{S}_{ABC}$ (see section 9).

8.1 Depth and weight

In this paragraph we show that the *weight* and the *depth* of a given node are closely related. Let us notice that T_B^-, T_B^+ are defined in such a way that:
if (U_{i+1}^-, U_{i+1}^+) is the result of a T_B^α-transformation with arguments $j < i$, then $i - j \geq 3$.Hence in every interval $[i+1, i+4]$, at most one integer j is such that T_B^- or T_B^+ occurs at j.

Lemma 8.1 *: Let τ be a proof-tree associated to the strategy $\mathcal{S}_{AB}$. Let $x, x' \in dom(\tau), x \preceq x'$. Then $\mid W(x') - W(x) \mid \leq |x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3$.*

(We recall the *depth* of a node x is just its length $|x|$). We denote by $W(x)$ the weight of x which we define as the first component of $\tau(x)$ i.e. the weight of the equation labelling x.
Proof: Let x, x' be such that $|x'| = |x| + 1$. Then $W(x') - W(x) \in \{-1, +1\}$, hence the inequality $\mid W(x') - W(x) \mid \leq |x'| - |x|$ is fulfilled by such nodes. The general case follows by induction on $(|x'| - |x|)$.
Let us prove now the other inequality. We distinguish two cases.
Case 1 : $|x'| - |x| \leq 3$.
Then $|x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3$ (because there is at most one T_B step in a sequence of length ≤ 3)).
Case 2 $|x'| - |x| \geq 4$.
Let $x = x_0, x_1, \cdots, x_q, x'$ be the sequence of nodes such that $d(x') - d(x) = 4 \cdot q + r, 0 \leq r < 4$ and $\forall i \in [0, q-1], |x_{i+1}| - |x_i| = 4$.

In every set $\{y \in \text{dom}(\tau) \mid x_i \prec y \preceq x_{i+1}\}$ at most one node z is such that a T_B transformation occurs at z. Hence $W(x_{i+1}) - W(x_i) \geq 2$.
It follows that :

$$\begin{aligned}
|x'| - |x| &= \sum_{i=0}^{q-1} [|x_{i+1}| - |x_i|] + |x'| - |x_q| \\
&\leq \sum_{i=0}^{q-1} 2(W(x_{i+1} - W(x_i)) + |x'| - |x_q| \\
&\leq 2(W(x_q) - W(x)) + 2(W(x') - W(x_q)) + 3 \text{ (by the first case)} \\
&\leq 2(W(x') - W(x)) + 3.
\end{aligned}$$

□

8.2 Stacking derivations

For every $u \in X^*$ we define the binary relation $\uparrow (u)$ over $\mathsf{DB} < V >$ by: for every $S, S' \in \mathsf{DB} < V >, S \uparrow (u) S' \Leftrightarrow \exists A \in Z, \omega \in Z^+, p, q \in Q, \Psi \in \mathsf{DB}_{Q,1} < V >$ such that

$$S = [pA] * \Psi, [pA] \odot u = [q\omega], S' = [q\omega] * \Psi.$$

It is clear that if $S \uparrow (u) S'$ then $S \odot u = S'$ and that the converse is not true in general. A sequence of deterministic series $S_0, S_1, \ldots, S_n$ is a *derivation* iff there exist $x_1, \ldots, x_n \in X$ such that $S_0 \odot x_1 = S_1, \ldots, S_{n-1} \odot x_n = S_n$. The *length* of this derivation is n. If $u = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ we call $S_0, S_1, \ldots, S_n$ the derivation *associated* with (S, u). We denote this derivation by $S_0 \xrightarrow{u} S_n$.
A derivation $S_0, S_1, \ldots, S_n$ is said to be *stacking* iff it is the derivation associated to a pair (S, u) such that $S = S_0$ and $S_0 \uparrow (u) S_n$. A derivation $S_0, S_1, \ldots, S_n$ is said to be a *sub-derivation* of a derivation $S'_0, S'_1, \ldots, S'_m$ iff there exists some $i \in [0, m]$ such that, $\forall j \in [1, n], S_j = S'_{i+j}$. (If we identify derivations with special words over the alphabet $\mathsf{DB} < V >$, a sub-derivation of $S'_0, S'_1, \ldots, S'_m$ is a factor of $S'_0, S'_1, \ldots, S'_m$).

Lemma 8.2 *Let $S, S' \in \mathsf{DRB} < V >, w \in X^*, k \in \mathbb{N}$, such that $S \odot w = S'$ and $\|S'\| \geq \|S\| + k|Q|$. Then the derivation $S \xrightarrow{w} S'$ contains some stacking sub-derivation of length k.*

Proof: Let $S = S_0, \ldots, S_i, \ldots, S_n = S'$ be the derivation associated to (S, w). Let $i_0 = \max\{i \in [0, n] \mid \|S_i\| = \min\{\|S_j\| \mid 0 \leq j \leq n\}\}$. Let $w = w'w''$ where $|w'| = i_0$. As $\|S_n\| - \|S_{i_0}\| \geq \|S_n\| - \|S_0\| \geq k|Q|$, using lemma 3 point (3) we must have $|w''| \geq k$. Let $w'' = w''_1 w''_2$ with $|w''_1| = k$. By definition of i_0, $\forall i \in [i_0 + 1, i_0 + k], \|S_i\| \geq \|S_{i_0}\| + 1$. We conclude that the sub-derivation $S_{i_0}, \ldots, S_{i_0+k}$ (associated to (S_{i_0}, w''_1)) is stacking. □

Lemma 8.3 *Let $S, S' \in \mathsf{DRB} < V >, w \in X^*, k, d, d' \in \mathbb{N}$, such that S is (d, d')-linear and:*

(1) the derivation $S \xrightarrow{w} S'$ contains no stacking sub-derivation of length k.
(2) $|w| \geq d \cdot k$.

Then S' is $(0, d')$-linear.

Proof: If S is $(0, d')$-linear, then the result follows from lemma 3, point (1). Otherwise

$$S = \Sigma_{q \in Q} [p\omega q][qeq] T_q$$

for some $\omega \in Z^*, 1 \leq |\omega| \leq d, (T_q)_{q \in Q} \in \mathsf{DRB}_{Q,1} < V >$. Let $S \xrightarrow{w} S' = (S_0, \ldots, S_n)$. By induction on l, using hypothesis (1), one can show that: for every $l \in [0, |\omega| - 1]$, either

$$\exists m \leq k \cdot l, \exists \omega_m \in Z^+, |\omega_m| \leq |\omega| - l \text{ and } S_m = \Sigma_{q \in Q} [p_m \omega_m q][qeq] T_q,$$

or

$$\exists m \leq k \cdot l, \exists q \in Q, S_m = \rho_\epsilon(T_q).$$

Similarly

$$\exists m_0 \leq k \cdot |\omega|, \exists q \in Q, S_{m_0} = \rho_\epsilon(T_q).$$

Hence S_{m_0} is $(0, d')$-linear. Using lemma 3 we obtain that $S_n = S'$ is $(0, d')$-linear too. $\square$

Let us recall the values of some constants (defined in section 6):

$$k_0 = \max\{\nu([pAq]) \mid p, q \in Q, A \in Z\},\ k_1 = \max\{2k_0 + 1, 3\},\ k_2 = 4k_1 + 2(k_1)^2 + k_0$$

$$d_0 = 2 \cdot |Q| \cdot \mathrm{Card}(X^{\leq k_1}), \qquad D_1 = 4k_0 + 2, \qquad D_2 = \Sigma_{d_0} + s_{d_0}$$

$$N_0 = 1 + (k_2 + 1)|Q| + D_2.$$

We fix throughout the remaining of this section a tree $\tau = \mathcal{T}(\mathcal{S}_{AB}, (\pi_0, U_0^-, U_0^+))$ (i.e. τ is the proof tree associated to the assertion $(\pi_0, U_0^-, U_0^+))$ by the strategy $\mathcal{S}_{AB}$). We suppose that

$$\mathrm{rd}(U_0^-) \leq D_2, \quad \mathrm{rd}(U_0^+) \leq D_2, \quad U_0^-, U_0^+ \text{ are both unmarked .} \qquad (88)$$

We recall that, formally, τ is a map $\mathrm{dom}(\tau) \longrightarrow \mathbb{N} \times \mathsf{DRB} < V > \times \mathsf{DRB} < V >$ such that $\mathrm{dom}(\tau) \subseteq \{1, \ldots, |X|\}^*$ is closed under prefix and under "left-brother" (i.e. $w \cdot (i+1) \in \mathrm{dom}(\tau) \Rightarrow w \cdot i \in \mathrm{dom}(\tau)$). We denote by $pr_{2,3} : \mathbb{N} \times \mathsf{DRB} < V > \times \mathsf{DRB} < V > \longrightarrow \mathsf{DRB} < V > \times \mathsf{DRB} < V >$ the projection $(\pi, U, U') \mapsto (U, U')$. By τ_s we denote the tree obtained from τ by forgetting the weights: $\tau_s = \tau \circ pr_{2,3}$.

8.3 Linearity

Lemma 8.4 *For every label $(\pi, U^-, U+)$ of τ,*

(1) U^α is (D_1, D_2)-linear (for $\alpha \in \{-, +\}$)
(2) if U^α is unmarked, then U^α is $(0, D_2)$-linear
(3) $\exists \alpha \in \{-, +\}, U^\alpha$ is unmarked.

Proof: Let $x \in \mathrm{dom}(\tau), \tau(x) = (\pi, U^-, U^+)$. We denote by $x(i)$ the prefix of length i of the node x. For every $0 \leq i \leq |x|$ we note $\tau(x(i)) = (\pi_i, U_i^-, U_i^+) = A_i$. (Hence $(A_i)_{0 \leq i \leq |x|}$ is the sequence of labels on the path from ϵ to x). We prove points (1-3) by induction on $|x|$ (the depth of node x).
$|x| = 0$: by hypothesis (88) points(1-3) are true.
$|x| = n + 1$:
case 1: (π, U^-, U^+) is the result of the application of T_B^α on $(\pi_i, U_i^-, U_i^+), (\pi_n, U_n^-, U_n^+)$ where $i = n - k_1$).
Then $U_i^{-\alpha}$ is unmarked (by definition of T_B^α) and $(0, D_2)$-linear (by induction hypothesis). It follows that U^α is $(2k_0 + 1, D_2)$-linear. On the other hand, there exists some $w \in X^{k_1}$ such that $U^{-\alpha} = U_i^{-\alpha} \odot w$. By induction hypothesis $\mathrm{rd}(U_i^{-\alpha}) \leq D_2$ and by lemma 3 point (1), $\mathrm{rd}(U^{-\alpha}) \leq D_2$. Moreover , as no letter $[qeq]$ can be introduced by the action $\odot$ (see hypothesis 9 in section 2.2)

, $U^{-\alpha}$ is unmarked.
case 2: $(\pi, U^-, U^+) = (\pi_n + 1, U_n^- \odot x, U_n^+ \odot x)$ for some $x \in X$.
If for every $1 \leq i \leq n$, (π_i, U_i^-, U_i^+) is not the result of an application of T_B^α (for any α), then U_{n+1}^-, U_{n+1}^+ are both $(0, D_2)$-linear. Otherwise, there is a maximal integer, m_0, such that $(\pi_{m_0}, U_{m_0}^-, U_{m_0}^+)$ is the result of an application of T_B^α (for some $\alpha \in \{-,+\}$). Then we have:

$$U_{m_0}^{-\alpha} \text{ is unmarked and } (0, D_2) - \text{linear}, \tag{89}$$

$$U_{m_0}^{+\alpha} = \Sigma_{q \in Q}[p\omega q][qeq]T_q, \tag{90}$$

for some $\omega \in Z^+, 1 \leq |\omega| \leq 2k_0 + 1, (T_q)_{q \in Q} \in \mathsf{DRB}_{Q,1} < V >$, where every T_q is unmarked and $(0, D_2)$-linear. As $U^{-\alpha} = U_{m_0}^{-\alpha} \odot w$ for some $w \in X^{n+1-m_0}$, property (89) implies that $U^{-\alpha}$ is unmarked and $(0, D_2) -$ linear .
If there exists some $i \in [m_0, n+1]$ such that $U_i^{+\alpha} = T_q$ then, by hypothesis $U_i^{+\alpha}$ is unmarked and $(0, D_2)$-linear. It follows that in this case $U^{+\alpha}$ is unmarked and $(0, D_2)$-linear.
If there exists no $i \in [m_0, n+1]$ such that $U_i^{+\alpha} = T_q$ then,

$$\forall i \in [m_0, n+1], \exists p_i \in Q, \omega_i \in Z^+, \;\; U_i^{+\alpha} = \Sigma_{q \in Q}[p_i \omega_i q][qeq]T_q.$$

Let $m_0 \leq i_0 \leq n+1$ such that $|\omega_{i_0}|$ is minimal. If $|\omega_{n+1}| - |\omega_{i_0}| > 2k_0 + 1$, then T_B^α would be defined at some word $A_0 \ldots A_i$ for some $i_0 \leq i < n+1$, contradicting the maximality of m_0. Hence $|\omega_{n+1}| - |\omega_{i_0}| \leq 2k_0 + 1$, and since $|\omega_{i_0}| \leq |\omega_{m_0}| \leq 2k_0 + 1$ we have $|\omega_{n+1}| \leq 4k_0 + 2$. Hence $U^{+\alpha}$ is (D_1, D_2)-linear. □

8.4 N-stacking sequences

We show here that the tree τ is somewhat "smooth" in the sense that its labels cannot be varying in a too chaotic way along a given branch. We shall establish that every "sufficiently long" branch must contain a "reasonnably short factor" (a "N-stacking sequence") where at least d_0 labels (U, U') are belonging to the same d-space V of dimension $\leq d_0$ with coordinates not greater than s_{d_0} (over some fixed generating family of cardinality $\leq d_0$). Let us define now this notion of stacking sequence which is , roughly speaking, an extension to sequences of *pairs* (U_i, U_i') appearing in τ_s, of the above notion of stacking derivation.
For every $U, U' \in \mathsf{DRB} < V >$ we set:

$$|||U||| = \max\{||\tilde{U}||, \;\; \tilde{U} \in \mathsf{Q}(U) \text{ and } \tilde{U} \text{ is unmarked } \}$$

$$|||(U, U')||| = \max\{|||U|||, |||U'|||\}.$$

Let $x \in \mathrm{dom}(\tau)$. We define now a kind of norm on nodes which in some sense "erases" the short-term variations of $|||(*, *)|||$:

$$\mathrm{N}(x) = \max\{|||\tau_s(x(j))|||, j \in \mathbb{N}, |x| - k_1 \leq j \leq |x|\}.$$

Lemma 8.5 *Let $(x_i)_{i \in I}$ (where I is a beginning section of $\mathbb{N}$) be some branch of τ.*

(1) The function $N : I \longrightarrow \mathbb{N}$ is $(k_0 \cdot |Q|)$-up-Lipschitz.
(2) The restriction $N : J \longrightarrow \mathbb{N}$ to any interval $J \subseteq I$ such that neither T_B^-, nor T_B^+ occur in J, is $|Q|$-up-Lipschitz.

Proof: Let us prove that for every i such that $i+1 \in I$

$$\mathrm{N}(i+1) \leq \mathrm{N}(i) + k_0|Q| \tag{91}$$

Let $\alpha \in \{-,+\}$. Let us consider four cases.
case 1: $U^\alpha_{i+1} = U^\alpha_i \odot x$ (for some $x \in X, U^\alpha_i$ unmarked).
Then $\|U^\alpha_{i+1}\| \leq \|U^\alpha_i\| + |Q| \leq \mathrm{N}(i) + |Q|$.
case 2: U^α_{i+1} is obtained by T^α_B transformation.

$$U^\alpha_{i+1} = \sum_{q \in Q'} [p_i\omega_i q][qeq](U^{-\alpha}_{i-k_1} \odot u_q)$$

where $Q' \subseteq Q, 1 \leq |u_q| \leq k_0$. Then we have

$$\begin{aligned}||| \; U^\alpha_{i+1} \; ||| = \max_{q \in Q'}\{\|U^{-\alpha}_{i-k_1} \odot u_q\|\} &\leq \|U^{-\alpha}_{i-k_1}\| + k_0|Q| \\ &\leq \mathrm{N}(i) + k_0|Q|\end{aligned}$$

case 3: $U^\alpha_{i+1} = U^\alpha_i \odot x$ (for some $x \in X, U^\alpha_i, U^\alpha_{i+1}$ are marked).

$$U^\alpha_i = \sum_{q \in Q} [p_i\omega_i q][qeq]V_q$$

where the V_q are unmarked

$$U^\alpha_{i+1} = \sum_{q \in Q} ([p_i\omega_i q] \odot x)[qeq]V_q$$

hence $||| \; U^\alpha_{i+1} \; ||| = \max_{q \in Q}\{\|V_q\|\} = ||| \; U^\alpha_i \; ||| \leq \mathrm{N}(i)$.
case 4: $U^\alpha_{i+1} = U^\alpha_i \odot x$ (for some $x \in X, U^\alpha_i$ marked, U^α_{i+1} unmarked).

$$U^\alpha_i = \sum_{q \in Q} [p_i\omega_i q][qeq]V_q$$

where the V_q are unmarked and ϵ-free $U^\alpha_{i+1} = V_{q_0}$, for some $q_0 \in Q$, hence

$$||| \; U^\alpha_{i+1} \; ||| = \|U^\alpha_{i+1}\| = \|V_{q_0}\| \leq \max_{q \in Q}\{\|V_q\|\} = ||| \; U^\alpha_i \; ||| \leq \mathrm{N}(i).$$

As in every case , $||| \; U^\alpha_{i+1} \; ||| \leq \mathrm{N}(i) + k_0|Q|$, inequality (91) and point (1) of the lemma are proved.
The discussion above also shows that, if no T_B- transformation occurs in J, then for every J such that $j \in J, j+1 \in J$, $\mathrm{N}(j+1) \leq \mathrm{N}(j) + |Q|$, which proves point (2) of the lemma. □
Let $\sigma = (x_i)_{i \in I}$ be a path in τ, where $I \subseteq \mathbb{N}$ is a non-empty interval and $i_0 = \min\{I\}$. We call σ a *N-stacking sequence* iff:

$$\forall i \in I, \mathrm{N}(x_i) \geq \mathrm{N}(x_{i_0}) \text{ and } \mathrm{N}(x_{i_0}) \geq N_0. \tag{92}$$

From now on and until lemma 8.15 , we fix a N-stacking sequence $\sigma = (x_i)_{i \in I}$. We call $\mathrm{Card}(I) - 1$ the *length* of σ (denoted $|\sigma|$) and we suppose $|\sigma| \geq k_1$. Let us use the simplified notations $\mathrm{N}(i)$ for $\mathrm{N}(x_i)$ and let us note $\tau_s(x_i) = (U^-_i, U^+_i)$.

Lemma 8.6 $\forall i \in \mathbb{N}$, *if* $i + k_1 \in I$, *then* $\exists j \in [i, i+k_1], |||\,(U_j^-, U_j^+)\,||| \geq \mathrm{N}(i_0)$.

Proof: This follows from the fact that $\mathrm{N}(i+k_1) \geq \mathrm{N}(i_0)$. □
We let $i_1 = \min\{i \in [i_0 - k_1, i_0] \cap \mathbb{N},\ |||\,((U_i^-, U_i^+)\,||| \geq \mathrm{N}(i_0)\}$, if no T_B^α occurs in $[i_0 - k_1, i_0]$ and i_1 is the unique element of $[i_0 - k_1, i_0]$ where some T_B^α occurs otherwise (we then define α_1 as the unique element of $\{-,+\}$ such that $T_B^{\alpha_1}$ occurs at i_1).
For every $\alpha \in \{-,+\}$, we define a Q-series $[p^\alpha \omega^\alpha]$ and a Q-forms Φ^α as follows.
case 1: $U_{i_1}^\alpha$ is unmarked and $\|U_{i_1}^\alpha\| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.
As $\mathrm{rd}(U_{i_1}^\alpha) \leq D_2$, $U_{i_1}^\alpha$ has a minimal decomposition:

$$U_{i_1}^\alpha = [p^\alpha \omega_1^\alpha] * \Phi_1^\alpha \text{ with } \|\Phi_1^\alpha\| \leq D_2.$$

Using the inequality

$$\|U_{i_1}^\alpha\| \geq \mathrm{N}(i_0) - k_2|Q| \geq 1 + |Q| + D_2,$$

we conclude that ω_1^α admits a decomposition $\omega_1^\alpha = \omega^\alpha \omega_2^\alpha$ such that

$$U_{i_1}^\alpha = [p^\alpha \omega^\alpha] * \Phi^\alpha,\ \ \Phi^\alpha = [\omega_2^\alpha] * \Phi_1^\alpha, \tag{93}$$

with

$$\mathrm{N}(i_0) - (k_2+1)|Q| < \|\Phi^\alpha\| \leq \mathrm{N}(i_0) - k_2|Q|,\ \ \|\Phi^\alpha\| \geq D_2 + |Q|,\ \ |\Phi^\alpha| \geq 1. \tag{94}$$

case 2: $U_{i_1}^\alpha$ is marked and $|||\,(U_{i_1}^\alpha)\,||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.
Hence $U_{i_1}^\alpha$ can be written as

$$U_{i_1}^\alpha = \Sigma_{q \in Q'} [p_1^\alpha \mu_1^\alpha q][qeq]\overline{U_{i_1}^\alpha} \odot u_q$$

for some $p_1^\alpha \in Q, Q' \subseteq Q, 1 \leq |\mu_1^\alpha| \leq D_1, |u_q| \leq k_0, \overline{U_{i_1}^\alpha} = U_j^{-\alpha}, j < i_1$.
Let $q \in Q'$ such that $|||\,U_{i_1}^\alpha\,||| = \|\overline{U_{i_1}^\alpha} \odot u_q\|$. We have:

$$\|\overline{U_{i_1}^\alpha}\| \geq \|\overline{U_{i_1}^\alpha} \odot u_q\| - |u_q||Q| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q| - k_0|Q| = \mathrm{N}(i_0) - k_2|Q|.$$

From this inequality we conclude, as in case 1, that $\overline{U_{i_1}^\alpha}$ has a decomposition:

$$\overline{U_{i_1}^\alpha} = [p^\alpha \omega^\alpha] * \Phi^\alpha, \tag{95}$$

with

$$\mathrm{N}(i_0) - (k_2+1)|Q| < \|\Phi^\alpha\| \leq \mathrm{N}(i_0) - k_2|Q|,\ \ \|\Phi^\alpha\| \geq D_2 + |Q|,\ \ |\Phi^\alpha| \geq 1. \tag{96}$$

case 3: $|||\,(U_{i_1}^\alpha)\,||| < \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.
In this case , $[p^\alpha \omega^\alpha], \Phi^\alpha$ are both undefined. Let us remark that, by the inequality $|||\,(U_{i_1}^-, U_{i_1}^+)\,||| \geq \mathrm{N}(i_0)$, there exists at least one $\alpha \in \{-,+\}$ such that $[p^\alpha \omega^\alpha], \Phi^\alpha$ are both defined. Let us define now the following families of series and d-spaces of series (i.e. in the universe $\mathsf{DRB} < V >$ and with coefficients in $\mathsf{DRB}_{1,n} < V >$, $n \in \mathbb{N}$):

$$W^\alpha = \mathsf{V}((\Phi_q^\alpha)_{q \in Q}),\ \ V^\alpha = \mathsf{V}((\Phi_q^\alpha \odot u)_{q \in Q, u \in X^*, 0 \leq |u| \leq k_0}), \tag{97}$$

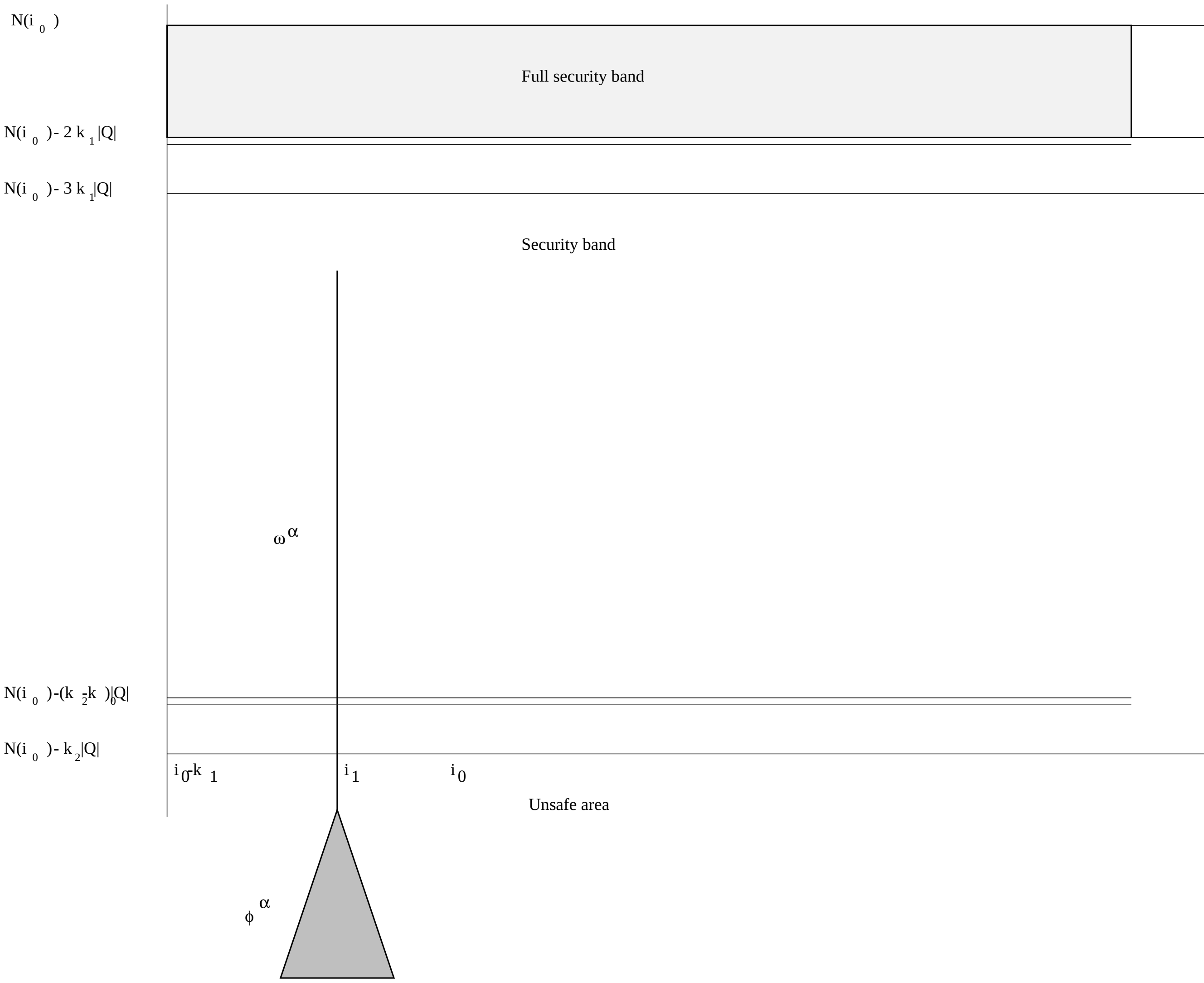

Fig. 2. the generating family.

for every $\alpha \in \{-, +\}$;

$$\bar{V} = V^- + V^+, \quad \tilde{W} = W^- \cup W^+, \quad \tilde{V} = V^- \cup V^+, \tag{98}$$

where , for every d-spaces V_1, V_2 in the same universe, $V_1 + V_2$ denotes the smallest d-space containing $V_1 \cup V_2$. We illustrate on figure 2 the above definitions. The band $\{U \in \mathsf{DRB} < V > |\mathrm{N}(i_0) - 2k_1|Q| \leq \mathrm{N}(U)\}$ is a *full security band* in the sense that every U_i^α in this band *belongs to* $\bar{V}$ (lemma 8.10). The band

$\{U \in \mathsf{DRB} < V > \mid \ \mathrm{N}(i_0) - (k_2 - k_0)|Q| \leq ||| \ U \ ||| < \mathrm{N}(i_0)\}$ is a *security band* in the sense that if U_j^α is in this band , $j < i$, U_i^α belongs to the full security band, and $U_j^\alpha \longrightarrow U_i^\alpha$, then U_j^α *belongs to* $\bar{V}$ (property $\mathcal{P}(\alpha, i, j)$ established in the proof of lemma 8.10).

Lemma 8.7 *Let* $i \in \mathbb{N}$, *such that* $i, i + k_1 + 1 \in I$. *If* T_B^α *occurs at node* x_{i+1} *then* $||| \ (U_{i+1}^-, U_{i+1}^+) \ ||| \geq \mathrm{N}(i_0) - k_1|Q|$.

Proof: Suppose that $||| \ (U_{i+1}^-, U_{i+1}^+) \ ||| \leq \mathrm{N}(i_0) - k_1|Q| - 1$. Then, $\forall j \in [i+1, i+1+k_1]$, $||| \ (U_j^-, U_j^+) \ ||| \leq \mathrm{N}(i_0) - 1$. Hence $\mathrm{N}(i+1+k_1) \leq \mathrm{N}(i_0) - 1$, contradicting the fact that σ is N-stacking. □

Lemma 8.8 *Suppose that* T_B^α *occurs at node* x_{i+1}. *Then* $U_{i+1}^\alpha = \sum_{q \in Q'} [p_i \mu_i q][qeq](\overline{U_{i+1}^\alpha} \odot u_q)$, *for some* $p_i \in Q, Q' \subseteq Q, 1 \leq |\mu_i|, |u_q| \leq k_0, ||\overline{U_{i+1}^\alpha}|| \geq \mathrm{N}(i_0) - 2k_1|Q|$.

Proof: We distinguish two cases.
case 1: $||| \ U_{i+1}^\alpha \ ||| = \max\{||| \ U_{i+1}^- \ |||, ||| \ U_{i+1}^+ \ |||\}$.
By lemma 8.7, $||| \ U_{i+1}^\alpha \ ||| \geq \mathrm{N}(i_0) - k_1|Q|$. So, $\exists q \in Q', ||\overline{U_{i+1}^\alpha} \odot u_q|| \geq \mathrm{N}(i_0) - k_1|Q|$, hence $||\overline{U_{i+1}^\alpha}|| \geq \mathrm{N}(i_0) - (k_1 + k_0)|Q| \geq \mathrm{N}(i_0) - 2k_1|Q|$.
case 2: $||| \ U_{i+1}^{-\alpha} \ ||| = \max\{||| \ U_{i+1}^- \ |||, ||| \ U_{i+1}^+ \ |||\}$.
Hence $||U_{i-k_1}^{-\alpha}|| \geq ||U_{i+1}^{-\alpha}|| - k_1|Q| \geq \mathrm{N}(i_0) - 2k_1|Q|$. As $\overline{U_{i+1}^\alpha} = U_{i-k_1}^{-\alpha}$, the lemma is proved. □

Lemma 8.9 *Let* $\alpha \in \{-, +\}$, $(i, j) \in \mathbb{N} \times \mathbb{N}, i \geq i_1, j \geq i_1$ *such that* $||| \ U_i^\alpha \ ||| \geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q|$ *and no* T_B^α *occurs in* $[j + 1, i]$. *Then* $||| \ U_j^\alpha \ ||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.

Proof: Let us suppose α, i, j fulfill the hypothesis of the lemma and let us suppose that the following inequality is realized:

$$||| \ U_j^\alpha \ ||| < \mathrm{N}(i_0) - (k_2 - k_0)|Q|. \tag{99}$$

As $||| \ U_i^\alpha \ ||| \geq \mathrm{N}(i_0) - 2k_1|Q|$, the following integers are well-defined:

$$i_2 = -1 + \min\{j' \in [j + k_1, i] \mid \mathrm{N}(i_0) - 3k_1|Q| \leq ||| \ U_i^\alpha \ ||| < \mathrm{N}(i_0) - (3k_1 - 1)|Q|\},$$

$$i_3 = -1 + \min\{j' \in [i_2, i] \mid \mathrm{N}(i_0) - 2k_1|Q| \leq ||| \ U_i^\alpha \ ||| < \mathrm{N}(i_0) - (2k_1 - 1)|Q|\}.$$

As for every $j' \in [j, i_2 - k_1]$, $||| \ U_{j'}^\alpha \ ||| < \mathrm{N}(i_0) - 3k_1|Q|$, by lemma 8.8, $T_B^{-\alpha}$ does not occur in $[j + k_1 + 1, i_2 + 1]$. But $||| \ U_{i_2}^\alpha \ ||| - ||| \ U_j^\alpha \ ||| > (k_2 - k_0 - 3k_1 - 1)|Q| = (2(k_1)^2 + k_1 - 1)|Q|$, hence $i_2 - j \geq 2(k_1)^2 + k_1$ so that the interval $[j + k_1, i_2]$ has a length greater or equal to $2(k_1)^2 = D_1 \cdot k_1$. Applying lemma 8.3 we conclude that $U_{i_2}^{-\alpha}$ is unmarked. (Let us notice that $U_{i_2}^\alpha$ is unmarked too, just because $||| \ U_{i_2}^\alpha \ ||| > ||| \ U_j^\alpha \ |||$ while $U_{i_2}^\alpha = U_j^\alpha \odot w$ for some $w \in X^{i_2 - j}$).
As $||U_{i_3}^\alpha|| \geq ||U_{i_2}^\alpha|| + k_1|Q|$, by lemma 8.2 the derivation $U_{i_2}^\alpha \longrightarrow U_{i_3}^\alpha$ must contain a stacking subderivation

$$U_{i_2+k}^\alpha \uparrow (u) U_{i_2+k+k_1}^\alpha,$$

for some $u \in X^{k_1}$. As $U_{i_2}^{-\alpha}$ is unmarked, there exists some $j' \in [i_2+k_1+1, i_3+1]$ such that either T_B^{α} occurs at j', which contradicts the hypothesis of the lemma, or $T_B^{-\alpha}$ occurs at j' which contradicts lemma 8.8. (We illustrate our argument on figure 3). □

Lemma 8.10 *Let* $i \geq i_1, \alpha \in \{-,+\}$ *such that* $|||\ U_i^{\alpha}\ ||| \geq \mathrm{N}(i_0) - 2k_1|Q|$. *Then* $U_i^{\alpha} \in \bar{V}$.

Proof: Let us consider the following property $\mathcal{P}(\alpha, i, j)$:

$$\{j \leq i \text{ and } |||\ U_i^{\alpha}\ ||| \geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q| \text{ and no } T_B^{\alpha} \text{ occurs in } [j+1, i]\} \quad (100)$$

$$\Longrightarrow$$

$$\{(\text{ if } U_j^{\alpha} \text{ is unmarked then } U_i^{\alpha} \in \tilde{W}) \text{ and (if } U_j^{\alpha} \text{ is marked then, there exists some } \alpha' \in \{-,+\}$$

$$\text{such that , every linear component of } U_j^{\alpha} \text{ is in } (W^{\alpha'} \cap \mathsf{DRBlin}^{D_2} < V >) \odot X^{<1,k_0>}\}. \quad (101)$$

(Here we denote by $X^{<m,m'>}$ the set $\{u \in X^*, m \leq |u| \leq m'\}$).

We prove by lexicographic induction on the pair (i, j) that:

$$\forall i \geq i_1, \forall j \geq i_1, [\forall \alpha \in \{-,+\}, \mathcal{P}(\alpha, i, j)]. \quad (102)$$

Let us consider a pair $(i, j) \in \mathbb{N} \times \mathbb{N}, i \geq i_1, j \geq i_1$ and some $\alpha \in \{-,+\}$, fulfilling the lefthand-side of the implication $\mathcal{P}(\alpha, i, j)$ (i.e. fulfilling condition (100)).

case 1: $j = i_1$.

Using the definitions of $\tilde{W}, \tilde{V}$, one easily sees that $\mathcal{P}(\alpha, i, j)$ is true.

case 2: $j > i_1$ and T_B^{α} occurs at j .

$$U_j^{\alpha} = \sum_{q \in Q'} [p_j \omega_j q][qeq](\overline{U_j^{\alpha}} \odot u_q),$$

for some $Q' \subseteq Q, \omega_j \in Z^+, p_j \in Q, |u_q| \leq k_0$, and $\overline{U_j^{\alpha}} = U_{j-k_1-1}^{-\alpha}$. By lemma 8.8

$$|||\ \overline{U_j^{\alpha}}\ ||| \geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q|.$$

It follows that $j - k_1 - 1 \geq i_1$ (by minimality of i_1 in the case where no T_B occurs in $[i_0 - k_1, i_0]$ and because two successive occurrences of T_B must be at distance $\geq k_1+1$ in the case where some T_B occurs at i_1). As $(j-k_1-1, j-k_1-1) < (i, j)$, by induction hypothesis, $\mathcal{P}(\alpha, j-k_1-1, j-k_1-1)$ is true: $U_{j-k_1-1}^{-\alpha}$ is unmarked (by definition of T_B^{α}) and $|||\ U_{j-k_1-1}^{-\alpha}\ ||| \geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q|$, hence $U_{j-k_1-1}^{-\alpha} \in W^{\alpha'}$ (for some α'). It follows that

$$U_j^{\alpha} \text{ is marked and for every } q \in Q', U_j^{\alpha} \odot u_q \in (W^{\alpha'} \cap \mathsf{DRBlin}^{D_2} < V >) \odot X^{<1,k_0>}.$$

case 3: $j > i_1$ and T_B^α does not occur at j .
subcase 1: $U_{j-1}^\alpha, U_j^\alpha$ are both marked.
$U_{j-1}^\alpha \odot x = U_j^\alpha$ for some $x \in X$. By lemma8.4,

$$U_{j-1}^\alpha = \sum_{q \in Q'} [p_{j-1}\omega_{j-1}q][qeq]T_q$$

for some $\omega_{j-1} \in Z^+, 1 \leq |\omega_{j-1}| \leq 2k_0+1, (T_q)_{q\in Q} \in \mathsf{DRB}_{Q,1} < V >$, where every T_q is unmarked and $(0, D_2)$-linear.It follows that

$$U_j^\alpha = \sum_{q \in Q'} ([p_{j-1}\omega_{j-1}q] \odot x)[qeq]T_q$$

By the induction hypothesis $\mathcal{P}(\alpha, i, j-1)$ we get: there exists $\alpha' \in \{-,+\}$ such that $\forall q \in Q', T_q \in (W^{\alpha'} \cap \mathsf{DRBlin}^{D_2} < V >) \odot X^{<1,k_0>}$. Hence (101) is true for (α, i, j).
subcase 2: $U_{j-1}^\alpha, U_j^\alpha$ are both unmarked.
By induction hypothesis $U_{j-1}^\alpha \in \tilde{W}$, i.e. $U_{j-1}^\alpha = [p_{j-1}\omega_{j-1}] * \Phi^{\alpha'}$ for some $\omega_{j-1} \in Z^*, \alpha' \in \{-,+\}$, and by lemma 8.9

$$||| \; U_{j-1}^\alpha \; ||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|.$$

As $\|U_{j-1}^\alpha\| > \|\Phi^{\alpha'}\|$, we must have $|\omega_{j-1}| \geq 1$. Let us apply lemma 3.33 on $U = U_{j-1}^\alpha, H = U_j^\alpha, u = x$. One can check that the hypothesis

$$\|H\| \geq 1 + (k-1)|Q| + \|\Phi\|$$

is fulfilled. Hence $U_j^\alpha = ([p_{j-1}\omega_{j-1}] \odot x) * \Phi^{\alpha'}$, which proves that $U_j^\alpha \in \tilde{W}$.
subcase 3: U_{j-1}^α is marked while U_j^α is unmarked.
By lemma 8.4

$$U_{j-1}^\alpha = \sum_{q \in Q} [p_{j-1}\omega_{j-1}q][qeq]T_q$$

for some $\omega_{j-1} \in Z^+, 1 \leq |\omega_{j-1}| \leq 2k_0+1, (T_q)_{q\in Q} \in \mathsf{DRB}_{Q,1} < V >$, where every T_q is unmarked and $(0, D_2)$-linear. It follows that

$$U_j^\alpha = \rho_\epsilon(T_{q_0})$$

for some $q_0 \in Q$. By induction-hypothesis there exists $\alpha' \in \{-,+\}$ such that $T_{q_0} \in (W^{\alpha'} \cap \mathsf{DRBlin}^{D_2} < V >) \odot X^{\leq k_0}$, i.e. there exist $\overline{T_{q_0}} \in W^{\alpha'} \cap \mathsf{DRBlin}^{D_2} < V >, u_{q_0} \in X^{<1,k_0>}$ such that:

$$T_{q_0} = \overline{T_{q_0}} \odot u_{q_0}.$$

(Let us notice that this equality implies that $T_{q_0} = \rho_\epsilon(T_{q_0})$).
$\overline{T_{q_0}} \in \mathsf{V}((\Phi_q^{\alpha'})_{q\in Q})$, and by definition of $\Phi^{\alpha'}$ we have:

(1) $\|\Phi^{\alpha'}\| \geq D_2 + |Q|, |\Phi^{\alpha'}| \geq 1$
(2) $\mathrm{rd}(\overline{T_{q_0}}) \leq D_2$.

Applying lemma 3.34, we obtain that

$$\exists \omega_0 \in Z^*, \exists p_0 \in Q, \overline{T_{q_0}} = [p_0\omega_0] * \Phi^{\alpha'}.$$

Using the inequality given by lemma 8.9 (for U_j^α) and inequality (94 or 96) we obtain that

$$\begin{aligned}||U_j^\alpha|| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q| &\geq [\mathrm{N}(i_0) - (k_2 - k_0)|Q| - 1 - (k_0 - 1)|Q|] + 1 + (k_0 - 1)|Q| \\ &\geq ||\Phi^{\alpha'}|| + 1 + (k_0 - 1)|Q|,\end{aligned}$$

which is equivalent to

$$||\overline{T_{q_0}} \odot u_{q_0}|| \geq ||\Phi^{\alpha'}|| + 1 + (k_0 - 1)|Q|.$$

Hence the hypothesis of lemma 3.33 are met by $U = \overline{T_{q_0}}, H = T_{q_0}, p = p_0, h = \omega_0, \Phi = \Phi^{\alpha'}, u = u_{q_0}, k = k_0$. Lemma 3.33 then concludes that

$$T_{q_0} = ([p_0\omega_0] \odot u_{q_0}) * \Phi^{\alpha'} \text{ where } [p_0\omega_0] \odot u_{q_0} = [q'\omega'], \text{ for some } q' \in Q, |\omega'| \geq k_0.$$

As $U_j^\alpha = T_{q_0}$, this establishes that $U_j^\alpha \in W^{\alpha'}$, hence that $\mathcal{P}(\alpha, i, j)$ is true. □

Lemma 8.11 *Suppose* $i \geq i_0, ||U_i^\alpha|| \geq \mathrm{N}(i_0) - 2k_1|Q|, U_i^\alpha$ *unmarked. Then* $\exists \alpha' \in \{-, +\}, p \in Q, \omega \in Z^*, U_i^\alpha = [p\omega] * \Phi^{\alpha'}$ *with* $1 \leq |\omega| \leq k_0(i - i_0) + (k_2 + 2)$.

Proof: By property $\mathcal{P}(\alpha, i, i)$ established in the proof of lemma 8.10, $\exists \alpha' \in \{-, +\}, p \in Q, \omega \in Z^*$,

$$U_i^\alpha = [p\omega] * \Phi^{\alpha'}.$$

By lemma 3.30

$$||U_i^\alpha|| = 1 + (|\omega| - 1)|Q| + ||\Phi^{\alpha'}||.$$

By lemma 8.5

$$||U_i^\alpha|| \leq \mathrm{N}(i) \leq (i - i_0)k_0|Q| + \mathrm{N}(i_0).$$

By inequality (94 or 96),

$$||\Phi^{\alpha'}|| > \mathrm{N}(i_0) - (k_2 + 1)|Q|.$$

Hence we get

$$\begin{aligned}(|\omega| - 1)|Q| \leq 1 + (|\omega| - 1)|Q| &\leq ||U_i^\alpha|| - ||\Phi^{\alpha'}|| \\ &< (i - i_0)k_0|Q| + \mathrm{N}(i_0) - (\mathrm{N}(i_0) - (k_2 + 1)|Q|) \\ &= (i - i_0)k_0|Q| + (k_2 + 1)|Q|\end{aligned}$$

Hence $|\omega| \leq (i - i_0)k_0 + (k_2 + 2)$. □

Let us define two additional spaces of series:

$$V_1^\alpha = \mathsf{V}((\Phi_q^\alpha \odot u)_{q \in Q, u \in X^{\leq k_1}}), \quad \bar{V}_1 = V_1^- + V_1^+ \tag{103}$$

We recall that:

$$K_1 = k_1 + 1, \quad K_2 = 2(k_1)^3 + 3(k_1)^2 + k_1 + 1.$$

Lemma 8.12 *Let $L > 0$. There exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ such that, $U_i^- \in \bar{V}_1, U_i^+ \in \bar{V}_1$.*

Proof: Let us establish that

$$\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2], \exists \alpha \in \{-,+\}, T_B^\alpha \text{ occurs at } i. \tag{104}$$

If a T_B^α occurs in $[i_0 + L, i_0 + L + 2(k_1)^2]$ then (104) is true.
Otherwise, by lemma 8.4 and lemma 8.3, both $U^\alpha_{i_0+L+2(k_1)^2}$ are unmarked. If some α-stacking sequence of length $\geq k_1$ occurs in the interval $I(i_0, L) = [i_0 + L + 2(k_1)^2, i_0 + K_1 L + K_2 - 1]$, then some $T_B^{\alpha'}$ occurs at some $i \in [i_0 + L + 2(k_1)^2 + 1, i_0 + K_1 L + K_2]$ and (104) is true.
Let us suppose now that no α-stacking sequence of length $\geq k_1$ occurs in the interval $I(i_0, L)$. By lemma 8.5 we must then have:

$$||| \, (U^-_{i_0+L+2(k_1)^2}, U^+_{i_0+L+2(k_1)^2)}) \, ||| \leq \mathrm{N}(i_0) + |Q|(L + 2(k_1)^2),$$

and , by induction on ℓ, using the fact that no α-stacking sequence of length $\geq k_1$ occurs in $I(i_0, L)$, we obtain that: $\forall \alpha \in \{-,+\}, \forall \ell \leq L + 2(k_1)^2 + k_1, \exists k \in [L + 2(k_1)^2, L + 2(k_1)^2 + \ell k_1]$,

$$||U^\alpha_{(i_0+k)}|| \leq ||U^\alpha_{i_0+L+2(k_1)^2}|| - |Q|\ell.$$

Taking $\bar{\ell} = L+2(k_1)^2+k_1$, we obtain some $k^-, k^+ \in [L+2(k_1)^2, L+2(k_1)^2+\bar{\ell}k_1]$ such that $\forall \alpha \in \{-,+\}$,

$$||U^\alpha_{i_0+k^\alpha}|| \leq ||U^\alpha_{i_0+L+2(k_1)^2}|| - |Q|\bar{\ell} \leq \mathrm{N}(i_0) - |Q|k_1. \tag{105}$$

Let $\hat{k} = \max\{k^-, k^+\}, \;\; \hat{i_0} = i_0 + \hat{k}$.

$$\begin{aligned}\hat{i_0} \leq i_0 + (L + 2(k_1)^2) + (L + 2(k_1)^2 + k_1)k_1 &= i_0 + L(k_1 + 1) + 2(k_1)^3 + 3(k_1)^2 \\ &= i_0 + K_1 L + K_2 - (k_1 + 1)\end{aligned}$$

As there is no α-stacking sequence of length $\geq k_1$ in $I(i_0, L)$, with the help of lemma 8.2, we obtain

$$\forall \alpha \in \{-,+\}, \forall k \in [0, k_1], ||U^\alpha_{\hat{i_0}+k}|| \leq \mathrm{N}(i_0) - |Q|,$$

hence $\mathrm{N}(\hat{i_0}+k_1) \leq \mathrm{N}(i_0) - |Q|$. As this inequality would contradict the hypothesis that σ is N-stacking, assertion (104) is proved.
Let us consider an integer i satisfying (104). Suppose that T_B^α occurs at i . By lemma 8.8, $||\overline{U_i^\alpha}|| \geq \mathrm{N}(i_0) - 2k_1|Q|$ and by the property $\mathcal{P}(\alpha, i', j')$ (where we choose $i' = j' = i - k_1 - 1$), established in the proof of lemma 8.10, $\overline{U_i^\alpha} \in \tilde{W}$. Hence $U_i^\alpha \in \tilde{V} \subseteq \overline{V_1}$ and $U_i^{-\alpha} \in \overline{V_1}$. □
Let us give now a stronger version of lemma 8.12 where we analyze the *size of the coefficients* of the linear combinations whose existence is proved in lemma 8.12. We recall that:

$$K_3 = k_0|Q|, \;\; K_4 = k_0|Q|^2 + (k_2 + 6)|Q|.$$

Lemma 8.13 *Let $L > 0$. There exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ such that, for every $\alpha \in \{-,+\}$, there exists $\alpha' \in \{-,+\}$ and a deterministic family $(\beta^{\alpha}_{q,u})_{q \in Q, u \in X^{\leq k_1}}$ fulfilling*

(1) $U^{\alpha}_i = \sum_{\substack{q \in Q \\ |u| \leq k_1}} \beta^{\alpha}_{q,u}(\Phi^{\alpha'}_q \odot u)$
(2) $\|\beta^{\alpha}_{*,*}\| \leq K_3 \cdot (i - i_0) + K_4$.

(here $(\beta^{\alpha}_{q,u})_{q \in Q, u \in X^{\leq k_1}}$ is seen as a row vector whose set of column indices is the set of pairs $\{(q,u) | q \in Q, u \in X^{\leq k_1}\}$).
Proof: We have already established property (104) i.e. $\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2], \exists \alpha \in \{-,+\}, T^{\alpha}_B$ occurs at i. Then

$$U^{-\alpha}_i = \bar{U} \odot u, \text{ for some } u \in X^{\leq k_1}$$

$$U^{+\alpha}_i = \sum_{q \in Q'} [phq][qeq](\bar{U} \odot u_q), \text{ for some } Q' \subseteq Q, u_q \in X^{<1,k_1>}$$

$$\bar{U} = U^{-\alpha}_{i-k_1-1} = [r\omega] * \Phi^{\alpha'}, \text{ for some } r \in Q, \omega \in Z^*, \alpha' \in \{-,+\}$$

and by lemma 8.8 and lemma 8.11

$$|\omega| \leq k_0(i - i_0 - k_1 - 1) + (k_2 + 2). \tag{106}$$

Let us analyze first the coefficients of $U^{-\alpha}_i$ expressed as a linear combination of the $\{\Phi^{\alpha'}_q \odot w | \; 0 \leq |w| \leq k_1\}$.
Either $U^{-\alpha}_i = ([r\omega] \odot u) * \Phi^{\alpha'}$ and then

$$\begin{aligned} \|[r\omega] \odot u\| \leq \|[r\omega]\| + |Q||u| &\leq 1 + (k_0(i - i_0 - k_1 - 1) + k_2 + 1 + k_1)|Q| \\ &\leq k_0|Q|(i - i_0) + (k_2 + 2)|Q|. \end{aligned}$$

or $U^{-\alpha}_i = \Phi^{\alpha'}_q \odot u''$ for some $q \in Q$, u'' suffix of u and then

$$\|(\emptyset, \ldots, \emptyset, \epsilon, \emptyset, \ldots, \emptyset)\| = 2 \leq k_0|Q|(i - i_0) + (k_2 + 2)|Q|.$$

Let us analyze now the coefficients of $U^{+\alpha}_i$. In order to deal with matrices we fix some total orderings of the sets $Q, \mathcal{G}^{\alpha'} = \{\Phi^{\alpha'}_q \odot w | \; q \in Q, w \in X^{\leq k_1}\}$:

$$Q = \{q_1, q_2, \ldots, q_n\}, \text{ where } n = |Q|,$$

$$\mathcal{G}^{\alpha'} = \{\psi_1, \psi_2, \ldots, \psi_m\}, \text{ where } m = |Q|\mathrm{Card}(X^{\leq k_1}).$$

Let us consider the following matrices $A \in \mathsf{DRB}_{1,n} < V >, B \in \mathsf{DRB}_{n,m} < V >$ where

$$a_{i,j} = [phq_j][q_jeq_j]$$

$$b_{i,l} = [r\omega q_j] \odot u_{q_i}, \quad \text{if } [r\omega] \odot u_{q_i} \neq (\emptyset, \ldots, \emptyset) \text{ and } \psi_l = \Phi^{\alpha'}_{q_j},$$

$$b_{i,l} = \emptyset \quad \text{if } u_{q_i} = u'_i u''_i, u''_i \neq \epsilon, [r\omega q_j] \odot u'_i = \epsilon \text{ and } \psi_l \neq \Phi^{\alpha'}_{q_j} \odot u''_i$$

$$b_{i,l} = \epsilon \quad \text{if } u_{q_i} = u'_i u''_i, u''_i \neq \epsilon, [r\omega q_j] \odot u'_i = \epsilon \text{ and } \psi_l = \Phi^{\epsilon'}_{q_j} \odot u''_i$$

Then

$$U_i^{-\alpha} = A \cdot B \cdot \bar{\psi}$$

where $\bar{\psi} \in \mathsf{DRB}_{m,1} < V >$ is defined by

$$\bar{\psi}_{\ell,1} = \psi_\ell(\text{ for } 1 \leq \ell \leq m).$$

Let us analyze the sizes of A, B.
It is clear that $\|A\| = \|[phe]\| = (1 + |h|)|Q| + 2 \leq (1 + k_1)|Q| + 2$.
Let us consider $B_{i,*} = (b_{i,\ell})_{1 \leq \ell \leq m}$. If $[r\omega] \odot u_{q_i} = [s\omega_i']$ then

$$\omega_i' = \mu_i \omega_i'' \text{ for some } |\mu_i| \leq k_0, \omega_i'' \text{ suffix of } \omega.$$

If $\exists j, [r\omega q_j] \odot u_i' = \epsilon, u_{q_i} = u_i' u_i''$ then

$$\mathsf{Q}(B_{i,*}) = \{(\emptyset, \ldots, \emptyset, \epsilon, \emptyset, \ldots, \emptyset), (\emptyset, \ldots, \emptyset)\}.$$

If none of these two cases is realized then

$$\mathsf{Q}(B_{i,*}) = \{(\emptyset, \ldots, \emptyset)\}.$$

It follows that

$$\text{Card}(\cup_i \mathsf{Q}(B_{i,*})) \leq |Q|^2(k_0 - 1) + |Q|(|\omega| + 2) + 1$$

Using the upper-bound (106) on $|\omega|$ we obtain:

$$\begin{aligned} \|B\| &\leq k_0|Q|^2 + |Q||\omega| + |Q| + 1 \\ &\leq (k_0|Q|^2 + |Q| + 1) + |Q|(k_0(i - i_0) + k_2 + 2 - k_0(k_1 + 1)) \end{aligned}$$

By lemma 3.17 we obtain

$$\begin{aligned} \|A \cdot B\| \leq \|A\| + \|B\| &\leq [(1 + k_1)|Q| + 2] + [(k_0|Q|^2 + |Q| + 1) + |Q|(k_0(i - i_0) + k_2 + 2 - k_0(k_1 + 1))] \\ &\leq k_0|Q|^2 + (k_2 + 6)|Q| + k_0|Q|(i - i_0) \\ &= K_3(i - i_0) + K_4. \end{aligned}$$

□

Let us denote by $\mathcal{G}$ the generating set of $\overline{V_1}$: $\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-$ (where the $\mathcal{G}^{\alpha'}$ are the sets defined in the above proof. We set $d = \text{Card}(\mathcal{G})$ and we fix a total ordering on $\mathcal{G}$: $\mathcal{G} = \{\theta_1, \theta_2, \ldots, \theta_d\}$.

Lemma 8.14 *There exists $i_0 \leq \kappa_1 < \kappa_2 < \ldots < \kappa_d$ and deterministic rational vectors $(\beta_{i,j}^\alpha)_{1 \leq j \leq d}$ (for every $i \in [1, d]$) such that*

(0)$W(\kappa_1) \geq 1$
(1)$\forall i, \forall \alpha, U_{\kappa_i}^\alpha = \sum_{j=1}^d \beta_{i,j}^\alpha \theta_j \in \overline{V_1}$
(2)$\forall i, \forall \alpha, \|\beta_{i,*}^\alpha\| \leq s_i$
(3)$\forall i, W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}$

where the sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \sigma_i)$ are those defined by relations (60, 61) in section 6.
Proof: Let us remark that $N(i_0) \geq D_2 + 2 \cdot k_0 \cdot |Q|$, hence $|x_{\kappa_1}| \geq |x_{i_0}| \geq 2$. By lemma 8.1 it follows that $W(\kappa_1) \geq 1$. Let us consider now the additional property
(4) $\kappa_i - i_0 \leq L_i$.
We prove by induction on i the conjunction $(1) \wedge (2) \wedge (3) \wedge (4)$.
i=1:
By lemma 8.13, there exists $\kappa_1 \in [i_0, i_0 + K_2]$ such that $\forall \alpha \in \{-,+\}, \exists$ a deterministic vector $(\beta^{\alpha}_{1,j})_{1 \leq j \leq d}$, such that

$$U^{\alpha}_{\kappa_1} = \sum_{j=1}^{d} \beta^{\alpha}_{1,j} \theta_j$$

and in addition $\|\beta^{\alpha}_{1,*}\| \leq K_3 K_2 + K_4 = s_1$.
i → i+1:
Suppose that $\kappa_1 < \kappa_2 < \ldots < \kappa_i$ are fulfilling $(1) \wedge (2) \wedge (3) \wedge (4)$. By lemma 8.13, there exists $\kappa_{i+1} \in [i_0 + L_i + \ell_{i+1}, i_0 + K_1(L_i + \ell_{i+1}) + K_2]$ such that $\forall \alpha \in \{-,+\}, \exists$ a deterministic vector $(\beta^{\alpha}_{i+1,j})_{1 \leq j \leq d}$, such that

$$U^{\alpha}_{\kappa_{i+1}} = \sum_{j=1}^{d} \beta^{\alpha}_{i+1,j} \theta_j \tag{107}$$

and in addition

$$\begin{aligned} \|\beta^{\alpha}_{i+1,*}\| \leq K_3(K_1(L_i + \ell_{i+1}) + K_2) + K_4 &= K_3 L_{i+1} + K_4 \\ &= s_{i+1} \end{aligned} \tag{108}$$

By lemma 8.1

$$2(W(\kappa_{i+1}) - W(\kappa_i)) + 3 \geq \kappa_{i+1} - \kappa_i \geq \ell_{i+1} = 2\delta_{i+1} + 3$$

hence

$$W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}. \tag{109}$$

At last

$$\kappa_{i+1} - i_0 \leq K_1(L_i + l_{i+1}) + K_2 = L_{i+1}. \tag{110}$$

The above properties (107-108-109-110) prove the required conjunction. □
We recall that:

$$C_2 = \mathrm{Card}\{U \in \mathsf{DRB} < V >, \|U\| \leq D_2\}.$$

$$K_6 = 7 \cdot [(C_2 + |Z|^{k_2+2})^{|Q|} \cdot |Z|^{D_1}]^2, \quad K_5 = (K_6 + 1) \cdot k_0 \cdot |Q|.$$

Lemma 8.15 *Let $(x_i)_{i \in I}$ be a path in τ (we suppose $I \subseteq \mathbb{N}$ is a non-empty interval). Let $L > 0$. One of the following cases is true:*

(0) $N(i_0) \geq N_0$, where $i_0 = \min(I)$

(1) $|I| \leq K_5 \cdot L + K_6$
(2) $(x_i)_{i \in I}$ *contains a* N*-stacking sequence of length* $\geq L$.

Proof: Suppose that neither (0) nor (2) is realized. The set $\{\tau_s(x_i) | i \in I\}$ can contain at most K_6 pairs (U^-, U^+) such that $|||\ (U^-, U^+)\ ||| < N_0$ (because they are (D_1, D_2)-linear and no pair can appear more than 7 times on a given path , due to the inequality of lemma 8.1, to the cut rule and to definition (??) of a proof-tree). Hence

$$\mathrm{Card}\{i \in I | \mathrm{N}(i) < N_0\} \leq K_6.$$

Let us consider now an interval $J = [i, i + \ell - 1] \subseteq I$ which is a maximal sub-interval of I on which N takes values in $[N_0, +\infty[$. As N is $k_0|Q|$-up-Lipschitz, and either $i = i_0$ or $\mathrm{N}(i-1) < N_0$ we have:

$$\mathrm{N}(i) < N_0 + k_0|Q|.$$

As J does not contain any N-stacking sequence of length L,

$$\ell \leq (L-1)k_0|Q|.$$

Finally, I contains at most $(K_6 + 1)$ maximal sub-intervals J on which N takes values in $[N_0, +\infty[$. It follows that $|I| \leq K_6 + (K_6 + 1)(L-1)k_0|Q|$ i.e.

$$|I| \leq K_5 \cdot L + K_6,$$

i.e. property (1) is realized. □

9 Completeness of $\mathcal{D}_0$

Lemma 9.1 *:* $\mathcal{S}_{ABC}$ *is terminating.*

Proof: Suppose A is true and $t = \mathcal{T}(\mathcal{S}_{ABC}, A)$ is infinite.

By Koenig's lemma, t contains an infinite branch whose (infinite) labelling word is $A_0 A_1 \cdots A_n \cdots A_s \cdots$. Lemma 5.8 applied to $m = 1$ and $d \leq d_0$ and combined with lemma 3.15 shows that the equations produced by T_C have size $\leq D_2$, hence that the number of possible results of T_C is *finite*. Hence T_C occurs only a finite number of times on this branch (otherwise T_{cut} would occur on this branch, which is impossible on an infinite branch). Let n_0 be the last point where T_C occurs. $(A_{n_0+i})_{i \geq 0}$ is a branch of the tree $t' = \mathcal{T}(\mathcal{S}_{AB}, A_{n_0})$. As every result of T_C is a weighted linear equation which has size bounded by D_2 (by the above arguments) and which is unmarked (by point (C4) in the definition of T_C), t' fulfills the hypothesis (88) stated in section 8.2 and assumed in section 8.4. By lemma 8.15, the branch $(A_{n_0+i})_{i \geq 0}$ must contain an infinite N-stacking sequence. Let us remark that, as $T_\emptyset$ does not occur (otherwise the branch would be finite) every equation (π, U^-, U^+) labelling this branch is such that $U^- \not\equiv \emptyset, U^+ \not\equiv \emptyset$. By lemma 8.14 such an N-stacking sequence contains a subsequence $(A_{i_1}, A_{i_2}, \cdots, A_{i_d})$ with $d \leq d_0$, fulfilling hypotheses (1,2) of lemma 5.8, and by the above remark it fulfills hypothesis (0) too. Hence some finite prefix of $(A_{n_0+i})_{i\geq 0}$ belongs to dom(T_C). The priority ordering given in the definition of $\mathcal{S}_{ABC}$ then implies that either $T_{cut}, T_\emptyset, T_\varepsilon$ or T_C occurs at some $n_0 + i + 1$. But the three first cases cannot occur on an infinite branch and the fourth one contradicts the maximality of n_0. □

Theorem 9.2 *The system $\mathcal{D}_0$ is complete.*

Proof: By lemma 7.1 $\mathcal{S}_{ABC}$ is a strategy for $\mathcal{D}_0$, by lemma (7.10) $\mathcal{S}_{ABC}$ is closed , by lemma (9.1) it is terminating and by lemma 4.9, $\mathcal{D}_0$ is complete. □

Theorem 9.3 *The equivalence problem for deterministic pushdown automata is decidable.*

Proof: Let $\mathcal{M}$ be some dpda. The equivalence relation $\equiv$ on $\mathsf{DRB} < V >$ (where V is the structured alphabet associated to the given $\mathcal{M}$) has a recursively enumerable complement (this is well-known). By theorem 9.2 and lemma 4.2 $\equiv$ is recursively enumerable too. Hence $\equiv$ is recursive.
Now, let $\mathcal{M}$ vary within the set of all dpda (on a given terminal alphabet X and with sets Q,Z included in some fixed denumerable set). The system $\mathcal{D}_0$ associated with $\mathcal{M}$ is computable from $\mathcal{M}$, hence theorem 9.2 and lemma 4.2 show that the following problem is decidable:

INSTANCE: a dpda $\mathcal{M}$ and $S, S' \in \mathsf{DRB} < V >$ (where V is the structured alphabet associated with $\mathcal{M}$).
QUESTION: $S \equiv S'$?

This shows that the equivalence problem for dpda is decidable. □

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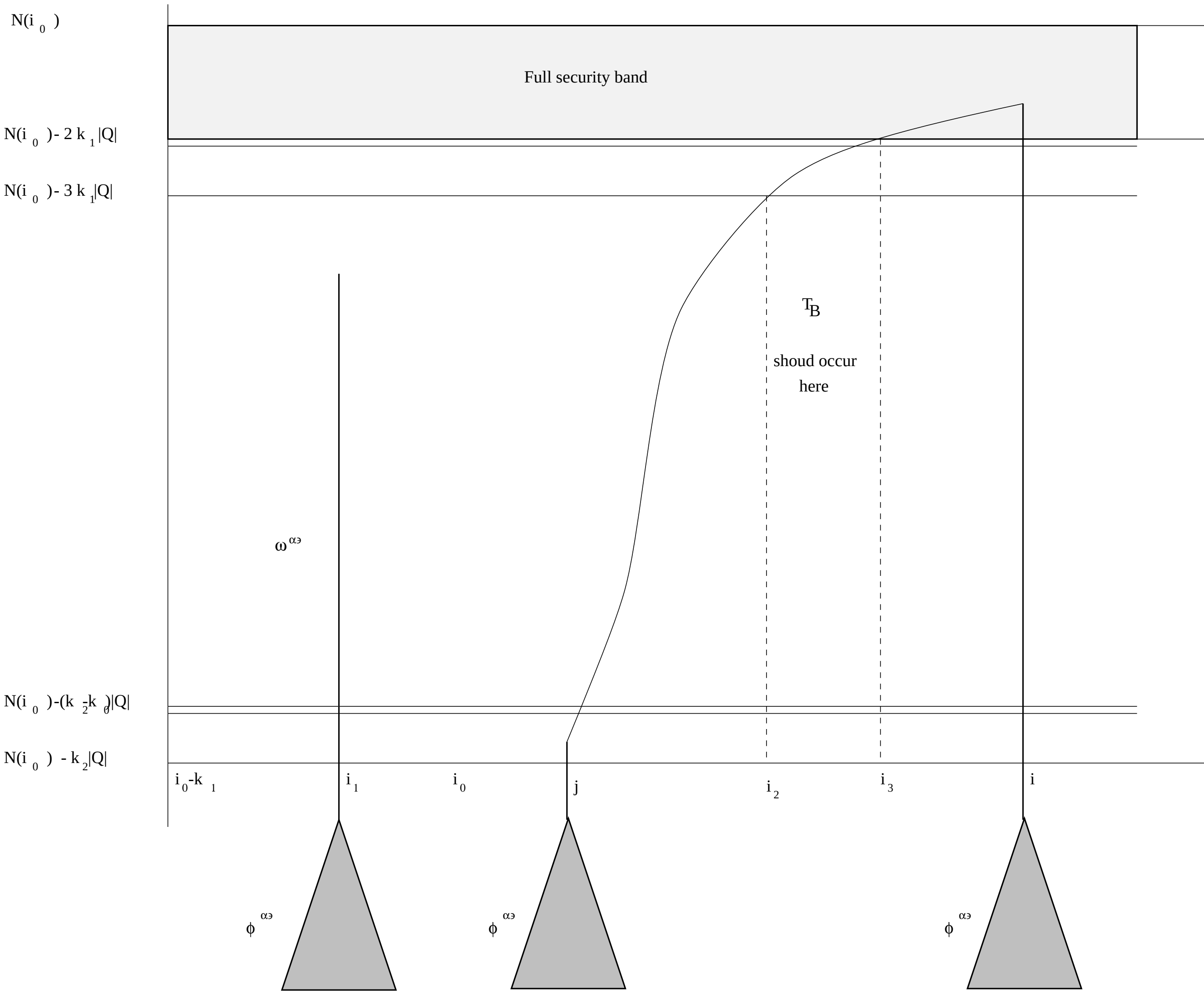

Fig. 3. U_j^α out of security band is impossible.