Rational subsets in HNN-extensions and amalgamated products

Markus Lohrey and Géraud Sénizergues

[1] Universität Stuttgart, FMI, Germany
[2] LABRI, Université Bordeaux I, France
lohrey@informatik.uni-stuttgart.de, ges@labri.fr

Abstract. Several transfer results for rational subsets and finitely generated subgroups of HNN-extensions $G = \langle H, t; t^{-1}at = \varphi(a)(a \in A)\rangle$ and amalgamated free products $G = H *_A J$ such that the associated subgroup A is finite. These transfer results allow to transfer decidability properties or structural properties from the subgroup H (resp. the subgroups H and J) to the group G.

1 Introduction

This paper is the first paper in a series of three papers dealing with algorithmic problems for equations over HNN-extensions and amalgamated free products of groups. HNN-extensions appeared in [HNN49] for the first time. These two operations are of fundamental importance in combinatorial group theory [LS77]. Recall that an amalgamated free product

$$G = \langle H_1 * H_2, a = \varphi(a)(a \in A_1)\rangle \tag{1}$$

of two groups H_1 and H_2 with subgroups $A_1 \leq H_1$, $A_2 \leq H_2$ and an isomorphism $\varphi : A_1 \to A_2$, results from the free product $H_1 * H_2$ by identifying every element $a \in A_1$ with $\varphi(a) \in A_2$. An HNN extension

$$G = \langle H, t; t^{-1}at = \varphi(a)(a \in A)\rangle \tag{2}$$

of a group H, where A and B are isomorphic subgroups of H and $\varphi : A \to B$ is an isomorphism, results from adding to H a new generator t such that the conjugation of the subgroup $A \leq H$ by t realizes the isomorphism φ.

One of the first important applications of HNN-extensions was a more transparent proof of the celebrated result of Novikov and Boone on the existence of a finitely presented group with an undecidable word problem, see e.g. [LS77]. Such a group can be constructed by a series of HNN-extensions starting from a free group. This shows that arbitrary HNN-extensions to not have good algorithmic properties. Therefore we mainly consider HNN-extensions and amalgamated free products, where the subgroups A and B in (1) and (2), respectively, are finite.

In this paper we prove several several transfer theorems about rational subsets and finitely generated subgroups of HNN-extensions and amalgamated products of groups. Some of these results are already known, others appear to be new. Apart from this, the purpose of this investigations is two-fold: First, we prepare the background for our further considerations on equations over HNN-extensions and amalgamated free products [LS06a,LS06b]. Second, we present a unified automata-theoretic framework, which is partly inspired by [Reu85,Sak87,Sak92], for reasoning about rational subsets of HNN-extensions and amalgamated free products. This framework allows to deduce known results as well as several new transfer theorems. Moreover, our approach allows to prove results about rational subsets even for HNN-extensions and amalgamated products of *monoids*, in case the associated parts A and B in (1) and (2) are groups. Let us now describe in more detail the content of this paper.

Section 2 contains necessary preliminaries concerning monoids, groups, HNN-extensions, amalgamated free products, and rational sets. At the beginning of Section 4, we introduce finite t-automata, which are a specific automaton model in order to recognize subsets of an HNN-extension $G = \langle H, t; t^{-1}at = \varphi(a)(a \in A)\rangle$, where H is a monoid and A and $B = \varphi(A)$ are finite groups. The transitions of such an automaton are either labelled with t, t^{-1}, or sets from some subset $\mathcal{F} \subseteq \mathcal{P}(H)$ (which has to contain every singleton subset consisting of an element from $A \cup B$. For our specific applications, $\mathcal{F}$ is either the set of rational subsets of H or the set of left cosets of finitely generated subgroups H. We prove several normalization results for finite t-automata (Proposition 1 and 2). Based on these results we can show Theorem 2, which is our main result for rational subsets of HNN-extensions. It says that the following four properties can be transfered from the monoid H to the HNN-extension $G = \langle H, t; t^{-1}at = \varphi(a)(a \in A)\rangle$:

(1) The set of rational subsets is closed under intersection.
(2) The set of rational subsets is closed under complement.
(3) The membership problem for rational subsets is decidable.
(4) The emptiness problem for boolean combinations of rational subsets is decidable.

Statement (3) is covered by [KSS06, Theorem 5.1], whereas (1), (2), and (4) are new to the knowledge of the authors. Theorem 3 states similar transfer results for finitely generated subgroups. More precisely, the following three properties can be transfered from a group H to the HNN-extension $G = \langle H, t; t^{-1}at = \varphi(a)(a \in A)\rangle$ (with A and B finite):

(1) Decidability of the generalized word problem.
(2) The Howson property.
(3) The LERF property.

Recall that a group X has the Howson property if the intersection of two finitely generated subgroups is again finitely generated. The group X has the LERF property [Hal49] if for every finitely generated subgroup $Y < X$ and every $x \in X \setminus Y$ there exists a normal subgroup Z of finite index in X with $Y \leq Z < X$ and $x \in X \setminus Z$. Statement (1) is covered by [KWM05, Corollary 5.15], statement (2) was shown in [Bez98,KS71], and statement (3) was shown in [MR04]. Finally, in Section 6 we briefly sketch the proof of analogous transfer results for amalgamated free products (Theorem 4 and 5).

2 Preliminaries

We recall in this section all needed definitions and classical results concerning groups, monoids, finite automata, and rational subsets of a monoid. We also recall some classical algorithmic problems about rational subsets of finitely generated groups (monoids).

We assume that the reader has some familiarity with monoid presentations and group theory. For a monoid homomorphism $\pi : \mathbb{M}_1 \to \mathbb{M}_2$ we denote with $\mathrm{Ker}(\pi) = \{(x, y) \mid \pi(x) = \pi(y)\}$ the *kernel* of π, it is a monoid congruence on $\mathbb{M}_1$. The neutral element 1 of a monoid $\mathbb{M}$ will be identified with the submonoid $\{1\}$.

The powerset of a set S is denoted by $\mathcal{P}(S)$. For an equivalence relation $\equiv$ on S and $s \in S$ we denote with $[s]_\equiv$ the equivalence class of $\equiv$ containing s. For a subset $T \subseteq S$ let $[T]_{\equiv} = \bigcup_{s \in T}[s]_{\equiv}$. The boolean closure of a set $F \subseteq \mathcal{P}(S)$ is denoted by $\mathrm{Bool}(F)$. It contains all subsets of S, which can be obtained by finitely many applications of complement and union, starting with subsets from F.

A *semi-Thue system* ST over the alphabet X is just a (possibly infinite) subset of $X^* \times X^*$. We associate with ST the one-step rewrite relation $\rightarrow_{\mathsf{ST}}$, where $x \rightarrow_{\mathsf{ST}} y$ for $x, y \in X^*$ if and only if there are $u, v \in X^*$ and $(\ell, r) \in \mathsf{ST}$ such that $x = u\ell v$ and $y = urv$. The set of irreducible words w.r.t. ST is

$$\mathrm{Irr}(\mathsf{ST}) = \{x \in X^* \mid \neg\exists y \in X^* : x \rightarrow_{\mathsf{ST}} y\}.$$

A binary relation $\rightarrow$ over a set E is said *confluent* iff, for every $e, f, f' \in E$,

$$(e \rightarrow^* f \wedge e \rightarrow^* f') \Rightarrow (\exists g \in E, f \rightarrow^* g \wedge f' \rightarrow^* g).$$

Suppose that a semi-Thue system ST has no lefthandside overlap i.e. for every $(\ell, r), (\ell', r') \in \mathsf{ST}, u, v \in X^*$,

$$(u\ell = \ell' v) \Rightarrow |u| \leq |\ell'|; \;\; (u\ell v = \ell') \Rightarrow (u = v = 1 \wedge r = r').$$

Then it can be easily shown that the relation $\rightarrow_{\mathsf{ST}}$ is confluent. See [DJ91] for more information about confluence and other properties of rewriting systems.

2.1 Finite automata over a free monoid

Let X be an alphabet (that we do not assume to be finite, in general). A finite automaton over the free monoid X^* is a 5-tuple

$$\mathcal{A} = \langle \mathcal{X}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle, \tag{3}$$

where $\mathcal{X}$ is a finite subset of X, Q is the finite set of states, $\mathsf{I} \subseteq \mathsf{Q}$ is the set of initial states, $\mathsf{T} \subseteq \mathsf{Q}$ is the set of terminal states, and δ, the set of transitions, is a subset of $\mathsf{Q} \times \mathcal{X} \times \mathsf{Q}$. Let

$$\mathsf{BR}(\mathsf{Q}) = \mathcal{P}(\mathsf{Q} \times \mathsf{Q})$$

be the monoid of binary relations over the set Q. We recall that, given two binary relations $m, m' \in \mathsf{BR}(\mathsf{Q})$, their composition is defined by

$$m \circ m' = \{(q, q'') \in \mathsf{Q} \times \mathsf{Q} \mid \exists q' \in \mathsf{Q}, (q, q') \in m \wedge (q', q'') \in m'\}.$$

The automaton (3) induces a representation map

$$\mu_{\mathcal{A}} : (X^*, \cdot, \varepsilon) \to (\mathsf{BR}(\mathsf{Q}), \circ, \mathrm{Id}_{\mathsf{Q}})$$

defined as the unique monoid homorphism such that for every $x \in X$:

$$\mu_{\mathcal{A}}(x) = \{(q, r) \in \mathsf{Q} \times \mathsf{Q} \mid (q, x, r) \in \delta\}.$$

We denote by $\odot$ the natural right-action of the monoid $\mathsf{BR}(\mathsf{Q})$ over $\mathcal{P}(\mathsf{Q})$, i.e., for every $P \subseteq \mathsf{Q}$ and $m \in \mathsf{BR}(\mathsf{Q})$ let

$$P \odot m = \{q \in \mathsf{Q} \mid \exists p \in P : (p, q) \in m\}. \tag{4}$$

The language recognized by $\mathcal{A}$ is

$$\mathrm{L}(\mathcal{A}) = \{u \in X^* \mid \mu_{\mathcal{A}}(u) \cap \mathsf{I} \times \mathsf{T} \neq \emptyset\}.$$

A subset $L \subseteq X^*$ is said *regular* iff there exists some finite automaton $\mathcal{A}$ such that $L = \mathrm{L}(\mathcal{A})$.

Given subsets $X' \subseteq X, \mathsf{I}' \subseteq \mathsf{Q}, \mathsf{T}' \subseteq \mathsf{Q}$, the automaton $\mathcal{A}[X', \mathsf{I}', \mathsf{T}']$ over X'^* is defined by

$$\mathcal{A}[X', \mathsf{I}', \mathsf{T}'] = \langle \mathcal{X} \cap X', \mathsf{Q}, \delta \cap (\mathsf{Q} \times X' \times \mathsf{Q}), \mathsf{I}', \mathsf{T}' \rangle. \tag{5}$$

2.2 Rational subsets of a monoid

Let $\mathbb{M}$ be a countably-generated monoid together with a generating set X. Thus $\mathbb{M} = X^*/\equiv$, for some monoid congruence $\equiv$ over X^*. We denote by

$$\pi : X^* \to X^*/\equiv$$

the canonical homomorphism with $\pi(u) = [u]_\equiv$ for $u \in X^*$.

The set

$$\mathrm{Rat}(\mathbb{M}) \subseteq \mathcal{P}(M)$$

is the smallest subset of $\mathcal{P}(M)$ which contains the finite subsets of $\mathbb{M}$ and which is closed under the operations $\cup$ (the union operation),

$\cdot$ (the product operation) and $*$ (the star operation, associating with a subset P the smallest submonoid of $\mathbb{M}$ containing P).

One can derive from Kleene's theorem about rational subsets of X^* that the rational subsets of $\mathbb{M}$ are exactly the subsets of the form $\pi(\mathrm{L}(\mathcal{A}))$, where $\mathcal{A}$ is a finite automaton over the free monoid X^*. For such an automaton $\mathcal{A} = \langle \mathcal{X}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle$ let the representation map

$$\mu_{\mathcal{A},\mathbb{M}} : \mathbb{M} \to (\mathsf{BR}(\mathsf{Q}), \circ, \mathrm{Id}_{\mathsf{Q}})$$

by defined by

$$\mu_{\mathcal{A},\mathbb{M}}(m) = \bigcup \{\mu_{\mathcal{A}}(u) \mid u \in X^*, \pi(u) = m\}$$

for $m \in \mathbb{M}$. Notice that

$$\mathrm{Id}_{\mathsf{Q}} \subseteq \mu_{\mathcal{A},\mathbb{M}}(1) \text{ and } \mu_{\mathcal{A},\mathbb{M}}(m) \circ \mu_{\mathcal{A},\mathbb{M}}(m') \subseteq \mu_{\mathcal{A},\mathbb{M}}(m \cdot m') \qquad (6)$$

but in general the map $\mu_{\mathcal{A},\mathbb{M}}$ is *not* a monoid homomorphism. In Section 4.1, we introduce in the particular case where $\mathbb{M}$ is an HNN-extension a new kind of finite automaton, called *finite t-automata*, which do not even fulfill property (6). We will show that, nevertheless, we can always normalize finite t-automata in such a way that the map μ fulfills some nice algebraic property, see Definition 4 and Proposition 2.

Theorem 1 ([AS75]). *Let G be a group. The rational subgroups of G are exactly the finitely generated subgroups of G.*

We present a proof sketch for Theorem 1, because the underlying idea will be used later in a slightly modified form.

Proof sketch for Theorem 1. Let $G = X^*/\equiv$ be a group, where X is some alphabet and $\equiv$ is a congruence over X^*. We suppose that X is endowed with an involution $x \mapsto x^{-1}$ such that for every $x \in X$, $xx^{-1} \equiv x^{-1}x \equiv \varepsilon$. We extend this involution to X^* by setting $(x_1 \cdots x_n)^{-1} = x_n^{-1} \cdots x_1^{-1}$, where $x_1, \ldots, x_n \in X$. We denote by $\pi : X^* \to G$ the canonical projection, i.e., $\pi(u) = [u]_\equiv$. Let $\mathcal{A} = \langle \mathcal{X}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle$ be some finite automaton over X^* such that

$$\pi(\mathrm{L}(\mathcal{A})) = K$$

is a subgroup of G. Without loss of generality, we can suppose that $\mathcal{A}$ has a unique initial state q_0 and that $\mathcal{A}$ is trim, i.e., that every state is accessible from $\mathsf{I} = \{q_0\}$ and co-accessible from T. Let us define $\delta^{-1} = \{(r, x^{-1}, q) \mid (q, x, r) \in \delta\}$ and $\delta' = \delta \cup \delta^{-1}$. Since $\pi(\mathrm{L}(\mathcal{A})) = K$ is a subgroup of G, it follows easily that

$$\pi(\mathrm{L}(\mathcal{A})) = \pi(\mathrm{L}(\langle \mathcal{X}, \mathsf{Q}, \delta', \mathsf{I}, \mathsf{T} \rangle)).$$

Let us fix some subset $\lambda \subseteq \delta'$ which is a covering tree in the sense that for every $q \in \mathsf{Q}$, there exists a *unique* path $\Lambda(q)$ from q_0 to q, using only edges in λ. We denote by $W(q) \in X^*$ the sequence of labels of this path. Note that $W(q_0) = \varepsilon$. Let us define

$$\mathcal{G} = \{W(q)xW(r)^{-1} \mid (q, x, r) \in \delta\} \cup \{W(q) \mid q \in \mathsf{T}\} \subseteq X^*.$$

One can check that $K = \pi((\mathcal{G} \cup \mathcal{G}^{-1})^*)$. Thus, K is finitely generated. $\square$

The idea above is reminiscent of that of defining a set of generators for the fundamental group of a graph.

Definition 1. *A group G is said to have the* Howson property *if for every two finitely generated subgroups K_1 and K_2 of G, the subgroup $K_1 \cap K_2$ is finitely generated, too.*

Definition 2. *A group G is said to have the* subgroup separation property *if for every finitely generated proper subgroup $K < G$, and every element $g \in G \setminus K$, there exists a normal subgroup K' which has finite index in G and such that $K \subseteq K'$ and $g \in G \setminus K'$.*

The subgroup separation property is also called the *local extended residual finiteness property* (*LERF* for short) and has been investigated in several works, see e.g. [Hal49,Sco78].

2.3 Algorithmic problems

Let $\mathbb{M} = X^*/\!\equiv$ be a monoid, where X is a *countable* generating set. Let $\pi : X^* \to X^*/\!\equiv$ be the canonical homomorphism.

The *word problem* for $\mathbb{M}$ is the following decision problem:

instance: Words $u, v \in X^*$

question: $\pi(u) = \pi(v)$?

Beside the word problem we shall investigate the following (more general) decision problems:

The *membership problem for rational subsets of* $\mathbb{M}$ is the following decision problem:

instance: A finite automaton $\mathcal{A}$ over X^* and a word $u \in X^*$.
question: $\pi(u) \in \pi(\mathrm{L}(\mathcal{A}))$?

The *emptiness problem for boolean combinations of rational subsets of* $\mathbb{M}$ is the following decision problem:

instance: A set $B \in \mathrm{Bool}(\mathrm{Rat}(\mathbb{M}))$ (represented by a boolean combination B of finite automata over X^*)
question: $B = \emptyset$?

Let us now assume that $\mathbb{M} = X^*/\equiv$ is a group. The *membership problem for finitely generated subgroups of* $\mathbb{M}$, also called the *generalized word problem* (GWP for short) is the following decision problem:

instance: Words $u_1, \ldots, u_n, u \in X^*$.
question: $\pi(u) \in \{\pi(u_1), \pi(u_1)^{-1}, \ldots, \pi(u_n), \pi(u_n)^{-1}\}^*$, i.e., does $\pi(u)$ belong to the subgroup of $\mathbb{M}$ generated by $\pi(u_1), \ldots, \pi(u_n)$?

It is easy to see that the decidability status of all these decision problems is independent of the chosen generating set X for $\mathbb{M}$, i.e., it is indeed a property of the monoid $\mathbb{M}$.

3 HNN-extensions

Let us fix throughout this section a monoid H (the base monoid), two finite *subgroups* $A \leq H, B \leq H$ and an isomorphism $\varphi : A \to B$. We also fix some presentation $\langle X; \equiv_H\rangle$ of H, i.e. an alphabet X and a congruence $\equiv_H$ over the free monoid X^* such that $H = X^*/\equiv_H$. Note that X is not supposed to be finite, in general. Since A and B are finite, we can without loss of generality assume that every element $h \in A \cup B$ is of the form $[x]_{\equiv_H}$ for some letter $x \in X$. In particular, there is a letter 1 for the neutral element of H. Of course $1 \equiv_H \varepsilon$. We consider the HNN-extension

$$G = \langle H, t; t^{-1}at = \varphi(a)(a \in A)\rangle. \tag{7}$$

This means that $G = (X \cup \{t, t^{-1}\})^*/\equiv$ where $\equiv$ is the smallest congruence over the free monoid $(X \cup \{t, t^{-1}\})^*$ such that:

$$\begin{aligned} \equiv_H \ &\subseteq \ \equiv \\ tt^{-1} &\equiv t^{-1}t \equiv \varepsilon \\ t^{-1}at &\equiv \varphi(a) \text{ for all } a \in A \end{aligned}$$

Let

$$\pi : (X \cup \{t, t^{-1}\})^* \to G \tag{8}$$

denote the canonical morphism defined by $\pi(w) = [w]_\equiv$ for $w \in (X \cup \{t, t^{-1}\})^*$. This morphism naturally factorizes as

$$\pi = \pi_t \circ \pi_G$$

where

$$\pi_t : (X \cup \{t, t^{-1}\})^* \to H * \{t, t^{-1}\}^* \text{ and} \tag{9}$$

$$\pi_G : H * \{t, t^{-1}\}^* \to G. \tag{10}$$

The kernel of π_G coincides with the smallest congruence $\approx$ over $H * \{t, t^{-1}\}^*$ such that:

$$tt^{-1} \approx t^{-1}t \approx 1 \tag{11}$$

$$at \approx t\varphi(a) \qquad \text{for all } a \in A \tag{12}$$

$$bt^{-1} \approx t^{-1}\varphi^{-1}(b) \text{ for all } b \in B \tag{13}$$

Note that (11) and (12) together imply (13) but below we will need (12) and (13) without assuming (11). Clearly, $G = H * \{t, t^{-1}\}^*/\approx$.

An element of $s \in H * \{t, t^{-1}\}^*$ can be viewed as a word over the alphabet $H \cup \{t, t^{-1}\}$ which has the form

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n, \tag{14}$$

where $n \geq 0, \alpha_i \in \{1, -1\}$, and $h_i \in H$. Such an element $s \in H * \{t, t^{-1}\}^*$ is also called a *t-sequence*. The t-sequence s is said to be a *reduced sequence* iff it does not contain a factor of the form $t^{-1}at$ (with $a \in A$) or tbt^{-1} (with $b \in B$). Let

$$\mathrm{Red}(H, t) = \{s \in H * \{t, t^{-1}\}^* \mid s \text{ is reduced}\}$$

We define the binary relation $\overset{1}{\sim}$ over $H * \{t, t^{-1}\}^*$ as follows: For all $s, s' \in H * \{t, t^{-1}\}^*$ let

$$s \overset{1}{\sim} s' \quad \Leftrightarrow \quad \exists s_1, s_2 \in H * \{t, t^{-1}\}^*, a \in A, b \in B : \left\{ \begin{array}{l} (s = s_1 t s_2 \wedge s' = s_1 a^{-1} t \varphi(a) s_2) \vee \\ (s = s_1 t^{-1} s_2 \wedge s' = s_1 b^{-1} t^{-1} \varphi^{-1}(b) s_2) \end{array} \right\}.$$

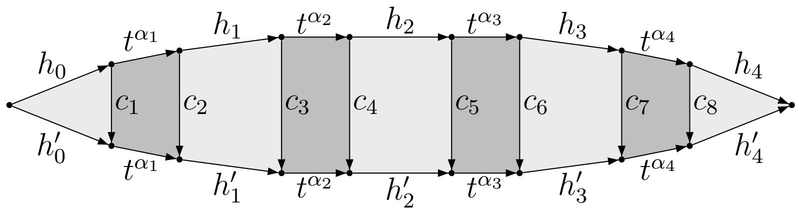
Let us denote by $\sim$ the congruence over $H * \{t, t^{-1}\}^*$ generated by all the rules of type (12) and (13) above. It coincides with the reflexive and transitive closure of $\overset{1}{\sim}$. Equivalently, if

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n \text{ and}$$
$$s' = h'_0 t^{\alpha'_1} h'_1 \cdots t^{\alpha'_m} h'_m$$

(with $n, m \geq 0$, $\alpha_i, \alpha'_j \in \{1, -1\}$ and $h_i, h'_j \in H$), then $s \sim s'$ if and only if $n = m$, $\alpha_i = \alpha'_i$ for $1 \leq i \leq n$, and there exist $c_1, \ldots, c_{2n} \in A \cup B$ such that:

- if $\alpha_i = 1$ then $c_{2i-1} \in A$ and $c_{2i} = \varphi(c_{2i-1}) \in B$ $(1 \leq i \leq n)$
- if $\alpha_i = -1$ then $c_{2i} \in A$ and $c_{2i-1} = \varphi(c_{2i}) \in B$ $(1 \leq i \leq n)$
- $h_i c_{2i+1} = c_{2i} h'_i$ in H for $0 \leq i \leq n$ (here we set $c_0 = c_{2n+1} = 1$)

This situation can be visualized by a diagram of the following form (also called a Van Kampen diagram, see [LS77] for more details), where $n = m = 4$. Light-shaded (resp. dark-shaded) areas represent relation in H (resp. relations of the form $at = t\varphi(a)$ $(a \in A)$ or $bt^{-1} = t^{-1}\varphi^{-1}(b)$ $(b \in B)$).



(†)

The elements $c_1, \ldots, c_{2n}$ in such a diagram are also called *connecting elements*.

The set $\mathrm{Red}(H, t)$ is saturated by the congruence $\sim$, i.e., $s \sim s'$ implies $s \in \mathrm{Red}(H, t) \Leftrightarrow s' \in \mathrm{Red}(H, t)$: just notice that, since A and B are groups, $aha' \in A \Leftrightarrow h \in A$ for all $h \in H, a, a' \in A$ and

$bhb' \in B \Leftrightarrow h \in B$ for all $h \in H, b, b' \in B$. This property would fail if A and B were assumed to be merely submonoids of H.

One has $\sim \subseteq \approx$. Moreover, for reduced sequences the following fundamental lemma holds.

Lemma 1. *Let s, s' be reduced sequences. Then $s \approx s'$ if and only if $s \sim s'$.*

In the case where H is a group, Lemma 1 is an immediate consequence of [LS77, Theorem 2.1 p.182]. The extension to monoids can be proved in the same way as for groups using an idea of Artin-Van der Waerden (see annex A for a detailed verification). Lemma 1 implies that H is a submonoid of G. Another consequence is:

Corollary 1. *If H has a decidable word problem, then also has G a decidable word problem.*

Proof. Let $w, w' \in (X \cup \{t, t^{-1}\})^*$. We describe a procedure for checking whether $w \equiv w'$ or not. The words w and w' have unique decompositions

$$w = u_0 t^{\alpha_1} u_1 \cdots t^{\alpha_i} u_i \cdots t^{\alpha_n} u_n \text{ and } w' = u'_0 t^{\alpha'_1} u'_1 \cdots t^{\alpha'_j} u'_j \cdots t^{\alpha'_m} u'_m,$$

where $n, m \geq 0, u_i, u'_j \in X^*$, and $\alpha_i, \alpha'_j \in \{1, -1\}$. Let the semi-Thue system ST over the alphabet $X \cup \{t, t^{-1}\}$ consist of the following rules:

$$t^{-1} u t \to \varphi([u]_{\equiv_H}) \quad \text{for } [u]_{\equiv_H} \in A \qquad (15)$$

$$t u t^{-1} \to \varphi^{-1}([u]_{\equiv_H}) \text{ for } [u]_{\equiv_H} \in B \qquad (16)$$

Note that $\mathrm{Red}(H, t) = \pi_t(\mathrm{Irr}(\mathsf{ST}))$. Since the word problem for H is decidable and A and B are finite, we can effectively compute from w and w' words z and z' such that $w \to^*_{\mathsf{ST}} z \in \mathrm{Irr}(\mathsf{ST})$ and $w \to^*_{\mathsf{ST}} z' \in \mathrm{Irr}(\mathsf{ST})$. Assume that

$$z = v_0 t^{\beta_1} v_1 \cdots t^{\beta_i} v_i \cdots t^{\beta_p} v_p \text{ and } z' = v'_0 t^{\beta'_1} v'_1 \cdots t^{\beta'_i} v'_i \cdots t^{\beta'_q} v'_q. \quad (17)$$

Note that $\to_{\mathsf{ST}} \subseteq \equiv$. Hence, $w \equiv w'$ iff $z \equiv z'$ iff $\pi_G(\pi_t(z)) = \pi_G(\pi_t(z'))$ iff $\pi_t(z) \approx \pi_t(z')$ iff $\pi_t(z) \sim \pi_t(z')$, where the last equivalence holds due to Lemma 1 and the fact that $\pi_t(z), \pi_t(z') \in \pi_t(\mathrm{Irr}(\mathsf{ST})) =$

$\mathrm{Red}(H,t)$. If $p \neq q$ or ($p = q$ and $\beta_i \neq \beta_i'$ for some $1 \leq i \leq p$) in (17), then $\pi_t(z) \sim \pi_t(z')$ cannot hold and we know that $w \not\equiv w'$. Hence, assume that

$$z = v_0 t^{\beta_1} v_1 \cdots t^{\beta_i} v_i \cdots t^{\beta_p} v_p \text{ and } z' = v_0' t^{\beta_1} v_1' \cdots t^{\beta_i} v_i' \cdots t^{\beta_p} v_p'.$$

We now guess the connecting elements between $\pi_t(z)$ and $\pi_t(z')$ in a diagram of the form (†). More precisely, we nondeterministically guess elements $c_1, \ldots, c_{2n} \in A \cup B$ such that:

- if $\beta_i = 1$ then $c_{2i-1} \in A$ and $c_{2i} = \varphi(c_{2i-1}) \in B$ ($1 \leq i \leq p$) and
- if $\beta_i = -1$ then $c_{2i} \in A$ and $c_{2i-1} = \varphi(c_{2i}) \in B$ ($1 \leq i \leq p$).

Since A and B are finite, this guessing can be done by a nondeterministic Turing machine. Finally, using the decidability of the word problem for H, we check whether $v_i c_{2i+1} \equiv_H c_{2i} v_i'$ for $0 \leq i \leq p$ (here we set $c_0 = c_{2p+1} = 1$). If this is true, then we know that indeed $w \equiv w'$.

3.1 HNN-extensions and morphisms

Suppose we are given a monoid homomorphism: $\pi' : H \to H'$ fulfilling

$$\mathrm{Ker}(\pi') \cap A \times A = \mathrm{Id}_A \quad \text{and} \quad \mathrm{Ker}(\pi') \cap B \times B = \mathrm{Id}_B\,. \tag{18}$$

Thus, A and B are subgroups of H'. Hence, we can define the HNN-extension

$$G' = \langle H', t; t^{-1}at = \varphi(a)(a \in A)\rangle. \tag{19}$$

In the presentation (19) every relation $t^{-1}at = \varphi(a)$ should be written as $t^{-1}\pi'(a)t = \pi'(\varphi(a))$, but (18) allows us to identify every $a \in A$ with its image $\pi'(a) \in \pi'(A)$. The map π' induces a map $\pi'_* : H * \{t, t^{-1}\}^* \to H' * \{t, t^{-1}\}^*$, which is the unique monoid homomorphism such that:

$$\forall h \in H : \pi'_*(h) = \pi'(h), \qquad \pi'_*(t) = t, \qquad \pi'_*(t^{-1}) = t^{-1}$$

The composite morphism $\pi'_* \circ \pi_{G'} : H * \{t, t^{-1}\}^* \to G'$ has a kernel which contains the congruence $\approx$. Hence it induces a map $\widetilde{\pi}' : G \to G'$ such that

$$\pi'_* \circ \pi_{G'} = \pi_G \circ \widetilde{\pi}'.$$

$$\begin{tikzcd}
H * \{t, t^{-1}\}^* \arrow[r, "\pi'_*"] \arrow[d, "\pi_G"] & H' * \{t, t^{-1}\}^* \arrow[d, "\pi_{G'}"] \\
G \arrow[r, "\pi'"'] & G'
\end{tikzcd}$$

Fig. 1. Maps induced by π'

One can check that

$$\forall h \in H : \widetilde{\pi}'(h) = \pi'(h), \quad \widetilde{\pi}'(t) = t, \quad \widetilde{\pi}'(t^{-1}) = t^{-1}.$$

Hence $\widetilde{\pi}'$ is an extension of π', therefore we shall (abusively) also use the symbol π' to denote $\widetilde{\pi}'$. We represent the situation by the commutative diagram in Figure 1.

4 Finite automata for HNN-extensions

Let us fix throughout this section an HNN-extension G of a base monoid H as described by (7) with A and $B = \varphi(A)$ finite. We define below a special kind of finite automata, which will be well-suited for our aim of showing that some decidability and structural properties of rational subsets of H are transfered to G.

4.1 Finite *t*-automata

We define here a kind of finite automata that recognize subsets of $H * \{t, t^{-1}\}^*$ via transitions that either read one of the letters t, t^{-1} or read an element of H, provided it belongs to some definite set $F \in \mathcal{F}$. The typical class $\mathcal{F}$ that we have in mind is the set $\mathrm{Rat}(H)$ of all rational subsets of H, or the set $\mathcal{S}$ of all right-cosets of finitely generated subgroups of H (in case H is a group), or some other families of subsets of H derived in a natural way from the two above families.

Let $\mathcal{F} \subseteq \mathcal{P}(H)$ be a set of subsets of H such that

$$\{\{c\} \mid c \in A \cup B\} \subseteq \mathcal{F}.$$

A *finite t-automaton*, briefly *fta* over $H * \{t, t^{-1}\}^*$ with labelling set $\mathcal{F}$ is a 5-tuple

$$\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle, \tag{20}$$

where

$$\mathcal{L} \subseteq \mathcal{F} \cup \{\{t\}, \{t^{-1}\}\}$$

is finite, Q is a finite set of states, $\mathsf{I} \subseteq \mathsf{Q}$ is the set of initial states, $\mathsf{T} \subseteq \mathsf{Q}$ is the set of terminal states, and δ (the set of transitions) is a subset of $\mathsf{Q} \times \mathcal{L} \times \mathsf{Q}$ such that

$$\forall q \in \mathsf{Q}\ \exists L \subseteq \mathcal{L} : 1 \in L\ \wedge\ (q, L, q) \in \delta. \tag{21}$$

Such an automaton induces a representation map

$$\mu_{\mathcal{A}} : H * \{t, t^{-1}\}^* \to \mathsf{BR}(\mathsf{Q})$$

defined as follows: First, define $\mu_{\mathcal{A},0} : H \cup \{t, t^{-1}\} \to \mathsf{BR}(\mathsf{Q})$ as follows, where $h \in H \cup \{t, t^{-1}\}$:

$$\mu_{\mathcal{A},0}(h) = \{(q, r) \in Q \times Q \mid \exists L \in \mathcal{L} : (q, L, r) \in \delta\ \wedge\ h \in L\} \tag{22}$$

(note that $\mathrm{Id}_{\mathsf{Q}} \subseteq \mu_{\mathcal{A},0}(1)$ due to (21)). Now, let $s \in H * \{t, t^{-1}\}^*$ be a sequence of the form (14). Then

$$\begin{multline}\mu_{\mathcal{A}}(s) = \mu_{\mathcal{A},0}(h_0) \circ \mu_{\mathcal{A},0}(t^{\alpha_1}) \circ \mu_{\mathcal{A},0}(h_1) \cdots \\ \mu_{\mathcal{A},0}(t^{\alpha_i}) \circ \mu_{\mathcal{A},0}(h_i) \cdots \mu_{\mathcal{A},0}(t^{\alpha_n}) \circ \mu_{\mathcal{A},0}(h_n). \tag{23}\end{multline}$$

Notice that the second property stated in (6) need not be fulfilled by this kind of automata in case $m, m' \in H$. The subset recognized by $\mathcal{A}$ is

$$\mathrm{L}(\mathcal{A}) = \{s \in H * \{t, t^{-1}\}^* \mid \mu_{\mathcal{A}}(s) \cap \mathsf{I} \times \mathsf{T} \neq \emptyset\}.$$

We also use the notation $q \xrightarrow{u}_{\mathcal{A}} r$ to mean that $(q, r) \in \mu_{\mathcal{A}}(u)$. Let us introduce a finite set of 6 types, which will induce a partition of the states of an fta:

$$\mathcal{T}_6 = \{(A, T), (B, T), (1, H), (1, 1), (A, H), (B, H)\} \tag{24}$$

The first component of the types will serve below for defining a normal form for automata; the second component suggests the possible labels going out of a state of that type. We define a graph

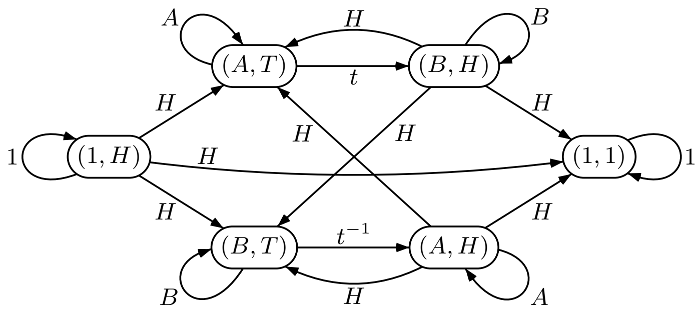


Fig. 2. The graph $\mathcal{G}_6 = (\mathcal{T}_6, \mathcal{E}_6)$

$\mathcal{G}_6 = (\mathcal{T}_6, \mathcal{E}_6)$ by

$$\begin{aligned}\mathcal{E}_6 =& \{((C,D),C,(C,D)) \mid (C,D) \in \mathcal{T}_6\}\\ &\cup \{((A,T),t,(B,H)),((B,T),t^{-1},(A,H))\}\\ &\cup \{((C,H),H,(C,T)),((1,H),H,(C,T)) \mid C \in \{A,B\}\}\\ &\cup \{((C,H),H,(1,1)) \mid C \in \{1,A,B\}\}\\ &\cup \{((B,H),H,(A,T)),((A,H),H,(B,T))\}.\end{aligned}$$

The graph $\mathcal{G}_6$ is represented in Figure 2. We sometimes use also the graph $\mathcal{R}_6 = (\mathcal{T}_6, \mathcal{E}'_6)$ where

$$\begin{aligned}\mathcal{E}'_6 =& \mathcal{E}_6 \setminus \{((A,H),H,(A,T)),((B,H),H,(B,T))\} \cup\\ &\{((A,H),H \setminus A,(A,T)),((B,H),H \setminus B,(B,T))\};\end{aligned}$$

it is shown in Figure 3. One can check that the graph $\mathcal{G}_6$ (resp. $\mathcal{R}_6$) endowed with the set of initial states $\mathsf{I}_6 = \{(1,H)\}$ and the set of final states $\mathsf{T}_6 = \{(1,1)\}$ is an fta recognizing $H * \{t,t^{-1}\}^*$ (resp. $\mathrm{Red}(H,t)$).[1]

Let us define from the finite edge set $\mathcal{E}_6$ the infinite set

$$\widehat{\mathcal{E}_6} = \{(p,\ell,q) \mid \exists (p,L,q) \in \mathcal{E}_6 : \ell \in L\}.$$

and similarly from $\mathcal{E}_6$ a corresponding set $\widehat{\mathcal{E}'}_6$. The reader can easily check the following:

[1] Formally, the label t^α ($\alpha \in \{1,-1\}$) of $\mathcal{G}_6$ (resp. $\mathcal{R}_6$) has to be replaced by a transition labelled with the singleton set $\{t^\alpha\}$ in order to make $\mathcal{G}_6$ (resp. $\mathcal{R}_6$) an fta. In the rest of this paper, we will repeatedly write t^α, when we formally have to write $\{t^\alpha\}$.

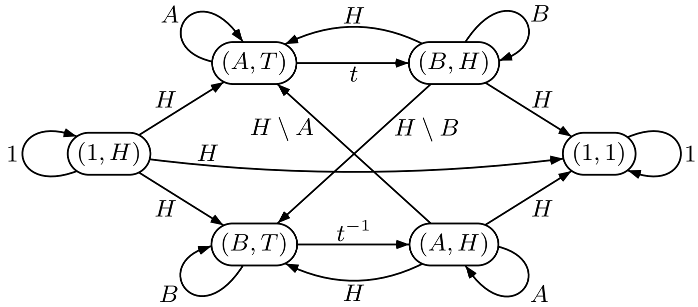


Fig. 3. The graph $\mathcal{R}_6 = (\mathcal{T}_6, \mathcal{E}'_6)$

Lemma 2. *For every sequence $s \in H * \{t, t^{-1}\}^*$ of the form (14) there is exactly one sequence $\Theta_0, \Theta_1, \ldots, \Theta_{2n+1} \in \mathcal{T}_6$ of types such that*

1. $\Theta_0 = (1, H)$*,* $\Theta_{2n+1} = (1, 1)$*,*
2. $(\Theta_{2i}, h_i, \Theta_{2i+1}) \in \widehat{\mathcal{E}}_6$ *for all* $0 \le i \le n$*, and*
3. $(\Theta_{2i-1}, t_i^{\alpha_i}, \Theta_{2i}) \in \widehat{\mathcal{E}}_6$ *for all* $1 \le i \le n$*.*

Moreover, we must have $\Theta_i \neq \Theta_{i+1}$ for all $0 \le i \le 2n$.

Lemma 2 implies that that every sequence s can be accepted in a unique way by the fta $\mathcal{G}_6$. The same statement also holds for $\mathcal{R}_6$ if we restrict to reduced sequences.

Definition 3. *A* partitionned fta *with labelling set $\mathcal{F}$ is a 6-tuple*

$$\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle, \tag{25}$$

where $\mathcal{L}$, Q, δ, I, and T are as in (20), and $\tau : Q \to \mathcal{T}_6$ assigns a type to every state, and the transitions of δ and the sets I and T respect the types in the following sense:

$$\forall (q, L, r) \in \delta : \{\tau(q)\} \times L \times \{\tau(r)\} \subseteq \widehat{\mathcal{E}}_6 \tag{26}$$
$$\tau(\mathsf{I}) = \{(1, H)\} \;\wedge\; \tau(\mathsf{T}) = \{(1, 1)\}. \tag{27}$$

Recall the equivalences $\sim$ and $\approx$ over $H * \{t, t^{-1}\}^*$ defined in Section 3.

Definition 4. *An fta $\mathcal{A}$ is said to be $\approx$-compatible if and only if*

$$[\mathrm{L}(\mathcal{A})]_{\approx} = [\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(H, t)]_{\approx}. \tag{28}$$

It is said to be $\sim$-saturated *if and only if*

$$\forall s, s' \in H * \{t, t^{-1}\}^* : s \sim s' \Rightarrow \mu_{\mathcal{A}}(s) = \mu_{\mathcal{A}}(s'). \tag{29}$$

Remark 1. The following observations are easy to verify:

- Property (28) is equivalent to $\mathrm{L}(\mathcal{A}) \subseteq [\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(H, t)]_{\approx}$, i.e., for every sequence $s \in \mathrm{L}(\mathcal{A})$ there must exist a reduced sequence $s' \in \mathrm{L}(\mathcal{A})$ such that $s \approx s'$.
- If $\mathrm{L}(\mathcal{A}) \subseteq \mathrm{Red}(H, t)$, then $\mathcal{A}$ is $\approx$-compatible.
- $\mathcal{A}$ is $\sim$-saturated if and only if for all states $q_1, q_2, q_3, q_4 \in \mathsf{Q}$, all $h_1, h_2 \in H$, and all $a \in A$, $b \in A$ such that

$$q_1 \xrightarrow{h_1}_{\mathcal{A}} q_2 \xrightarrow{t}_{\mathcal{A}} q_3 \xrightarrow{h_2}_{\mathcal{A}} q_4$$
$$(\text{resp. } q_1 \xrightarrow{h_1}_{\mathcal{A}} q_2 \xrightarrow{t^{-1}}_{\mathcal{A}} q_3 \xrightarrow{h_2}_{\mathcal{A}} q_4)$$

  there exist $q_2', q_3' \in \mathsf{Q}$ with

$$q_1 \xrightarrow{h_1 a^{-1}}_{\mathcal{A}} q_2' \xrightarrow{t}_{\mathcal{A}} q_3' \xrightarrow{\varphi(a) h_2}_{\mathcal{A}} q_4$$
$$(\text{resp. } q_1 \xrightarrow{h_1 b^{-1}}_{\mathcal{A}} q_2' \xrightarrow{t^{-1}}_{\mathcal{A}} q_3' \xrightarrow{\varphi^{-1}(b) h_2}_{\mathcal{A}} q_4).$$

- Notice that $\approx$-compatibility depends on $\mathrm{L}(\mathcal{A})$ only, while $\sim$-saturation does really depend on the automaton $\mathcal{A}$ itself.
- If $\mathcal{A}$ is $\sim$-saturated, then the set $\mathrm{L}(\mathcal{A})$ is $\sim$-saturated, i.e., if $s \sim s'$, then $s \in \mathrm{L}(\mathcal{A})$ if and only if $s' \in \mathrm{L}(\mathcal{A})$.
- The partitioned fta $\mathcal{G}_6$ and $\mathcal{R}_6$ are both $\approx$-compatible and $\sim$-saturated. For $\sim$-saturation, the A- and B-labelled edges in Figure 2 and 3 are crucial.

Definition 5. *The partitionned fta $\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$ is said to be* unitary *if and only if for every type $\Theta \in \mathcal{T}_6$ we have*

$$\mu_{\mathcal{A},0}(1) \cap (\tau^{-1}(\Theta) \times \tau^{-1}(\Theta)) = \mathrm{Id}_{\mathsf{Q}} \cap (\tau^{-1}(\Theta) \times \tau^{-1}(\Theta)).$$

This last technical condition turns out to be useful in [LS06a]. It means that no edge between two different states of the same type is labelled with a set containing 1.

Definition 6. *The partitionned fta* $\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$ *is said to be* deterministic *if and only if*

$$Card(\mathsf{I}) = 1 \tag{30}$$

(i.e. it has exactly one initial state) and for all $q, r, r' \in \mathsf{Q}$ *and all* $L, L' \in \mathcal{L}$*,*

$$((q, L, r), (q, L', r') \in \delta \wedge L \cap L' \neq \emptyset \wedge \tau(r) = \tau(r')) \Rightarrow \\ L = L' \ \wedge \ r = r'. \tag{31}$$

Clearly, $\mathcal{G}_6$ and $\mathcal{R}_6$ are both deterministic, since these fta both have only one initial state and for every type there exists only one state of that type.

Finally, we define the notion of a complete partitionned fta. The intuitive idea is that of a local condition over the labels of transitions, which ensures that for every reduced sequence $s \in \mathrm{Red}(H, t)$ and every state $q \in \tau^{-1}(1, H)$ we have $q \xrightarrow{s}_{\mathcal{A}} r$ for some state $r \in \tau^{-1}(1, 1)$. For every $q \in \mathsf{Q}$ and every $\Theta \in \mathcal{T}_6$ we thus introduce the set

$$\mathrm{L}^1(q, \Theta) = \bigcup \{L \mid \exists r \in \tau^{-1}(\Theta) : (q, L, r) \in \delta\}. \tag{32}$$

Definition 7. *The partitionned fta* $\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$ *is said to be* complete *if and only if*

$$\forall (\Theta, C, \Theta') \in \mathcal{E}_6' \forall q \in \tau^{-1}(\Theta) : C \subseteq \mathrm{L}^1(q, \Theta').$$

Note that $\mathcal{G}_6$ and $\mathcal{R}_6$ are both complete.

4.2 Operations on fta

For a partitionned, deterministic, and complete fta $\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$ we define the *complement fta*

$$\overline{\mathcal{A}} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \tau_{\mathcal{A}}^{-1}(1, 1) \setminus \mathsf{T} \rangle.$$

It is again a partitionned, deterministic, and complete fta.

Lemma 3. *Let $\mathcal{A}$ be a partitionned, deterministic, and complete fta such that* $\mathrm{L}(\mathcal{A}) \subseteq \mathrm{Red}(H,t)$*. Then* $\mathrm{L}(\overline{\mathcal{A}}) = \mathrm{Red}(H,t) \setminus \mathrm{L}(\mathcal{A})$*. Moreover, if $\mathcal{A}$ is $\sim$-saturated, then $\overline{\mathcal{A}}$ is again $\sim$-saturated.*

Proof. The first statement holds, since for a partitionned, deterministic, and complete fta $\mathcal{A}$, there exists for every reduced sequence $s \in \mathrm{Red}(H,t)$ exactly one state $q \in \tau_{\mathcal{A}}^{-1}(1,1)$ such that $(q_0, q) \in \mu_{\mathcal{A}}(s)$, where q_0 is the unique initial state of $\mathcal{A}$. For the second statement just note that $\mu_{\overline{\mathcal{A}}} = \mu_{\mathcal{A}}$. $\square$

Given two fta

$$\mathcal{A}_i = \langle \mathcal{L}_i, \mathsf{Q}_i, \delta_i, \mathsf{I}_i, \mathsf{T}_i \rangle$$

($i \in \{1,2\}$), we define the *product fta* $\mathcal{A}_1 \times \mathcal{A}_2 = \langle \mathcal{L}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle$, where:

$$\begin{aligned}
\mathcal{L} &= \{L_1 \cap L_2 \mid L_1 \in \mathcal{L}_1, L_2 \in \mathcal{L}_2\} \setminus \{\emptyset\} \\
\mathsf{Q} &= \mathsf{Q}_1 \times \mathsf{Q}_2, \\
\mathsf{I} &= \mathsf{I}_1 \times \mathsf{I}_2 \\
\mathsf{T} &= \mathsf{T}_1 \times \mathsf{T}_2 \\
\delta &= \{((p_1,p_2), L_1 \cap L_2, (q_1,q_2)) \mid (p_1, L_1, q_1) \in \delta_1, (p_2, L_2, q_2) \in \delta_2\}
\end{aligned}$$

For two finite *partitionned* fta $\mathcal{A}_i = \langle \mathcal{L}_i, \mathsf{Q}_i, \tau_i, \delta_i, \mathsf{I}_i, \mathsf{T}_i \rangle$ the product automaton is the partitionned fta $\mathcal{A}_1 \times \mathcal{A}_2 = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$, where $\mathcal{L}, \mathsf{I}$, and T are defined as above, but

$$\begin{aligned}
\mathsf{Q} = &\bigcup_{\theta \in \mathcal{T}_6} \tau_1^{-1}(\Theta) \times \tau_2^{-1}(\Theta), \\
\delta = &\{((p_1,p_2), L_1 \cap L_2, (q_1,q_2)) \mid (p_1, L_1, q_1) \in \delta_1, (p_2, L_2, q_2) \in \delta_2\} \cap \\
&\mathsf{Q} \times \mathcal{L} \times \mathsf{Q},
\end{aligned}$$

and $\tau(p_1,p_2) = \tau_1(p_1) = \tau_2(p_2)$ for $(p_1,p_2) \in \mathsf{Q}$.

Lemma 4. *For two fta (resp. partitionned fta) $\mathcal{A}_1$ and $\mathcal{A}_2$ we have* $\mathrm{L}(\mathcal{A}_1 \times \mathcal{A}_2) = \mathrm{L}(\mathcal{A}_1) \cap \mathrm{L}(\mathcal{A}_2)$.

Proof. For ordinary (non-partitionned) fta the statement of the lemma is obvious. For partitionned fta one has to use Lemma 2. $\square$

Lemma 4 implies:

Lemma 5. *Let $\mathcal{A}$ be some fta with labelling set $\mathcal{F}$. The fta $\mathcal{A} \times \mathcal{G}_6$ endowed with the map $\tau : (p, \Theta) \mapsto \Theta$ is a partitionned fta with labelling set $\mathcal{F}$, which recognizes the same language as $\mathcal{A}$.*

Lemma 6. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be fta (resp. partitionned fta). Then we have:*

(a) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are both $\sim$-saturated, then $\mathcal{A}_1 \times \mathcal{A}_2$ is $\sim$-saturated, too.
(b) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are both $\approx$-compatible and $\sim$-saturated, then $\mathcal{A}_1 \times \mathcal{A}_2$ is $\approx$-compatible and $\sim$-saturated, too.

Proof. For statement (a) and non-partitionned fta assume that in the product automatom $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ we have

$$(q_1, p_1) \xrightarrow{h_1}_{\mathcal{A}} (q_2, p_2) \xrightarrow{t}_{\mathcal{A}} (q_3, p_3) \xrightarrow{h_2}_{\mathcal{A}} (q_4, p_4).$$

Hence,

$$\begin{aligned} &q_1 \xrightarrow{h_1}_{\mathcal{A}_1} q_2 \xrightarrow{t}_{\mathcal{A}_1} q_3 \xrightarrow{h_2}_{\mathcal{A}_1} q_4 \text{ and} \\ &p_1 \xrightarrow{h_1}_{\mathcal{A}_2} p_2 \xrightarrow{t}_{\mathcal{A}_2} p_3 \xrightarrow{h_2}_{\mathcal{A}_2} p_4. \end{aligned}$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are both $\sim$-saturated, we have, for every $a \in A$,

$$\begin{aligned} &q_1 \xrightarrow{h_1 a^{-1}}_{\mathcal{A}_1} q_2' \xrightarrow{t}_{\mathcal{A}_1} q_3' \xrightarrow{\varphi(a) h_2}_{\mathcal{A}_1} q_4 \text{ and} \\ &p_1 \xrightarrow{h_1 a^{-1}}_{\mathcal{A}_2} p_2' \xrightarrow{t}_{\mathcal{A}_2} p_3' \xrightarrow{\varphi(a) h_2}_{\mathcal{A}_2} p_4 \end{aligned}$$

for states q_2' and q_3' (resp. p_2' and p_3') of $\mathcal{A}_1$ (resp. $\mathcal{A}_2$). Hence,

$$(q_1, p_1) \xrightarrow{h_1 a^{-1}}_{\mathcal{A}} (q_2', p_2') \xrightarrow{t}_{\mathcal{A}} (q_3', p_3') \xrightarrow{\varphi(a) h_2}_{\mathcal{A}} (q_4, p_4).$$

Analogous arguments for t^{-1} show that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is indeed $\sim$-saturated. For partitionned fta we can reason similarly, we just have to note that the states q_2, q_2', p_2, p_2' (resp. q_3, q_3', p_3, p_3') have to be of type (A, T) (resp. (B, H)). This, ensures that (q_2', p_2') and (q_3', p_3') are states of the partitionned fta $\mathcal{A}$.

For (b) it suffices by Lemma 4 and statement (a) to prove that $\mathrm{L}(\mathcal{A}_1) \cap \mathrm{L}(\mathcal{A}_2) \subseteq [\mathrm{L}(\mathcal{A}_1) \cap \mathrm{L}(\mathcal{A}_2) \cap \mathrm{Red}(H, t)]_{\approx}$. So, assume that

$$s \in \mathrm{L}(\mathcal{A}_1) \cap \mathrm{L}(\mathcal{A}_2).$$

Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are both $\approx$-compatible, we have

$$s \in [\mathrm{L}(\mathcal{A}_1) \cap \mathrm{Red}(H,t)]_\approx \cap [\mathrm{L}(\mathcal{A}_2) \cap \mathrm{Red}(H,t)]_\approx.$$

Thus, there exist sequences s_1 and s_2 such that

$$s \approx s_1 \in \mathrm{L}(\mathcal{A}_1) \cap \mathrm{Red}(H,t) \text{ and } s \approx s_2 \in \mathrm{L}(\mathcal{A}_2) \cap \mathrm{Red}(H,t).$$

Hence, $s_1 \approx s_2$ and Lemma 1 implies $s_1 \sim s_2 \in \mathrm{L}(\mathcal{A}_2)$. Since $\mathcal{A}_2$ is $\sim$-saturated, we have $s_1 \in \mathrm{L}(\mathcal{A}_2)$. Thus,

$$s \approx s_1 \in \mathrm{L}(\mathcal{A}_1) \cap \mathrm{L}(\mathcal{A}_2) \cap \mathrm{Red}(H,t),$$

i.e., $s \in [\mathrm{L}(\mathcal{A}_1) \cap \mathrm{L}(\mathcal{A}_2) \cap \mathrm{Red}(H,t)]_\approx$. □

The next lemma is straightforward to prove.

Lemma 7. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be partitionned fta. Then we have:*

(a) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are both deterministic, then $\mathcal{A}_1 \times \mathcal{A}_2$ is deterministic, too.
(b) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are both complete, then $\mathcal{A}_1 \times \mathcal{A}_2$ is complete, too.

4.3 Automata and morphisms

Let us consider another HNN-extension G' defined by an homomorphism $\pi' : H \to H'$ satisfying (18) as described in Section 3.1 by (19). Given an fta $\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle$, we define

$$\pi'_*(\mathcal{A}) = \langle \mathcal{L}', \mathsf{Q}, \delta', \mathsf{I}, \mathsf{T} \rangle \tag{33}$$

with $\mathcal{L}' = \{\pi'(L) \mid L \in \mathcal{L}\}$ and

$$\delta' = \{(q, \pi'(L), r) \in \mathsf{Q} \times \mathcal{L}' \times \mathsf{Q} \mid (q, L, r) \in \delta\}$$

We denote the congruence on $H' * \{t, t^{-1}\}$, which is defined in the same way as $\sim$ on $H * \{t, t^{-1}\}$, with $\sim'$.

Lemma 8. *Let $\mathcal{A}$ be some fta. Then we have:*

(a) $\mathrm{L}(\pi'_*(\mathcal{A})) = \pi'_*(\mathrm{L}(\mathcal{A}))$
(b) If $\mathcal{A}$ is $\sim$-saturated, then $\pi'_*(\mathcal{A})$ is $\sim'$-saturated.

Proof. Statement (a) is obvious. For statement (b) assume that $\mathcal{A}$ is $\sim$-saturated and that

$$q_1 \xrightarrow{h_1}_{\pi'_*(\mathcal{A})} q_2 \xrightarrow{t}_{\pi'_*(\mathcal{A})} q_3 \xrightarrow{h_2}_{\pi'_*(\mathcal{A})} q_4.$$

Hence, there exist $k_1, k_2 \in H$ with $h_1 = \pi'(k_1)$, $h_2 = \pi'(k_2)$, and

$$q_1 \xrightarrow{k_1}_{\mathcal{A}} q_2 \xrightarrow{t}_{\mathcal{A}} q_3 \xrightarrow{k_2}_{\mathcal{A}} q_4.$$

Since $\mathcal{A}$ is $\sim$-saturated, there are states $p_2, p_3 \in \mathsf{Q}$ such that

$$q_1 \xrightarrow{k_1 a^{-1}}_{\mathcal{A}} p_2 \xrightarrow{t}_{\mathcal{A}} p_3 \xrightarrow{\varphi(a) k_2}_{\mathcal{A}} q_4$$

and hence

$$q_1 \xrightarrow{\pi'(k_1 a^{-1})}_{\pi'_*(\mathcal{A})} p_2 \xrightarrow{t}_{\pi'_*(\mathcal{A})} p_3 \xrightarrow{\pi'(\varphi(a) k_2)}_{\pi'_*(\mathcal{A})} q_4.$$

Since $\pi'(c) = c$ for all $c \in A \cup B$ we obtain

$$q_1 \xrightarrow{h_1 a^{-1}}_{\pi'_*(\mathcal{A})} p_2 \xrightarrow{t}_{\pi'_*(\mathcal{A})} p_3 \xrightarrow{\varphi(a) h_2}_{\pi'_*(\mathcal{A})} q_4.$$

An analogous argument for t^{-1} shows that $\pi'_*(\mathcal{A})$ is $\sim'$-saturated. $\square$

4.4 Normalization of automata

The following proposition is crucial for our further considerations.

Proposition 1. *The following holds:*

(1) From a given partitionned and $\approx$-compatible fta $\mathcal{A}$ with labelling set $\mathcal{F} \subseteq \mathcal{P}(H)$ one can effectively construct a partitionned, $\approx$-compatible, $\sim$-saturated, and unitary fta $\mathcal{B}$ with labelling set $\{cFc' \mid F \in \mathcal{F}, c, c' \in A \cup B\}$ such that

$$[\mathrm{L}(\mathcal{A})]_\approx \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t).$$

(2) From a given partitionned, $\approx$-compatible, and $\sim$-saturated fta $\mathcal{B}$ with labelling set $\mathcal{F} \subseteq \mathcal{P}(H)$ one can effectively construct a partitionned, $\sim$-saturated, deterministic, and complete fta $\mathcal{C}$ with labelling set $\mathrm{Bool}(\mathcal{F})$ such that

$$\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{C}).$$

Proof. Let $\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$ be some partitionned and $\approx$-compatible fta with labeling set $\mathcal{F} \subseteq \mathcal{P}(H)$. Thus, we have

$$[\mathrm{L}(\mathcal{A})]_{\approx} = [\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(H, t)]_{\approx}. \tag{34}$$

If necessary, we can replace the set of transitions δ by the set

$$\delta \cap \{(p, L, q) \in \mathsf{Q} \times \mathcal{L} \times \mathsf{Q} \mid \tau(p) \neq \tau(q)\} \cup \{(q, 1, q) \mid q \in \mathsf{Q}\}$$

and preserve the recognized language. The second term of the above union ensures that property (21) still holds. Such a modification might modify the mappings $\mu_{\mathcal{A},0}$ and $\mu_{\mathcal{A}}$, but for every sequence s it does not modify the relation $\mu_{\mathcal{A}}(s) \cap \tau^{-1}((1, H)) \times \tau^{-1}((1, 1))$, i.e., after this modification, $\mathcal{A}$ is still a partitionned and $\approx$-compatible fta. Hence, w.l.o.g. we assume that

$$\forall (p, L, q) \in \delta : \tau(p) \neq \tau(q) \vee p = q. \tag{35}$$

Note that property (35) implies that $\mathcal{A}$ is unitary.

Proof of (1). We define the fta $\mathcal{B} = \langle \mathcal{L}_{\mathcal{B}}, \mathsf{Q}_{\mathcal{B}}, \tau_{\mathcal{B}}, \delta_{\mathcal{B}}, \mathsf{I}, \mathsf{T} \rangle$ in the following way: Let us denote for every $q \in \mathsf{Q}$ by $\gamma(q)$ the subgroup

$$\gamma(q) = \pi_1(\tau(q)),$$

where π_1 denotes the projection onto the first component of a type. Note that $\gamma(q)$ is one of the three subgroups $1, A, B$. Moreover, since $\mathcal{A}$ is partitionned, if $(q, t, r) \in \delta$ (resp. $(q, t^{-1}, r) \in \delta$) then $\gamma(q) = A$ (resp. $\gamma(r) = B$) and $\gamma(q) = B$ (resp. $\gamma(r) = A$). We set

$$\mathsf{Q}_{\mathcal{B}} = \{(q, g) \in \mathsf{Q} \times (A \cup B) \mid g \in \gamma(q)\}.$$

We shall identify $(q, 1)$ with q and denote (q, g) by qg in the sequel. Note that the set of initial (resp. terminal) states of $\mathcal{B}$ is the same as for $\mathcal{A}$. The typing map $\tau_{\mathcal{B}}$ is inherited from $\mathcal{A}$:

$$\tau_{\mathcal{B}}(qg) = \tau(q)$$

The set $\delta_{\mathcal{B}}$ consists of all the following transitions, where $q, r \in \mathsf{Q}$:

$$(qg, g', q(gg')) \text{ for } g, g' \in \gamma(q) \tag{36}$$

$$(qg, g^{-1}Lg', rg') \text{ for } g \in \gamma(q), g' \in \gamma(r), L \subseteq H, (q, L, r) \in \delta \tag{37}$$

$$(qa, t, r\varphi(a)) \text{ for } a \in A, (q, t, r) \in \delta \tag{38}$$

$$(qb, t^{-1}, r\varphi^{-1}(b)) \text{ for } b \in B, (q, t^{-1}, r) \in \delta \tag{39}$$

$$
\begin{tikzcd}
q_1 \arrow[r, "h"] \arrow[d, "g"'] & r_1 \arrow[d, "g'"] \\
q_1 g \arrow[r, "g^{-1}hg'"'] & r_1 g'
\end{tikzcd}
\qquad
\begin{tikzcd}
q \arrow[r, "t"] \arrow[d, "a"'] & r \arrow[d, "\varphi(a)"] \\
qa \arrow[r, "t"'] & r\varphi(a)
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{q} \arrow[r, "t^{-1}"] \arrow[d, "b"'] & \overline{r} \arrow[d, "\varphi^{-1}(b)"] \\
\overline{q}b \arrow[r, "t^{-1}"'] & \overline{r}\varphi^{-1}(b)
\end{tikzcd}
$$

Fig. 4. Some subgraphs of $\mathcal{B}$

For all $(q_1, L, r_1), (q, t, r), (\overline{q}, t^{-1}, \overline{r}) \in \delta$, $g \in \gamma(q_1)$, $g' \in \gamma(r_1)$, $h \in L$, $a \in A$, and $b \in B$, the automaton $\mathcal{B}$ contains the three basic subgraphs exhibited in Figure 4. We cut the proof that $\mathcal{B}$ has the required properties into five claims.

Claim 1. $[\mathrm{L}(\mathcal{A})]_{\approx} \cap \mathrm{Red}(H, t) = [\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(H, t)]_{\sim}$

By intersecting both sides of (34) with $\mathrm{Red}(H, t)$, we obtain

$$[\mathrm{L}(\mathcal{A})]_{\approx} \cap \mathrm{Red}(H, t) = [\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(H, t)]_{\approx} \cap \mathrm{Red}(H, t).$$

But by Lemma 1, the righthandside above equals $[\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(H, t)]_{\sim}$, which proves Claim 1.

Claim 2. $\mathrm{L}(\mathcal{A}) \subseteq \mathrm{L}(\mathcal{B})$

Let us notice that every $(q, L, r) \in \delta$ has a copy in $\delta_{\mathcal{B}}$ under the form of $(q1, L, r1)$, since whatever $\gamma(q)$ and $\gamma(r)$ are, $1 \in \gamma(q) \cap \gamma(r)$. Thus, up to the identification of every state $q \in \mathsf{Q}$ with the state $q1 \in \mathsf{Q}_{\mathcal{B}}$, $\delta \subseteq \delta_{\mathcal{B}}$, which proves Claim 2.

Claim 3. The fta $\mathcal{B}$ is $\sim$-saturated and hence, $\mathrm{L}(\mathcal{B}) = [\mathrm{L}(\mathcal{B})]_{\sim}$.

Let us assume that $q_0, r_0 \in \mathsf{Q}_{\mathcal{B}}, h, h' \in H$, and

$$q_0 \xrightarrow{hth'}_{\mathcal{B}} r_0.$$

Such a computation can be decomposed as

$$q_0 \xrightarrow{h}_{\mathcal{B}} qa_1 \xrightarrow{t}_{\mathcal{B}} r\varphi(a_1) \xrightarrow{h'}_{\mathcal{B}} r_0,$$

where $q, r \in \mathsf{Q}, a_1 \in A$, and $(q, t, r) \in \delta$. Note that we must have $\gamma(q) = A$ since a t-transition is enabled in qa_1. The transition $q_0 \xrightarrow{h}_{\mathcal{B}}$

qa_1 is either of type (36) or (37). In both cases, we can derive for every $a \in A$ the transition

$$q_0 \xrightarrow{ha^{-1}}_{\mathcal{B}} q(a_1 a^{-1}). \tag{40}$$

Analogously, we get

$$r\varphi(a_1 a^{-1}) \xrightarrow{\varphi(a)h'}_{\mathcal{B}} r_0.$$

Finally, $qa_1 \xrightarrow{t}_{\mathcal{B}} r\varphi(a_1)$ and (38) implies

$$q(a_1 a^{-1}) \xrightarrow{t}_{\mathcal{B}} r\varphi(a_1 a^{-1}).$$

Altogether, it follows

$$q_0 \xrightarrow{ha^{-1}t\varphi(a)h'}_{\mathcal{B}} r_0. \tag{41}$$

Similarly, if $q_0 \xrightarrow{ht^{-1}h'}_{\mathcal{B}} r_0$ and $b \in B$, then

$$q_0 \xrightarrow{hb^{-1}t^{-1}\varphi^{-1}(b)h'}_{\mathcal{B}} r_0.$$

This proves Claim 3.

Claim 4. $\mathrm{L}(\mathcal{B}) \subseteq [\mathrm{L}(\mathcal{A})]_\sim$

Let $s \in H * \{t, t^{-1}\}^*, q, r \in \mathsf{Q}, g \in \gamma(q)$, and $h \in \gamma(r)$ such that

$$qg \xrightarrow{s}_{\mathcal{B}} rh. \tag{42}$$

By gluing squares from Figure 4 along their vertical edges, one can prove by induction on the integer n of the decomposition (14) that, under the hypothesis (42), there exists $s' \sim gsh^{-1}$ such that

$$q \xrightarrow{s'}_{\mathcal{A}} r. \tag{43}$$

If $q \in \mathsf{I}$ and $r \in \mathsf{T}$, then the type-conditions (27) imply $\gamma(q) = \gamma(r) = 1$, i.e., $g = h = 1$ and $s \in \mathrm{L}(\mathcal{B})$. Then (43) says that

$$\exists s' : s' \sim s \ \wedge \ s' \in \mathrm{L}(\mathcal{A}),$$

which proves Claim 4.

Claim 5. $\mathrm{L}(\mathcal{B}) \subseteq [\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)]_{\approx}$

Claim 5 can be deduced as follows:

$$\begin{aligned}
\mathrm{L}(\mathcal{B}) &\subseteq [\mathrm{L}(\mathcal{A})]_{\sim} && \text{(by Claim 4)}\\
&\subseteq [\mathrm{L}(\mathcal{A})]_{\approx} && \text{(since } \sim \subseteq \approx\text{)}\\
&= [\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(H,t)]_{\approx} && \text{(since } \mathcal{A} \text{ is } \approx\text{-compatible)}\\
&\subseteq [\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)]_{\approx} && \text{(by Claim 2)}
\end{aligned}$$

Let us now use Claim 1–5 in order to show point (1) of Proposition 1: First, we have

$$\begin{aligned}
[\mathrm{L}(\mathcal{A})]_{\approx} \cap \mathrm{Red}(H,t) &= [\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(H,t)]_{\sim} && \text{(by Claim 1)}\\
&\subseteq [\mathrm{L}(\mathcal{A})]_{\sim} &&\\
&\subseteq [\mathrm{L}(\mathcal{B})]_{\sim} && \text{(by Claim 2)}\\
&= \mathrm{L}(\mathcal{B}) && \text{(by Claim 3),}
\end{aligned}$$

and thus

$$[\mathrm{L}(\mathcal{A})]_{\approx} \cap \mathrm{Red}(H,t) \subseteq \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t).$$

On the other hand, we have

$$\begin{aligned}
\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t) &\subseteq [\mathrm{L}(\mathcal{A})]_{\sim} \cap \mathrm{Red}(H,t) && \text{(by Claim 4)}\\
&\subseteq [\mathrm{L}(\mathcal{A})]_{\approx} \cap \mathrm{Red}(H,t) && \text{(since } \sim \subseteq \approx\text{).}
\end{aligned}$$

Hence, we obtain

$$[\mathrm{L}(\mathcal{A})]_{\approx} \cap \mathrm{Red}(H,t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t),$$

as was required. Moreover:

- $\mathcal{B}$ is partitionned, since $\mathcal{A}$ is partitionned.
- By Claim 5, $\mathcal{B}$ is $\approx$-compatible.
- By Claim 3, $\mathcal{B}$ is $\sim$-saturated.
- $\mathcal{B}$ is unitary: If $g' = 1$ in (36), then we have $qg = qgg'$. If $1 \in g^{-1}Lg'$ and $(q, L, r) \in \delta$, then property (35) guaranties that $\tau(q) \neq \tau(r)$ or $q = r$. Hence, no $\mathcal{B}$-transition of type (36) or (37) has the form (p, L, q) with $\tau(p) = \tau(q)$, $1 \in L$, and $p \neq q$.

Point (1) of Proposition 1 is thus established.

Proof of (2). Let $\mathcal{B} = \langle \mathcal{L}_\mathcal{B}, \mathsf{Q}_\mathcal{B}, \tau_\mathcal{B}, \delta_\mathcal{B}, \mathsf{I}_\mathcal{B}, \mathsf{T}_\mathcal{B} \rangle$ be a partitionned, $\approx$-compatible, and $\sim$-*saturated* fta. We first define an fta

$$\mathcal{D} = \langle \mathcal{L}_\mathcal{D}, \mathsf{Q}_\mathcal{D}, \tau_\mathcal{D}, \delta_\mathcal{D}, \mathsf{I}_\mathcal{D}, \mathsf{T}_\mathcal{D} \rangle$$

in the following way: Let

$$\mathcal{L}_\mathcal{D} = \{\{t\}, \{t^{-1}\}\} \cup \Big\{ \bigcap_{i \in J} L_i \cap \bigcap_{i \in I \setminus J} H \setminus L_i \mid J \subseteq I \Big\} \setminus \{\emptyset\},$$

where $\{L_i \mid i \in I\} = (\mathcal{L}_\mathcal{B} \cap \mathcal{P}(H)) \cup \{\{c\} \mid c \in A \cup B\}$. Thus, $\mathcal{L}_\mathcal{D} \cap \mathcal{P}(H)$ is the set of atoms of the boolean algebra generated by $\{L_i \mid i \in I\}$. Let

$$\mathsf{Q}_\mathcal{D} = \{(P, \Theta) \mid P \subseteq \mathsf{Q}_\mathcal{B}, \Theta \in \mathcal{T}_6, P \subseteq \tau_\mathcal{B}^{-1}(\Theta)\}.$$

The type function is the projection onto the second component:

$$\tau_\mathcal{D}(P, \Theta) = \Theta.$$

The set of transitions $\delta_\mathcal{D}$ consists of all triples $((P, \Theta), K, (P', \Theta')) \in \mathsf{Q}_\mathcal{D} \times \mathcal{L}_\mathcal{D} \times \mathsf{Q}_\mathcal{D}$ such that

$$P' = \{p' \in \mathsf{Q}_\mathcal{B} \mid \exists p \in P \, \exists L : (p, L, p') \in \delta_\mathcal{B}, K \subseteq L\} \cap \tau_\mathcal{B}^{-1}(\Theta')$$
$$\text{and } \{\Theta\} \times K \times \{\Theta'\} \subseteq \widehat{\mathcal{E}}_6.$$

Finally, let

$$\mathsf{I}_\mathcal{D} = \{(\mathsf{I}, (1, H))\} \text{ and } \mathsf{T}_\mathcal{D} = \{(P, (1, 1)) \in \mathsf{Q}_\mathcal{D} \mid P \cap \mathsf{T} \neq \emptyset\}$$

(recall that I (resp. T) is the set of initial (resp. final) states of $\mathcal{A}$ and $\mathcal{B}$). The definition of $\delta_\mathcal{D}$ implies that $\mathcal{D}$ is partitioned and complete. Moreover, the fta $\mathcal{D}$ is deterministic: if we have two transitions $((P, \Theta), K_i, (P_i, \Theta')) \in \delta_\mathcal{D}$ ($i \in \{1, 2\}$) with $K_1 \cap K_2 \neq \emptyset$ then we must have $K_1 = K_2$ and hence $P_1 = P_2$.

Remark 2. For every $P \subseteq \mathsf{Q}_\mathcal{B}$ fulfilling $\emptyset \subsetneq P \subseteq \tau^{-1}(\Theta)$, (P, Θ) is a state of $\mathcal{D}$ and no pair (P, Θ') with $\Theta \neq \Theta'$ is a state of $\mathcal{D}$. But $(\emptyset, \Theta)$ is a state of $\mathcal{D}$ for every type Θ. This is in some sense the price to pay for the completeness of $\mathcal{D}$.

Claim 6. Let $(P, \Theta), (P', \Theta') \in \mathsf{Q}_{\mathcal{D}}$ and $s \in H \cup \{t, t^{-1}\}$. Then

$$((P, \Theta), (P', \Theta')) \in \mu_{\mathcal{D},0}(s) \quad \Leftrightarrow \quad \begin{array}{l} P' = (P \odot \mu_{\mathcal{B},0}(s)) \cap \tau_{\mathcal{B}}^{-1}(\Theta') \text{ and} \\ (\Theta, s, \Theta') \in \widehat{\mathcal{E}}_6. \end{array}$$

Remark 3. If $P' = (P \odot \mu_{\mathcal{B},0}(s)) \cap \tau_{\mathcal{B}}^{-1}(\Theta') \neq \emptyset$, then the condition $(\Theta, s, \Theta') \in \widehat{\mathcal{E}}_6$ follows automatically, since the fta $\mathcal{B}$ is partitioned. But since $P' = \emptyset$ is possible, we have to add the condition $(\Theta, s, \Theta') \in \widehat{\mathcal{E}}_6$ explicitly.

In order to prove Claim 6, we distinguish three cases.

Case 1: $s = t$. Note that $((A, T), t, (B, H))$ is the only t-labelled edge of $\mathcal{G}_6$. We get:

$$\begin{array}{l} ((P, \Theta), (P', \Theta')) \in \mu_{\mathcal{D},0}(t) \Leftrightarrow ((P, \Theta), t, (P', \Theta')) \in \delta_{\mathcal{D}} \Leftrightarrow \\ P' = P \odot \mu_{\mathcal{B},0}(t) \cap \tau_{\mathcal{B}}^{-1}(\Theta') \ \wedge \ \Theta = (A, T) \ \wedge \ \Theta' = (B, H). \end{array}$$

Case 2: $s = t^{-1}$. Same kind of arguments as for $s = t$.

Case 3: $s = h \in H$. Let $K(h)$ be the unique atom of $\mathcal{L}_{\mathcal{D}}$ such that $h \in K(h)$. For every transition $(p, L, q) \in \delta_{\mathcal{B}}$, one has

$$h \in L \Leftrightarrow K(h) \subseteq L.$$

It follows that

$$\begin{array}{rcl} ((P, \Theta), (P', \Theta')) \in \mu_{\mathcal{D},0}(h) & \Leftrightarrow & ((P, \Theta), K(h), (P', \Theta')) \in \delta_D \\ & \Leftrightarrow & P' = (P \odot \mu_{\mathcal{B},0}(h)) \cap \tau_{\mathcal{B}}^{-1}(\Theta') \text{ and} \\ & & \{\Theta\} \times K(h) \times \{\Theta'\} \subseteq \widehat{\mathcal{E}}_6 \\ & \Leftrightarrow & P' = (P \odot \mu_{\mathcal{B},0}(h)) \cap \tau_{\mathcal{B}}^{-1}(\Theta') \text{ and} \\ & & (\Theta, h, \Theta') \in \widehat{\mathcal{E}}_6. \end{array}$$

For the last equivalence note that $\{\Theta\} \times K(h) \times \{\Theta'\} \subseteq \widehat{\mathcal{E}}_6$ if and only if $(\Theta, h, \Theta') \in \widehat{\mathcal{E}}_6$. For this, it is important that if $h \in A \cup B$ then $K(h) = \{h\}$; note that every singleton $\{c\}$ with $c \in A \cup B$ belongs to $\{L_i \mid i \in I\}$.

Claim 7. Let $(P, \Theta), (P', \Theta') \in \mathsf{Q}_{\mathcal{D}}$ and $s \in H * \{t, t^{-1}\}^*$. Then

$$((P, \Theta), (P', \Theta')) \in \mu_{\mathcal{D}}(s) \quad \Leftrightarrow \quad P' = (P \odot \mu_{\mathcal{B}}(s)) \cap \tau_{\mathcal{B}}^{-1}(\Theta') \text{ and } (\Theta, \Theta') \in \mu_{\mathcal{G}_6}(s).$$

Let us prove this claim by induction over the integer n appearing in decomposition (14) of the t-sequence s.

Base case: $n = 0$. For $s = h \in H$ the assertion of Claim 7 is equivalent with that of Claim 6, since $\mu_{\mathcal{A}}(h) = \mu_{\mathcal{A},0}(h)$ for every fta $\mathcal{A}$.

Induction step: Let s be of the form (14) with $n \geq 1$. We suppose that $\alpha_n = 1$ and let

$$s_1 = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_{n-1}} h_{n-1} \text{ and } s = s_1 t h_n$$

(where $s_1 = h_0$ if $n = 1$). Let us first assume that

$$((P, \Theta), (P', \Theta')) \in \mu_{\mathcal{D}}(s). \tag{44}$$

The definition of $\delta_{\mathcal{D}}$ implies $(\Theta, \Theta') \in \mu_{\mathcal{G}_6}(s)$. Moreover, the structure of the graph $\mathcal{G}_6$ implies that there exist $P_1 \subseteq \tau_{\mathcal{B}}^{-1}(A, T)$ and $P_2 \subseteq \tau_{\mathcal{B}}^{-1}(B, H)$ with

$$((P, \Theta), (P_1, (A, T))) \in \mu_{\mathcal{D}}(s_1), \tag{45}$$

$$((P_1, (A, T)), (P_2, (B, H))) \in \mu_{\mathcal{D},0}(t), \text{ and} \tag{46}$$

$$((P_2, (B, H)), (P', \Theta')) \in \mu_{\mathcal{D},0}(h_n). \tag{47}$$

As Claim 7 is fulfilled by s_1 (by the induction hypothesis) and also by sequences in $H \cup \{t, t^{-1}\}$ (by Claim 6), equations (45)–(47) imply:

$$(P \odot \mu_{\mathcal{B}}(s_1)) \cap \tau_{\mathcal{B}}^{-1}(A, T) = P_1 \tag{48}$$

$$(P_1 \odot \mu_{\mathcal{B},0}(t)) \cap \tau_{\mathcal{B}}^{-1}(B, H) = P_2 \tag{49}$$

$$(P_2 \odot \mu_{\mathcal{B},0}(h_n)) \cap \tau_{\mathcal{B}}^{-1}(\Theta') = P' \tag{50}$$

Omitting the type intersections in (48) and (49), we obtain

$$(P \odot \mu_{\mathcal{B}}(s_1 t h_n)) \cap \tau_{\mathcal{B}}^{-1}(\Theta) \supseteq P'. \tag{51}$$

Conversely, by the structure of $\mathcal{G}_6$, every path in the partitioned fta $\mathcal{B}$ from a state $p \in P$ to a state $q' \in \tau_{\mathcal{B}}^{-1}(\Theta')$, reading $s_1 t h_n$, must decompose into

$$p \xrightarrow{s_1}_{\mathcal{B}} p_1 \xrightarrow{t}_{\mathcal{B}} p_2 \xrightarrow{h_n}_{\mathcal{B}} q'$$

with $\tau_{\mathcal{B}}(p_1) = (A, T)$ and $\tau_{\mathcal{B}}(p_2) = (B, H)$. Hence, (48)–(50) imply

$$(P \odot \mu_{\mathcal{B}}(s_1 t h_n)) \cap \tau_{\mathcal{B}}^{-1}(\Theta') \subseteq P'. \tag{52}$$

From (51) and (52) we obtain $P \odot \mu_{\mathcal{B}}(s_1 t h_n) \cap \tau_{\mathcal{B}}^{-1}(\Theta') = P'$. This proves implication $\Rightarrow$ from Claim 7.

Now, suppose conversely that $P \odot \mu_{\mathcal{B}}(s_1 t h_n) \cap \tau_{\mathcal{B}}^{-1}(\Theta') = P'$ and $(\Theta, \Theta') \in \mu_{\mathcal{G}_6}(s)$. By the same kind of arguments as above, there must exist $P_1 \subseteq \tau_{\mathcal{B}}^{-1}(A, T)$ and $P_2 \subseteq \tau_{\mathcal{B}}^{-1}(B, H)$ fulfilling equations (48)–(50). Moreover, from $(\Theta, \Theta') \in \mu_{\mathcal{G}_6}(s)$ we get $(\Theta, (A, T)) \in \mu_{\mathcal{G}_6}(s_1)$ and $((B, H), \Theta') \in \mu_{\mathcal{G}_6}(h_n)$. By the induction hypothesis and Claim 6 this leads to (45)–(47), and hence to (44).

The induction step in the case where $\alpha_n = -1$ can be treated analogously. This proves Claim 7.

From Claim 7 and the fact that the fta $\mathcal{B}$ and $\mathcal{G}_6$ are $\sim$-saturated, it follows immediately that also $\mathcal{D}$ is $\sim$-saturated. Moreover,

$$\mathrm{L}(\mathcal{D}) = \mathrm{L}(\mathcal{B}).$$

Let us finally set $\mathcal{C} = \mathcal{D} \times \mathcal{R}_6$, where $\mathcal{R}_6$ is the partitionned fta from Figure 3. The partitioned fta $\mathcal{C}$ has the following properties:

- $\mathrm{L}(\mathcal{C}) = \mathrm{L}(\mathcal{D}) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t)$ by Lemma 4.
- $\mathcal{C}$ is deterministic and complete by Lemma 7 and the fact that $\mathcal{R}_6$ and $\mathcal{D}$ are both deterministic and complete.
- $\mathcal{C}$ is $\sim$-saturated by Lemma 6 and the fact that $\mathcal{R}_6$ and $\mathcal{D}$ are $\sim$-saturated.

Hence, $\mathcal{C}$ fulfills point (2) of Proposition 1. $\square$

Recall that $\pi : (X \cup \{t, t^{-1}\})^* \to G$ denotes the canonical morphism from (8).

Proposition 2. *Let $K \subseteq G$. The following are equivalent:*

(1) K is a rational subset of G.

(2) *There exists some finite automaton* $\mathcal{A}$ *over the free monoid* $(X \cup \{t, t^{-1}\})^*$ *such that* $K = \pi(\mathrm{L}(\mathcal{A}))$.
(3) $\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t)$ *for some partitionned,* $\approx$*-compatible, and* $\sim$*-saturated fta* $\mathcal{B}$ *with labelling set* $\mathrm{Rat}(H)$.

Moreover,

- *in point (3),* $\mathcal{B}$ *can always be chosen unitary, and*
- *in point (3), if the membership problem for languages in* $\mathrm{Rat}(H)$ *is decidable, then* $\mathcal{B}$ *can be computed effectively from* $\mathcal{A}$ *in point (2).*

Proof. The equivalence of (1) and (2) is trivial.

(2) $\Rightarrow$ (3): We cut the proof of this implication into three claims. Let us start with a finite automaton

$$\mathcal{A} = \langle \mathcal{X}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle$$

over the free monoid $(X \cup \{t, t^{-1}\})^*$ such that

$$K = \pi(\mathrm{L}(\mathcal{A})).$$

Let us recall that ST is the semi-Thue system consisting of all the rules (15) and (16), where $u \in X^*$ and $\varphi([u]_\equiv) \in B$ and $\varphi^{-1}([u]_\equiv) \in A$ denote the corresponding letters in X (we choose such a letter for every element of $A \cup B$).

For a set E, a binary relation $\rightarrow \subseteq E \times E$, and a subset $R \subseteq E$, let $R\!\rightarrow = \{e \in E \mid \exists r \in R : r \rightarrow e\}$.

Claim 8. There exists a finite automaton $\mathcal{A}_1$ over the free monoid $(X \cup \{t, t^{-1}\})^*$, such that

$$\mathrm{L}(\mathcal{A}_1) = \mathrm{L}(\mathcal{A})\!\rightarrow^*_{\mathsf{ST}} .$$

Moreover, if the membership problem is decidable for rational subsets of H, then $\mathcal{A}_1$ can be constructed effectively from $\mathcal{A}$.

For the proof of Claim 8, notice that ST is *monadic*, i.e., every righthand-side of a rule has length one or zero. It is known that for every monadic semi-Thue system S over Y^* and every rational subset $R \subseteq Y^*$, $R\!\rightarrow^*_S \in \mathrm{Rat}(Y^*)$, see [BJW82,Sén94]. Taking

$R = \mathrm{L}(\mathcal{A})$ and $S = \mathsf{ST}$ in this result, we obtain the first statement of Claim 8. In order to check the effectiveness statement, let us consider the following algorithm:

1 $\delta_1 := \delta$;
2 `while` there are $p, q, q', r \in \mathsf{Q}, u \in X^*, a \in A, b \in B$ `such that`
`case 1`: $p \xrightarrow{t^{-1}}_{\delta_1} q \xrightarrow{u}_{\delta_1} q' \xrightarrow{t}_{\delta_1} r$, $[u]_{\equiv} = a$, and not $p \xrightarrow{\varphi(a)}_{\delta_1} r$
`or`
`case 2`: $p \xrightarrow{t}_{\delta_1} q \xrightarrow{u}_{\delta_1} q' \xrightarrow{t^{-1}}_{\delta_1} r$, $[u]_{\equiv} = b$, and not $p \xrightarrow{\varphi^{-1}(b)}_{\delta_1} r$
`do`
3 `in case 1`: $\delta_1 := \delta_1 \cup \{(p, \varphi(a), r)\}$
4 `in case 2`: $\delta_1 := \delta_1 \cup \{(p, \varphi^{-1}(b), r)\}$
5 `od`

This algorithm constructs from the finite automaton $\mathcal{A}$ a finite automaton

$$\mathcal{A}_1 = \langle \mathcal{X} \cup A \cup B, \mathsf{Q}, \delta_1, \mathsf{I}, \mathsf{T} \rangle \tag{53}$$

satisfying the first statement of Claim 8. In order to make line 2 above effective, it is enough that we can decide for every $a \in A$ (resp. $b \in B$) and every $q, q' \in \mathsf{Q}$, whether $a \in \pi_t(\mathrm{L}(\mathcal{A}[X, q, q']))$ (resp. $b \in \pi_t(\mathrm{L}(\mathcal{A}[X, q, q']))$), where $\pi_t : (X \cup \{t, t^{-1}\})^* \to H * \{t, t^{-1}\}^*$ is the canonical morphism from (9). But this is an instance of the membership problem for rational subsets of H. This proves the effectiveness statement in Claim 8.

Claim 9. There exists some $\approx$-compatible fta $\mathcal{A}_2$ over $H * \{t, t^{-1}\}^*$ with labelling set $\mathrm{Rat}(H)$ such that

$$\mathrm{L}(\mathcal{A}_2) = \pi_t(\mathrm{L}(\mathcal{A}_1)).$$

Moreover, $\mathcal{A}_2$ can be constructed effectively from $\mathcal{A}_1$.

Recall the finite automaton $\mathcal{A}_1$ from (53). We set

$$\mathcal{A}_2 = \langle \mathcal{L}, \mathsf{Q}, \delta_2, \mathsf{I}, \mathsf{T} \rangle$$

where

$$\delta_2 = \{(q, t^\alpha, r) \mid \alpha \in \{1, -1\}, (q, t^\alpha, r) \in \delta_1\} \cup \{(q, \pi_t(\mathrm{L}(\mathcal{A}_1[X, q, r])), r) \mid q, r \in \mathsf{Q}\}$$

(the label set $\mathcal{L}$ is given implicitly by δ_2). In some sense the transitions of δ_2 are collecting into the rational labels $\mathrm{L}(\mathcal{A}_1[X,q,r])$ all possible finite paths with labels in X^*, while the transitions labelled by t or t^{-1} are just copied. Clearly, $\mathrm{L}(\mathcal{A}_2) = \pi_t(\mathrm{L}(\mathcal{A}_1))$ and $\mathcal{A}_2$ can be constructed effectively from $\mathcal{A}_1$. So it remains to show that $\mathcal{A}_2$ is $\approx$-compatible.

Recall that $\pi_t(\mathrm{Irr}(\mathsf{ST})) = \mathrm{Red}(H,t)$. Since, $\mathrm{L}(\mathcal{A}_1) = \mathrm{L}(\mathcal{A}){\to}^*_{\mathsf{ST}}$, the language $\mathrm{L}(\mathcal{A}_1)$ is closed under the terminating rewrite relation $\to_{\mathsf{ST}} \subseteq \equiv$. It follows

$$\mathrm{L}(\mathcal{A}_1) \subseteq [\mathrm{L}(\mathcal{A}_1) \cap \mathrm{Irr}(\mathsf{ST})]_{\equiv}.$$

By applying the morphism π_t to this inclusion, we obtain

$$\begin{aligned}
\mathrm{L}(\mathcal{A}_2) &= \pi_t(\mathrm{L}(\mathcal{A}_1)) \\
&\subseteq \pi_t([\mathrm{L}(\mathcal{A}_1) \cap \mathrm{Irr}(\mathsf{ST})]_{\equiv}) \\
&= [\pi_t(\mathrm{L}(\mathcal{A}_1) \cap \mathrm{Irr}(\mathsf{ST}))]_{\approx} \\
&\subseteq [\pi_t(\mathrm{L}(\mathcal{A}_1)) \cap \pi_t(\mathrm{Irr}(\mathsf{ST}))]_{\approx} \\
&= [\mathrm{L}(\mathcal{A}_2) \cap \mathrm{Red}(H,t)]_{\approx},
\end{aligned}$$

which proves that $\mathcal{A}_2$ is $\approx$-compatible.

Claim 10. There exists some partitionned and $\approx$-compatible fta $\mathcal{A}_3$ over $H * \{t, t^{-1}\}^*$ with labelling set $\mathrm{Rat}(H)$ such that

$$\mathrm{L}(\mathcal{A}_3) = \mathrm{L}(\mathcal{A}_2).$$

Moreover, $\mathcal{A}_3$ can be constructed effectively from $\mathcal{A}_2$.

Just set $\mathcal{A}_3 = \mathcal{A}_2 \times \mathcal{G}_6$. By Lemma 5, we have $\mathrm{L}(\mathcal{A}_3) = \mathrm{L}(\mathcal{A}_2)$ (and hence, since $\mathcal{A}_2$ is $\approx$-compatible, also $\mathcal{A}_3$ is $\approx$-compatible).

By point (1) of Proposition 1 there exists a partitionned, $\approx$-compatible, $\sim$-*saturated*, and unitary fta $\mathcal{B}$ with labelling set $\mathrm{Rat}(H)$ and such that

$$\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t) = [\mathrm{L}(\mathcal{A}_3)]_{\approx} \cap \mathrm{Red}(H,t).$$

Let us summarize the connections between the rational set $K \subseteq G$ and the successive automata $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$, and $\mathcal{B}$:

$$\begin{aligned}
K &= \pi(\mathrm{L}(\mathcal{A})) && \subseteq G && (54)\\
\mathrm{L}(\mathcal{A}_1) &= \mathrm{L}(\mathcal{A}) \rightarrow^*_{\mathsf{ST}} && \subseteq (X \cup \{t, t^{-1}\})^* && (55)\\
\mathrm{L}(\mathcal{A}_2) &= \pi_t(\mathrm{L}(\mathcal{A}_1)) && \subseteq H * \{t, t^{-1}\}^* && (56)\\
\mathrm{L}(\mathcal{A}_3) &= \mathrm{L}(\mathcal{A}_2) && \subseteq H * \{t, t^{-1}\}^* && (57)\\
\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t) &= [\mathrm{L}(\mathcal{A}_3)]_{\approx} \cap \mathrm{Red}(H, t) && \subseteq H * \{t, t^{-1}\}^* && (58)
\end{aligned}$$

By (54),

$$\pi_G^{-1}(K) = \pi_G^{-1}(\pi_G(\pi_t(\mathrm{L}(\mathcal{A})))) = [\pi_t(\mathrm{L}(\mathcal{A}))]_{\approx} .$$

By (55), $[\mathrm{L}(\mathcal{A})]_{\equiv} = [\mathrm{L}(\mathcal{A}_1)]_{\equiv}$, hence $[\pi_t(\mathrm{L}(\mathcal{A}))]_{\approx} = [\pi_t(\mathrm{L}(\mathcal{A}_1))]_{\approx}$ and thus

$$\pi_G^{-1}(K) = [\pi_t(\mathrm{L}(\mathcal{A}_1))]_{\approx} .$$

Using (56) and (57) we get

$$\pi_G^{-1}(K) = [\mathrm{L}(\mathcal{A}_2)]_{\approx} = [\mathrm{L}(\mathcal{A}_3)]_{\approx}.$$

Finally, by (58) we have

$$\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t).$$

We have obtained point (3) of Proposition 2.

(3) $\Rightarrow$ (1): Let $\mathcal{B}$ be some partitionned, $\approx$-compatible, and $\sim$-saturated fta with labelling set $\mathrm{Rat}(H)$ such that

$$\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t). \qquad (59)$$

Let $\mathcal{L}_{\mathcal{B}} \subseteq \mathrm{Rat}(H) \cup \{\{t\}, \{t^{-1}\}\}$ be the set of labels of the transitions of the fta $\mathcal{B}$. We can substitute in $\mathcal{B}$ every L-labelled edge with $L \in \mathrm{Rat}(H)$ by a finite automaton over the generating set X of H. In this way, we see that

$$\mathrm{L}(\mathcal{B}) \in \mathrm{Rat}(H * \{t, t^{-1}\}^*). \qquad (60)$$

As every t-sequence is equivalent (modulo $\approx$) with at least one reduced sequence, we have

$$K = \pi_G(\pi_G^{-1}(K) \cap \mathrm{Red}(H, t)). \qquad (61)$$

With (59) we get

$$K = \pi_G(\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)). \tag{62}$$

Since $\mathcal{B}$ is $\approx$-compatible, we have

$$\pi_G(\mathrm{L}(\mathcal{B})) \subseteq \pi_G([\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)]_\approx) = \pi_G(\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)).$$

Since also $\pi_G(\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)) \subseteq \pi_G(\mathrm{L}(\mathcal{B}))$, (62) implies

$$K = \pi_G(\mathrm{L}(\mathcal{B})). \tag{63}$$

As any homomorphic image of a rational subset is rational, K is a rational subset of G. $\square$

4.5 Transfer results for rational sets

In this section we prove the following transfer theorem for several properties of rational sets of H (resp. G):

Theorem 2. *The following holds:*

(1) If $\mathrm{Rat}(H)$ *is closed under intersection, then so is* $\mathrm{Rat}(G)$.
(2) If $\mathrm{Rat}(H)$ *is closed under complement, then so is* $\mathrm{Rat}(G)$.
(3) If the membership problem for rational subsets of H is decidable, then so is it for G.
(4) If the emptiness problem for boolean combinations of rational subsets of H is decidable, then so is it for G.

Proof. For statement (1), let $K_1, K_2 \in \mathrm{Rat}(G)$ and $K = K_1 \cap K_2$. By Proposition 2 there are partitionned, $\approx$-compatible, and $\sim$-saturated fta $\mathcal{A}_1$ and $\mathcal{A}_2$ with labelling set $\mathrm{Rat}(H)$ such that

$$\pi_G^{-1}(K_i) \cap \mathrm{Red}(H,t) = \mathrm{L}(\mathcal{A}_i) \cap \mathrm{Red}(H,t)$$

for $i \in \{1,2\}$. Hence,

$$\begin{aligned}\pi_G^{-1}(K) \cap \mathrm{Red}(H,t) &= \pi_G^{-1}(K_1) \cap \pi_G^{-1}(K_2) \cap \mathrm{Red}(H,t)\\ &= \mathrm{L}(\mathcal{A}_1) \cap \mathrm{L}(\mathcal{A}_2) \cap \mathrm{Red}(H,t).\end{aligned}$$

Let us consider the partitionned fta

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2.$$

By Lemma 4, $L(\mathcal{A}) = L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$, and by Lemma 6, $\mathcal{A}$ is $\approx$-compatible and $\sim$-saturated. Moreover, since $\mathrm{Rat}(H)$ is closed under intersection, the labelling set of $\mathcal{A}$ is still $\mathrm{Rat}(H)$. Hence K fulfills point (3) of Proposition 2. By point (1) of Proposition 2, K is a rational subset of G.

For statement (2), let $K \in \mathrm{Rat}(G)$. By point (3) of Proposition 2 we have

$$\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = L(\mathcal{B}) \cap \mathrm{Red}(H, t)$$

for a partitionned, $\approx$-compatible, and $\sim$-saturated fta $\mathcal{B}$ with labelling set $\mathrm{Rat}(H)$. The proof of the implication (3) $\Rightarrow$ (1) from Proposition 2 shows that $K = \pi_G(L(\mathcal{B}))$. By applying point (2) of Proposition 1 to this automaton $\mathcal{B}$, we obtain

$$\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = L(\mathcal{C})$$

for some partitionned, $\sim$-saturated, deterministic, and complete fta $\mathcal{C}$. The labelling set of this fta is $\mathrm{Bool}(\mathrm{Rat}(H))$, which equals $\mathrm{Rat}(H)$, since $\mathrm{Rat}(H)$ is supposed to to be closed under complement.

Now take the complement automaton $\overline{\mathcal{C}}$ of $\mathcal{C}$. By Lemma 3 it is a partitionned and $\sim$-saturated fta with $L(\overline{\mathcal{C}}) = \mathrm{Red}(H, t) \setminus L(\mathcal{C}) \subseteq \mathrm{Red}(H, t)$. In particular, $\overline{\mathcal{C}}$ is also $\approx$-compatible. We obtain

$$\begin{aligned}
\pi_G^{-1}(G \setminus K) \cap \mathrm{Red}(H, t) &= (H * \{t, t^{-1}\}^* \setminus \pi_G^{-1}(K)) \cap \mathrm{Red}(H, t) \\
&= \mathrm{Red}(H, t) \setminus (\pi_G^{-1}(K) \cap \mathrm{Red}(H, t)) \\
&= \mathrm{Red}(H, t) \setminus L(\mathcal{C}) \\
&= L(\overline{\mathcal{C}}) \\
&= L(\overline{\mathcal{C}}) \cap \mathrm{Red}(H, t).
\end{aligned}$$

Hence, Proposition 2 implies that $G \setminus K$ is a rational subset of G. This proves statement (2).

For statement (3) suppose that the membership problem for rational subsets of H is decidable. Let $K = \pi(L(\mathcal{A}))$ for a given finite automaton $\mathcal{A}$ over the free monoid $(X \cup \{t, t^{-1}\})^*$. By point (3) of Proposition 2 we have

$$\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = L(\mathcal{B}) \cap \mathrm{Red}(H, t)$$

for a partitionned, $\approx$-compatible, and $\sim$-saturated fta $\mathcal{B}$ with labelling set $\mathrm{Rat}(H)$. Moreover, since the membership problem for languages in $\mathrm{Rat}(H)$ is decidable, $\mathcal{B}$ can be constructed effectively from $\mathcal{A}$.

Now, to a given word $u \in (X \cup \{t, t^{-1}\})^*$ we first apply (effectively) as long as possible the rules of ST in order to obtain a word $v \in (X \cup \{t, t^{-1}\})^*$ of the form

$$v = v_0 t^{\alpha_1} v_1 \cdots t^{\alpha_i} v_i \cdots t^{\alpha_p} v_p$$

where $p \geq 0, \alpha_i \in \{1, -1\}, v_i \in X^*$, and $\pi_t(v) \in \mathrm{Red}(H, t)$. Since the membership problem is decidable for the labelling sets of $\mathcal{B}$, we can compute all the binary relations $\mu_{\mathcal{B}}(\pi_t(v_i))$, $\mu_{\mathcal{B}}(t)$, and $\mu_{\mathcal{B}}(t^{-1})$. This allows to compute the binary relation $\mu_{\mathcal{B}}(\pi_t(v))$. Since $u \equiv v$, we have $\pi(u) \in K$ iff $\pi(v) \in K$ iff $\pi_t(v) \in \pi_G^{-1}(K)$ iff $\pi_t(v) \in \mathrm{L}(\mathcal{B})$ iff $\mu_{\mathcal{B}}(\pi_t(v)) \cap \mathsf{I}_{\mathcal{B}} \times \mathsf{T}_{\mathcal{B}} \neq \emptyset$.

Let us finally prove statement (4). Assume that the emptiness problem for boolean combinations of rational subsets of H is decidable. Let $K_1, \ldots, K_p$ be rational subsets of G defined as $K_i = \pi(\mathrm{L}(\mathcal{A}_i))$ for some given finite automaton $\mathcal{A}_i$ over the free monoid $(X \cup \{t, t^{-1}\})^*$. Following the construction described for statement (2) of Theorem 2, we can construct for every $i \in \{1, \ldots, p\}$ a partitionned, $\sim$-saturated, deterministic, and complete fta $\mathcal{C}_i$, whose labelling set is $\mathrm{Bool}(\mathrm{Rat}(H))$, and such that

$$\mathrm{L}(\mathcal{C}_i) = \pi_G^{-1}(K_i) \cap \mathrm{Red}(H, t).$$

The boolean operations $\cap$ and complement over $\sim$-saturated subsets of $\mathrm{Red}(H, t)$ can be translated into the product and complement operation, respectively, over partitionned, $\sim$-saturated, deterministic, and complete fta reading only reduced sequences (for products, we have to use Lemma 6 and 7, for complements we use Lemma 3). Hence, from a given boolean combination K of the sets K_i, one can compute a corresponding fta $\mathcal{C}$, whose labelling set is $\mathrm{Bool}(\mathrm{Rat}(H))$, and such that

$$\mathrm{L}(\mathcal{C}) = \pi_G^{-1}(K) \cap \mathrm{Red}(H, t).$$

Emptiness of the set K reduces to emptiness of $\mathrm{L}(\mathcal{C})$, which itself reduces to the emptiness problem for the label sets of $\mathcal{C}$. Hence, under the hypothesis that emptiness is decidable for languages in $\mathrm{Bool}(\mathrm{Rat}(H))$, we can test whether $\mathrm{L}(\mathcal{C}) = \emptyset$ or not. $\square$

Remark 4. For a given monoid $\mathbb{M}$, closure of $\mathrm{Rat}(\mathbb{M})$ under complement implies closure of $\mathrm{Rat}(\mathbb{M})$ under intersection, but the converse does not hold in general: we give examples of such monoids $\mathbb{M}$ where $\mathrm{Rat}(\mathbb{M})$ is closed under intersection but not under complement in the appendix B. This shows that points (1) and (2) of theorem 2 cannot be deduced in a straightforward manner one from the other.

4.6 Transfer results for finitely generated subgroups

We suppose in this section that H (and hence G) is a group. We assume moreover that X is endowed with an involution $x \mapsto x^{-1}$ such that $xx^{-1} \equiv x^{-1}x \equiv \varepsilon$. The following subset of $\mathcal{P}(H)$ will serve for labelling fta in the sequel:

$$\mathcal{S} = \{Kh \mid K \text{ is a finitely generated subgroup of } H, h \in H\}$$

Note that every singleton subset $\{h\}$ belongs to $\mathcal{S}$.

Normalization of automata Let us prove first a normalization lemma for finitely generated subgroups of G, which parallels Proposition 2:

Proposition 3. *Let K be a subgroup of G. The following are equivalent:*

(1) K is finitely generated.
(2) K is rational.
(3) There exists some finite automaton $\mathcal{A}$ over the free monoid $(X \cup \{t, t^{-1}\})^*)$ such that $K = \pi(\mathrm{L}(\mathcal{A}))$.
(4) $\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t)$ for some partitionned, $\approx$-compatible, and $\sim$-saturated fta $\mathcal{B}$ with labelling set $\mathcal{S}$.

Moreover, in (4), if the generalized word problem of H is decidable, then $\mathcal{B}$ can be constructed effectively from $\mathcal{A}$.

Proof. We follow the arguments from the proof of Proposition 2.

(1) $\Leftrightarrow$ (2): This is just Theorem 1.

(2) $\Leftrightarrow$ (3): This has been shown in Proposition 2.

(3) $\Rightarrow$ (4): We must here slightly adapt the corresponding proof given for Proposition 2. Let us start with a finite automaton $\mathcal{A} = \langle \mathcal{X}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle$ over the free monoid $(X \cup \{t, t^{-1}\})^*$ such that

$$K = \pi(\mathrm{L}(\mathcal{A})).$$

Claim 11. There exists a finite automaton $\mathcal{A}_1$ over the free monoid $(X \cup \{t, t^{-1}\})^*$ such that

$$\mathrm{L}(\mathcal{A})\rightarrow^*_{\mathcal{R}_\delta} \subseteq \mathrm{L}(\mathcal{A}_1) \subseteq [\mathrm{L}(\mathcal{A})]_\equiv. \tag{64}$$

Moreover, if the generalized word problem of H is decidable, then $\mathcal{A}_1$ can be constructed effectively from $\mathcal{A}$.

We assume that $\mathcal{A}$ is trim. Let

$$\delta^{-1} = \{(r, c^{-1}, q) \mid c \in X \cup \{t, t^{-1}\}, (q, c, r) \in \delta\}.$$

We start with $\delta_1 = \delta \cup \delta^{-1}$. Note that

$$\mathrm{L}(\mathcal{A}) \subseteq \mathrm{L}(\langle \mathcal{X}, \mathsf{Q}, \delta_1, \mathsf{I}, \mathsf{T} \rangle) \subseteq [\mathrm{L}(\mathcal{A})]_\equiv.$$

For all states q, q' we define $\ell(\delta_1, q, q')$ as a shortest word over X which labels a δ_1-path from q to q'. We saturate δ_1 under the following inference rules, where $u \in X^*$:

$$\frac{p \xrightarrow{t^{-1}}_{\delta_1} q \xrightarrow{u}_{\delta_1} q \xrightarrow{\ell(\delta_1,q,q')}_{\delta_1} q' \xrightarrow{t}_{\delta_1} r \;\wedge\; u\ell(\delta_1, q, q') \equiv_H a \in A}{p \xrightarrow{\varphi(a)}_{\delta_1} r} \tag{65}$$

$$\frac{p \xrightarrow{t}_{\delta_1} q \xrightarrow{u}_{\delta_1} q \xrightarrow{\ell(\delta_1,q,q')}_{\delta_1} q' \xrightarrow{t^{-1}}_{\delta_1} r \;\wedge\; u\ell(\delta_1, q, q') \equiv_H b \in B}{p \xrightarrow{\varphi^{-1}(b)}_{\delta_1} r} \tag{66}$$

Let δ_1 be the resulting transition relation and let

$$\mathcal{A}_1 = \langle \mathcal{X} \cup A \cup B, \mathsf{Q}, \delta_1, \mathsf{I}, \mathsf{T} \rangle.$$

It satisfies (64). For this notice that if $q \xrightarrow{v}_{\delta_1} q'$ for a word $v \in X^*$ with $v \equiv_H a \in A$ then $v\ell(\delta_1, q, q')^{-1}\ell(\delta_1, p, q) \equiv_H a$ and $q \xrightarrow{v\ell(\delta_1,q,q')^{-1}}_{\delta_1} q$, i.e., the word $v\ell(\delta_1, q, q')^{-1}$ is a canditate for u in the above rule (65).

Let us consider for every value of δ_1 and every $q \in \mathsf{Q}_0$ the set

$$P_1(\delta_1, q) = \{u \in (X \cup \overline{X})^* \mid q \xrightarrow{u}_{\delta_1} q\}.$$

By Theorem 1, $\pi_t(P_1(\delta_1, q))$ is a finitely generated subgroup of H, and by the proof given in Section 2.2, a finite set of generators for it can be explicitly computed. If the generalized word problem of H is decidable, then the validity of the hypothesis of the inference rules (65) and (66) can be algorithmically tested so that δ_1 can be algorithmically computed. This proves Claim 11.

Claim 12. There exists a $\approx$-compatible fta $\mathcal{A}_2$ over $H * \{t, t^{-1}\}^*$ with labelling set $\mathcal{S}$ such that

$$\mathrm{L}(\mathcal{A}_2) = \pi_t(\mathrm{L}(\mathcal{A}_1)).$$

Moreover, $\mathcal{A}_2$ can be constructed effectively from $\mathcal{A}_1$.

Analogously to the proof of Claim 9, we do a collecting process:

$$\begin{aligned} \delta_2 = & \{(q, t^\alpha, r) \mid \alpha \in \{1, -1\}, (q, t^\alpha, r) \in \delta_1\} \cup \\ & \{(q, \pi_t(P_1(\delta_1, q)\ell(\delta_1, q, r)), r) \mid q, r \in \mathsf{Q}\}. \end{aligned} \tag{67}$$

This step can be shown to be effective by the arguments already used for the proof of Claim 11. Moreover, (64) implies that $\mathrm{L}(\mathcal{A}_1) \subseteq [\mathrm{L}(\mathcal{A}_1) \cap \mathrm{Irr}(\mathsf{ST})]_\equiv$. Now $\mathrm{L}(\mathcal{A}_2) \subseteq [\mathrm{L}(\mathcal{A}_2) \cap \mathrm{Red}(H, t)]_\approx$ can be shown in the same way as in the proof of Claim 9. Hence, $\mathcal{A}_2$ is $\approx$-compatible.

Claim 13. There exists a partitionned and $\approx$-compatible fta $\mathcal{A}_3$ over $H * \{t, t^{-1}\}^*$ with labelling set $\mathcal{S}$ and such that

$$\mathrm{L}(\mathcal{A}_3) = \mathrm{L}(\mathcal{A}_2).$$

Moreover, $\mathcal{A}_3$ can be constructed effectively from $\mathcal{A}_2$.

As for Claim 10 set $\mathcal{A}_3 = \mathcal{A}_2 \times \mathcal{G}_6$.

Let us now apply point (1) of Proposition 1 to the set $\mathcal{F} = \mathcal{S}$ and the fta $\mathcal{A}_3$, and let $\mathcal{B}$ be the resulting partitionned, $\approx$-compatible, and $\sim$-saturated fta. Note that the labelling set of $\mathcal{B}$ is still $\mathcal{S}$: If Kh labels a transition of $\mathcal{A}_3$, where K is a finitely generated subgroup of

H and $h \in H$, then a transition of $\mathcal{B}$ may be labelled with $c(Kh)c'$, where $c, c' \in A \cup B$. But we have $c(Kh)c' = (cKc^{-1})(chc') \in \mathcal{S}$.

We obtain the following relationships between the automata $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{B}$:

$$\begin{aligned}
K &= \pi(\mathrm{L}(\mathcal{A})) && \subseteq G && (68)\\
\mathrm{L}(\mathcal{A}){\to}^*_{\mathsf{ST}} \subseteq \mathrm{L}(\mathcal{A}_1) & && \subseteq [\mathrm{L}(\mathcal{A})]_{\equiv} && (69)\\
\mathrm{L}(\mathcal{A}_2) &= \pi_t(\mathrm{L}(\mathcal{A}_1)) && \subseteq H * \{t, t^{-1}\}^* && (70)\\
\mathrm{L}(\mathcal{A}_3) &= \mathrm{L}(\mathcal{A}_2) && \subseteq H * \{t, t^{-1}\}^* && (71)\\
\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t) &= [\mathrm{L}(\mathcal{A}_3)]_{\approx} \cap \mathrm{Red}(H, t) && \subseteq H * \{t, t^{-1}\}^* && (72)
\end{aligned}$$

From (69) we obtain $[\mathrm{L}(\mathcal{A})]_{\equiv} = [\mathrm{L}(\mathcal{A}_1)]_{\equiv}$. Now, we can deduce $\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t)$ in the same way as in the corresponding part of the proof of Proposition 2.

(4) $\Rightarrow$ (2): This follows from Proposition 2 since, $\mathcal{S} \subseteq \mathrm{Rat}(H)$. $\square$

Using Proposition 3, we can next prove a transfer theorem for finitely generated subgroups:

Theorem 3. *The following holds:*

(1) If H has a decidable generalized word problem, then so has G.
(2) If H has the Howson property, then so has G.
(3) If H has the LERF property, then so has G.

Proof. For property (1), let $u_1, \ldots, u_n, u \in (X \cup \{t, t^{-1}\})^*$. Let $K \subseteq G$ be the subgroup generated by $\{\pi(u_1), \ldots, \pi(u_n)\}$. We give an algorithm for deciding whether $u \in K$ or not:

Step 1: We construct from $u_1, \ldots, u_n$ a finite automaton $\mathcal{A}$ over the free monoid $(X \cup \{t, t^{-1}\})^*$ such that

$$K = \pi(\mathrm{L}(\mathcal{A})).$$

Step 2: Using Proposition 3 we can compute from $\mathcal{A}$ effectively a partitionned, $\approx$-compatible, and $\sim$-saturated fta $\mathcal{B}$ with labelling set $\mathcal{S}$ such that

$$\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t).$$

Step 3: Starting from the word $u \in (X \cup \{t, t^{-1}\})^*$ and applying (effectively) the rules of ST as long as possible, one computes a word $v \in (X \cup \{t, t^{-1}\})^*$ of the form

$$v = v_0 t^{\alpha_1} v_1 \cdots t^{\alpha_i} v_i \cdots t^{\alpha_p} v_p$$

where $p \geq 0, \alpha_i \in \{1, -1\}, v_i \in X^*$, and $\pi_t(v) \in \mathrm{Red}(H, t)$.

Step 4: From $\mathcal{B}$ one can compute all the binary relations $\mu_{\mathcal{B}}(\pi_t(v_i))$, $\mu_{\mathcal{B}}(t)$, and $\mu_{\mathcal{B}}(t^{-1})$. This allows to compute the binary relation $\mu_{\mathcal{B}}(\pi_t(v))$. Since $u \equiv v$, we have $\pi(u) \in K$ iff $\pi(v) \in K$ iff $\pi_t(v) \in \pi_G^{-1}(K)$ iff $\pi_t(v) \in \mathrm{L}(\mathcal{B})$ iff $\mu_{\mathcal{B}}(\pi_t(v)) \cap \mathsf{I}_{\mathcal{B}} \times \mathsf{T}_{\mathcal{B}} \neq \emptyset$.

Let us now prove (2): Suppose that H has the Howson property. Let K_1 and K_2 be two finitely generated subgroups of G. By Proposition 3, there exist partitionned, $\approx$-compatible, and $\sim$-saturated fta $\mathcal{B}_1$ and $\mathcal{B}_2$ with labelling set $\mathcal{S}$ such that

$$\pi_G^{-1}(K_i) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}_i) \cap \mathrm{Red}(H, t)$$

for $i \in \{1, 2\}$. Let

$$\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2.$$

By the same arguments as for point (1) of Theorem 2, $\mathcal{B}$ is partitionned, $\approx$-compatible, and $\sim$-saturated. Let us check that its labels belong to $\mathcal{S}$: Every label L of $\mathcal{B}$ either belongs to $\{\{t\}, \{t^{-1}\}\}$ or has the form

$$L = F_1 r_1 \cap F_2 r_2 \neq \emptyset$$

for finitely generated subgroups $F_1, F_2 \leq H$ and $r_1, r_2 \in H$. Since $F_1 r_1 \cap F_2 r_2 \neq \emptyset$, there exist $f_1 \in F_1$ and $f_2 \in F_2$ such that $f_1 r_1 = f_2 r_2$ in H. We thus have

$$\begin{aligned} F_1 r_1 \cap F_2 r_2 &= ((F_1 f_1^{-1} f_2) \cap F_2) r_2 \\ &= (F_1 f_2 \cap F_2) r_2 \\ &= (F_1 \cap F_2) f_2 r_2. \end{aligned}$$

By the Howson property for H, $F_1 \cap F_2$ is a finitely generated subgroup of H. Hence $(F_1 \cap F_2) f_2 r_2$ belongs to $\mathcal{S}$. Thus

$$\pi_G^{-1}(K_1 \cap K_2) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t)$$

for a partitionned, $\approx$-compatible, and $\sim$-saturated fta $\mathcal{B}$ with labelling set $\mathcal{S}$. By Proposition 3 this implies that $K_1 \cap K_2$ is finitely generated.

Finally, we prove point (3). In order to treat the LERF property we have to introduce a technical notion about finite t-automata, which is stronger than $\approx$-compatibility:

Definition 8. *Let $\mathcal{B} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$ be a partitionned fta. It is said to be* downwards-closed *(briefly d-closed) if for all q_1, q_2, q_3, q_4, q_5, $q_6 \in \mathsf{Q}$, $h_2, h_6 \in H$, $a \in A$, and $b \in B$ such that $\tau(q_1) \neq \tau(q_2)$ and $\tau(q_5) \neq \tau(q_6)$:*

$$q_1 \xrightarrow{h_2}_{\mathcal{B}} q_2 \xrightarrow{t^{-1}}_{\mathcal{B}} q_3 \xrightarrow{a}_{\mathcal{B}} q_4 \xrightarrow{t}_{\mathcal{B}} q_5 \xrightarrow{h_6}_{\mathcal{B}} q_6 \;\Rightarrow\; q_1 \xrightarrow{h_2\varphi(a)h_6}_{\mathcal{B}} q_6 \quad (73)$$

$$q_1 \xrightarrow{h_2}_{\mathcal{B}} q_2 \xrightarrow{t}_{\mathcal{B}} q_3 \xrightarrow{b}_{\mathcal{B}} q_4 \xrightarrow{t^{-1}}_{\mathcal{B}} q_5 \xrightarrow{h_6}_{\mathcal{B}} q_6 \;\Rightarrow\; q_1 \xrightarrow{h_2\varphi^{-1}(b)h_6}_{\mathcal{B}} q_6 \quad (74)$$

Lemma 9. *A partitionned and d-closed fta is $\approx$-compatible.*

Proof. Assume that the partitionned fta $\mathcal{B} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$ is d-closed and let $s \in \mathrm{L}(\mathcal{B})$. We have to show that $s \in [\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)]_{\approx}$. For this, assume that s is not reduced, i.e., $s = s_1 t^{-1} a t s_2$. Since $s \in \mathrm{L}(\mathcal{B})$, there exists a path in $\mathcal{B}$, labelled with s, leading from a state of type $(1, H)$ to a state of type $(1, 1)$. Since $\mathcal{B}$ is partitionned, the sequences s_1 and s_2 cannot be empty: note that no t-labelled (resp. t^{-1}-labelled) transitions exits (resp. enters) a state of type $(1, H)$ (resp. $(1, 1)$) in a partitionned fta. Hence we have $s = s_1' h_2 t^{-1} a t h_6 s_2'$ for some $h_2, h_6 \in H$. Moreover, there exist states $q_1, q_2, q_3, q_4, q_5, q_6 \in \mathsf{Q}$ such that (73) holds. Since $\mathcal{B}$ is partitionned and the path labelled with s leads from a state of type $(1, H)$ to a state of type $(1, 1)$, we have $\tau(q_1) \neq \tau(q_2)$ and $\tau(q_5) \neq \tau(q_6)$ (see the last statement in Lemma 2). Hence, since $\mathcal{B}$ is d-closed, we have $q_1 \xrightarrow{h_2\varphi(a)h_6}_{\mathcal{B}} q_6$, i.e., $s_1'(h_2\varphi(a)h_6)s_2' \in \mathrm{L}(\mathcal{B})$. By continuing in this way, we finally arrive at a reduced sequence $s' \in \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)$ such that $s \approx s'$. $\square$

Let us now continue with the proof for point (3) in Theorem 3. Assume that H has the LERF property. Let $g \in G$, let $K \leq G$ be some finitely generated subgroup of G, and let us suppose that

$$g \notin K. \quad (75)$$

By Proposition 3, there exists a partitionned, $\approx$-compatible, and $\sim$-saturated fta

$$\mathcal{B} = \langle \mathcal{L}_{\mathcal{B}}, \mathsf{Q}_{\mathcal{B}}, \delta_{\mathcal{B}}, \mathsf{I}_{\mathcal{B}}, \mathsf{T}_{\mathcal{B}} \rangle$$

with labelling set $\mathcal{S}$ such that

$$\pi_G^{-1}(K) \cap \mathrm{Red}(H, t) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H, t). \tag{76}$$

Claim 14. The fta $\mathcal{B}$ can be chosen to be d-closed.

Let us look at the fta $\mathcal{A}_2$ produced in the proof of Claim 12 by equation (67). This automaton is not partitionned but has both properties (73) and (74). The partitionned automaton $\mathcal{A}_3 = \mathcal{A}_2 \times \mathcal{G}_6$ still satisfies (73) and (74), since for all $q_1, \ldots, q_6$ fulfilling the hypothesis of (73) and (74), respectively, the edge $(\tau(q_1), H, \tau(q_6))$ belongs to $\mathcal{E}_6$. Let us look finally at the fta $\mathcal{B}$ obtained from $\mathcal{A}_3$ by the construction in the proof of point (1) of Proposition 1. Suppose that the hypothesis of implication (73) is fulfilled, i.e.,

$$q_1 g_1 \xrightarrow{h_2}_{\mathcal{B}} q_2 g_2 \xrightarrow{t^{-1}}_{\mathcal{B}} q_3 g_3 \xrightarrow{a}_{\mathcal{B}} q_4 g_4 \xrightarrow{t}_{\mathcal{B}} q_5 g_5 \xrightarrow{h_6}_{\mathcal{B}} q_6 g_6 \tag{77}$$

where $q_i \in \mathsf{Q}_{\mathcal{A}_3}, g_i \in \gamma(q_i)$, $\tau_{\mathcal{B}}(q_1 g_1) \neq \tau_{\mathcal{B}}(q_2 g_2)$, $\tau_{\mathcal{B}}(q_5 g_5) \neq \tau_{\mathcal{B}}(q_6 g_6)$, and hence $\tau_{\mathcal{A}_3}(q_1) \neq \tau_{\mathcal{A}_3}(q_2)$, $\tau_{\mathcal{A}_3}(q_5) \neq \tau_{\mathcal{A}_3}(q_6)$. Assembling five of the diagrams described in Figure 4, we obtain from (77) in the fta $\mathcal{A}_3$ the path

$$q_1 \xrightarrow{g_1 h_2 g_2^{-1}}_{\mathcal{A}_3} q_2 \xrightarrow{t^{-1}}_{\mathcal{A}_3} q_3 \xrightarrow{g_3 a g_4^{-1}}_{\mathcal{A}_3} q_4 \xrightarrow{t}_{\mathcal{A}_3} q_5 \xrightarrow{g_5 h_6 g_6^{-1}}_{\mathcal{A}_3} q_6,$$

where

$$\varphi^{-1}(g_2) = g_3 \quad \text{and} \quad \varphi(g_4) = g_5. \tag{78}$$

Since $\mathcal{A}_3$ is d-closed, we get

$$q_1 \xrightarrow{(g_1 h_2 g_2^{-1})\varphi(g_3 a g_4^{-1})(g_5 h_6 g_6^{-1})}_{\mathcal{A}_3} q_6,$$

i.e., with (78),

$$q_1 \xrightarrow{(g_1 h_2)\varphi(a)(h_6 g_6^{-1})}_{\mathcal{A}_3} q_6.$$

Using the left diagram in Figure 4, we obtain

$$q_1 g_1 \xrightarrow{h_2 \varphi(a) h_6}_{\mathcal{B}} q_6 g_6.$$

$$
\begin{tikzcd}
q_1g_1 \arrow[r, "h_2"] & q_2g_2 \arrow[r, "t^{-1}"] & q_3g_3 \arrow[r, "a"] & q_4g_4 \arrow[r, "t"] & q_5g_5 \arrow[r, "h_6"] & q_6g_6 \\
q_1 \arrow[u, "g_1"] \arrow[r, "g_1h_2g_2^{-1}"] \arrow[rrrrr, bend right, "(g_1h_2g_2^{-1})\varphi(g_3ag_4^{-1})(g_5h_6g_6^{-1})"'] \arrow[d, "g_1"'] & q_2 \arrow[u, "g_2"] \arrow[r, "t^{-1}"] & q_3 \arrow[u, "g_3"] \arrow[r, "g_3ag_4^{-1}"] & q_4 \arrow[u, "g_4"] \arrow[r, "t"] & q_5 \arrow[u, "g_5"] \arrow[r, "g_5h_6g_6^{-1}"] & q_6 \arrow[u, "g_6"'] \arrow[d, "g_6"] \\
q_1g_1 \arrow[rrrrr, "h_2\varphi(a)h_6"] & & & & & q_6g_6
\end{tikzcd}
$$

Fig. 5. $\mathcal{B}$ is d-closed

Thus, implication (73) holds, see also Figure 5. A similar verification can be done for implication (74). Claim 14 is thus established.

Let

$$\mathcal{L}_{\mathcal{B}} \cap \mathcal{P}(H) = \{F_0r_0, \ldots, F_\ell r_\ell\}$$

be the set of H-labels of $\mathcal{B}$, where $F_i \leq H$ is finitely generated and $r_i \in H$. Let

$$s = h_0t^{\alpha_1}h_1 \cdots t^{\alpha_n}h_n \in \mathrm{Red}(H, t)$$

be a reduced sequence such that

$$\pi_G(s) = g \notin K. \tag{79}$$

By (76), (79), and the fact that s is reduced, we have

$$s \notin \mathrm{L}(\mathcal{B}). \tag{80}$$

For every subgroup $\widehat{H} \leq H$ let us consider the following properties:

$$\widehat{H} \cap A = 1 = \widehat{H} \cap B \tag{81}$$

$$\forall i \in \{0, \ldots, \ell\}\ \forall c \in A \cup B : c \in \widehat{H}F_ir_i \iff c \in F_ir_i \tag{82}$$

$$\forall i \in \{0, \ldots, \ell\}, j \in \{0, \ldots, n\} : h_j \in \widehat{H}F_ir_i \iff h_j \in F_ir_i \tag{83}$$

$$\forall j \in \{0, \ldots, n\} : h_j \in \widehat{H}A \Leftrightarrow h_j \in A \quad \wedge \quad h_j \in \widehat{H}B \Leftrightarrow h_j \in B \tag{84}$$

Let us notice that every property among (81)–(84) is *hereditary* in the sense that if $\widehat{H}_1 \leq \widehat{H}_2 \leq H$ and $\widehat{H}_2$ meets the property, then $\widehat{H}_1$

meets the same property as well. As H meets the LERF property, for each value of $i \in \{0, \ldots, \ell\}, j \in \{0, \ldots, n\}$, and each property P among the above properties (81)–(84), there exists some normal subgroup of finite index $\widehat{H}(i,j,P) \trianglelefteq H$ fulfilling the given property. Let us define

$$\widehat{H} = \bigcap_{i,j,P} \widehat{H}(i,j,P).$$

This subgroup $\widehat{H}$ is normal, has finite index in H, and fulfills simultaneously all the properties (81)–(84). Let us set

$$G' = \langle H, t;\ \widehat{h} = 1\,(\widehat{h} \in \widehat{H}),\ t^{-1}at = \varphi(a)\,(a \in A)\rangle.$$

Let $\pi' : H \to H' = H/\widehat{H}$ be the canonical projection. Due to (81), this morphism π' fulfills condition (18) from Section 3.1. Thus, the group G' is also the HNN-extension of $H' = H/\widehat{H}$ with stable letter t, associated subgroups A, B, and the partial isomorphism φ:

$$G' = \langle H/\widehat{H}, t; t^{-1}at = \varphi(a)(a \in A)\rangle.$$

Let us use the notation from Section 3.1. Let $\pi'_*(\mathcal{B})$ be the $\sim'$-saturated fta from Lemma 8, which satisfies

$$\mathrm{L}(\pi'_*(\mathcal{B})) = \pi'_*(\mathrm{L}(\mathcal{B})).$$

Let us define:

$$s' = \pi'_*(s) = \pi'(h_0)t^{\alpha_1}\pi'(h_1)\cdots t^{\alpha_n}\pi'(h_n). \tag{85}$$

Since s is a reduced sequence in $H * \{t, t^{-1}\}^*$, condition (84) implies

$$s' \in \mathrm{Red}(H', t). \tag{86}$$

By (80) and (83) we have

$$s' \notin \mathrm{L}(\pi'_*(\mathcal{B})). \tag{87}$$

By (76) and the fact that $\mathcal{B}$ is $\approx$-compatible, we get

$$\begin{aligned}
\pi_G(\mathrm{L}(\mathcal{B})) &= \pi_G([\mathrm{L}(\mathcal{B})]_\approx)\\
&= \pi_G([\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t)]_\approx)\\
&= \pi_G(\mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H,t))\\
&= \pi_G(\pi_G^{-1}(K) \cap \mathrm{Red}(H,t))\\
&= K,
\end{aligned}$$

and thus,

$$\pi_{G'}^{-1}(\pi'(K)) = \pi_{G'}^{-1}(\pi'(\pi_G(\mathrm{L}(\mathcal{B})))).$$

The commutative diagram in Figure 1 yields

$$\pi_{G'}^{-1}(\pi'(K)) = \pi_{G'}^{-1}(\pi_{G'}(\pi'_*(\mathrm{L}(\mathcal{B})))) = [\pi'_*(\mathrm{L}(\mathcal{B}))]_{\approx'} = [\mathrm{L}(\pi'_*(\mathcal{B}))]_{\approx'}.$$

By Claim 14 and (82), $\pi'_*(\mathcal{B})$ is d-closed and hence $\approx'$-compatible by Lemma 9. Hence,

$$\begin{aligned}\pi_{G'}^{-1}(\pi'(K)) \cap \mathrm{Red}(H',t) &= [\mathrm{L}(\pi'_*(\mathcal{B}))]_{\approx'} \cap \mathrm{Red}(H',t)\\ &= [\mathrm{L}(\pi'_*(\mathcal{B})) \cap \mathrm{Red}(H',t)]_{\approx'} \cap \mathrm{Red}(H',t)\\ &= [\mathrm{L}(\pi'_*(\mathcal{B})) \cap \mathrm{Red}(H',t)]_{\sim'}\\ &= \mathrm{L}(\pi'_*(\mathcal{B})) \cap \mathrm{Red}(H',t),\end{aligned}$$

where the last equality holds since $\pi'_*(\mathcal{B})$ is $\sim'$-saturated by Lemma 8. Together with (86) and (87) we get

$$s' \notin \pi_{G'}^{-1}(\pi'(K)).$$

This shows that $\pi_{G'}(s') \notin \pi'(K)$, i.e., by (85), $\pi_{G'}(\pi'_*(s)) \notin \pi'(K)$. The commutative diagram in Figure 1 implies $\pi'(\pi_G(s)) \notin \pi'(K)$, i.e., by (79),

$$\pi'(g) \notin \pi'(K). \tag{88}$$

The group G' is the fundamental group of a graph of groups (with just one vertex and one edge) with finite vertex groups (namely $H/\widehat{H}$), hence it is a virtually free group, i.e., it has a free subgroup of finite index, see e.g. [DD90, Corollary IV.1.9, p.104]. By the result of [Hal49] every free group is LERF and by [Sco78, Lemma 1.1, p. 556] every group with a finite index subgroup which is LERF is itself LERF. Hence G' is LERF. In particular, (88) implies the existence of a finite group G'' and a homorphism $\pi'' : G' \to G''$ such that

$$\pi''(\pi'(g)) \notin \pi''(\pi'(K)). \tag{89}$$

This proves that G is LERF. $\square$

5 Amalgamated products

In the rest of this paper, we will prove results similar to those for HNN-extensions from Section 4.

Let us consider two monoids H_1 and H_2, two finite *subgroups* $A_1 \leq H_1, A_2 \leq H_2$, and an isomorphism $\varphi : A_1 \to A_2$. The corresponding amalgamated product

$$G = \langle H_1, H_2; a = \varphi(a)(a \in A_1)\rangle$$

is defined by $G = (H_1 * H_2)/\approx$, where $\approx$ is the congruence on the free product $H_1 * H_2$ generated by the identities $a = \varphi(a)$ for $a \in A_1$. An *a-sequence* is an element $s \in H_1 * H_2$; it has a unique decomposition of the form

$$s = h_0 k_1 h_1 \cdots k_i h_i \cdots k_n h_n, \tag{90}$$

where $n \geq 0, h_1, \ldots, h_{n-1} \in H_2 \setminus 1$, $k_1, \ldots, k_n \in H_1 \setminus 1$, and $h_0, h_n \in H_2$. A *reduced a-sequence* is an element $s \in H_1 * H_2$ of the form

$$s = h_0 k_1 h_1 \cdots k_i h_i \cdots k_n h_n, \tag{91}$$

where $n \geq 0, h_1, \ldots, h_{n-1} \in H_2 \setminus A_2$, $k_1, \ldots, k_n \in H_1 \setminus A_1$ and $h_0, h_n \in H_2$. We denote by $\mathrm{Red}(H_1, H_2)$ the set of all reduced a-sequences.

It is well-known that G can be embedded into the HNN-extension

$$\widehat{G} = \langle H_1 * H_2, t; t^{-1} a t = \varphi(a)(a \in A_1)\rangle$$

by the map with

$$h_1 \mapsto t^{-1} h_1 t \text{ for } h_1 \in H_1 \text{ and } h_2 \mapsto h_2 \text{ for } h_2 \in H_2 \tag{92}$$

(see [LS77, Th. 2.6. p. 187]).

Assume now that $H_1 = X_1^*/\equiv_{H_1}$ and $H_2 = X_2^*/\equiv_{H_2}$ for generating sets X_1 and X_2 such that $X_1 \cap X_2 = \emptyset$. Then $G = (X_1 \cup X_2)^*/\equiv$ where $\equiv$ is the smallest congruence over the free monoid $(X_1 \cup X_2)^*$ such that

$$\begin{gathered} \equiv_{H_1} \cup \equiv_{H_2} \subseteq \equiv \text{ and} \\ a \equiv \varphi(a) \text{ for all } a \in A_1. \end{gathered}$$

Let

$$\pi : (X_1 \cup X_2)^* \to G$$

denote the canonical morphism defined by $\pi(w) = [w]_\equiv$ for $w \in (X_1 \cup X_2)^*$. This morphism naturally factorizes as

$$\pi = \pi_a \circ \pi_G$$

where

$$\begin{aligned} \pi_a &: (X_1 \cup X_2)^* \to H_1 * H_2 \text{ and} \\ \pi_G &: H_1 * H_2 \to G. \end{aligned}$$

The kernel of π_G coincides with the congruence $\approx$ over $H_1 * H_2$ defined above.

We define the binary relation $\stackrel{1}{\sim}$ over $H_1 * H_2$ as follows: For all $s, s' \in H_1 * H_2$ let $s \stackrel{1}{\sim} s'$ if and only if there are sequences s_1 and s_2 such that either

$$\exists h, h' \in H_1, a, a' \in A_1 : \left\{ \begin{array}{l} s = s_1 h s_2 \ \wedge \ ha' = ah' \\ s' = s_1 \varphi(a) h' \varphi(a') s_2 \end{array} \right\}$$

or

$$\exists h, h' \in H_2, a, a' \in A_2 : \left\{ \begin{array}{l} s = s_1 h s_2 \ \wedge \ ha' = ah' \\ s' = s_1 \varphi^{-1}(a) h' \varphi^{-1}(a') s_2 \end{array} \right\}$$

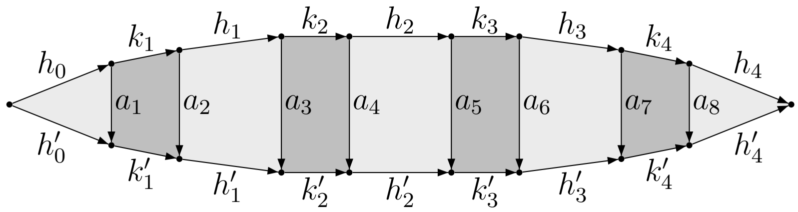
Let us denote by $\sim \subseteq \approx$ the congruence over $H_1 * H_2$ generated by $\stackrel{1}{\sim}$; it coincides with the reflexive and transitive closure of $\stackrel{1}{\sim}$. Equivalently, if

$$\begin{aligned} s &= h_0 k_1 h_1 k_2 \cdots h_{n-1} k_n h_n \text{ and} \\ s' &= h'_0 k'_1 h'_1 k'_2 \cdots h'_{m-1} k'_m h'_m \end{aligned}$$

(with $n, m \geq 0$, and $k_1, \ldots, k_n, k'_1, \ldots, k'_m \in H_1$ $h_0, \ldots, h_n, h'_0, \ldots, h'_m \in H_2$) are reduced sequences, then $s \sim s'$ if and only if $n = m$ and there exist $a_1, \ldots, a_{2n} \in A_1$ such that:

- $h_i \varphi(a_{2i+1}) = \varphi(a_{2i}) h'_i$ in H_2 for $0 \leq i \leq n$ (here we set $a_0 = a_{2n+1} = 1$)
- $k_i a_{2i} = a_{2i-1} k'_i$ in H_1 for $1 \leq i \leq n$

In other words, there exists a Van Kampen diagram of the following kind, where $n = m = 4$. Here, we omit the $\varphi(a_i)$-labelled edge, which runs parallel to the a_i-labelled vertical edge ($1 \le i \le 2n$).[2] Light-shaded (resp. dark-shaded) areas represent relation in H_2 (resp. relations in H_1.



The elements $a_1, \dots, a_{2n}$ in such a diagram are also called *connecting elements*.

The set $\mathrm{Red}(H_1, H_2)$ is again saturated by the congruence $\sim$, i.e., $s \sim s'$ implies $s \in \mathrm{Red}(H_1, H_2) \Leftrightarrow s' \in \mathrm{Red}(H_1, H_2)$. Here it is important that A_1 and A_2 are groups. With these definitions of the relations $\approx$ and $\sim$ on the free product $H_1 * H_2$ and the new definition of a reduced sequence, Lemma 1 again holds.

6 Finite automata for amalgamated products

We fix throughout this section two monoids H_1, H_2, two finite *subgroups* $A_1 \le H_1, A_2 \le H_2$, an isomorphism $\varphi : A_1 \to A_2$ and the corresponding amalgamated product

$$G = \langle H_1, H_2; a = \varphi(a)(a \in A_1)\rangle.$$

6.1 Finite a-automata

The classical embedding (92) leads naturally to the following adaptations of the definitions from Section 4 to the case of an amalgamated product.

Let $\mathcal{F}_i \subseteq \mathcal{P}(H_i)$ be a set of subsets of H_i ($i \in \{1, 2\}$) such that every singleton $\{a\}$ with $a \in A_i$ belongs to $\mathcal{F}_i$. A finite a-automaton over $H_1 * H_2$, with labelling sets $(\mathcal{F}_1, \mathcal{F}_2)$ is a 5-tuple

$$\mathcal{A} = \langle \mathsf{\mathcal{L}}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T}\rangle,$$

[2] Alternatively, one may assume that that $A_1 = A_2 = A$, $H_1 \cap H_2 = A$, where A is a subgroup of H_1 and H_2, and that φ is the identity map. Then one obtains the above Van Kampen diagram.

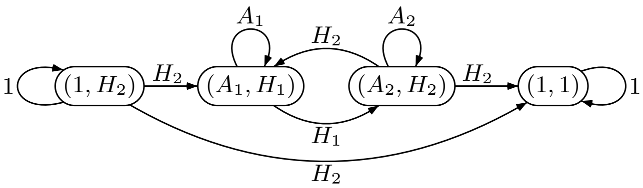


Fig. 6. The graph $\mathcal{G}_4 = (\mathcal{T}_4, \mathcal{E}_4)$

where $\mathcal{L}$ is finite and

$$\mathcal{L} \subseteq \mathcal{F}_1 \cup \mathcal{F}_2,$$

Q is a finite set of states, $\mathsf{I} \subseteq \mathsf{Q}$ is the set of initial states, $\mathsf{T} \subseteq \mathsf{Q}$ is the set of terminal states, and δ (the set of transitions) is a subset of $\mathsf{Q} \times \mathcal{L} \times \mathsf{Q}$ such that for every state $q \in \mathsf{Q}$ there exists $L \in \mathcal{L}$ with $(q, L, q) \in \delta$ and $1 \in L$. For the last point, 1 denotes both the neutral element of H_1 and H_2, The representation map $\mu_{\mathcal{A}}$ is defined as follows. First define for every $h \in H_1 \cup H_2$:

$$\mu_{\mathcal{A},0}(h) = \{(q, r) \in Q \times Q \mid \exists L \in \mathcal{L} : (q, L, r) \in \delta \wedge h \in L\}. \quad (93)$$

Now, for every $s \in H_1 * H_2$ of the form (90), let

$$\mu_{\mathcal{A}}(s) = \mu_{\mathcal{A},0}(h_0) \circ \mu_{\mathcal{A},0}(k_1) \circ \mu_{\mathcal{A},0}(h_1) \circ \cdots \circ \mu_{\mathcal{A},0}(k_n) \circ \mu_{\mathcal{A},0}(h_n). \quad (94)$$

The set $\mathrm{L}(\mathcal{A}) \subseteq H_1 * H_2$ recognized by $\mathcal{A}$ is defined as for HNN-extensions.

Let us introduce a finite set of 4 types, which will induce a partition of the set of states of a finite a-automaton:

$$\mathcal{T}_4 = \{(A_1, H_1), (A_2, H_2), (1, H_2), (1, 1)\}. \quad (95)$$

We define a graph $\mathcal{G}_4 = (\mathcal{T}_4, \mathcal{E}_4)$ by:

$$\begin{aligned}\mathcal{E}_4 =& \{((C, D), C, (C, D)) \mid (C, D) \in \mathcal{T}_4\} \\ & \cup \{((A_i, H_i), H_i, (A_{3-i}, H_{3-i})) \mid i \in \{1, 2\}\} \\ & \cup \{((A_2, H_2), H_2, (1, 1)), ((1, H_2), H_2, (A_1, H_1)), ((1, H_2), H_2, (1, 1))\}.\end{aligned}$$

The graph $\mathcal{G}_4$ is represented in Figure 2. An analogous graph $\mathcal{R}_4 = (\mathcal{T}_4, \mathcal{E}'_4)$ is defined by

$$\begin{aligned}\mathcal{E}'_4 =& \mathcal{E}_4 \setminus \{((A_1, H_1), H_1, (A_2, H_2)), ((A_2, H_2), H_2, (A_1, H_1))\} \\ & \cup \{((A_1, H_1), H_1 \setminus A_1, (A_2, H_2)), ((A_2, H_2), H_2 \setminus A_2, (A_1, H_1))\},\end{aligned}$$

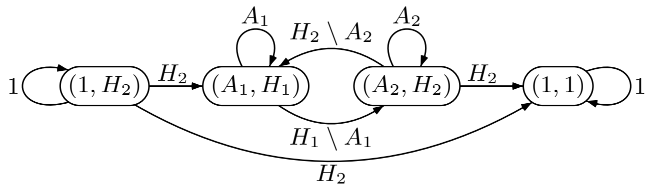


Fig. 7. The graph $\mathcal{R}_4 = (\mathcal{T}_4, \mathcal{E}'_4)$

see Figure 7. A *partitionned finite a-automaton* with labelling sets $(\mathcal{F}_1, \mathcal{F}_2)$ is a 6-tuple

$$\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle,$$

where $\mathcal{L}$, Q, δ, I, and T are as above and $\tau : \mathsf{Q} \to \mathcal{T}_4$ assigns a type with every state such that:

$$\forall (q, L, r) \in \delta : \{q\} \times L \times \{r\} \subseteq \widehat{\mathcal{E}_4}$$
$$\tau(\mathsf{I}) = \{(1, H_2)\} \quad \wedge \quad \tau(\mathsf{T}) = \{(1, 1)\}$$

The reader might be surprised by the asymetry of this definition: it corresponds to our choice for the definition of an a-sequence in (90), where we require $h_0, h_n \in H_2$. This choice was made arbitrarily; we could have also required $h_0, h_n \in H_1$.

Let us consider the equivalences $\sim$ and $\approx$ over $H_1 * H_2$ defined in Section 5. The notions of $\sim$-saturated and $\approx$-compatible finite a-automata are defined as in Definition 4, but with the equivalences $\approx$ and $\sim$ being those defined over $H_1 * H_2$.

The notion of a *deterministic* and partitionned finite a-automaton $\mathcal{A}$ with labelling sets $(\mathcal{F}_1, \mathcal{F}_2)$ is still defined by the properties (30) and (31). For every $q \in \mathsf{Q}$, and $\Theta \in \mathcal{T}_4$ we again define the set $\mathrm{L}^1(q, \Theta) \subseteq H_1 \cup H_2$ by (32). Then, a partitionned finite a-automaton $\mathcal{A}$ is *complete* if

$$\forall (\Theta, C, \Theta') \in \mathcal{E}'_4 \forall q \in \tau^{-1}(\Theta) : C \subseteq \mathrm{L}^1(q, \Theta').$$

6.2 Normalization of automata

The following two propositions can be now proved in exactly the same way as the corresponding statements for HNN-extensions (Proposition 1 and 2).

Proposition 4. *The following holds:*

(1) From a given partitionned and $\approx$-compatible finite a-automaton $\mathcal{A}$ with labelling set $(\mathcal{F}_1, \mathcal{F}_2)$ one can effectively construct a partitionned, $\approx$-compatible, $\sim$-saturated, and unitary fta $\mathcal{B}$ with labelling set $(\mathcal{F}_1', \mathcal{F}_2')$, where

$$\mathcal{F}_i' = \mathcal{F} \cup \{aFa' \mid F \in \mathcal{F}_i, a, a' \in A_i\},$$

and such that

$$[\mathrm{L}(\mathcal{A})]_{\approx} \cap \mathrm{Red}(H_1, H_2) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H_1, H_2).$$

(2) From a given partitionned, $\approx$-compatible, and $\sim$-saturated finite a-automaton $\mathcal{B}$ with labelling set $(\mathcal{F}_1, \mathcal{F}_2)$ one can effectively construct a partitionned, $\sim$-saturated, deterministic, and complete finite a-automaton $\mathcal{C}$ with labelling set $(\mathcal{F}_1', \mathcal{F}_2')$, where $\mathcal{F}_i' = \mathrm{Bool}(\mathcal{F}_i)$, and such that

$$[\mathrm{L}(\mathcal{A})]_{\approx} \cap \mathrm{Red}(H_1, H_2) = \mathrm{L}(\mathcal{C}).$$

Proposition 5. *Let $K \subseteq G$ The following are equivalent:*

(1) K is a rational subset of G.
(2) $K = \pi(R)$ for some rational subset $R \subseteq (X_1 \cup X_2)^*$.
(3) $\pi_G^{-1}(K) \cap \mathrm{Red}(H_1, H_2) = \mathrm{L}(\mathcal{B}) \cap \mathrm{Red}(H_1, H_2)$ for some partitionned, $\approx$-compatible, and $\sim$-saturated finite a-automaton $\mathcal{B}$ with labelling set $(\mathrm{Rat}(H_1), \mathrm{Rat}(H_2))$.

6.3 Transfer results for rational sets

Theorem 4. *The following holds:*

(1) If $\mathrm{Rat}(H_1)$ and $\mathrm{Rat}(H_2)$ are closed under intersection, then also $\mathrm{Rat}(G)$ is closed under intersection.
(2) If $\mathrm{Rat}(H_1)$ and $\mathrm{Rat}(H_2)$ are closed under complement, then also $\mathrm{Rat}(G)$ is closed under complement.
(3) If the membership problem is decidable for languages in $\mathrm{Rat}(H_1)$ and $\mathrm{Rat}(H_2)$, then the same is true for languages in $\mathrm{Rat}(G)$.
(4) If the emptiness problem is decidable for languages in the boolean closure of $\mathrm{Rat}(H_1)$ and $\mathrm{Rat}(H_2)$, then the same is true for languages in the boolean closure of $\mathrm{Rat}(G)$.

Statement (3) is covered by [KSS06, Theorem 5.1], whereas (1), (2), and (4) are new to the knowledge of the authors.

6.4 Transfer results for finitely generated subgroups

We assume throuthout this section that H_1 and H_2 are groups.

Theorem 5. *The following holds:*

(1) If H_1 and H_2 have a decidable generalized word problem, then also G has a decidable generalized word problem.
(2) If H_1 and H_2 have the Howson property, then also G has the Howson property.
(3) If H_1 and H_2 have the LERF property, then also G has the LERF property.

Statement (1) is covered by [KWM05, Corollary 5.15], statement (2) was shown in [Bez98,KS70], and statement (3) was shown in [AG73].

Acknowledgments We would like to thank Volker Diekert and Benjamin Steinberg for many useful comments on the topic of this paper.

References

[AG73] R. B. J. T. Allenby and R. J. Gregorac. On locally extended residually finite groups. In *Conference on Group Theory (Univ. Wisconsin-Parkside, Kenosha, Wis., 1972)*, number 319 in Lecture Notes in Mathematics, pages 9–17. Springer, Berlin, 1973.

[AS75] Anatolij W. Anissimow and Franz D. Seifert. Zur algebraischen Charakteristik der durch kontext-freie Sprachen definierten Gruppen. *Elektron. Informationsverarbeit. Kybernetik*, 11(10–12):695–702, 1975.

[Bez98] V. N. Bezverkhniĭ. On the intersection of subgroups in HNN-groups. *Fundam. Prikl. Mat.*, 4(1):199–222, 1998.

[BJW82] R.V. Book, M. Jantzen, and C. Wrathall. Monadic Thue systems. *TCS 19*, pages 231–251, 1982.

[DJ91] N. Dershowitz and J.P. Jouannaud. Rewrite systems. In *Handbook of theoretical computer science*, vol.B, Chapter 2, pages 243-320. Elsevier, 1991.

[DD90] W. Dicks and M.J. Dunwoody. *Groups acting on graphs.* Cambridge University Press, 1990.

[Die05] V. Diekert. *Personnal communication.* 2005.

[Hal49] Marshall Hall, Jr. Coset representations in free groups. *Trans. Amer. Math. Soc.*, 67:421–432, 1949.

[HNN49] Graham Higman, B. H. Neumann, and Hanna Neumann. Embedding theorems for groups. *J. London Math. Soc.*, 24:247–254, 1949.

[KS70] A. Karrass and D. Solitar. The subgroups of a free product of two groups with an amalgamated subgroup. *Transactions of the American Mathematical Society*, 150:227–255, 1970.

[KS71] A. Karrass and D. Solitar. Subgroups of HNN groups and groups with one defining relation. *Canadian Journal of Mathematics*, 23:627–643, 1971.

[KSS06] Mark Kambites, Pedro V. Silva, and Benjamin Steinberg. On the rational subset problem for groups. Technical report, arXiv.org, 2006. `http://arxiv.org/abs/math.GR/0602454`.

[KWM05] Ilya Kapovich, Richard Weidmann, and Alexei Myasnikov. Foldings, graphs of groups and the membership problem. *International Journal of Algebra and Computation*, 15(1):95–128, 2005.

[LS77] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory.* Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89.

[LS06a] M. Lohrey and G. Sénizergues. Equations in HNN-extensions. Manuscript, 2006.

[LS06b] M. Lohrey and G. Sénizergues. Positive theories of HNN-extensions and amalgamated free products. Manuscript, 2006.

[MR04] V. Metaftsis and E. Raptis. Subgroup separability of graphs of abelian groups. *Proc. Amer. Math. Soc.*, 132(7):1873–1884 (electronic), 2004.

[Reu85] Christophe Reutenauer. Sur les semi-groupes vérifiant le théorème de Kleene. *RAIRO Inform. Théor.*, 19(3):281–291, 1985.

[Sak87] Jacques Sakarovitch. On regular trace languages. *Theoretical Computer Science*, 52:59–75, 1987.

[Sak92] Jacques Sakarovitch. The "last" decision problem for rational trace languages. In I. Simon, editor, *Proceedings of the 1st Latin American Symposium on Theoretical Informatics (LATIN'92)*, number 583 in Lecture Notes in Computer Science, pages 460–473. Springer, 1992.

[Sco78] Peter Scott. Subgroups of surface groups are almost geometric. *J. London Math. Soc. (2)*, 17(3):555–565, 1978.

[Sén94] G. Sénizergues. Formal languages and word-rewriting. In H. Comon and J.P. Jouannaud, editors, *Term Rewriting, Font-Romeu 93, advanced course*, pages 75–94. Lect. Notes. Comput. Sci. 909, 1994.

A Normal forms

We check here that the classical notion of normal form for HNN-extensions of *groups* adapts to the case of *monoids*, provided that A, B are subgroups of the base monoid. This will result in a proof of Lemma 1.

Let us fix a monoid H, subgroups A, B of H and an isomorphism $\varphi : A \to B$, as considered in section 3 and let G be the corresponding HNN-extension defined by (7).

For every subgroup C of H, the left-congruence $\equiv_C$ is defined by:

$$h \equiv_C h' \Leftrightarrow \exists c \in C,\, h = h' \cdot c.$$

The classes for this equivalence are the sets hC for $h \in H$. Let us choose some systems of representatives R_A, (resp. R_B), for the equivalence $\equiv_A$ (resp. $\equiv_B$). We suppose that $1 \in R_A \cap R_B$. We define now the set of normal forms.

Definition 9. *A* normal form *is an element of* $H * \{t, t^{-1}\}^*$

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n, \tag{96}$$

where $n \geq 0$*, such that, for every* $i \in [1, n]$
(i) h_n *is an arbitrary element of* H
(ii) if $\alpha_i = +1$ *then* $h_{i-1} \in R_A$
(iii)if $\alpha_i = -1$ *then* $h_{i-1} \in R_B$
(iv) if $\alpha_i = -\alpha_{i+1}$ *then* $h_i \neq 1$.

One can notice that, due to conditions (ii) (iii) (iv), every normal form is a *reduced* sequence. We denote by $\mathcal{N}$ the set of all normal forms. We define a right-action $\odot$ of the monoid $H * \{t, t^{-1}\}^*$ over $\mathcal{N}$ in the following way:
for every $h \in H$ and s of the form (96)

$$s \odot h = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} (h_n \cdot h)$$

the right-action of t on s of the form (96) is defined through 3 cases:
case $t\mathbf{1}$:$n = 0$ or $\alpha_n = +1, h_n = r_n a$ with $r_n \in R_A, a \in A$

$$s \odot t = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} r_n t \varphi(a)$$

case $t\mathbf{2}$:$\alpha_n = -1, h_n = r_n a$ with $r_n \in R_A - \{1\}, a \in A$

$$s \odot t = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} r_n t \varphi(a)$$

case $t\mathbf{3}$:$\alpha_n = -1, h_n \in A$

$$s \odot t = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_{n-1}} (h_{n-1} \cdot \varphi(a))$$

the right-action of t^{-1} on s of the form (96) is defined similarly through 3 cases:
case $t^{-1}\mathbf{1}$:$n = 0$ or $\alpha_n = -1, h_n = r_n b$ with $r_n \in R_B, b \in B$
case $t^{-1}\mathbf{2}$:$\alpha_n = +1, h_n = r_n b$ with $r_n \in R_B - \{1\}, b \in B$
case $t^{-1}\mathbf{3}$:$\alpha_n = +1, h_n \in B$
One can easily check that, for every $s \in \mathcal{N}$:

$$s \odot (t t^{-1}) = s \odot (t^{-1} t) = s. \tag{97}$$

Let us compute the sequence $s \odot tbt^{-1}$, when s fulfills case $t1$ or $t2$:

$$\begin{aligned}
s \odot tbt^{-1} &= (h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} r_n t\varphi(a)) \quad \odot (bt^{-1}) \\
&= (h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} r_n t(\varphi(a) \cdot b)) \quad \odot t^{-1} \\
&= h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} (r_n \varphi^{-1}(\varphi(a) \cdot b)) \\
&= h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} (r_n a \varphi^{-1}(b)) \\
&= h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} (h_n \varphi^{-1}(b)) \\
&= (h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n) \odot \varphi^{-1}(b).
\end{aligned}$$

When the sequence s fulfills case $t3$ we obtain:

$$\begin{aligned}
s \odot tbt^{-1} &= (h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_{n-1}} (h_{n-1} \varphi(h_n))) \quad \odot (bt^{-1}) \\
&= (h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_{n-1}} (h_{n-1} \varphi(h_n) b)) \quad \odot t^{-1} \\
&= h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_{n-1}} h_{n-1} t^{-1} (h_n \varphi^{-1}(b)) \\
&= (h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n) \odot \varphi^{-1}(b).
\end{aligned}$$

We have thus established that, for every $s \in \mathcal{N}$:

$$s \odot (tbt^{-1}) = s \odot \varphi^{-1}(b). \tag{98}$$

The set of relations $\{(tt^{-1}, 1), (t^{-1}t, 1)\} \cup \{tbt^{-1}, \varphi^{-1}(b) \mid b \in B\}$ generates the congruence $\approx$ over $H * \{t, t^{-1}\}^*$. Relations (97)(98) thus show that the action $\odot$ induces a right-action of the monoid G over $\mathcal{N}$ that we still denote by $\odot$.

Lemma 10. *For every $s \in \mathcal{N}$, $1 \odot \pi_G(s) = s$.*

This lemma can be proved by induction over the integer n in the description (96) of a normal form s.

Theorem 6. *The set $\mathcal{N}$ is a system of representatives for the congruence $\approx$.*

Proof. 1- Let us show that every sequence s is equivalent, modulo $\approx$, with some normal form.
As we already saw in section 3 every t-sequence is equivalent, modulo $\approx$, with some *reduced* sequence. Let s be some reduced sequence given by (14). Let us show, by induction over n, that s is $\sim$-equivalent to some normal form.

$n = 0$: Every sequence with $n = 0$ is a normal form.
$n = m+1$: By induction hypothesis, there exist $h'_0, \cdots, h'_i, \cdots, h'_m \in H$ such that

$$h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_m} h_m \sim h'_0 t^{\alpha_1} h'_1 \cdots t^{\alpha_i} h'_i \cdots t^{\alpha_m} h'_m.$$

Suppose that α_n=1. Let us decompse $h'_m = r_m a$ with $r_m \in R_A, a \in A$, and let us set

$$s' = h'_0 t^{\alpha_1} h'_1 \cdots t^{\alpha_i} h'_i \cdots t^{\alpha_m} r_m t^{\alpha_n} (\varphi(a) \cdot h_n).$$

We have $s \sim s' \in \mathcal{N}$.
If α_n=-1, we decompose $h'_m = r_m b$ with $r_m \in R_B, b \in B$, we set

$$s' = h'_0 t^{\alpha_1} h'_1 \cdots t^{\alpha_i} h'_i \cdots t^{\alpha_m} r_m t^{\alpha_n} (\varphi^{-1}(b) \cdot h_n),$$

and we again claim that $s \sim s' \in \mathcal{N}$.
2- Let us show that two different normal forms are not equivalent modulo $\approx$. Let $s, s' \in \mathcal{N}$ and $s \neq s'$. By lemma 10,

$$1 \odot \pi_G(s) = s, \;\; 1 \odot \pi_G(s') = s',$$

hence $\pi_G(s) \neq \pi_G(s')$ i.e. s, s' are not equivalent modulo $\approx$.

We are ready to give the announced
proof of lemma 1.
Suppose that s_1, s_2 are reduced t-sequences and that $s_1 \approx s_1$. By the above proof of point 1 of theorem 6, there exist $s'_1, s'_2 \in \mathcal{N}$ such that $s_i \sim s'_i$ (for $i \in \{1, 2\}$). Since $s'_1 \approx s'_2$, by theorem 6 $s'_1 = s'_2$. Finally, $s_1 \sim s'_1 = s'_2 \sim s_2$, which proves that $s_1 \sim s_2$.

B Examples of monoids

We give here some examples of monoids $\mathbb{M}$ such that $\mathrm{Rat}(\mathbb{M})$ is closed under intersection but not closed under complement.

Example 1 (Die05). Let $\mathbb{M} = \langle X^*, \cdot, 1\rangle$ be the free monoid over some infinite alphabet X.

Since every rational expression uses only a finite number of symbols, every rational set of $\mathbb{M}$ is included in some finitely generated submonoid of $\mathbb{M}$. Hence, for every $R \in \mathrm{Rat}(\mathbb{M})$, $X^* \setminus R \notin \mathrm{Rat}(\mathbb{M})$.
On the other hand, if R, R' are rational sets of $\mathbb{M}$, they are rational subsets of some free submonoid $\mathcal{X}^*$ generated by a finite subalphabet $\mathcal{X} \subseteq X$. By the well-known Kleene's theorem about rational sets of a finitely generated free monoid, $R \cap R' \in \mathrm{Rat}(\mathcal{X}^*)$, and finally, $R \cap R' \in \mathrm{Rat}(\mathbb{M})$.
Let us construct now examples of *finitely generated* monoids.

Example 2. Let $X = \{a, b, c, d, e\}$. Let $L \subseteq \{b, c\}^*$ be some context-free language with non context-free complement. Such examples are abundant, for example one could take

$$L = \{b^p c b^q c b^r \mid p, q, r \in \mathbb{N}, p \neq q \vee q \neq r\}.$$

We then define a semi-Thue system S and a monoid $\mathbb{M}$ by:

$$\mathsf{S} = aLd \times \{e\}, \quad \mathbb{M} = X^*/ \leftrightarrow_{\mathsf{S}}^* .$$

Lemma 11. *The set* $\mathrm{Rat}(\mathbb{M})$ *is closed under intersection.*

Proof. Let us denote by $\pi : X^* \to \mathbb{M}$ the canonical projection. Let $R, R' \in \mathrm{Rat}(\mathbb{M})$. There exist $K, K' \in \mathrm{Rat}(X^*)$ such that

$$R = \pi(K), R' = \pi(K').$$

Since the system S has no overlap of lefthand-sides nor two different lefthand-sides with the same righthand-side, the relation $\to_{\mathsf{S}}^*$ is confluent (see our recalls in section 2). It follows that

$$R \cap R' = \pi(K \to_{\mathsf{S}}^* \cap K' \to_{\mathsf{S}}^*). \tag{99}$$

Since the semi-Thue system is monadic, both sets of descendants $K \to_{\mathsf{S}}^*$, $K' \to_{\mathsf{S}}^*$ are rational subsets of X^* ([BJW82],[Sén94]). Equality (99) thus shows that $R \cap R'$ is a rational subset of $\mathbb{M}$.

Lemma 12. *For every* $R \in \mathrm{Rat}(\mathbb{M})$ *, the language* $\pi^{-1}(R)$ *is context-free.*

Proof. Let us assume that $R = \pi(K)$ for some $K \in \mathrm{Rat}(X^*)$. By the confluence property of S,

$$\pi^{-1}(R) = \{u \in X^*, \exists v \in K \rightarrow^*_{\mathsf{S}} \mid u \rightarrow^*_{\mathsf{S}} v\}.$$

We already observed that $K \rightarrow^*_{\mathsf{S}}$ is rational. Since the set aLd is context-free, for every rational set Q, $Q \leftarrow^*_{\mathsf{S}}$ is context-free. Applying this property to $Q = K \rightarrow^*_{\mathsf{S}}$ we get that $\pi^{-1}(R)$ is context-free.

Lemma 13. *The set* $\mathrm{Rat}(\mathbb{M})$ *is not closed under complement.*

Proof. Let $R = \{\pi(e)\}$. Clearly R is a rational subset of $\mathbb{M}$. Let $R' = M \setminus R$. One can check that

$$\pi^{-1}(R') \cap a\{b, c\}^* d = a(\{b, c\}^* \setminus L)d.$$

Since $\{b, c\}^* \setminus L$ is non context-free, $\pi^{-1}(R')$ must be non context-free (by the closure property of context-free languages by intersection with rational subsets). By Lemma 12 we can conclude that R' is not rational.