Equations in HNN-extensions.

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Abstract. Let $\mathbb{H}$ be a cancellative monoid and let $\mathbb{G}$ be an HNN-extension of $\mathbb{H}$ with finite associated subgroups $A, B \leq \mathbb{H}$. We show that, if equations are algorithmically solvable in $\mathbb{H}$, then they are also algorithmically solvable in $\mathbb{G}$. The result also holds for equations with rational constraints and for the existential first order theory. Analogous results are derived for amalgamated product with finite amalgamated subgroups. Finally, a transfer theorem is shown for the fundamental group of a finite graph of groups where the edge groups are finite and the vertex groups have a decidable existential first-order theory.

Keywords Equations; groups and monoids; HNN-extensions; amalgamated product.

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1 Introduction

General context Theories of equations over groups are a classical research topic at the borderline between algebra, mathematical logic, and theoretical computer science. This line of research was initiated by the work of Lyndon, Tarski, and others in the first half of the

20th century. A major driving force for the development of this field was a question that was posed by Tarski around 1945: Is the first-order theory of a free group F of rank two, i.e, the set of all statements of first-order logic with equations as atomic propositions that are true in F, decidable. Decidability results for fragments of this theory were obtained by Makanin (for the existential theory of a free group) [Mak83] and Merzlyakov and Makanin (for the positive theory of a free group) [Mak84,Mer66]. A complete (positive) solution of Tarski's problem was finally announced in [KM98]; the complete solution is spread over a series of papers. The complexity of Makanin's algorithm for deciding the existential theory of a free group was shown to be not primitive recursive in [KP98]. Based on [Pla99], a new PSPACE algorithm for the existential theory of a free group, which also allows to include rational constraints for variables, was presented in [DHG05]. Beside these results for free groups, also extensions to larger classes of groups were obtained in the past: [DL04,DM06,KMS05,RS95]. In [DL03], a general transfer theorem for existential theories was shown: the decidability of the existential theory is preserved by graph products over groups — a construction that generalizes both free and direct products, see e.g. [Gre90]. Moreover, it is shown in [DL03] that for a large class of graph products, the positive theory can be reduced to the existential theory.

The aim of this paper is to prove similar transfer theorems for HNN-extensions (this kind of extension was introduced in [HNN49], its definition is recalled by eq. (1) of Section 2) and amalgamated free products (which is a classical tool in algebraic topology, its definition is recalled by eq. (9) of Section 2). These two operations are of fundamental importance in combinatorial group theory [LS77]. One of the first important applications of HNN-extensions was a more transparent proof of the celebrated result of Novikov and Boone on the existence of a finitely presented group with an undecidable word problem, see e.g. [LS77]. Such a group can be constructed by a series of HNN-extensions starting from a free group. This shows that there is no hope to prove a transfer theorem for HNN-extensions, similar to the one for graph products from [DL03]. Therefore we mainly consider HNN-extensions and amalgamated free products, where the subgroup A in (1) and (9), respectively, is finite. These restrictions appear also in other contexts in combinatorial group theory: A seminal result of Stallings [Sta71] states that every group G with more than one end can be either written in the form (9) with A finite or in the form (1) with A finite. Those groups which can be built up from finite groups using the operations of amalgamated free products and HNN-extensions, both subject to the finiteness restrictions above, are precisely the virtually-free groups [DD90] (i.e., those groups with a free subgroup of finite index). Virtually-free groups also have strong connections to formal language theory and infinite graph theory [MS83].

Main results of the paper This paper is part of a sequence of three articles dealing with transfer theorems for HNN-extensions and free products with amalgamation where the associated subgroups are finite (see in [LS06] a survey of the full sequence). In [LS08] we studied the membership problem and other related algorithmic problems about rational subsets of monoids and subgroups of groups. Here we study the satisfiability problem for systems of equations (possibly together with disequations and rational constraints) in monoids. In a forthcoming paper ([LS05]) we study the validity of positive first-order formulas in groups.

The main result of this paper is Theorem 4: it states that the satisfiability problem for equations with rational constraints in a monoid $\mathbb{G}$, which is an HNN-extension of a cancellative monoid $\mathbb{H}$, with finite associated subgroups, is Turing-reducible to the same problem over the base monoid $\mathbb{H}$. Several variations and corollaries of this result are also derived:

- The same transfer property is shown for systems of equations with constants (Theorem 7) and for systems of equations and disequations with rational constraints (Theorem 8);
- A similar theorem is proved for systems of equations and disequations with rational constraints in an amalgamated product (Theorem 11).

As a corollary, the satisfiability problem for equations and disequations with rational constraints in the fundamental group of a finite graph of groups with finite edge groups is Turing-reducible to the join of the same problems in the vertex groups (Theorem 12).

Contents Section 2 consists of recalls and notation on the subject of monoids and groups, rational subsets of monoids, equations in monoids and algorithmic problems in general. In Section 3 we introduce the main technical tool allowing us to deal with equations in HNN-extensions: the notion of AB-algebra. This notion is defined in general. We then study two particular AB-algebras, denoted by $\mathbb{H}_t$ and $\mathbb{W}$, which are crucial in our reductions. The domain of the AB-algebra $\mathbb{H}_t$ is essentially the set of reduced sequences of the HNN-extension; the domain of $\mathbb{W}$ is a free product of the finite associated subgroups A,B with a free monoid generated by all the possible "types" of sequences that might occur in $\mathbb{H}_t$, quotiented by relations expressing the pseudo-commutations between each type of sequence and the elements of $A \cup B$. In Section 4 we make precise our notion of system of equations with rational constraints in a monoid and give normal forms for such systems. In Section 5 we provide a reduction of equations over an HNN-extension $\mathbb{G}$ of a monoid $\mathbb{H}$ to equations over the AB-algebra $\mathbb{H}_t$ and equations over the base monoid $\mathbb{H}$. In Section 6 we reduce equations over the AB-algebra $\mathbb{H}_t$ to equations over the AB-algebra $\mathbb{W}$. The essential statement of this section, which is also the key-step of the present paper, is Lemma 31 asserting that any solution σ_t of an equation in $\mathbb{H}_t$ factorizes through a solution of the same equation, but in the structure $\mathbb{W}$. Most of the technical lemmas of section 3 establish some algebraic properties of $\mathbb{W}$ and of $\mathbb{H}_t$ in order to enable the proof of this factorization lemma. In Section 7.1 we reduce equations over the the AB-algebra $\mathbb{W}$ to equations over some finitely generated group $\mathbb{E}$ which turns out to be virtually-free. In Section 9 we show, by induction over the size of the associated subgroups A,B, that equations in $\mathbb{E}$ are algorithmically solvable. We then prove a transfer theorem concerning groups only and prove finally our main Theorem 4 concerning cancellative monoids. In Section 10 we adapt the proof-techniques developed in the previous sections in order to prove some variants of Theorem 4 where the constraints can be positive only, in particular to the case of equations with constants. In Section 11 we extend to equations and disequations the reduction exposed in Section 5. We thus prove some transfer theorems for systems of equations and disequations in HNN-extensions of cancellative monoids. In Section 12 we deduce from our previous results on HNN-extensions a transfer theorem for systems of equations and disequations with rational constraints in free products with amalgamation of cancellative monoids (with finite amalgamated subgroups). In Section 13 we synthesize two transfer theorems obtained previously, one for HNN-extensions and the other for free products with amalgamation, into a single theorem about finite graphs of groups.

2 Preliminaries

The power set of a set A is denoted by 2^A. For $C \subseteq 2^A$ we denote with $\mathcal{B}(C)$ the set of all boolean combinations of sets from C, i.e., it is the smallest boolean subalgebra of 2^A containing C. Let us denote by $\mathsf{B}(\mathsf{Q})$ the monoid $2^{\mathsf{Q}\times\mathsf{Q}}$ of binary relations over the set Q and let $\mathsf{B}^2(\mathsf{Q}) = \mathsf{B}(\mathsf{Q}) \times \mathsf{B}(\mathsf{Q})$. For a string $s \in \Sigma^*$ and $\Gamma \subseteq \Sigma$ we denote with $|s|_\Gamma$ the number of occurrences of symbols from Γ in s. For a tuple $\overline{a} = (a_1, \dots, a_n)$ we denote with $p_i(\overline{a}) = a_i$ the projection onto the i-th component.

In the following subsections, we recall all the needed definitions and classical results concerning partial semigroups, semigroups, monoids, and groups.

2.1 Partial semigroups

Let $\langle P, * \rangle$ be a set endowed with a partial operation $* : P \times P \to P$. By $\mathrm{dom}(*) \subseteq P \times P$ we denote the domain of $*$. The structure $\langle P, * \rangle$ is a *partial semigroup* if for all $p, q, r \in P$:

$$(p,q) \in \mathrm{dom}(*) \wedge ((p*q), r) \in \mathrm{dom}(*) \iff (q,r) \in \mathrm{dom}(*) \wedge (p, (q*r)) \in \mathrm{dom}(*) \text{ and}$$
$$(p,q) \in \mathrm{dom}(*) \wedge ((p*q), r) \in \mathrm{dom}(*) \implies (p*q)*r = p*(q*r).$$

Let us notice that, when P is a partial semigroup, the structure $\langle 2^P, * \rangle$ with

$$\forall R, S \in \mathcal{P}(P) : R * S = \{r * s \mid (r,s) \in (R \times S) \cap \mathrm{dom}(*)\}$$

is a semigroup.

Let A, B be two groups. We denote by $\mathrm{PGI}(A, B)$ the set of all partial group isomorphisms φ with $\mathrm{dom}(\varphi) \subseteq A$ and $\mathrm{im}(\varphi) \subseteq B$. Let

$$\mathrm{PGI}\{A, B\} = \mathrm{PGI}(A, A) \cup \mathrm{PGI}(A, B) \cup \mathrm{PGI}(B, A) \cup \mathrm{PGI}(B, B).$$

The pair $\langle \mathrm{PGI}\{A, B\}, \circ \rangle$, where $\circ$ is the composition operator, is a partial semigroup.

2.2 Monoids and groups

Let $\mathbb{M}$ be a monoid with identity element $1 \in \mathbb{M}$. The product of two elements $x, y \in \mathbb{M}$ is just written as xy. The monoid $\mathbb{M}$ is called *cancellative* if ($xy = xz$ or $yx = zx$) implies $y = z$ for all $x, y, z \in \mathbb{M}$. A *submonoid* of $\mathbb{M}$ is given by a subset $A \subseteq \mathbb{M}$ such that (i) $1 \in A$ and (ii) $xy \in A$ for all $x, y \in A$. If for every $x \in A$ there exists $y \in A$ with $xy = yx = 1$ (i.e., A is a group), then we call A a *subgroup* of $\mathbb{M}$.[1] With $\mathrm{Hom}(\mathbb{M}_1, \mathbb{M}_2)$ we denote the set of all monoid homomorphims from the monoid $\mathbb{M}_1$ to the monoid $\mathbb{M}_2$. An *anti-homomorphism* between monoids $\mathbb{M}_1$ and $\mathbb{M}_2$ is a mapping $h : \mathbb{M}_1 \to \mathbb{M}_2$ that maps the identity of $\mathbb{M}_1$ to the identity of $\mathbb{M}_2$ and moreover satisfies $h(xy) = h(y)h(x)$. If h is a bijection, then it is called an *anti-isomorphism*. If moreover $h^2(x) = x$ for all $x \in \mathbb{M}_1$ then it is called an *involutory anti-isomorphism*.

Given a subset $P \subseteq \mathbb{M} \times \mathbb{M}$, we denote by $\equiv_P$ the smallest monoid congruence over $\mathbb{M}$ that contains P.

Lemma 1. *Let $h : \mathbb{M}_1 \to \mathbb{M}_2$ be some monoid homomorphism and $P \subseteq \mathbb{M}_1 \times \mathbb{M}_1$. Then, $h(\equiv_P) \subseteq \equiv_{h(P)}$.*

[1] Note that usually, a subgroup A of $\mathbb{M}$ is defined to be a subsemigroup of $\mathbb{M}$, which forms a group. In particular, the identity element of A may be an idempotent of $\mathbb{M}$ different from the identity of e of $\mathbb{M}$. Nevertheless, we prefer to use the term subgroup for our definition in order to avoid too many different notions.

HNN-extensions Let us fix throughout this section a monoid $\mathbb{H}$ (the base monoid) and two *finite*, isomorphic *subgroups* $A \le \mathbb{H}, B \le \mathbb{H}$ and an isomorphism $\varphi : A \to B$. The *HNN-extension*

$$\mathbb{G} = \langle \mathbb{H}, t;\ t^{-1}at = \varphi(a)\ (a \in A)\rangle. \tag{1}$$

can be defined as the quotient monoid $\mathbb{H} * \{t, t^{-1}\}^*/_{\approx}$, of the free product of $\mathbb{H}$ and the free monoid $\{t, t^{-1}\}^*$ by the smallest congruence $\approx$ over $\mathbb{H} * \{t, t^{-1}\}^*$ such that:

$$tt^{-1} \approx t^{-1}t \approx 1 \tag{2}$$

$$at \approx t\varphi(a) \qquad \text{for all } a \in A \tag{3}$$

$$bt^{-1} \approx t^{-1}\varphi^{-1}(b) \text{ for all } b \in B \tag{4}$$

Note that (2) and (3) together imply (4) but below we will need (3) and (4) without assuming (2). There exists a canonical morphism

$$\pi_{\mathbb{G}} : \mathbb{H} * \{t, t^{-1}\}^* \to \mathbb{G},$$

where the kernel of $\pi_{\mathbb{G}}$ coincides with $\approx$.

An element of $s \in \mathbb{H} * \{t, t^{-1}\}^*$ can be viewed as a word over the alphabet $\mathbb{H} \cup \{t, t^{-1}\}$ which has the form

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n, \tag{5}$$

where $n \ge 0, \alpha_i \in \{1, -1\}$, and $h_i \in \mathbb{H}$. Such an element $s \in \mathbb{H} * \{t, t^{-1}\}^*$ is also called a *t-sequence*. The t-sequence s is said to be a *reduced sequence* if it neither contains a factor of the form $t^{-1}at$ (with $a \in A$) nor tbt^{-1} (with $b \in B$). Let

$$\mathrm{Red}(\mathbb{H}, t) = \{s \in \mathbb{H} * \{t, t^{-1}\}^* \mid s \text{ is reduced}\}.$$

For the further considerations, it is useful to define

$$A(+1) = B(-1) = A, \quad A(-1) = B(+1) = B.$$

Note that $\varphi^\alpha : A(\alpha) \to B(\alpha)$ for $\alpha \in \{1, -1\}$.

Definition 1. *Let $\sim$ be the congruence on the monoid $\mathbb{H} * \{t, t^{-1}\}^*$ generated by all the rules of type (3) and (4) above.*

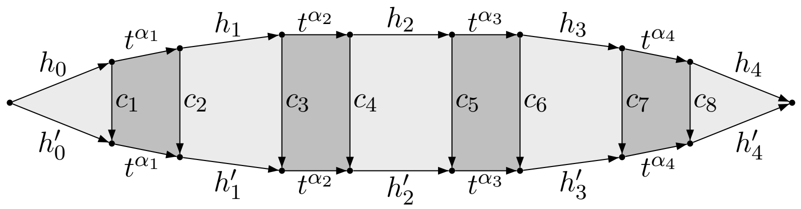
The congruence $\sim$ coincides with reflexive and transitive closure of the binary relation $\overset{1}{\sim}$ over $\mathbb{H} * \{t, t^{-1}\}^*$, which is defined as follows, where $s, s' \in \mathbb{H} * \{t, t^{-1}\}^*$:

$$s \overset{1}{\sim} s' \iff \exists u, v \in \mathbb{H} * \{t, t^{-1}\}^*, \alpha \in \{1, -1\}, c \in A(\alpha) : s = ut^\alpha v \wedge s' = uc^{-1}t^\alpha \varphi^\alpha(c)v.$$

Equivalently, if $s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n$ and $s' = h'_0 t^{\alpha'_1} h'_1 \cdots t^{\alpha'_m} h'_m$ (with $n, m \ge 0$, $\alpha_i, \alpha'_j \in \{1, -1\}$ and $h_i, h'_j \in \mathbb{H}$), then $s \sim s'$ if and only if $n = m$, $\alpha_i = \alpha'_i$ for $1 \le i \le n$, and there exist $c_1, \ldots, c_{2n} \in A \cup B$ such that:

- $h_k c_{2k+1} = c_{2k} h'_k$ in $\mathbb{H}$ for $0 \le k \le n$ (here we set $c_0 = c_{2n+1} = 1$)
- $c_{2k-1} \in A(\alpha_k)$ and $c_{2k} = \varphi^{\alpha_k}(c_{2k-1}) \in A(-\alpha_k)$ for $1 \le k \le n$.

This situation can be visualized by a diagram of the following form (also called a van Kampen diagram, see [LS77] for more details), where $n = m = 4$. Light-shaded (resp. dark-shaded) areas represent relations in $\mathbb{H}$ (resp. relations of the form $at = t\varphi(a)$ $(a \in A)$ or $bt^{-1} = t^{-1}\varphi^{-1}(b)$ $(b \in B)$).



(†)

The elements $c_1, \ldots, c_{2n}$ in such a diagram are also called *connecting elements.*

The set $\mathrm{Red}(\mathbb{H}, t)$ is saturated by the congruence $\sim$, i.e.,

$$s \sim s' \quad \Longrightarrow \quad (s \in \mathrm{Red}(\mathbb{H}, t) \Longleftrightarrow s' \in \mathrm{Red}(\mathbb{H}, t)).$$

Just notice that, since A and B are groups, $aha' \in A \Longleftrightarrow h \in A$ for all $h \in \mathbb{H}, a, a' \in A$ and $bhb' \in B \Longleftrightarrow h \in B$ for all $h \in \mathbb{H}, b, b' \in B$. This property would fail if A and B were assumed to be merely submonoids of $\mathbb{H}$.

One has $\sim \subseteq \approx$. Moreover, for reduced sequences the following fundamental lemma holds.

Lemma 2. *Let $s, s' \in \mathrm{Red}(\mathbb{H}, t)$. Then $s \approx s'$ if and only if $s \sim s'$.*

See [LS08, Appendix A] for a proof. Let us define the quotient monoid $\mathbb{H}_t = \mathbb{H} * \{t, \bar{t}\}^*/_\sim$.

Lemma 3. *If the monoid $\mathbb{H}$ is cancellative then the monoid $\mathbb{H}_t$ is cancellative too.*

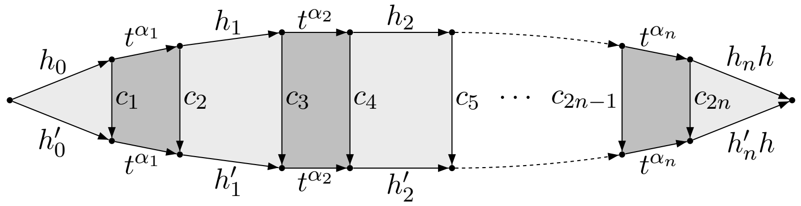
Proof. We will only prove right-cancellativity, left-cancellativity can be shown analogously. We first prove that $sh \sim s'h$ implies $s \sim s'$ for all $s, s' \in \mathbb{H} * \{t, \bar{t}\}^*$ and $h \in \mathbb{H}$. Assume that

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n \quad \text{and} \quad s' = h'_0 t^{\alpha'_1} h'_1 \cdots t^{\alpha'_m} h'_m.$$

Hence, $sh \sim s'h$ implies

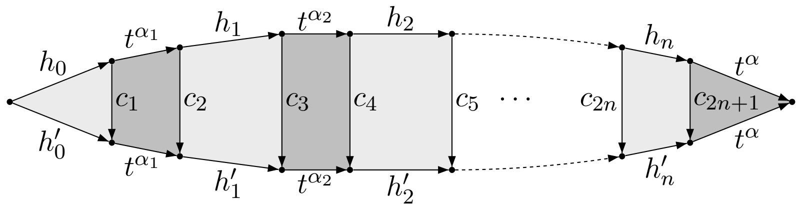
$$h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n h \sim h'_0 t^{\alpha'_1} h'_1 \cdots t^{\alpha'_m} h'_m h.$$

This implies $n = m$ and $\alpha_i = \alpha'_i$ for $1 \leq i \leq n$. Moreover, there exists a van Kampen diagram of the following form:



In particular, we obtain the $\mathbb{H}$-identity $h_n h = c_{2n} h'_n h$ Since $\mathbb{H}$ is cancellative, we get $h_n = c_{2n} h'_n$. Thus, the above van Kampen diagram can be turned into a diagram for $s \sim s'$.

Next, assume that $st^\alpha \sim s't^\alpha$. Again, we get $n = m$ and $\alpha_i = \alpha'_i$ for $1 \leq i \leq n$ and a van Kampen diagram of the form



Since we must have $\varphi^\alpha(c_{2n+1}) = 1$ we get $c_{2n+1} = 1$. Hence, again the above van Kampen diagram can be turned into a diagram for $s \sim s'$.

Finally, since $\mathbb{H}_t$ is generated by $\mathbb{H} \cup \{t, t^{-1}\}$, it follows that $\mathbb{H}_t$ is indeed right-cancellative. $\square$

A direct consequence of Lemma 3 and Lemma 2 is:

Lemma 4. *If the monoid $\mathbb{H}$ is cancellative then the HNN-extension $\mathbb{G}$ is cancellative too.*

We define the norm of a given t-sequence $s \in \mathbb{H} * \{t, \bar{t}\}^*$ by

$$\|s\| = |s|_{\{t,t^{-1}\}}. \tag{6}$$

Clearly, for all $s, s' \in \mathbb{H} * \{t, \bar{t}\}^*$, we have

$$\|ss'\| = \|s\| + \|s'\| \quad \text{and} \quad \|s\| = 0 \Longleftrightarrow s \in \mathbb{H}. \tag{7}$$

The boolean norm of a t-sequence s is the boolean value defined by

$$\|s\|_b = \min\{1, \|s\|\}. \tag{8}$$

Amalgamated free products Let us consider two monoids $\mathbb{H}_1$ and $\mathbb{H}_2$, two *finite subgroups* $A_1 \leq \mathbb{H}_1$ and $A_2 \leq \mathbb{H}_2$, and an isomorphism $\varphi : A_1 \to A_2$. The corresponding amalgamated free product

$$\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2;\ a = \varphi(a)\ (a \in A_1)\rangle \tag{9}$$

is defined by $\mathbb{G} = (\mathbb{H}_1 * \mathbb{H}_2)/_{\approx}$, where $\approx$ is the congruence on the free product $\mathbb{H}_1 * \mathbb{H}_2$ generated by the equations $a = \varphi(a)$ for $a \in A_1$. An $(\mathbb{H}_1, \mathbb{H}_2)$-*sequence* is an element $s \in \mathbb{H}_1 * \mathbb{H}_2$; it has a unique decomposition of the form

$$s = h_0 k_1 h_1 \cdots k_n h_n, \tag{10}$$

where $n \geq 0, h_1, \ldots, h_{n-1} \in \mathbb{H}_2 \setminus \{1\}$, $k_1, \ldots, k_n \in \mathbb{H}_1 \setminus \{1\}$ and $h_0, h_n \in \mathbb{H}_2$. If, in addition, we have $h_1, \ldots, h_{n-1} \in \mathbb{H}_2 \setminus A_2$ and $k_1, \ldots, k_n \in \mathbb{H}_1 \setminus A_1$, then s is a *reduced* $(\mathbb{H}_1, \mathbb{H}_2)$-sequence. We denote by $\mathrm{Red}(\mathbb{H}_1; \mathbb{H}_2)$ the set of all reduced $(\mathbb{H}_1, \mathbb{H}_2)$-sequences.

It is well-known that $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2; a = \varphi(a)(a \in A_1)\rangle$ embeds into the HNN-extension

$$\langle \mathbb{H}_1 * \mathbb{H}_2, t; t^{-1}at = \varphi(a)(a \in A_1)\rangle$$

by the map with

$$h_1 \mapsto t^{-1}h_1 t \text{ for all } h_1 \in \mathbb{H}_1 \quad \text{and} \quad h_2 \mapsto h_2 \text{ for all } h_2 \in \mathbb{H}_2 \tag{11}$$

(see [LS77, Theorem 2.6. p. 187] in the case where $\mathbb{H}_1, \mathbb{H}_2$ are groups and [LS08] for the adaptation to amalgamated free products of monoids with amalgamation over subgroups). Further basic terminology and results on amalgamated free products are given in [LS08, Section 5].

2.3 Rational subsets of a monoid

Let $\mathbb{M}$ be some monoid. The set

$$\mathrm{Rat}(\mathbb{M}) \subseteq 2^{\mathbb{M}}$$

is the smallest subset of $2^{\mathbb{M}}$ which contains all finite subsets of $\mathbb{M}$ and which is closed under the operations $\cup$ (the union operation), $\cdot$ (the product operation), and $*$ (the star operation, associating with a subset P the smallest submonoid of $\mathbb{M}$ containing P). Note that in case $\mathbb{M}$ is finitely generated, $\mathbb{M}$ itself is rational.

We introduced in [LS08], in the particular case where $\mathbb{M}$ is an HNN-extension, a kind of finite automata called *t-automata*, which recognize exactly the rational subsets of $\mathbb{M}$. In Section 2.5, we will introduce the basic facts concerning these automata.

2.4 Types

For this section, we assume all definitions from Section 2.2 on HNN-extensions. Following [LS08], let

$$\mathcal{T}_6 = \{(A,T),(B,T),(1,H),(1,1),(A,H),(B,H)\} \tag{12}$$

Elements of $\mathcal{T}_6$ are called *vertex types*. Note that the first component $p_1(\theta)$ of a vertex type $\theta \in \mathcal{T}_6$ is one of the three groups $1, A, B$.

We define a finite partial semigroup

$$\mathcal{T} = \mathcal{T}_6 \times \mathbb{B} \times \mathcal{T}_6$$

where $\mathbb{B} = \langle \{0,1\}, \vee \rangle$ is the monoid of booleans with OR. The partial product on $\mathcal{T}$ is defined as follows, where $(\theta, b, \rho), (\theta', b', \rho') \in \mathcal{T}_6 \times \mathbb{B} \times \mathcal{T}_6$:

$$(\theta,b,\rho)(\theta',b',\rho') = \begin{cases} (\theta, b \vee b', \rho') & \text{if } \rho = \theta' \\ \text{undefined} & \text{otherwise.} \end{cases}$$

As we noticed in Section 2.1, $2^{\mathcal{T}}$ is thus a semigroup. In fact, $2^{\mathcal{T}}$ is a monoid, the neutral element is $\{(\theta, 0, \theta) \mid \theta \in \mathcal{T}_6\}$.

We define an involution $\mathbb{I}_{\mathcal{R}} : \mathcal{T}_6 \to \mathcal{T}_6$ by:

$$(A,T) \mapsto (A,H) \mapsto (A,T) \tag{13}$$

$$(B,T) \mapsto (B,H) \mapsto (B,T) \tag{14}$$

$$(1,H) \mapsto (1,1) \quad \mapsto (1,H). \tag{15}$$

We then define an involution $\mathbb{I}_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ by:

$$\mathbb{I}_{\mathcal{T}}(\theta, b, \rho) = (\mathbb{I}_{\mathcal{R}}(\rho), b, \mathbb{I}_{\mathcal{R}}(\theta)). \tag{16}$$

One can check that $\mathbb{I}_{\mathcal{T}}$ is an involutory anti-automorphism of the partial semigroup $\mathcal{T}$. This involution induces an involutory monoid anti-automorphism of $2^{\mathcal{T}}$ that will be denoted by $\mathbb{I}_{\mathcal{T}}$ too. We associate to every element $\boldsymbol{\theta}$ of $\mathcal{T}$ an *initial type* $\tau\mathrm{i}(\boldsymbol{\theta}) \in \mathcal{T}_6$, an *end type* $\tau\mathrm{e}(\boldsymbol{\theta}) \in \mathcal{T}_6$, an *initial group* $\mathrm{Gi}(\boldsymbol{\theta}) \in \{1, A, B\}$ and an *end group* $\mathrm{Ge}(\boldsymbol{\theta}) \in \{1, A, B\}$:

$$\tau\mathrm{i}(\theta_1, b, \theta_2) = \theta_1, \quad \mathrm{Gi}(\theta_1, b, \theta_2) = p_1(\theta_1), \quad \tau\mathrm{e}(\theta_1, b, \theta_2) = \theta_2, \quad \mathrm{Ge}(\theta_1, b, \theta_2) = p_1(\theta_2).$$

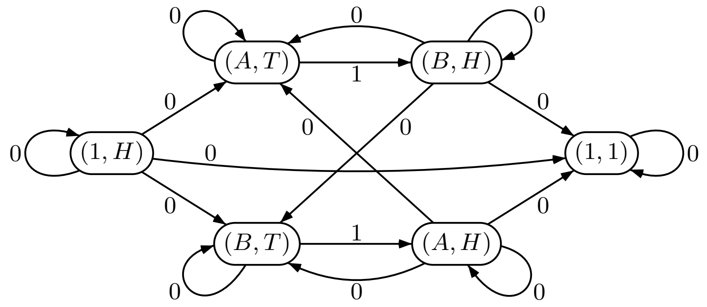


Fig. 1. The graph $\mathcal{B}_6$

Note that

$$\mathrm{Gi}(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta})) = \mathrm{Ge}(\boldsymbol{\theta}) \quad \text{and} \quad \mathrm{Ge}(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta})) = \mathrm{Gi}(\boldsymbol{\theta}). \tag{17}$$

Elements of $\mathcal{T}$ are called a *path type*. This terminology refers to the graph $\mathcal{B}_6$ exhibited in Figure 1. The following 17 types are called *atomic types*; they correspond to the edges of the graph $\mathcal{B}_6$:

$$(A(\alpha), T, 1, A(-\alpha), H) \text{ for } \alpha \in \{+1, -1\} \tag{18}$$

$$(C, H, 0, D, T) \text{ for } C \in \{1, A, B\},\ D \in \{A, B\} \tag{19}$$

$$(C, H, 0, 1, 1) \text{ for } C \in \{1, A, B\}, \tag{20}$$

$$(\theta, 0, \theta) \text{ for } \theta \in \mathcal{T}_6 \tag{21}$$

This set is closed under $\mathbb{I}_{\mathcal{T}}$. The partial submonoid of $\mathcal{T}$ generated by the set of atomic path types will be denoted by $\mathcal{TR}$. It is closed under the involution $\mathbb{I}_{\mathcal{T}}$. The only path types used in this work are those from $\mathcal{TR}$. The following four types are called *T-types*:

$$(A, T, 1, B, H), (B, T, 1, A, H), (A, T, 1, A, H), (B, T, 1, B, H). \tag{22}$$

H-types are all atomic types listed in (19) and (20). Finally, $(A \cup B)$*-types* are all atomic types listed in (21). Note that some types from $\mathcal{TR}$ are not atomic: for example the T-type $(A, T, 1, A, H) = (A, T, 1, B, H)(B, H, 0, B, T)(B, T, 1, A, H)$ is not atomic.

2.5 Finite t-automata

Given an HNN-extension of the form (1), we have defined in Section 4.1 of [LS08] finite automata that recognize subsets of $\mathbb{H} * \{t, t^{-1}\}^*$. We recall here the main definitions and refer to [LS08] for further details and proofs.

Let $\mathcal{F} \subseteq 2^{\mathbb{H}}$ be a set of subsets of $\mathbb{H}$ such that

$$\forall c \in A \cup B : \{c\} \in \mathcal{F}.$$

A *finite t-automaton*, briefly *fta*, over $\mathbb{H} * \{t, t^{-1}\}^*$ with labelling set $\mathcal{F}$ is a 5-tuple

$$\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle, \tag{23}$$

where $\mathcal{L} \subseteq \mathcal{F} \cup \{\{t\}, \{t^{-1}\}\}$ is finite, Q is a finite set of states, $\mathsf{I} \subseteq \mathsf{Q}$ is the set of initial states, $\mathsf{T} \subseteq \mathsf{Q}$ is the set of terminal states, and δ (the set of transitions) is a finite subset of $\mathsf{Q} \times \mathcal{L} \times \mathsf{Q}$ such that

$$\forall q \in \mathsf{Q}\ \exists L \subseteq \mathcal{L} : 1 \in L\ \wedge\ (q, L, q) \in \delta. \tag{24}$$

We define

$$\widehat{\delta} = \{(p, x, q) \in \mathsf{Q} \times (\mathbb{H} \cup \{t, t^{-1}\}) \times \mathsf{Q} \mid \exists L \subseteq \mathcal{L} : x \in L \ \wedge \ (p, L, q) \in \delta\}. \tag{25}$$

The automaton $\mathcal{A}$ induces a representation map

$$\mu_{\mathcal{A}} : \mathbb{H} * \{t, t^{-1}\}^* \to \mathsf{B}(\mathsf{Q})$$

defined as follows: First, define $\mu_{\mathcal{A},0} : \mathbb{H} \cup \{t, t^{-1}\} \to \mathsf{B}(\mathsf{Q})$ as follows, where $x \in \mathbb{H} \cup \{t, t^{-1}\}$:

$$\mu_{\mathcal{A},0}(x) = \{(q, r) \in Q \times Q \mid (q, x, r) \in \widehat{\delta}\,\} \tag{26}$$

(note that $\mathrm{Id}_{\mathsf{Q}} \subseteq \mu_{\mathcal{A},0}(1)$ due to (24)). For $x \in \mathbb{H} \cup \{t, t^{-1}\}$ and $q, r \in \mathsf{Q}$ we will also write $q \xrightarrow{x}_{\mathcal{A}} r$ instead of $(q, r) \in \mu_{\mathcal{A},0}(x)$.

Now, let $s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n$ be a t-sequence of the form (5). Then

$$\mu_{\mathcal{A}}(s) = \mu_{\mathcal{A},0}(h_0) \circ \mu_{\mathcal{A},0}(t^{\alpha_1}) \circ \mu_{\mathcal{A},0}(h_1) \cdots \mu_{\mathcal{A},0}(t^{\alpha_i}) \circ \mu_{\mathcal{A},0}(h_i) \cdots \mu_{\mathcal{A},0}(t^{\alpha_n}) \circ \mu_{\mathcal{A},0}(h_n).$$

The subset of $\mathbb{H} * \{t, t^{-1}\}^*$ recognized by $\mathcal{A}$ is

$$\mathrm{L}(\mathcal{A}) = \{s \in \mathbb{H} * \{t, t^{-1}\}^* \mid \mu_{\mathcal{A}}(s) \cap (\mathsf{I} \times \mathsf{T}) \neq \emptyset\}.$$

On the set $\mathcal{T}_6$ of vertex types, we define a directed, edge-labeled graph $\mathcal{G}_6 = (\mathcal{T}_6, \mathcal{E}_6)$ by

$$\begin{aligned}\mathcal{E}_6 = \quad & \{((A(\alpha), T), t^{\alpha}, (A(-\alpha), H)) \mid \alpha \in \{+1, -1\}\} \\ & \cup \{(C, H, \mathbb{H}, D, T) \mid C, D \in \{A, B\}\} \\ & \cup \{(1, H, \mathbb{H}, C, T) \mid C \in \{A, B\}\} \\ & \cup \{(C, H, \mathbb{H}, 1, 1) \mid C \in \{1, A, B\}\} \\ & \cup \{(\theta, p_1(\theta), \theta) \mid \theta \in \mathcal{T}_6\}.\end{aligned}$$

The graph $\mathcal{G}_6$ is represented in Figure 2. We sometimes use also the graph $\mathcal{R}_6 = (\mathcal{T}_6, \mathcal{E}'_6)$ where

$$\begin{aligned}\mathcal{E}'_6 = (\mathcal{E}_6 \setminus \{(A, H, \mathbb{H}, A, T), (B, H, \mathbb{H}, B, T))\}) \cup \\ \{(A, H, \mathbb{H} \setminus A, A, T), (B, H, \mathbb{H} \setminus B, B, T)\};\end{aligned}$$

it is shown in Figure 3. One can check that the graph $\mathcal{G}_6$ (resp. $\mathcal{R}_6$) endowed with the set of initial states $\mathsf{I}_6 = \{(1, H)\}$ and the set of final states $\mathsf{T}_6 = \{(1, 1)\}$ is an fta recognizing $\mathbb{H} * \{t, t^{-1}\}^*$ (resp. $\mathrm{Red}(\mathbb{H}, t)$).

Definition 2 (partitionned fta). *A* partitioned fta *with labelling set $\mathcal{F}$ is a 6-tuple*

$$\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle, \tag{27}$$

where $\langle \mathcal{L}, \mathsf{Q}, \delta, \mathsf{I}, \mathsf{T} \rangle$ is an fta with labelling set $\mathcal{F}$, $\tau : Q \to \mathcal{T}_6$ assigns a vertex type to every state, and the transitions in δ and the sets I and T respect the types in the following sense:

$$\forall (q, h, r) \in \widehat{\delta} : (\tau(q), h, \tau(r)) \in \widehat{\mathcal{E}}_6 \tag{28}$$

$$\tau(\mathsf{I}) = \{(1, H)\} \ \wedge \ \tau(\mathsf{T}) = \{(1, 1)\}. \tag{29}$$

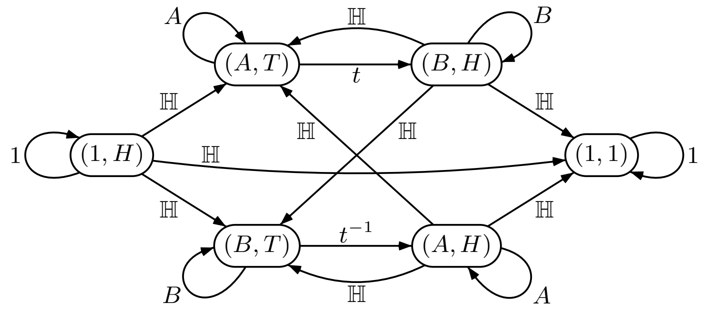


Fig. 2. The graph $\mathcal{G}_6 = (\mathcal{T}_6, \mathcal{E}_6)$

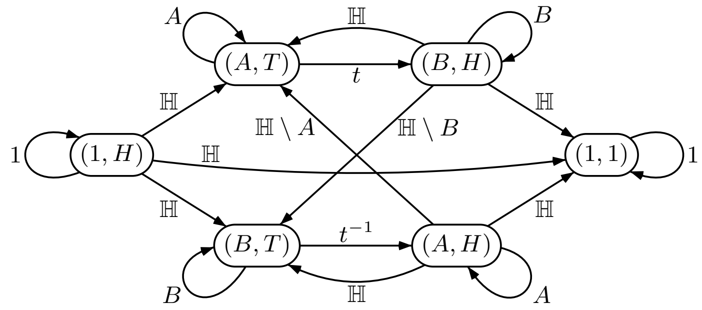


Fig. 3. The graph $\mathcal{R}_6 = (\mathcal{T}_6, \mathcal{E}'_6)$

For a partitioned fta $\mathcal{A}$ as in (27) and a state $q \in \mathsf{Q}$, we denote by $\gamma(q)$ the subgroup

$$\gamma(q) = p_1(\tau(q)) \in \{1, A, B\}.$$

The following lemma follows directly from the structure of the graph $\mathcal{G}_6$:

Lemma 5. *Let $\mathcal{A}$ be a partitioned fta, let q_1, q_2, q_3 be states of $\mathcal{A}$, and let $h, h' \in \mathbb{H}$ such that $q_1 \xrightarrow{h}_{\mathcal{A}} q_2 \xrightarrow{h'}_{\mathcal{A}} q_3$. Then either $\tau(q_1) = \tau(q_2)$ and $h \in \gamma(q_2)$ or $\tau(q_2) = \tau(q_3)$ and $h' \in \gamma(q_2)$.*

We define an additional function $\mu_{\mathcal{A},1} : \mathcal{T} \times \mathbb{H} * \{t, \bar{t}\}^* \to \mathsf{B}(\mathsf{Q})$ as follows: For all $s \in \mathbb{H} * \{t, \bar{t}\}^*$ and $(\theta, b, \theta') \in \mathcal{T}$ let

$$\mu_{\mathcal{A},1}((\theta, b, \theta'), s) = \mu_{\mathcal{A}}(s) \cap (\tau^{-1}(\theta) \times \tau^{-1}(\theta')). \tag{30}$$

Note that the boolean component b from (θ, b, θ') has no influence on the value of $\mu_{\mathcal{A},1}((\theta, b, \theta'), s)$. Let us define maps γ_+ and γ_t that associate to every sequence $s \in \mathbb{H} * \{t, \bar{t}\}^*$ the set of all path types that can be realized by s in the automaton $\mathcal{G}_6$ (resp. $\mathcal{R}_6$):

$$\gamma_+(s) = \{(\theta, \|s\|_b, \theta') \in \mathcal{T} \mid (\theta, \theta') \in \mu_{\mathcal{G}_6}(s)\} \in 2^{\mathcal{T}} \tag{31}$$
$$\gamma_t(s) = \{(\theta, \|s\|_b, \theta') \in \mathcal{T} \mid (\theta, \theta') \in \mu_{\mathcal{R}_6}(s)\} \in 2^{\mathcal{T}} \tag{32}$$

Clearly, we have $\gamma_t(s) \subseteq \gamma_+(s)$. Moreover, note that

$$\gamma_+(h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n) = \gamma_+(h_0)\gamma_+(t^{\alpha_1})\gamma_+(h_1) \cdots \gamma_+(t^{\alpha_n})\gamma_+(h_n) \text{ and} \tag{33}$$
$$\gamma_t(h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n) = \gamma_t(h_0)\gamma_t(t^{\alpha_1})\gamma_t(h_1) \cdots \gamma_t(t^{\alpha_n})\gamma_t(h_n). \tag{34}$$

Recall the equivalences $\sim$ and $\approx$ over $\mathbb{H} * \{t, t^{-1}\}^*$ defined in Section 2.2.

Definition 3 ($\approx$-compatible, $\sim$-saturated, strict). *An fta $\mathcal{A}$ is said to be $\approx$-compatible if and only if*

$$[\mathrm{L}(\mathcal{A})]_{\approx} = [\mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(\mathbb{H}, t)]_{\approx}. \tag{35}$$

It is said to be $\sim$-saturated *if and only if*

$$\forall s, s' \in \mathbb{H} * \{t, t^{-1}\}^* : s \sim s' \Rightarrow \mu_{\mathcal{A}}(s) = \mu_{\mathcal{A}}(s'). \tag{36}$$

Finally, $\mathcal{A}$ is strict*, if*

$$\mathrm{L}(\mathcal{A}) \subseteq \mathrm{Red}(\mathbb{H}, t). \tag{37}$$

Note that every strict fta is $\approx$-compatible.

Definition 4 (unitary, subgroup compatible, multiplicative). *The partitioned fta $\mathcal{A} = \langle \mathcal{L}, \mathsf{Q}, \tau, \delta, \mathsf{I}, \mathsf{T} \rangle$ is* unitary*, if for every vertex type $\Theta \in \mathcal{T}_6$ we have*

$$\mu_{\mathcal{A},0}(1) \cap (\tau^{-1}(\theta) \times \tau^{-1}(\theta)) = \mathrm{Id}_{\mathsf{Q}} \cap (\tau^{-1}(\theta) \times \tau^{-1}(\theta)).$$

$\mathcal{A}$ is subgroup-compatible *if for every vertex type $\theta \in \mathcal{T}_6$, there exists a right-action $\odot$ of the subgroup $p_1(\theta)$ on the set of states $\tau^{-1}(\theta) \subseteq \mathsf{Q}$ such that for all $q, r \in \mathsf{Q}$, $c \in p_1(\theta)$, and $h \in \mathbb{H}$,*

$$\tau(q) = \theta \implies q \xrightarrow{c}_{\mathcal{A}} q \odot c \tag{38}$$

$$(\tau(q) = \theta \wedge q \xrightarrow{h}_{\mathcal{A}} r) \implies q \odot c^{-1} \xrightarrow{ch}_{\mathcal{A}} r \tag{39}$$

$$(\tau(r) = \theta \wedge q \xrightarrow{h}_{\mathcal{A}} r) \implies q \xrightarrow{hc}_{\mathcal{A}} r \odot c \tag{40}$$

Note that in case $p_1(\theta) = 1$, these conditions are trivially satisfied by setting $q \odot 1 = q$ for all q with $\tau(q) = \theta$.

The partitioned fta $\mathcal{A}$ is multiplicative*, if for all $\boldsymbol{\theta} \in \gamma_+(s), \boldsymbol{\theta'} \in \gamma_+(s')$ such that $\boldsymbol{\theta}\boldsymbol{\theta'}$ is defined in the semigroup $\mathcal{T}$:*

$$\mu_{\mathcal{A},1}(\boldsymbol{\theta}\boldsymbol{\theta'}, ss') = \mu_{\mathcal{A},1}(\boldsymbol{\theta}, s) \circ \mu_{\mathcal{A},1}(\boldsymbol{\theta'}, s') \tag{41}$$

Definition 5 (normal fta). *A partitioned fta $\mathcal{A}$ is said to be* normal*, if it is $\approx$-compatible, $\sim$-saturated, unitary and multiplicative.*

Lemma 6. *Let $\mathcal{A}$ be a partitioned fta. If $\mathcal{A}$ is unitary and subgroup-compatible, then $\mathcal{A}$ is multiplicative.*

Proof. Let Q be the state set of the fta $\mathcal{A}$. Assume that $\mathcal{A}$ is subgroup-compatible and unitary, and let $\boldsymbol{\theta} \in \gamma_+(s)$, $\boldsymbol{\theta'} \in \gamma_+(s')$ such that $\boldsymbol{\theta}\boldsymbol{\theta'}$ is defined in $\mathcal{T}$. We have to show that

$$\mu_{\mathcal{A},1}(\boldsymbol{\theta}\boldsymbol{\theta'}, ss') = \mu_{\mathcal{A},1}(\boldsymbol{\theta}, s) \circ \mu_{\mathcal{A},1}(\boldsymbol{\theta'}, s').$$

We prove this identity by induction over $\|ss'\| = \|s\| + \|s'\|$.

First assume that $\|s\| = \|s'\| = 0$, i.e., $s = h \in \mathbb{H}$ and $s' = h' \in \mathbb{H}$. We must have $\boldsymbol{\theta} = (\theta, 0, \theta')$ and $\boldsymbol{\theta'} = (\theta', 0, \theta'')$ for some vertex types $\theta, \theta', \theta'' \in \mathcal{T}_6$. An inspection of the graph $\mathcal{G}_6$ (Figure 2) shows that either $\theta = \theta'$ or $\theta' = \theta''$. Let us first assume that $\theta = \theta'$. Thus, $(\theta, 0, \theta) \in \gamma_+(h)$, i.e., $(\theta, \theta) \in \mu_{\mathcal{G}_6}(h)$. Then $h \in p_1(\theta)$. We obtain for all $(p, q) \in \mathsf{Q} \times \mathsf{Q}$:

$$\begin{aligned}
(p,q) \in \mu_{\mathcal{A},1}((\theta,0,\theta''),hh') &\Longrightarrow p \xrightarrow{hh'}_{\mathcal{A}} q,\ \tau(p)=\theta,\ \tau(\theta'')=q \\
&\Longrightarrow p \xrightarrow{h}_{\mathcal{A}} p \odot h = p \odot (h^{-1})^{-1} \xrightarrow{h^{-1}hh'}_{\mathcal{A}} q, \\
&\qquad\quad \tau(p \odot h) = \tau(p) = \theta,\ \tau(\theta'') = q \\
&\Longrightarrow (p,q) \in \mu_{\mathcal{A},1}((\theta,0,\theta),h) \circ \mu_{\mathcal{A},1}((\theta,0,\theta''),h') \\
&\Longrightarrow \exists r \in \tau^{-1}(\theta) : (p,r) \in \mu_{\mathcal{A},1}((\theta,0,\theta),h), (r,q) \in \mu_{\mathcal{A},1}((\theta,0,\theta''),h') \\
&\Longrightarrow \tau(p)=\theta, \tau(\theta'')=q, \exists r \in \tau^{-1}(\theta) : p \xrightarrow{h}_{\mathcal{A}} r \xrightarrow{h'}_{\mathcal{A}} q \\
&\Longrightarrow \tau(p)=\theta, \tau(\theta'')=q, \exists r \in \tau^{-1}(\theta) : p \xrightarrow{hh^{-1}}_{\mathcal{A}} r \odot h^{-1} \xrightarrow{hh'}_{\mathcal{A}} q
\end{aligned}$$

Since $\mathcal{A}$ is unitary, $p \xrightarrow{1}_{\mathcal{A}} r \odot h^{-1}$, and $\tau(p) = \tau(r) = \tau(r \odot h^{-1}) = \theta$, we must have $p = r \odot h^{-1}$. Thus, we get:

$$\tau(p) = \theta, \quad \tau(\theta'') = q, \quad p \xrightarrow{hh'}_{\mathcal{A}} q,$$

i.e. $(p, q) \in \mu_{\mathcal{A},1}((\theta, 0, \theta''), hh')$. The case that $\theta' = \theta''$ can be dealt analogously. This settles the induction base.

Now assume that $\|s\| > 0$ or $\|s'\| > 0$. Let $\|s'\| > 0$; the case that $\|s\| > 0$ can be dealt analogously). We can factorize the sequence s' as $s' = s_1 t^\alpha h$ with $\alpha \in \{-1, 1\}$ and $h \in \mathbb{H}$. The path type $\boldsymbol{\theta'}$ must be of the form $\boldsymbol{\theta'} = (\theta', 1, \theta'')$. Let $\boldsymbol{\theta} = (\theta, \|s\|_b, \theta')$. Let us assume that $\alpha = 1$, again the case $\alpha = -1$ can be dealt analogously. By induction, we get:

$$\begin{aligned}
\mu_{\mathcal{A},1}(\boldsymbol{\theta}, s) \circ \mu_{\mathcal{A},1}(\boldsymbol{\theta'}, s') &= \mu_{\mathcal{A},1}((\theta, \|s\|_b, \theta'), s) \circ \mu_{\mathcal{A},1}((\theta', 1, \theta''), s_1 t h) \\
&= \mu_{\mathcal{A},1}((\theta, \|s\|_b, \theta'), s) \circ \mu_{\mathcal{A},1}((\theta', \|s_1\|_b, A, T), s_1) \circ \\
&\qquad \mu_{\mathcal{A},1}((A, T, 1, B, H), t) \circ \mu_{\mathcal{A},1}((B, H, 0, \theta''), h) \\
&= \mu_{\mathcal{A},1}((\theta, \|ss_1\|_b, A, T), ss_1) \circ \\
&\qquad \mu_{\mathcal{A},1}((A, T, 1, B, H), t) \circ \mu_{\mathcal{A},1}((B, H, 0, \theta''), h) \\
&= \mu_{\mathcal{A},1}((\theta, 1, \theta''), ss_1 t h) \\
&= \mu_{\mathcal{A},1}(\boldsymbol{\theta\theta'}, ss')
\end{aligned}$$

□

Proposition 1. *Let $R \subseteq \mathbb{G}$.*

(1) $R \in \mathrm{Rat}(\mathbb{G})$ if and only if $R = \pi_{\mathbb{G}}(\mathrm{L}(\mathcal{A}))$ for some normal *partitioned fta $\mathcal{A}$ with labeling set $\mathrm{Rat}(\mathbb{H})$.*

(2) If $R \in \mathcal{B}(\mathrm{Rat}(\mathbb{G}))$ then $R = \pi_{\mathbb{G}}(\mathrm{L}(\mathcal{A}))$ for some strict normal *partitioned fta $\mathcal{A}$ with labeling set $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$.*

Proof. It is proved in [LS08, Prop. 33, points (1), (3), and (a)] that R is a rational subset of $\mathbb{G}$ if and only if $\pi_{\mathbb{G}}^{-1}(R) \cap \mathrm{Red}(\mathbb{H}, t) = \mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(\mathbb{H}, t)$ for some partitioned, $\approx$-compatible, $\sim$-saturated, and unitary fta $\mathcal{A}$ with labelling set $\mathrm{Rat}(\mathbb{H})$. Moreover, the automaton $\mathcal{A}$ can be chosen in such a way that it is subgroup compatible: this follows from equations (39)-(40) in the proof of [LS08, Prop. 22]. By Lemma 6, $\mathcal{A}$ is multiplicative and hence normal. Finally, $\pi_{\mathbb{G}}^{-1}(R) \cap \mathrm{Red}(\mathbb{H}, t) = \mathrm{L}(\mathcal{A}) \cap \mathrm{Red}(\mathbb{H}, t)$ implies $R = \pi_{\mathbb{G}}(\mathrm{L}(\mathcal{A}))$. This shows point (1).

For (2), let R be a boolean combination of rational subsets $K_1, \ldots, K_p$ of $\mathbb{G}$. By [LS08, Prop. 33, point (3)], for every $1 \leq i \leq p$ there exists a partitionned, $\sim$-saturated, deterministic, complete, fta $\mathcal{C}_i$, whose labelling set is $\mathcal{B}(\mathrm{Rat}(H))$, and such that

$$\mathrm{L}(\mathcal{C}_i) = \pi_G^{-1}(K) \cap \mathrm{Red}(\mathbb{H}, t).$$

Since every boolean operation can be translated over partitioned, $\sim$-saturated, deterministic, complete, finite fta (see [LS08, Lemma 17, 20]), there exists a partitioned, $\sim$-saturated, deterministic, complete, fta $\mathcal{A}$, whose labelling set is $\mathcal{B}(\mathrm{Rat}(H))$, and such that

$$\mathrm{L}(\mathcal{A}) = \pi_G^{-1}(K) \cap \mathrm{Red}(\mathbb{H}, t). \tag{42}$$

Applying [LS08, Prop. 22] to this fta $\mathcal{A}$ and the set $\mathcal{F} = \mathcal{B}(\mathrm{Rat}(H))$, we obtain a partitioned,$\approx$-compatible, $\sim$-saturated, unitary fta $\mathcal{B}$ with labelling set $\mathcal{B}(\mathrm{Rat}(H))$ such that

$$[\mathrm{L}(\mathcal{B})]_{\approx} \cap \mathrm{Red}(\mathbb{H}, t) = \pi_G^{-1}(K) \cap \mathrm{Red}(\mathbb{H}, t).$$

Moreover, by equations (39)–(40) of [LS08], $\mathcal{B}$ is subgroup compatible. Thus, by Lemma 6, $\mathcal{B}$ is multiplicative. By [LS08, Claim 26], $\mathrm{L}(\mathcal{B}) \subseteq \mathrm{L}(\mathcal{A})$ while by (42) $\mathrm{L}(\mathcal{A}) \subseteq \mathrm{Red}(\mathbb{H}, t)$, hence $\mathcal{B}$ is also strict. Thus, $\mathcal{B}$ is strict and normal. $\square$

The fta $\mathcal{G}_6$ and $\mathcal{R}_6$ are clearly partionned with τ the identity mapping. Moreover, these fta are $\approx$-compatible, $\sim$-saturated, unitary, and subgroup compatible. For the latter, we choose for $\odot$ the trivial action. Hence, by Lemma 6, $\mathcal{G}_6$ and $\mathcal{R}_6$ are also multiplicative. As an corollary, we obtain:

Lemma 7. *For all $s_1, s_2 \in \mathbb{H} * \{t, t^{-1}\}^*$ we have $\gamma_+(s_1)\gamma_+(s_2) \subseteq \gamma_+(s_1 s_2)$ and $\gamma_t(s_1)\gamma_t(s_2) \subseteq \gamma_t(s_1 s_2)$.*

Proof. Let us prove the statement for γ_+, the same proof also works for γ_t. Let $s_1, s_2 \in \mathbb{H} * \{t, t^{-1}\}^*$, $\boldsymbol{\theta_1} \in \gamma_+(s_1)$ and $\boldsymbol{\theta_2} \in \gamma_+(s_2)$ such that $\boldsymbol{\theta_1\theta_2}$ is defined. We have to show that $\boldsymbol{\theta_1\theta_2} \in \gamma_+(s_1 s_2)$. Since $\mathcal{G}_6$ is multiplicative, we have

$$\mu_{\mathcal{G}_6,1}(\boldsymbol{\theta_1\theta_2}, s_1 s_2) = \mu_{\mathcal{G}_6,1}(\boldsymbol{\theta_1}, s_1) \circ \mu_{\mathcal{G}_6,1}(\boldsymbol{\theta_2}, s_2). \tag{43}$$

Let $\boldsymbol{\theta_1} = (\theta_1, \|s_1\|_b, \theta_2)$, $\boldsymbol{\theta_2} = (\theta_2, \|s_2\|_b, \theta_3)$, and $\boldsymbol{\theta_1\theta_2} = (\theta_1, \|s_1 s_2\|_b, \theta_3)$. Since $\boldsymbol{\theta_1} \in \gamma_+(s_1)$, we have $(\theta_1, \theta_2) \in \mu_{\mathcal{G}_6}(s_1)$ and similarly $(\theta_2, \theta_3) \in \mu_{\mathcal{G}_6}(s_2)$. Hence, $(\theta_1, \theta_2) \in \mu_{\mathcal{G}_6,1}(\boldsymbol{\theta_1}, s_1)$ and $(\theta_2, \theta_3) \in \mu_{\mathcal{G}_6,1}(\boldsymbol{\theta_2}, s_2)$. Thus, (43) implies $(\theta_1, \theta_3) \in \mu_{\mathcal{G}_6,1}(\boldsymbol{\theta_1\theta_2}, s_1 s_2)$. This implies $\boldsymbol{\theta_1\theta_2} = (\theta_1, \|s_1 s_2\|_b, \theta_3) \in \gamma_+(s_1 s_2)$. $\square$

Given a normal fta $\mathcal{A}$ and an element $g \in \mathbb{G}$ we set

$$\mu_{\mathcal{A},\mathbb{G}}(g) = \mu_{\mathcal{A},1}((1, H, \|s\|_b, 1, 1), s), \tag{44}$$

where s is any reduced t-sequence representing g. Since $\mathcal{A}$ is $\sim$-saturated, the value of $\mu_{\mathcal{A},1}((1, H, \|s\|_b, 1, 1), s)$ does not depend of the chosen reduced representative s of g.

2.6 Equations and disequations over a monoid

Let $\mathbb{M}$ be some monoid and let
$$\mathcal{C} \subseteq 2^{\mathbb{M}}$$
be a class of constraints sets. A *system of equations and disequations* over the monoid $\mathbb{M}$ is a subset $\mathcal{S} \subseteq \mathcal{U}^* \times \{=, \neq\} \times \mathcal{U}^*$, where $\mathcal{U}$ is a set of variables of $\mathcal{S}$. Instead of $(u, =, u') \in \mathcal{S}$ (resp. $(u, \neq, u') \notin \mathcal{S}$) we just write $(u = u') \in \mathcal{S}$ (resp. $(u \neq u') \notin \mathcal{S}$). The system $\mathcal{S}$ is said to be *quadratic* if

- for every $(u = u') \in \mathcal{S}$ we have $|u| = 1$ and $|u'| = 2$, and
- for every $(u \neq u') \in \mathcal{S}$ we have $|u| = |u'| = 1$.

A system of equations and disequations over the monoid $\mathbb{M}$ with *$\mathcal{C}$-constraints* is a pair $(\mathcal{S}, \mathsf{C})$, where $\mathcal{S}$ is a system of equations and disequations over the monoid $\mathbb{M}$ and C is a map $\mathsf{C} : \mathcal{U} \to \mathcal{C}$. An $\mathbb{M}$-solution of the system $(\mathcal{S}, \mathsf{C})$ is any monoid homomorphism
$$\sigma_{\mathbb{M}} : \mathcal{U}^* \to \mathbb{M}$$
fulfilling the following conditions:
$$\forall (u = u') \in \mathcal{S} : \sigma_{\mathbb{M}}(u) = \sigma_{\mathbb{M}}(u') \tag{45}$$
$$\forall (u \neq u') \in \mathcal{S} : \sigma_{\mathbb{M}}(u) \neq \sigma_{\mathbb{M}}(u') \tag{46}$$
$$\forall U \in \mathcal{U} : \sigma_{\mathbb{M}}(U) \in \mathsf{C}(U). \tag{47}$$

The system $\mathcal{S}$ is a *system of equations* over $\mathbb{M}$ (with $\mathcal{C}$-constraints), if
$$\mathcal{S} \subseteq \mathcal{U}^* \times \{=\} \times \mathcal{U}^*.$$

In this paper, we will consider three particular sets of constraints.

- $\mathcal{C} = \{\{m\} \mid m \in \mathbb{M}\} \cup \{\mathbb{M}\}$: In this case, a system of equations (and disequations) with $\mathcal{C}$-constraints is called a system of equations (and disequations) with *constants*, since the variables $U \in \mathcal{U}$ with $|\mathsf{C}(U)| = 1$ can be seen as constants from $\mathbb{M}$, while the variables $U \in \mathcal{U}$ with $\mathsf{C}(U) = \mathbb{M}$ can be seen as variables without any constraint.
- $\mathcal{C} = \mathcal{B}(\mathit{Rat}(\mathbb{M}))$, i.e. the boolean closure of the set of rational subsets of $\mathbb{M}$: In this case, a system of equations (and disequations) with $\mathcal{C}$-constraints is called a system of equations with *rational constraints*.
- $\mathcal{C} = \mathit{Rat}(\mathbb{M})$, i.e. the set of rational subsets of $\mathbb{M}$: In this case, in order to emphasize the fact that we do not use constraints that are complements of rational subsets, a system of equations (and disequations) with $\mathcal{C}$-constraints is called a system of equations (and disequations) with *positive* rational constraints.

The following proposition can be shown by standard techniques:

Proposition 2. *Assume that $\mathbb{M}$ is finitely generated. Let $(\mathcal{S}, \mathsf{C})$ be a system of equations and disequations over $\mathbb{M}$ with variables from $\mathcal{U}$ and rational constraints. From $(\mathcal{S}, \mathsf{C})$ we can compute a system $(\mathcal{S}', \mathsf{C}')$ of equations and disequations over $\mathbb{M}$ with variables from $\mathcal{U}'$ and rational constraints such that:*

- *$\mathcal{S}'$ is quadratic.*

– $\mathcal{U} \subseteq \mathcal{U}'$
– *The solutions of* $(\mathcal{S}, \mathsf{C})$ *are exactly the restrictions of the solutions of* $(\mathcal{S}', \mathsf{C}')$ *to* $\mathcal{U}^*$
– *If* $\mathcal{S}$ *is a system of equations (without disequations), then also* $\mathcal{S}'$ *is a system of equations (without disequations).*
– *If* $(\mathcal{S}, \mathsf{C})$ *is a system with* positive *rational constraints, then also* $(\mathcal{S}', \mathsf{C}')$ *is a system with* positive *rational constraints.*

One sometimes also considers systems of equations and disequations with a partial involution. This means that some variables $u \in \mathcal{U}$ have a formal inverse $u^{-1} \in \mathcal{U}$ and that a solution $\sigma_{\mathbb{M}}$ of the system has to respect the partial involution i.e. to map the variable u^{-1} onto the inverse of $\sigma_{\mathbb{M}}(u)$ (in particular, $\sigma_{\mathbb{M}}(u)$ has to be contained in the group of units of $\mathbb{M}$). Such a system "with partial involution" can be reduced to a system of the form above by adding the equations $uu^{-1} = 1$ and $u^{-1}u = 1$ to the system and removing the explicit constraint that $\sigma_{\mathbb{M}}$ respects some partial involution.

Let $\mathbb{M}$ be a monoid and let $\mathcal{C} \subseteq 2^{\mathbb{M}}$. We denote by $\mathrm{EQ}(\mathbb{M}, \mathcal{C})$ the set of all subsets of $\mathbb{M}$ which can be defined by a system of equations with constants and with constraints in $\mathcal{C}$. We denote by $\mathrm{Def}_{\exists^+}(\mathbb{M}, \mathcal{C})$ the set of all subsets of $\mathbb{M}$ which can be defined by a logical formula of the form $\exists x_1 \exists x_2 \cdots \exists x_n \Psi$, where Ψ is a positive boolean combination of statements which are either equations with constants or of the form $x_i \in P$ (for some variable x_i and some $P \in \mathcal{C}$). The elements of $\mathrm{EQ}(\mathbb{M}, \mathcal{C})$ are called the *equational* subsets (with constraints in $\mathcal{C}$); the elements of $\mathrm{Def}_{\exists^+}(\mathbb{M}, \mathcal{C})$ are called the *positively definable* subsets (with constraints in $\mathcal{C}$).

Lemma 8. *For every group* $\mathbb{H}$, $\mathcal{B}(\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))) \subseteq \mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$.

In words: a boolean combination of equational subsets of $\mathbb{H}$ (with rational constraints) is positively definable (with rational contraints).

Proof. Every disequation $u \neq v$ in $\mathbb{H}$ can be translated into the formula

$$\exists v' \, \exists w : vv' = 1 \wedge uv' = w \wedge w \in \mathbb{H} \setminus \{1\}. \tag{48}$$

Hence, if P is defined by a positive boolean combination of equations and rational contraints, its complement is definable by a positive boolean combination of statements of the form (48) above, of equations and of rational constraints. This positive boolean combination can be put in existential prenex form, showing that $\mathbb{H} \setminus P$ belongs to $\mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$.

2.7 Reductions among decision problems

A Turing machine M *with oracle* L_2 is a Turing machine with a distinguished oracle tape and three distinguished states $q_?, q_y, q_n$ such that, when the current state of M is $q_?$, M makes a transition that does not move the tape heads, does not print anything, and enters the state q_y or the state q_n according to whether the word on the oracle tape belongs to L_2 or not (see, for example [HU79, §8.9, p. 209-210] or [Rog87, p. 128] for a more precise definition). Such an oracle machine M is called a (Turing-)reduction from L_1 to L_2 if the language recognized by M, using the oracle L_2, is the language L_1. When such a reduction exists, we say that L_1 is Turing-reducible to L_2. The underlying idea is that, if somebody knows both M and a machine M_2 that recognizes L_2, then he can combine them into a Turing machine M_1 that recognizes L_1. Let us consider three languages L_1, L_2, L_3. A Turing-reduction from L_1 to (L_2, L_3) is a Turing-reduction from L_1 to $(L_2 \times \{0\}) \cup (L_3 \times \{1\})$. Given decision problems

P_1, P_2, P_3, we say that P_1 is Turing-reducible to P_2 (or to (P_2, P_3)) when the formal languages L_1, L_2, L_3 that encode the set of instances where the correct answer is "yes" are fulfiling the above definition of Turing-reducibility.

2.8 Submonoids; free products

The following lemma is straightforward.

Lemma 9. *If $\mathbb{M}$ is a finitely generated submonoid of the monoid $\mathbb{M}'$ and if equations with rational constraints in $\mathbb{M}'$ are algorithmically solvable, then equations with rational constraints in $\mathbb{M}$ are algorithmically solvable.*

In [LS08, section 6.1] we defined a notion of $(\mathbb{H}_1, \mathbb{H}_2)$-automata which is the direct adaptation to the operation of free product (possibly with amalgamation) of the notion of t-automaton. Let us formulate with the help of these automata a result about free-products which is a particular case of [DL03, Theorem 2] (where the more general operation of *graph product* is treated [2]). Given two sets $\mathcal{C}_{\mathbb{H}_1} \in \mathcal{P}(\mathbb{H}_1), \mathcal{C}_{\mathbb{H}_2} \in \mathcal{P}(\mathbb{H}_2)$, we denote by $\mathcal{L}(\mathcal{C}_{\mathbb{H}_1}, \mathcal{C}_{\mathbb{H}_2})$ the set of subsets of $\mathrm{Red}(\mathbb{H}_1, \mathbb{H}_2)$ which are recognized by partitioned, $\approx$-compatible and finite $(\mathbb{H}_1, \mathbb{H}_2)$-automata with labelling set $(\mathcal{C}_{\mathbb{H}_1}, \mathcal{C}_{\mathbb{H}_2})$. Informally, such an automaton $\mathcal{A}$ is a finite automaton over the alphabet $\mathcal{C}_{\mathbb{H}_1} \cup \mathcal{C}_{\mathbb{H}_2}$ which has transitions labeled alternatively by elements of $\mathcal{C}_{\mathbb{H}_1}$ and by elements of $\mathcal{C}_{\mathbb{H}_2}$; $\approx$-compatibility means that, if it recognizes some $(\mathbb{H}_1, \mathbb{H}_2)$-sequence s it also recognizes the reduced sequence s' such that $s \approx s'$; the language recognized by $\mathcal{A}$ is obtained by substituting, in the ordinary language recognized by $\mathcal{A}$, each letter by its value in $\mathcal{P}(\mathbb{H}_1) \cup \mathcal{P}(\mathbb{H}_2)$.

Theorem 1. *Let us consider two monoids $\mathbb{H}_1, \mathbb{H}_2$. The satisfiability problem for systems of equations and disequations with constraints in $\mathcal{B}(\mathcal{L}(\mathcal{C}_{\mathbb{H}_1}, \mathcal{C}_{\mathbb{H}_2}))$ over the free product $\mathbb{H}_1 * \mathbb{H}_2$ is Turing-reducible to the pair of problems (S_1, S_2) where*

- *S_1 is the SAT-problem for systems of equations and disequations with constraints in $\mathcal{B}(\mathcal{C}_{\mathbb{H}_1})$ and*
- *S_2 is the SAT-problem for systems of equations and disequations with constraints in $\mathcal{B}(\mathcal{C}_{\mathbb{H}_2})$.*

Note that when $\mathcal{C}_{\mathbb{H}_i} = \mathrm{Rat}(\mathbb{H}_i)$ (for $i \in \{1, 2\}$), then $\mathcal{L}(\mathcal{C}_{\mathbb{H}_1}, \mathcal{C}_{\mathbb{H}_2}) = \mathrm{Rat}(\mathbb{H}_1 * \mathbb{H}_2)$.

3 AB-algebras

We define here the notion of AB-algebra, which we devised for handling equations with rational constraints in an HNN-extension.

[2] the original formulation of [DL03, Theorem 2] deals with the existential first order theory, with rational constraints, of a group $\mathbb{G} = \mathbb{H}_1 * \mathbb{H}_2$. But the decidability of this fragment is equivalent (see, in section 9, the trick of formula (48)) to the decidability of the SAT-problem for systems of equations with rational constraints in $\mathbb{G}$.

3.1 AB-algebra axioms

Let A, B be two groups (what we have in mind are the two subgroups A, B of $\mathbb{H}$ leading to the HNN-extension $\mathbb{G}$ defined by (1)) and let Q be some finite set (we have in mind the set of states of some fta $\mathcal{A}$ over $\mathbb{H} * \{t, \bar{t}\}^*$). Given a binary relation $r \in \mathsf{B}(\mathsf{Q}), r^{-1}$ is the binary relation $r^{-1} = \{(p,q) \in \mathsf{Q} \times \mathsf{Q} \mid (q,p) \in r\}$. We consider the involutory monoid anti-isomorphism $\mathbb{I}_{\mathsf{Q}} : \mathsf{B}^2(\mathsf{Q}) \to \mathsf{B}^2(\mathsf{Q})$ defined by

$$\forall r_1, r_2 \in \mathsf{B}(\mathsf{Q}) : \mathbb{I}_{\mathsf{Q}}(r_1, r_2) = (r_2^{-1}, r_1^{-1}).$$

An AB-algebra is a structure of the form

$$\langle \mathbb{M}, \iota_A, \iota_B, \mathbb{I}, \gamma, \mu, \delta \rangle \tag{49}$$

where

- $\mathbb{M}$ is a monoid,
- $\iota_A : A \to \mathbb{M}, \iota_B : B \to \mathbb{M}$ are injective monoid homomorphisms,
- $\mathbb{I} : \mathbb{M} \to \mathbb{M}$ is a partial map,
- $\gamma : \mathbb{M} \to 2^{\mathcal{T}}$ is a total map,
- $\mu : \mathcal{T} \times \mathbb{M} \to \mathsf{B}^2(\mathsf{Q})$ is a partial map with $\mathrm{dom}(\mu) = \{(\boldsymbol{\theta}, m) \mid m \in \mathbb{M}, \boldsymbol{\theta} \in \gamma(m)\}$,
- $\delta : \mathcal{T} \times \mathbb{M} \to \mathrm{PGI}\{A, B\}$ is a partial map with $\mathrm{dom}(\delta) = \{(\boldsymbol{\theta}, m) \mid m \in \mathbb{M}, \boldsymbol{\theta} \in \gamma(m)\}$.

such that the following twelve axioms (AB1)–(AB12) are satisfied.

(AB1) $\mathrm{dom}(\mathbb{I}) = \mathrm{im}(\mathbb{I})$ is a submonoid of $\mathbb{M}$.

(AB2) $\mathbb{I}$ is an involutive anti-automorphism on $\mathrm{dom}(\mathbb{I})$.

(AB3) $\iota_A(A) \cup \iota_B(B) \subseteq \mathrm{dom}(\mathbb{I})$

(AB4) $\forall a \in A : \mathbb{I}(\iota_A(a)) = \iota_A(a^{-1}), \quad \forall b \in B : \mathbb{I}(\iota_B(b)) = \iota_B(b^{-1})$

For all $m, m_1, m_2 \in \mathbb{M}$:

(AB5) $\gamma(m_1)\gamma(m_2) \neq \emptyset$ and $m_1 m_2 \in \mathrm{dom}(\mathbb{I}) \;\Rightarrow\; m_1, m_2 \in \mathrm{dom}(\mathbb{I})$

(AB6) $\gamma(m_1)\gamma(m_2) \subseteq \gamma(m_1 m_2)$

(AB7) $\forall \boldsymbol{\theta} \in \gamma(m) : \delta(\boldsymbol{\theta}, m) \subseteq \mathrm{Gi}(\boldsymbol{\theta}) \times \mathrm{Ge}(\boldsymbol{\theta})$

(AB8) $\forall \boldsymbol{\theta}_1 \in \gamma(m_1), \boldsymbol{\theta}_2 \in \gamma(m_2) : \boldsymbol{\theta}_1\boldsymbol{\theta}_2$ defined $\;\Rightarrow\; \mu(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, m_1 m_2) = \mu(\boldsymbol{\theta}_1, m_1) \circ \mu(\boldsymbol{\theta}_2, m_2)$

(AB9) $\forall \boldsymbol{\theta}_1 \in \gamma(m_1), \boldsymbol{\theta}_2 \in \gamma(m_2) : \boldsymbol{\theta}_1\boldsymbol{\theta}_2$ defined $\;\Rightarrow\; \delta(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, m_1 m_2) = \delta(\boldsymbol{\theta}_1, m_1) \circ \delta(\boldsymbol{\theta}_2, m_2)$

(AB10) $m \in \mathrm{dom}(\mathbb{I}) \;\Rightarrow\; \gamma(\mathbb{I}(m)) = \mathbb{I}_{\mathcal{T}}(\gamma(m))$

(AB11) $m \in \mathrm{dom}(\mathbb{I}) \;\Rightarrow\; \forall \boldsymbol{\theta} \in \gamma(m) : \mu(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}(m)) = \mathbb{I}_{\mathsf{Q}}(\mu(\boldsymbol{\theta}, m))$

(AB12) $m \in \mathrm{dom}(\mathbb{I}) \;\Rightarrow\; \forall \boldsymbol{\theta} \in \gamma(m) : \delta(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}(m)) = \delta(\boldsymbol{\theta}, m)^{-1}$

Quite often, we will identify an AB-algebra with its underlying monoid.

3.2 AB-homomorphisms

For $i \in \{1,2\}$ let $\mathcal{M}_i = \langle \mathbb{M}_i, \iota_{A,i}, \iota_{B,i}, \mathbb{I}_i, \gamma_i, \mu_i, \delta_i \rangle$ be an AB-algebra with the underlying groups A, B and set Q. An AB-homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a map $\psi : \mathbb{M}_1 \to \mathbb{M}_2$ fulfilling the following seven properties (Hom1)–(Hom7):

(Hom1) $\psi : \mathbb{M}_1 \to \mathbb{M}_2$ is a monoid homomorphism.

(Hom2) $\forall a \in A : \psi(\iota_{A,1}(a)) = \iota_{A,2}(a),\ \forall b \in B : \psi(\iota_{B,1}(b)) = \iota_{B,2}(b)$

(Hom3) $\forall m \in \mathbb{M}_1 \setminus \gamma_1^{-1}(\{\emptyset\}) : m \in \mathrm{dom}(\mathbb{I}_1) \Leftrightarrow \psi(m) \in \mathrm{dom}(\mathbb{I}_2)$

(Hom4) $\forall m \in \mathrm{dom}(\mathbb{I}_1) : \mathbb{I}_2(\psi(m)) = \psi(\mathbb{I}_1(m))$

(Hom5) $\forall m \in \mathbb{M}_1 : \gamma_1(m) \subseteq \gamma_2(\psi(m))$

(Hom6) $\forall m \in \mathbb{M}_1 : \forall \boldsymbol{\theta} \in \gamma_1(m) : \mu_2(\boldsymbol{\theta}, \psi(m)) = \mu_1(\boldsymbol{\theta}, m)$

(Hom7) $\forall m \in \mathbb{M}_1 : \forall \boldsymbol{\theta} \in \gamma_1(m) : \delta_2(\boldsymbol{\theta}, \psi(m)) = \delta_1(\boldsymbol{\theta}, m)$

With $\mathrm{Hom}_{AB}(\mathcal{M}_1, \mathcal{M}_2)$ we denote the set of all AB-homomorphisms from $\mathcal{M}_1$ to $\mathcal{M}_2$. The following lemma is easy and widely (though implicitely) used.

Lemma 10. *Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be AB-algebras and let $\psi_2 \in \mathrm{Hom}_{AB}(\mathcal{M}_1, \mathcal{M}_2), \psi_3 \in \mathrm{Hom}_{AB}(\mathcal{M}_2, \mathcal{M}_3)$. Then $\psi_2 \circ \psi_3 : \mathcal{M}_1 \to \mathcal{M}_3$ is an AB-homomorphism.*

Proof. The only non-trivial check is for (Hom 3). Let $\psi_2 \in \mathrm{Hom}_{AB}(\mathcal{M}_1, \mathcal{M}_2), \psi_3 \in \mathrm{Hom}_{AB}(\mathcal{M}_2, \mathcal{M}_3)$. Let $m \in \mathbb{M}_1$ such that $\gamma_1(m) \neq \emptyset$. Since ψ_2 fulfills (Hom3):

$$m \in \mathrm{dom}(\mathbb{I}_1) \Leftrightarrow \psi_2(m) \in \mathrm{dom}(\mathbb{I}_2) \tag{50}$$

Since ψ_2 fulfills (Hom5):

$$\gamma_1(m) \subseteq \gamma_2(\psi_2(m))$$

hence $\gamma_2(\psi_2(m)) \neq \emptyset$. Since ψ_3 fulfills (Hom3):

$$\psi_2(m) \in \mathrm{dom}(\mathbb{I}_2) \Leftrightarrow \psi_3(\psi_2(m)) \in \mathrm{dom}(\mathbb{I}_3) \tag{51}$$

The conjunction of (50)(51) implies that

$$m \in \mathrm{dom}(\mathbb{I}_1) \Leftrightarrow \psi_3(\psi_2(m)) \in \mathrm{dom}(\mathbb{I}_3)$$

i.e. $\psi_2 \circ \psi_3$ fulfills axiom (Hom3).

Lemma 11. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be AB-algebras as above and let $\pi, \psi' \in \mathrm{Hom}_{AB}(\mathcal{M}_1, \mathcal{M}_2)$ such that*

(a) π is surjective,
(b) $\gamma_1(k) = \gamma_2(\pi(k))$ for all $k \in \mathbb{M}_1$ (this strengthens (Hom5) for π), and
(c) $\mathrm{dom}(\mathbb{I}_1) = \pi^{-1}(\mathrm{dom}(\mathbb{I}_2))$ (this strengthens (Hom3) for π).

If $\psi : \mathbb{M}_2 \to \mathbb{M}_2$ is a monoid homomorphism with $\psi' = \pi \circ \psi$, then $\psi \in \mathrm{Hom}_{AB}(\mathcal{M}_2, \mathcal{M}_2)$.

Proof. We have to check properties (Hom2)–(Hom7) for ψ.

(Hom2): For all $a \in A$ we have

$$\psi(\iota_{A,2}(a)) = \psi(\pi(\iota_{A,1}(a))) = \psi'(\iota_{A,1}(a)) = \iota_{A,2}(a).$$

For B the same argument holds.

For the following properties let $m \in \mathbb{M}_2$ be arbitrary. Since π is surjective, there exists $k \in \mathbb{M}_1$ with $\pi(k) = m$.

(Hom3): Assume in addition that $\gamma_2(m) \neq \emptyset$. By assumption (b) form the lemma we have $\gamma_1(k) = \gamma_2(\pi(k)) = \gamma_2(m) \neq \emptyset$. Hence we get:

$$\begin{aligned}
\psi(m) \in \mathrm{dom}(\mathbb{I}_2) &\iff \psi(\pi(k)) \in \mathrm{dom}(\mathbb{I}_2) \\
&\iff \psi'(k) \in \mathrm{dom}(\mathbb{I}_2) \\
&\iff k \in \mathrm{dom}(\mathbb{I}_1) && \text{((Hom3) for } \psi' \text{ and } \gamma_1(k) \neq \emptyset) \\
&\iff \pi(k) \in \mathrm{dom}(\mathbb{I}_2) && \text{((Hom3) for } \pi \text{ and } \gamma_1(k) \neq \emptyset) \\
&\iff m \in \mathrm{dom}(\mathbb{I}_2)
\end{aligned}$$

(Hom4): Assume that $m = \pi(k) \in \mathrm{dom}(\mathbb{I}_2)$. By assumption (c) from the lemma we have $k \in \mathrm{dom}(\mathbb{I}_1)$. Thus, with (Hom4) for π and ψ' we get:

$$\psi(\mathbb{I}_2(m)) = \psi(\mathbb{I}_2(\pi(k))) = \psi(\pi(\mathbb{I}_1(k))) = \psi'(\mathbb{I}_1(k)) = \mathbb{I}_2(\psi'(k)) = \mathbb{I}_2(\psi(\pi(k))) = \mathbb{I}_2(\psi(m)).$$

(Hom5): With assumption (b) from the lemma and (Hom5) for ψ', we get:

$$\gamma_2(m) = \gamma_2(\pi(k)) = \gamma_1(k) \subseteq \gamma_2(\psi'(k)) = \gamma_2(\psi(\pi(k))) = \gamma_2(\psi(m)).$$

(Hom6): Let $\boldsymbol{\theta} \in \gamma_2(m) = \gamma_1(k)$. With (Hom6) for π and ψ' we get:

$$\mu_2(\boldsymbol{\theta}, m) = \mu_2(\boldsymbol{\theta}, \pi(k)) = \mu_1(\boldsymbol{\theta}, k) = \mu_2(\boldsymbol{\theta}, \psi'(k)) = \mu_2(\boldsymbol{\theta}, \psi(\pi(k))) = \mu_2(\boldsymbol{\theta}, \psi(m)).$$

(Hom7) for ψ can be verified with a similar calculation as for (Hom6). $\square$

Later, it will be convenient to check the conditions for an AB-homomorphism only on generators. The following lemma gives conditions under which this is possible.

Lemma 12. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be AB-algebras as above and assume that $\mathcal{M}_1$ satisfies the following:*

(A) The monoid $\mathbb{M}_1$ is generated by the set Γ and $A \cup B \subseteq \Gamma$
(B) $\gamma_1(g) \neq \emptyset$ for all $g \in \Gamma$
(C) For every $m \in \mathbb{M}_1$ there exists a decomposition $m = g_1 \cdots g_n$ with $g_1, \ldots, g_n \in \Gamma$ such that: $\gamma_1(m) = \gamma_1(g_1) \cdots \gamma_1(g_n)$ and $[m \in \mathrm{dom}(\mathbb{I}_1) \iff g_1, \ldots, g_n \in \mathrm{dom}(\mathbb{I}_1)]$.[3]

Let $\psi : \mathbb{M}_1 \to \mathbb{M}_2$ be a monoid homomorphism. Then $\psi \in \mathrm{Hom}_{AB}(\mathcal{M}_1, \mathcal{M}_2)$ if and only if for every $g \in \Gamma$ and $\boldsymbol{\theta} \in \gamma_1(g)$ we have:

(a) $\forall a \in A : \psi(\iota_{A,1}(a) = \iota_{A_2}(a)$ and $\forall b \in B : \psi(\iota_{B,1}(b) = \iota_{B,2}(b)$
(b) $g \in \mathrm{dom}(\mathbb{I}_1) \Leftrightarrow \psi(g) \in \mathrm{dom}(\mathbb{I}_2)$

[3] If $n = 0$, i.e., $m = 1$, then $\gamma_1(g_1) \cdots \gamma_1(g_n)$ is the neutral element $\{(\theta, 0, \theta) \mid \theta \in \mathcal{T}_6\}$ of the monoid $2^{\mathcal{T}}$.

(c) If $g \in \mathrm{dom}(\mathbb{I}_1)$ *then* $\mathbb{I}_2(\psi(g)) = \psi(\mathbb{I}_1(g))$
(d) $\gamma_1(g) \subseteq \gamma_2(\psi(g))$
(e) $\mu_2(\boldsymbol{\theta}, \psi(g)) = \mu_1(\boldsymbol{\theta}, g)$
(f) $\delta_2(\boldsymbol{\theta}, \psi(g)) = \delta_1(\boldsymbol{\theta}, g)$

Proof. First, suppose that ψ is an AB-homomorphism. By definition it must fulfilll conditions (Hom2)–(Hom7). Since by (B), $\gamma_1(g) \neq \emptyset$ for all $g \in \Gamma$, (Hom3) implies condition (b) from the lemma. The other five conditions (a), (c)-(f) follow immediately from (Hom2), (Hom4)–(Hom7).

Conversely, let us suppose that ψ fulfills conditions (a)–(f) of the lemma. By (a), condition (Hom2) is fulfilled. For the following points consider an element $m \in \mathbb{M}_1$ and choose a decomposition

$$m = g_1 g_2 \cdots g_n \tag{52}$$

that satisfies (C) from the lemma ($g_1, \ldots, g_n \in \Gamma$).

Extending (b) to $\mathbb{M}_1$. Suppose that

$$\gamma_1(m) \neq \emptyset. \tag{53}$$

Let us first suppose that

$$m \in \mathrm{dom}(\mathbb{I}_1). \tag{54}$$

By (C) from the lemma we have $\gamma_1(m) = \gamma_1(g_1) \cdots \gamma_1(g_n)$ Hence, (53) implies that

$$\gamma_1(g_1) \cdots \gamma_1(g_n) \neq \emptyset. \tag{55}$$

Together with axiom (AB5) of AB-algebras (which can be easily extended to products of arbitrary length using axiom (AB6)), (54) and (55) imply

$$g_i \in \mathrm{dom}(\mathbb{I}_1) \text{ for all } 1 \leq i \leq n. \tag{56}$$

By condition (b) of the lemma, (56) implies

$$\psi(g_i) \in \mathrm{dom}(\mathbb{I}_2) \text{ for all } 1 \leq i \leq n. \tag{57}$$

Since $\mathrm{dom}(\mathbb{I}_2)$ is by axiom (AB1) a submonoid of $\mathbb{M}_2$, we get

$$\psi(g_1) \cdots \psi(g_n) \in \mathrm{dom}(\mathbb{I}_2). \tag{58}$$

But ψ is a monoid homomorphism, hence

$$\psi(m) = \psi(g_1) \cdots \psi(g_n) \in \mathrm{dom}(\mathbb{I}_2). \tag{59}$$

We have proved that under the hypothesis that $\gamma_1(m) \neq \emptyset$, (54) implies (59).

Let us now establish the converse. Assume that again $\gamma_1(m) \neq \emptyset$ and let $\psi(m) \in \mathrm{dom}(\mathbb{I}_2)$. From $\gamma_1(m) = \gamma_1(g_1) \cdots \gamma_1(g_n) \neq \emptyset$ (assumption (C)), we obtain with condition (d) of the lemma:

$$\gamma_2(\psi(g_1)) \cdots \gamma_2(\psi(g_n)) \neq \emptyset.$$

With $\psi(m) = \psi(g_1) \cdots \psi(g_n) \in \mathrm{dom}(\mathbb{I}_2)$ and axiom (AB5) we get $\psi(g_i) \in \mathrm{dom}(\mathbb{I}_2)$ for all $1 \leq i \leq n$. Hence, (b) from the lemma implies $g_i \in \mathrm{dom}(\mathbb{I}_1)$ for all $1 \leq i \leq n$. Since $\mathrm{dom}(\mathbb{I}_1)$ is a submonoid of $\mathbb{M}_1$, we finally get $m = g_1 \cdots g_n \in \mathrm{dom}(\mathbb{I}_1)$.

Extending (c) to $\mathbb{M}_1$. Assume that $m \in \mathrm{dom}(\mathbb{I}_1)$. Hence, by (C) from the lemma we have $g_1, \dots, g_n \in \mathrm{dom}(\mathbb{I}_1)$. Condition (c) from the lemma implies that $\mathbb{I}_2(\psi(g_i)) = \psi(\mathbb{I}_1(g_i))$ for all $1 \le i \le n$. Since ψ, $\mathbb{I}_1$, and $\mathbb{I}_2$ are monoid homomorphisms, we obtain $\mathbb{I}_2(\psi(m)) = \psi(\mathbb{I}_1(m))$ as required.

Extending (d) to $\mathbb{M}_1$. For $m = g_1 \cdots g_n$ we obtain

$$\begin{aligned}
\gamma_1(g_1 \cdots g_n) &= \gamma_1(g_1) \cdots \gamma_1(g_n) && \text{((C) from the lemma)} \\
&\subseteq \gamma_2(\psi(g_1)) \cdots \gamma_2(\psi(g_n)) && \text{(condition (d) from the lemma)} \\
&\subseteq \gamma_2(\psi(g_1) \cdots \psi(g_n)) && \text{(axiom (AB6)) for } \mathcal{M} \\
&= \gamma_2(\psi(g_1 \cdots g_n)) && (\psi \text{ is a monoid homomorphism})
\end{aligned}$$

as required.

Extending (e) to $\mathbb{M}_1$. Assume that $\boldsymbol{\theta} \in \gamma_1(m) = \gamma_1(g_1) \cdots \gamma_1(g_n)$. Hence, the path type $\boldsymbol{\theta}$ must have the form

$$\boldsymbol{\theta} = \boldsymbol{\theta}_1 \cdots \boldsymbol{\theta}_n \text{ with } \boldsymbol{\theta}_i \in \gamma_1(g_i) \subseteq \gamma_2(\psi(g_i)) \text{ for } 1 \le i \le n. \tag{60}$$

We obtain

$$\begin{aligned}
\mu_2(\boldsymbol{\theta}, \psi(m)) &= \mu_2(\prod_{i=1} \boldsymbol{\theta}_i, \prod_{i=1}^{n} \psi(g_i)) && (\psi \text{ is a monoid homomorphism}) \\
&= \prod_{i=1}^{n} \mu_2(\boldsymbol{\theta}_i, \psi(g_i)) && \text{(axiom (AB8) for } \mathcal{M}_2) \\
&= \prod_{i=1}^{n} \mu_1(\boldsymbol{\theta}_i, g_i) && \text{(by condition (e) from the lemma)} \\
&= \mu_1(\boldsymbol{\theta}, m) && \text{(axiom (AB8) for } \mathcal{M}_1)
\end{aligned}$$

as required.

Extending (f) to $\mathbb{M}_1$. Consider some $\boldsymbol{\theta} \in \gamma_1(m)$, hence of the form (60). We obtain

$$\begin{aligned}
\delta_2(\boldsymbol{\theta}, \psi(m)) &= \delta_2(\prod_{i=1} \boldsymbol{\theta}_i, \prod_{i=1}^{n} \psi(g_i)) && (\psi \text{ is a monoid homomorphism}) \\
&= \prod_{i=1}^{n} \delta_2(\boldsymbol{\theta}_i, \psi(g_i)) && \text{(axiom (AB9) for } \mathcal{M}_2) \\
&= \prod_{i=1}^{n} \delta_1(\boldsymbol{\theta}_i, g_i) && \text{(by condition (f) from the lemma)} \\
&= \delta_1(\boldsymbol{\theta}, m) && \text{(axiom (AB9) for } \mathcal{M}_1)
\end{aligned}$$

as required. $\square$

3.3 AB-subalgebras and quotients

For $i \in \{1, 2\}$ let $\mathcal{M}_i = \langle \mathbb{M}_i, \iota_{A,i}, \iota_{B,i}, \mathbb{I}_i, \gamma_i, \mu_i, \delta_i \rangle$ be an AB-algebra with the underlying groups A, B and set Q. Then, $\mathcal{M}_1$ is said to be an *AB-subalgebra* of $\mathcal{M}_2$ if $\mathbb{M}_1 \subseteq \mathbb{M}_2$ and the inclusion map $\iota : \mathbb{M}_1 \to \mathbb{M}_2$ is an AB-homomorphism. This means that $\iota_{A,1} = \iota_{A,2}$, $\iota_{B,1} = \iota_{B,2}$, $\mathbb{I}_1 = \mathbb{I}_2 \restriction \mathbb{M}_1$, $\gamma_1 = \gamma_2 \restriction \mathbb{M}_1$, $\mu_1 = \mu_2 \restriction \mathcal{T} \times \mathbb{M}_1$, and $\delta_1 = \delta_2 \restriction \mathcal{T} \times \mathbb{M}_1$. Vice versa, if $\mathbb{M}_1$ is a submonoid of $\mathbb{M}_2$ which contains $\iota_{A,2}(A) \cup \iota_{B,2}(B)$ and which is closed under $\mathbb{I}_2$, then there is a natural way to endow $\mathbb{M}_1$ with the structure of an AB-subalgebra of $\mathcal{M}_2$: it suffices to set $\iota_{A,1} = \iota_{A,2}$, $\iota_{B,1} = \iota_{B,2}$, $\mathbb{I}_1 = \mathbb{I}_2 \restriction \mathbb{M}_1$, $\gamma_1 = \gamma_2 \restriction \mathbb{M}_1$, $\mu_1 = \mu_2 \restriction \mathcal{T} \times \mathbb{M}_1$, and $\delta_1 = \delta_2 \restriction \mathcal{T} \times \mathbb{M}_1$. This structure is called the AB-structure *induced* by $\mathcal{M}_2$ on $\mathcal{M}_1$.

Definition 6. *Let $\equiv$ be a monoid congruence on the monoid $\mathbb{M}$ and $\mathcal{M} = \langle \mathbb{M}, \iota_A, \iota_B, \mathbb{I}, \gamma, \mu, \delta \rangle$ be an AB-algebra. We say that $\equiv$ is* compatible *w.r.t. the AB-algebra $\mathcal{M}$, if the following holds for all $m, m' \in \mathbb{M}$ with $m \equiv m'$ and all $a, a' \in A$ and $b, b' \in B$:*

(a) $m \in \mathrm{dom}(\mathbb{I}) \Longleftrightarrow m' \in \mathrm{dom}(\mathbb{I})$ and if $m, m' \in \mathrm{dom}(\mathbb{I})$ then $\mathbb{I}(m) \equiv \mathbb{I}(m')$,
(b) $\iota_A(a) \equiv \iota_A(a') \Longrightarrow a = a'$ and $\iota_B(b) \equiv \iota_B(b') \Longrightarrow b = b'$
(c) $\gamma(m) = \gamma(m')$
(d) For all $\boldsymbol{\theta} \in \gamma(m)$: $\mu(\boldsymbol{\theta}, m) = \mu(\boldsymbol{\theta}, m')$ and $\delta(\boldsymbol{\theta}, m) = \delta(\boldsymbol{\theta}, m')$.

By (a), $\equiv$ is also a congruence on $\mathrm{dom}(\mathbb{I})$, i.e. we can consider the quotient monoid $\mathrm{dom}(\mathbb{I})/_{\equiv}$. By (a), (c) and (d), the mappings $\mathbb{I}$, γ, μ, and δ induce quotient mappings (we use the same symbols for these quotient mappings)

$$\begin{aligned} \mathbb{I} : \mathrm{dom}(\mathbb{I})/_{\equiv} &\to \mathrm{dom}(\mathbb{I})/_{\equiv} \\ \gamma : \mathbb{M}/_{\equiv} &\to 2^{\mathcal{T}} \\ \mu : \mathcal{T} \times \mathbb{M}/_{\equiv} &\to \mathsf{B}^2(\mathsf{Q}) \\ \delta : \mathcal{T} \times \mathbb{M}/_{\equiv} &\to \mathrm{PGI}\{A, B\}. \end{aligned}$$

Moreover, by (b), ι_A and ι_B can be viewed as embeddings of A and B, respectively, into $\mathbb{M}/_{\equiv}$. It is now easy to check that

$$\mathcal{M}/_{\equiv} \;=\; \langle \mathbb{M}/_{\equiv}, \iota_A, \iota_B, \mathbb{I}, \gamma, \mu, \delta \rangle$$

is again an AB-algebra; it is called the quotient of $\mathcal{M}$ w.r.t. $\equiv$. Moreover, the projection morphism $\pi_{\equiv}$ with $\pi_{\equiv}(m) = [m]_{\equiv}$ becomes a surjective AB-homomorphism $\pi_{\equiv} : \mathcal{M} \to \mathcal{M}/_{\equiv}$ such that moreover for all $m \in \mathbb{M}$:

$$\gamma(m) = \gamma(\pi_{\equiv}(m)) \quad \text{and} \quad (m \in \mathrm{dom}(\mathbb{I}) \Longleftrightarrow \pi_{\equiv}(m) \in \mathrm{dom}(\mathbb{I})) \tag{61}$$

Hence, with Lemma 11 we get:

Lemma 13. *Let $\mathcal{M}$ be an AB-algebra with underlying monoid $\mathbb{M}$ and let the monoid congruence $\equiv$ on $\mathbb{M}$ be compatible w.r.t. $\mathcal{M}$. Let $\psi' \in \mathrm{Hom}_{AB}(\mathcal{M}, \mathcal{M}/_{\equiv})$. If $\psi : \mathbb{M}/_{\equiv} \to \mathbb{M}/_{\equiv}$ is a monoid homomorphism with $\psi' = \pi_{\equiv} \circ \psi$, then $\psi \in \mathrm{Hom}_{AB}(\mathcal{M}/_{\equiv}, \mathcal{M}/_{\equiv})$.*

Lemma 14. *Let $\mathcal{M}$ be an AB-algebra with underlying monoid $\mathbb{M}$ and let the monoid congruence $\equiv$ on $\mathbb{M}$ be compatible w.r.t. $\mathcal{M}$. If $\mathcal{M}$ fulfills hypotheses (A)(B)(C) of Lemma 12 for the set of generators Γ, then $\mathcal{M}/_{\equiv}$ fulfills (A)(B)(C) for the set of generators $\pi_{\equiv}(\Gamma)$.*

(this can be easily checked).

3.4 The AB-algebra $\mathbb{H}_t$

From now on, we assume that $\mathbb{H}$ is a *cancellative monoid*. In this section, we will define an AB-algebra $\mathbb{H}_t$. In a first step, we extend the monoid $\mathbb{H} * \{t, \bar{t}\}^*$ to an AB-algebra that we denote with $\mathbb{H} * \{t, \bar{t}\}^*$ again. The AB-algebra $\mathbb{H}_t$ will be a quotient of $\mathbb{H} * \{t, \bar{t}\}^*$.

Recall the definition of a normal fta from Definition 5. Given an HNN-extension (1) and a normal fta $\mathcal{A}$ with state set Q, we define an AB-algebra

$$\langle \mathbb{H} * \{t, \bar{t}\}^*, \iota_A, \iota_B, \mathbb{I}_t, \gamma_t, \mu_t, \delta_t \rangle \tag{62}$$

with underlying monoid $\mathbb{H} * \{t, \bar{t}\}^*$ and set of states Q as follows:

- ι_A and ι_B are the natural injections from A (resp. B) into $\mathbb{H} * \{t, \bar{t}\}^*$.
- $\mathrm{dom}(\mathbb{I}_t) = \mathsf{I}(\mathbb{H}) * \{t, t^{-1}\}^*$, where $\mathsf{I}(\mathbb{H})$ is the subgroup of invertible elements of $\mathbb{H}$
- $\mathbb{I}_t$ is the unique involutive anti-isomorphism $\mathbb{I}_t : \mathrm{dom}(\mathbb{I}_t) \to \mathrm{dom}(\mathbb{I}_t)$ such that

$$\forall h \in \mathsf{I}(\mathbb{H}) : \mathbb{I}_t(h) = h^{-1},\ \mathbb{I}_t(t) = t^{-1},\ \mathbb{I}_t(t^{-1}) = t.$$

- The map γ_t is the one previously defined in (32).
- The map $\mu_t : \mathcal{T} \times \mathbb{H} * \{t, \bar{t}\}^* \to \mathsf{B}^2(\mathsf{Q})$ is defined by:

$$\mu_t(\boldsymbol{\theta}, s) = \begin{cases} \langle \mu_1(\boldsymbol{\theta}, s), \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}_t(s))^{-1} \rangle & \text{if } s \in \mathrm{dom}(\mathbb{I}_t) \\ \langle \mu_1(\boldsymbol{\theta}, s), \emptyset \rangle & \text{if } s \notin \mathrm{dom}(\mathbb{I}_t). \end{cases} \tag{63}$$

  Here, μ_1 is the representation map $\mu_{\mathcal{A},1}$ associated with the partitioned fta $\mathcal{A}$ fixed above, see (30).
- The map $\delta_t : \mathcal{T} \times \mathbb{H} * \{t, \bar{t}\}^* \to \mathrm{PGI}\{A, B\}$ is defined by:

$$\delta_t(\boldsymbol{\theta}, s) = \{(c, c') \in \mathrm{Gi}(\boldsymbol{\theta}) \times \mathrm{Ge}(\boldsymbol{\theta}) \mid cs \sim sc'\},$$

where $\sim$ is the congruence on $\mathbb{H} * \{t, \bar{t}\}^*$ from Definition 1. Note that $\delta_t(\boldsymbol{\theta}, s)$ is indeed a partial isomorphism from $\mathrm{PGI}\{A, B\}$: It is a partial injection, since the quotient $\mathbb{H}_t = \mathbb{H} * \{t, \bar{t}\}^*/_\sim$ is cancellative (Lemma 3). Moreover, if $c_1 s \sim sc_1'$ and $c_2 s \sim sc_2'$, then $(c_1c_2)s \sim c_1 s c_2' \sim s(c_1'c_2')$.

It is noteworthy that for every $s \in \mathbb{H} * \{t, \bar{t}\}^*$:

$$\gamma_t(s) \neq \emptyset \iff s \in \mathrm{Red}(\mathbb{H}, t). \tag{64}$$

Before we show that $\langle \mathbb{H} * \{t, \bar{t}\}^*, \iota_A, \iota_B, \mathbb{I}_t, \mu_t, \gamma_t, \delta_t \rangle$ is indeed an AB-algebra, let us prove the following lemma:

Lemma 15. *Let $s_1, s_2 \in \mathbb{H} * \{t, \bar{t}\}^*$, $\boldsymbol{\theta}_1 \in \gamma_+(s_1)$, $\boldsymbol{\theta}_2 \in \gamma_+(s_2)$, $\boldsymbol{\theta}_1\boldsymbol{\theta}_2$ defined. Assume that $(c, c') \in \mathrm{Gi}(\boldsymbol{\theta}_1\boldsymbol{\theta}_2) \times \mathrm{Ge}(\boldsymbol{\theta}_1\boldsymbol{\theta}_2) = \mathrm{Gi}(\boldsymbol{\theta}_1) \times \mathrm{Ge}(\boldsymbol{\theta}_2)$ and $cs_1s_2 \sim s_1s_2c'$. Then there exists $c'' \in \mathrm{Ge}(\boldsymbol{\theta}_1) = \mathrm{Gi}(\boldsymbol{\theta}_2)$ such that $cs_1 \sim s_1c''$ and $c''s_2 \sim s_2c'$.*

Proof. Assume that $s_1 = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n$ and $s_2 = k_0 t^{\beta_1} k_1 \cdots t^{\beta_m} k_m$.

Case 1. $n = m = 0$: Thus, $ch_0k_0 = h_0k_0c'$. Since $\boldsymbol{\theta}_1\boldsymbol{\theta}_2$ is defined, Lemma 5 (applied to $\mathcal{G}_6$) implies that either (i) $\tau\mathrm{i}(\boldsymbol{\theta}_1) = \tau\mathrm{e}(\boldsymbol{\theta}_1)$ and $s_1 = h_0 \in \mathrm{Ge}(\boldsymbol{\theta}_1) = \mathrm{Gi}(\boldsymbol{\theta}_2)$ or (ii) $\tau\mathrm{i}(\boldsymbol{\theta}_2) = \tau\mathrm{e}(\boldsymbol{\theta}_2)$ and $s_2 = k_0 \in \mathrm{Ge}(\boldsymbol{\theta}_1) = \mathrm{Gi}(\boldsymbol{\theta}_2)$. In case (i) we can set $c'' = h_0^{-1}ch_0 \in \mathrm{Ge}(\boldsymbol{\theta}_1)$. In case (ii) we set $c'' = k_0c'k_0^{-1} \in \mathrm{Ge}(\boldsymbol{\theta}_1)$.

Case 2. $n = 0$ and $m > 0$. Thus,

$$(ch_0k_0)t^{\beta_1}k_1 \cdots t^{\beta_m}k_m \sim (h_0k_0)t^{\beta_1}k_1 \cdots t^{\beta_m}(k_mc').$$

We obtain a van Kampen diagram of the form

$$\begin{tikzcd}
\bullet \arrow[r, "h_0"] \arrow[d, "c"'] & \bullet \arrow[r, "k_0"] & \bullet \arrow[r, "t^{\beta_1}k_1\cdots t^{\beta_m}k_m"] \arrow[d, "c_0"] & \bullet \arrow[d, "c'"] \\
\bullet \arrow[r, "h_0"'] & \bullet \arrow[r, "k_0"'] & \bullet \arrow[r, "t^{\beta_1}k_1\cdots t^{\beta_m}k_m"'] & \bullet
\end{tikzcd}$$

where $c_0 \in A(\beta_1)$. As in case 1, since $\boldsymbol{\theta}_1\boldsymbol{\theta}_2$ is defined, Lemma 5 implies that either (i) $\tau\mathrm{i}(\boldsymbol{\theta}_1) = \tau\mathrm{e}(\boldsymbol{\theta}_1)$ and $s_1 = h_0 \in \mathrm{Ge}(\boldsymbol{\theta}_1) = \mathrm{Gi}(\boldsymbol{\theta}_2)$ or (ii) $k_0 \in \mathrm{Ge}(\boldsymbol{\theta}_1) = \mathrm{Gi}(\boldsymbol{\theta}_2) = A(\beta_1)$. In case (i) we can set $c'' = h_0^{-1}ch_0 \in \mathrm{Ge}(\boldsymbol{\theta}_1)$; in case (ii) we set $c' = k_0c_0k_0^{-1} \in \mathrm{Ge}(\boldsymbol{\theta}_1)$.

Case 3. $n > 0$ and $m = 0$. This case can be dealt analogously to case 2.

Case 4. $n, m > 0$. Hence, we have

$$(ch_0)t^{\alpha_1}h_1\cdots t^{\alpha_n}(h_nk_0)t^{\beta_1}k_1\cdots t^{\beta_m}k_m = h_0t^{\alpha_1}h_1\cdots t^{\alpha_n}(h_nk_0)t^{\beta_1}k_1\cdots t^{\beta_m}(k_mc').$$

We obtain a van Kampen diagram of the form

$$\begin{tikzcd}
& \bullet \arrow[r, "h_n"] \arrow[dd, "c_{2n}"] & \bullet \arrow[r, "k_0"] & \bullet \arrow[dr, bend left, "t^{\beta_1}k_1\cdots t^{\beta_m}k_m"] \arrow[dd, "c_{2n+1}"] & \\
\bullet \arrow[ur, bend left, "(ch_0)t^{\alpha_1}h_1\cdots t^{\alpha_n}"] \arrow[dr, bend right, "h_0t^{\alpha_1}h_1\cdots t^{\alpha_n}"'] & & & & \bullet \\
& \bullet \arrow[r, "h_n"'] & \bullet \arrow[r, "k_0"'] & \bullet \arrow[ur, bend right, "t^{\beta_1}k_1\cdots t^{\beta_m}k_mc'"'] &
\end{tikzcd}$$

where $c_{2n} \in B(\alpha_n)$ and $c_{2n+1} \in A(\beta_1)$. Again, with Lemma 5 we get either (i) $h_n, c_{2n} \in B(\alpha_n) = \mathrm{Ge}(\boldsymbol{\theta}_1) = \mathrm{Gi}(\boldsymbol{\theta}_2)$ or (ii) $k_0, c_{2n+1} \in A(\beta_1) = \mathrm{Ge}(\boldsymbol{\theta}_1) = \mathrm{Gi}(\boldsymbol{\theta}_2)$. In case (i) we set $c'' = h_n^{-1}c_{2n}h_n$; in case (ii) we set $c'' = k_0c_{2n+1}k_0^{-1}$. $\square$

Proposition 3. *The structure* $\langle \mathbb{H} * \{t, \bar{t}\}^*, \iota_A, \iota_B, \mathbb{I}_t, \mu_t, \gamma_t, \delta_t\rangle$ *is an AB-algebra.*

Proof. We check the properties (AB1)–(AB12). Properties (AB1)–(AB4) are obvious.

(AB5): Assume that $\gamma_t(s_1)\gamma_t(s_2) \neq \emptyset$ and $s_1s_2 \in \mathrm{dom}(\mathbb{I})$. Let $s_1 = h_0t^{\alpha_1}h_1\cdots t^{\alpha_n}h_n$ and $s' = h'_0t^{\alpha'_1}h'_1\cdots t^{\alpha'_m}h'_m$. Thus

$$s_1s_2 = h_0t^{\alpha_1}h_1\cdots t^{\alpha_n}(h_nh'_0)t^{\alpha'_1}h'_1\cdots t^{\alpha'_m}h'_m \in \mathrm{dom}(\mathbb{I})$$

Hence, $h_0, \ldots, h_{n-1}, h_nh'_0, h'_1, \ldots, h'_m$ are invertible elements of $\mathbb{H}$. It suffices to show that h_n and h'_0 are both invertible. This is clearly the case if $h_n = 1$ or $h'_0 = 1$. Hence, we can assume that $h_n \neq 1 \neq h'_0$. Since $\gamma_t(s_1)\gamma_t(s_2) \neq \emptyset$, Lemma 5 implies $h_n \in A \cup B$ or $h'_0 \in A \cup B$. The claim follows.

(AB6): This is stated in Lemma 7.

(AB7): Follows directly from the definition of δ_t.

(AB10): Note that by (34), it suffices to show $\gamma_t(\mathbb{I}_t(s)) = \mathbb{I}_{\mathcal{T}}(\gamma_t(s))$ only for 1 and all generators $s \in \mathsf{I}(\mathbb{H}) \cup \{t, t^{-1}\}$ of $\mathrm{dom}(\mathbb{I}_t)$. This follows easily from the definition of $\mathbb{I}_{\mathcal{T}}$ in (16).

(AB8): Let $\boldsymbol{\theta}_1 \in \gamma_t(s_1)$ and $\boldsymbol{\theta}_2 \in \gamma_t(s_2)$ such that $\boldsymbol{\theta}_1\boldsymbol{\theta}_2$ is defined. First assume that $s_1s_2 \in \mathrm{dom}(\mathbb{I}_t)$. Then, by (AB5), also $s_1, s_2 \in \mathrm{dom}(\mathbb{I}_t)$. Moreover, by (AB10), $\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1) \in \mathbb{I}_{\mathcal{T}}(\gamma_t(s_1)) =$

$\gamma_t(\mathbb{I}_t(s_1))$ and $\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_2) \in \gamma_t(\mathbb{I}_t(s_2))$. Since the fta $\mathcal{A}$ is multiplicative (see Definition 4), we obtain:

$$\begin{aligned}
\mu_t(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, s_1s_2) &= \langle \mu_1(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, s_1s_2), \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1\boldsymbol{\theta}_2), \mathbb{I}_t(s_1s_2))^{-1}\rangle \\
&= \langle \mu_1(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, s_1s_2), \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_2)\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1), \mathbb{I}_t(s_2)\mathbb{I}_t(s_1))^{-1}\rangle \\
&= \langle \mu_1(\boldsymbol{\theta}_1, s_1) \circ \mu_1(\boldsymbol{\theta}_2, s_2), (\mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_2), \mathbb{I}_t(s_2)) \circ \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1), \mathbb{I}_t(s_1)))^{-1}\rangle \\
&= \langle \mu_1(\boldsymbol{\theta}_1, s_1) \circ \mu_1(\boldsymbol{\theta}_2, s_2), \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1), \mathbb{I}_t(s_1))^{-1} \circ \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_2), \mathbb{I}_t(s_2))^{-1}\rangle \\
&= \langle \mu_1(\boldsymbol{\theta}_1, s_1), \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1), \mathbb{I}_t(s_1))^{-1}\rangle \circ \langle \mu_1(\boldsymbol{\theta}_2, s_2), \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_2), \mathbb{I}_t(s_2))^{-1}\rangle \\
&= \mu_t(\boldsymbol{\theta}_1, s_1) \circ \mu_t(\boldsymbol{\theta}_2, s_2)
\end{aligned}$$

On the other hand, if $s_1s_2 \notin \mathrm{dom}(\mathbb{I}_t)$, then either $s_1 \notin \mathrm{dom}(\mathbb{I}_t)$ or $s_2 \notin \mathrm{dom}(\mathbb{I}_t)$. Assume w.l.o.g. that $s_1 \notin \mathrm{dom}(\mathbb{I}_t)$, i.e. $\mu_t(\boldsymbol{\theta}_1, s_1) = \langle \mu_1(\boldsymbol{\theta}_1, s_1), \emptyset\rangle$. Again, by the fact that $\mathcal{A}$ is multiplicative, we obtain:

$$\begin{aligned}
\mu_t(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, s_1s_2) &= \langle \mu_1(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, s_1s_2), \emptyset\rangle \\
&= \langle \mu_1(\boldsymbol{\theta}_1, s_1) \circ \mu_1(\boldsymbol{\theta}_2, s_2), \emptyset\rangle \\
&= \langle \mu_1(\boldsymbol{\theta}_1, s_1), \emptyset\rangle \circ \mu_t(\boldsymbol{\theta}_2, s_2) \\
&= \mu_t(\boldsymbol{\theta}_1, s_1) \circ \mu_t(\boldsymbol{\theta}_2, s_2)
\end{aligned}$$

(AB9): The inclusion

$$\delta_t(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, s_1s_2) \subseteq \delta_t(\boldsymbol{\theta}_1, s_1) \circ \delta_t(\boldsymbol{\theta}_2, s_2)$$

follows directly from Lemma 15. For the reverse inclusion assume that $(c_1, c_2) \in \delta_t(\boldsymbol{\theta}_1, s_1)$ and $(c_2, c_3) \in \delta_t(\boldsymbol{\theta}_2, s_2)$. Hence, $c_1s_1 \sim s_1c_2$ and $c_2s_2 \sim s_2c_3$. This implies $c_1(s_1s_2) \sim s_1c_2s_2 \sim (s_1s_2)c_3$. Hence, $(c_1, c_3) \in \delta_t(\boldsymbol{\theta}_1\boldsymbol{\theta}_2, s_1s_2)$.

(AB11): Assume that $s \in \mathrm{dom}(\mathbb{I}_t) = \mathsf{I}(\mathbb{H}) * \{t, t^{-1}\}^*$ and let $\boldsymbol{\theta} \in \gamma_t(s)$. We obtain

$$\begin{aligned}
\mathbb{I}_{\mathsf{Q}}(\mu_t(\boldsymbol{\theta}, s)) &= \mathbb{I}_{\mathsf{Q}}(\langle \mu_1(\boldsymbol{\theta}, s), \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}_t(s))^{-1}\rangle) \\
&= \langle \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}_t(s)),\ \mu_1(\boldsymbol{\theta}, s)^{-1}\rangle \\
&= \langle \mu_1(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}_t(s)),\ \mu_1(\mathbb{I}_{\mathcal{T}}(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta})), \mathbb{I}_t(\mathbb{I}_t(s)))^{-1}\rangle \\
&= \mu_t(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}_t(s)).
\end{aligned}$$

(AB12): Assume that $s \in \mathrm{dom}(\mathbb{I}_t) = \mathsf{I}(\mathbb{H}) * \{t, t^{-1}\}^*$ and let $\boldsymbol{\theta} \in \gamma_t(s)$. We obtain

$$\begin{aligned}
\delta_t(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}_t(s)) &= \{(c, d) \in \mathrm{Gi}(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta})) \times \mathrm{Ge}(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta})) \mid c\mathbb{I}_t(s) \sim \mathbb{I}_t(s)d\} \\
&\overset{(17)}{=} \{(c, d) \in \mathrm{Ge}(\boldsymbol{\theta}) \times \mathrm{Gi}(\boldsymbol{\theta}) \mid \mathbb{I}_t(sc^{-1}) \sim \mathbb{I}_t(d^{-1}s)\} \\
&= \{(c, d) \in \mathrm{Ge}(\boldsymbol{\theta}) \times \mathrm{Gi}(\boldsymbol{\theta}) \mid sc^{-1} \sim d^{-1}s\} \\
&= \{(c, d) \in \mathrm{Ge}(\boldsymbol{\theta}) \times \mathrm{Gi}(\boldsymbol{\theta}) \mid ds \sim sc\} \\
&= \{(d, c) \in \mathrm{Gi}(\boldsymbol{\theta}) \times \mathrm{Ge}(\boldsymbol{\theta}) \mid ds \sim sc\}^{-1} \\
&= \delta_t(\boldsymbol{\theta}, s)^{-1}.
\end{aligned}$$

This concludes the proof of Proposition 3. □

Another AB-structure over $\mathbb{H} * \{t, \bar{t}\}^*$ can be defined by choosing, in place of the map γ_t defined in (32), the map γ_+ defined in (31). Assertion (64) is now replaced by

$$\forall s \in \mathbb{H} * \{t, \bar{t}\}^* : \gamma_+(s) \neq \emptyset. \tag{65}$$

We call the resulting structure

$$\langle \mathbb{H} * \{t, \bar{t}\}^*, \iota_A, \iota_B, \mathbb{I}_t, \mu_t, \gamma_+, \delta_t \rangle \tag{66}$$

the *positive* AB-algebra over $\mathbb{H} * \{t, \bar{t}\}^*$. This variant will be used in Section 10 where we deal with *positive* rational constraints.

One can check that the monoid-congruence $\sim$ on $\mathbb{H} * \{t, \bar{t}\}^*$ is compatible (see Definition 6) w.r.t. to the AB-algebra $\langle \mathbb{H} * \{t, \bar{t}\}^*, \iota_A, \iota_B, \mathbb{I}_t, \mu_t, \gamma_t, \delta_t \rangle$. Here, for $\gamma_t(s) = \gamma_t(s')$ and $\mu_t(\boldsymbol{\theta}, s) = \mu_t(\boldsymbol{\theta}, s')$ (if $s \sim s'$) it is important that the ftas $\mathcal{R}_6$ and $\mathcal{A}$ are $\sim$-saturated, see Definition 3. Hence, on the quotient monoid $\mathbb{H}_t = \mathbb{H} * \{t, \bar{t}\}^*/_\sim$, we obtain an AB-algebra $\langle \mathbb{H}_t, \iota_A, \iota_B, \mathbb{I}_t, \mu_t, \gamma_t, \delta_t \rangle$ that we will denote just with $\mathbb{H}_t$ in the following. Since $s \sim s'$ implies $\|s\| = \|s'\|$, the notion of norm remains well-defined in the quotient $\mathbb{H}_t$.

Similarly, we can define an AB-algebra

$$\langle \mathbb{H}_t, \iota_A, \iota_B, \mathbb{I}_t, \mu_t, \gamma_+, \delta_t \rangle, \tag{67}$$

where γ_+ is defined via the automaton $\mathcal{G}_6$ instead of $\mathcal{R}_6$, see (31).

3.5 Algebraic properties of $\mathbb{H}_t$

The monoid $\mathbb{H}_t$ has some properties which resemble equidivisibility. We detail here these properties.

Lemma 16. *Let $P, P', S, S' \in \mathbb{H}_t$, $\boldsymbol{\theta} \in \gamma_+(P) \cap \gamma_+(P')$, $\boldsymbol{\rho} \in \gamma_+(S) \cap \gamma_+(S')$ such that $\boldsymbol{\theta\rho}$ is defined, $\boldsymbol{\theta}$ is an H-type and $PS = P'S'$ in $\mathbb{H}_t$. Then, there exists $c \in \mathrm{Ge}(\boldsymbol{\theta})$ such that $P = P'c$ and $cS = S'$.*

Proof. Let $p, p', s, s' \in \mathbb{H} * \{t, \bar{t}\}^*$ such that $P = [p]_\sim$, $P' = [p']_\sim$, $S = [s]_\sim$, and $S' = [s']_\sim$. Since $\boldsymbol{\theta}$ is an H-type, we must have $p, p' \in \mathbb{H}$ and since the product $\boldsymbol{\theta\rho}$ is defined, $\mathrm{Ge}(\boldsymbol{\theta}) = \mathrm{Gi}(\boldsymbol{\rho})$. An inspection of the H-types in (19) and (20) shows that the vertex type $\tau\mathrm{e}(\boldsymbol{\theta}) = \tau\mathrm{i}(\boldsymbol{\rho})$ must be either (A, T), (B, T), or $(1, 1)$.

Case 1. $\tau\mathrm{e}(\boldsymbol{\theta}) = \tau\mathrm{i}(\boldsymbol{\rho}) = (1, 1)$. The structure of $\mathcal{G}_6$ implies $\boldsymbol{\rho} = (1, 1, 0, 1, 1)$. Since $\boldsymbol{\rho} = (1, 1, 0, 1, 1) \in \gamma_+(s) \cap \gamma_+(s')$, we have $s = s' = 1$, i.e., $S = S' = 1$. Hence, $P = P'$. Choosing $c = 1$, we obtain $c \in \{1\} = \mathrm{Ge}(\boldsymbol{\theta})$, $P = P'c$ and $cS = S'$.

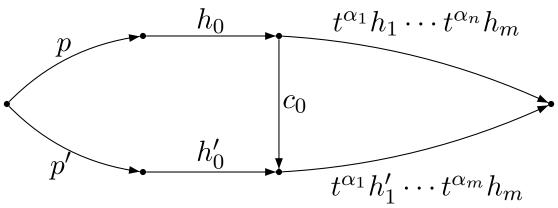
Case 2. $\tau\mathrm{e}(\boldsymbol{\theta}) = \tau\mathrm{i}(\boldsymbol{\rho}) = (C, T)$ for $C \in \{A, B\}$. Thus, $\mathrm{Gi}(\boldsymbol{\rho}) = \mathrm{Ge}(\boldsymbol{\theta}) = C$. Suppose that

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_n \text{ and } s' = h'_0 t^{\alpha'_1} h'_1 \cdots t^{\alpha'_m} h_m.$$

where $h_i, h'_j \in \mathbb{H}, \alpha_i, \alpha'_j \in \{-1, +1\}$. Since $ps \sim p's'$ and $p, p' \in \mathbb{H}$, we have $m = n$ and $\alpha_i = \alpha'_i$ for $1 \leq i \leq m$, i.e.,

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_n} h_m \text{ and } s' = h'_0 t^{\alpha_1} h'_1 \cdots t^{\alpha_m} h_m.$$

Since these sequences can be read into $\mathcal{G}_6$ starting from (C, T), we must have $h_0, h'_0 \in C$. Moreover, if $m \geq 1$, then $C = A(\alpha_1)$. Since $ps \sim p's'$ there exists a van Kampen diagram of the form



with $c_0 \in C$ (if $m = 0$, then $c_0 = 1$). Let us choose $c = h'_0 c_0^{-1} h_0^{-1} \in C = \mathrm{Ge}(\boldsymbol{\theta})$. We obtain $P = P'c$ and $cS = S'$. $\square$

Lemma 17. *Let $P, P', S, S' \in \mathbb{H}_t$, $\boldsymbol{\theta} \in \gamma_+(P)$, $\boldsymbol{\theta}' \in \gamma_+(P')$, $\boldsymbol{\rho} \in \gamma_+(S)$, $\boldsymbol{\rho}' \in \gamma_+(S')$ such that $\boldsymbol{\theta\rho} = \boldsymbol{\theta}'\boldsymbol{\rho}'$ is defined, $\boldsymbol{\theta}, \boldsymbol{\theta}'$ are T-types (see (22)) and $PS = P'S'$ in $\mathbb{H}_t$. Then, one of the following cases must occur:*

(1) $\|P\| = \|P'\|$, $\boldsymbol{\theta} = \boldsymbol{\theta}'$, and there exists $c \in \mathrm{Ge}(\boldsymbol{\theta})$ such that $Pc = P'$ and $S = cS'$.

(2) $\|P\| < \|P'\|$ and there exist $c \in \mathrm{Ge}(\boldsymbol{\theta})$, $P'_1, P'_2, P'_3 \in \mathbb{H} * \{t, \bar{t}\}^*$ such that P'_1 has a T-type $\boldsymbol{\theta}'_1$, P'_3 has a T-type $\boldsymbol{\theta}'_3$, P'_2 has a H-type $\boldsymbol{\theta}'_2$, and

$$Pc = P'_1, \quad P' = P'_1P'_2P'_3, \quad S = cP'_2P'_3S', \quad \boldsymbol{\theta}' = \boldsymbol{\theta}'_1\boldsymbol{\theta}'_2\boldsymbol{\theta}'_3, \quad \boldsymbol{\rho} = \boldsymbol{\theta}'_2\theta'_3\boldsymbol{\rho}'.$$

(3) $\|P\| > \|P'\|$ and there exists $c \in \mathrm{Ge}(\boldsymbol{\theta}')$, $P_1, P_2, P_3 \in \mathbb{H} * \{t, \bar{t}\}^*$ such that P_1 has a T-type $\boldsymbol{\theta}_1$, P_3 has a T-type $\boldsymbol{\theta}_3$, P_2 has a H-type $\boldsymbol{\theta}_2$ and

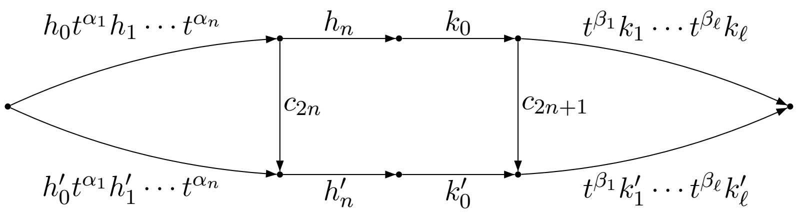
$$P = P_1P_2P_3, \quad P_1 = P'c, \quad cP_2P_3S = S', \quad \boldsymbol{\theta} = \boldsymbol{\theta}_1\boldsymbol{\theta}_2\boldsymbol{\theta}_3, \quad \boldsymbol{\rho}' = \boldsymbol{\theta}_2\theta_3\boldsymbol{\rho}.$$

Proof. Let $p, p', s, s' \in \mathbb{H} * \{t, \bar{t}\}^*$ such that $P = [p]_\sim$, $P' = [p']_\sim$, $S = [s]_\sim$, and $S' = [s']_\sim$. Since the products $\boldsymbol{\theta\rho}, \boldsymbol{\theta}'\boldsymbol{\rho}'$ are defined, $\mathrm{Ge}(\boldsymbol{\theta}) = \mathrm{Gi}(\boldsymbol{\rho})$ and $\mathrm{Ge}(\boldsymbol{\theta}') = \mathrm{Gi}(\boldsymbol{\rho}')$. Suppose that

$$p = h_0t^{\alpha_1}h_1 \cdots t^{\alpha_n}h_n \qquad s = k_0t^{\beta_1}k_1 \cdots t^{\beta_\ell}k_\ell$$
$$p' = h'_0t^{\alpha'_1}h'_1 \cdots t^{\alpha'_\nu}h'_\nu \qquad s' = k'_0t^{\beta'_1}k'_1 \cdots t^{\beta_\lambda}k'_\lambda$$

where $h_i, k_j, h'_\iota, k'_\kappa \in \mathbb{H}$, $\alpha_i, \beta_j, \alpha'_\iota, \beta'_\kappa \in \{-1, +1\}$. Since $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ are T-types, the structure of $\mathcal{G}_6$ implies that $n, \nu \geq 1$, $h_n \in \mathrm{Ge}(\boldsymbol{\theta})$, and $h'_\nu \in \mathrm{Ge}(\boldsymbol{\theta}')$ (note that the only arrow entering (A, H) (resp. (B, H)) in $\mathcal{G}_6$ that is labeled with a subset of $\mathbb{H}$ is the A-loop (resp. B-loop)). Since $ps \sim p's'$, we get $n + \ell = \nu + \lambda$ and $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_\ell) = (\alpha'_1, \ldots, \alpha'_\nu, \beta'_1, \ldots, \beta'_\lambda)$. Moreover, $\tau\mathrm{i}(\boldsymbol{\theta}) = \tau\mathrm{i}(\boldsymbol{\theta}')$ and the Boolean component of both $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ is 1, since they are both T-types.

Case 1. $\|p\| = \|p'\|$: Thus, $n = \nu$, $\ell = \lambda$, $\tau\mathrm{e}(\boldsymbol{\theta}) = \tau\mathrm{e}(\boldsymbol{\theta}')$ (and hence $\boldsymbol{\theta} = \boldsymbol{\theta}'$), and there exist a van Kampen diagram of the form

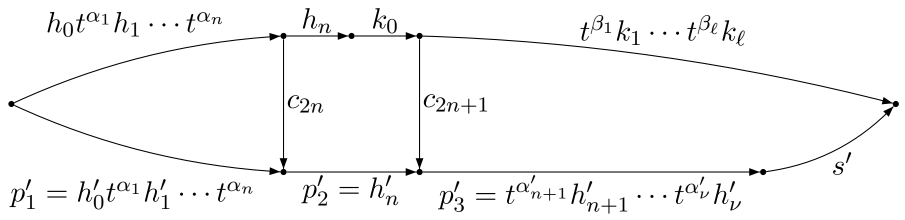


with $c_{2n} \in B(\alpha_n) = \mathrm{Ge}(\boldsymbol{\theta}) = \mathrm{Ge}(\boldsymbol{\theta}')$. Hence, we can set $c = h_n^{-1}c_{2n}h'_n \in \mathrm{Ge}(\boldsymbol{\theta})$.

Case 2. $\|p\| < \|p'\|$: Hence $n < \nu$. Let us set

$$p'_1 = h'_0t^{\alpha_1}h'_1 \cdots h'_{n-1}t^{\alpha_n}, \quad p'_2 = h'_n, \quad p'_3 = t^{\alpha'_{n+1}}h'_{n+1} \cdots t^{\alpha'_\nu}h'_\nu.$$

Thus, $p' = p'_1p'_2p'_3$ and there exists a van Kampen diagram of the following form:



Since $h_n \in \mathrm{Ge}(\boldsymbol{\theta})$ and $c_{2n} \in B(\alpha_n) = \mathrm{Ge}(\boldsymbol{\theta})$, we have $c = h_n^{-1} c_{2n} \in \mathrm{Ge}(\boldsymbol{\theta})$. We obtain $p'_1 = pc$ and $s \sim cp'_2 p'_3 s'$.

For the T-types $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ we get

$$\boldsymbol{\theta} = (A(\alpha_1), T, 1, B(\alpha_n), H) \quad \text{and} \quad \boldsymbol{\theta}' = (A(\alpha_1), T, 1, B(\alpha'_\nu), H).$$

A possible type for p'_1 is

$$\boldsymbol{\theta}'_1 = (A(\alpha_1), T, 1, B(\alpha_n), H),$$

a possible type for p'_2 is

$$\boldsymbol{\theta}'_2 = (B(\alpha_n), H, 0, A(\alpha'_{n+1}), T)$$

and a possible type for p'_3 is

$$\boldsymbol{\theta}'_3 = (A(\alpha'_{n+1}), T, 1, B(\alpha'_\nu), H).$$

Thus $\boldsymbol{\theta}'_1, \boldsymbol{\theta}'_3$ are T-types and $\boldsymbol{\theta}'_2$ is an H-type such that $\boldsymbol{\theta}' = \boldsymbol{\theta}'_1 \boldsymbol{\theta}'_2 \boldsymbol{\theta}'_3$. It remains to show $\boldsymbol{\rho} = \boldsymbol{\theta}'_2 \boldsymbol{\theta}'_3 \boldsymbol{\rho}'$, i.e., $\boldsymbol{\rho} = (B(\alpha_n), H, 1, B(\alpha'_\nu), H)\boldsymbol{\rho}'$. Since $\|s\| = \ell > \lambda \geq 0$, the Boolean component of $\boldsymbol{\rho} \in \gamma_+(s)$ must be 1, which is the Boolean component of $(B(\alpha_n), H, 1, B(\alpha'_\nu), H)\boldsymbol{\rho}'$. Moreover, $\tau\mathrm{e}(\boldsymbol{\rho}) = \tau\mathrm{e}(\boldsymbol{\rho}') = \tau\mathrm{e}((B(\alpha_n), H, 1, B(\alpha'_\nu), H)\boldsymbol{\rho}')$. Finally, $\tau\mathrm{i}(\boldsymbol{\rho}) = \tau\mathrm{e}(\boldsymbol{\theta}) = (B(\alpha_n), H) = \tau\mathrm{i}((B(\alpha_n), H, 1, B(\alpha'_\nu), H)$.

Case 3. $\|p\| > \|p'\|$: This case is similar to Case 2. $\square$

3.6 The AB-algebra $\mathcal{W}^* * A * B$

Let $\mathcal{S}$ be a system of equations over $\mathbb{H}_t$ with involution and rational constraints. The rational constraints are expressed via the map μ_t defined by (63) in Section 3.4. We define an alphabet of "generic" symbols $\mathcal{W}$ with the underlying idea of representing inside each symbol the values of the functions $\gamma_t, \mu_t, \delta_t$ for the "concrete" value (i.e. in $\mathbb{H} * \{t, \bar{t}\}^*$) of that variable that leads to a solution of the system of equations.

Let $\mathcal{V}_0$ be some starting set that will be made precise later. Let us denote by $\mathcal{T}_{HT} \subseteq \mathcal{T}$ the set of all types which are either H-types or T-types. The set $\mathcal{W}$ of generic symbols is the set of all 5-tuples

$$(V, \epsilon, \boldsymbol{\theta}, r, \varphi) \in \mathcal{V}_0 \times \{-1, 0, 1\} \times \mathcal{T}_{HT} \times \mathsf{B}^2(\mathsf{Q}) \times \mathrm{PGI}\{A, B\}$$

such that:

- $r = (r_1, r_2)$ with $r_1 \subseteq \tau^{-1}(\tau\mathrm{i}(\boldsymbol{\theta})) \times \tau^{-1}(\tau\mathrm{e}(\boldsymbol{\theta}))$ and $r_2 \subseteq \tau^{-1}(\mathbb{I}_{\mathcal{R}}(\tau\mathrm{i}(\boldsymbol{\theta}))) \times \tau^{-1}(\mathbb{I}_{\mathcal{R}}(\tau\mathrm{e}(\boldsymbol{\theta})))$,
- $\varphi \subseteq \mathrm{Gi}(\boldsymbol{\theta}) \times \mathrm{Ge}(\boldsymbol{\theta})$, and
- $\forall (c, d) \in \varphi : \mu_t((\tau\mathrm{i}(\boldsymbol{\theta}), 0, \tau\mathrm{i}(\boldsymbol{\theta})), c) \circ r = r \circ \mu_t((\tau\mathrm{e}(\boldsymbol{\theta}), 0, \tau\mathrm{e}(\boldsymbol{\theta})), d)$

Let
$$\widehat{\mathcal{W}} = \{(V, \epsilon, \boldsymbol{\theta}, r, \varphi) \in \mathcal{W} \mid \epsilon \neq 0\}.$$

Let us consider the free product $\mathcal{W}^* * A * B$. We denote by $\iota_A : A \to \mathcal{W}^* * A * B$ (resp. $\iota_B : B \to \mathcal{W}^* * A * B$ the natural embedding of A (resp. B) into $\mathcal{W}^* * A * B$. Note that

$$\iota_A(A) \cap \iota_B(B) = \{1\}.$$

Most of the time, we will identify $\iota_A(a)$ with a and $\iota_B(b)$ with b. For every $s \in \mathcal{W}^* * A * B$ let $\|s\|$ be the number of occurrences of symbols from $\mathcal{W}$ in s. Clearly

$$\|uv\| = \|u\| + \|v\| \qquad \text{and} \qquad \|s\| = 0 \iff s \in A * B.$$

In this section, we will define an AB-algebra

$$\langle \mathcal{W}^* * A * B, \iota_A, \iota_B, \mathbb{I}_w, \gamma_w, \mu_w, \delta_w \rangle \tag{68}$$

with underlying monoid $\mathcal{W}^* * A * B$ and set of states Q. Let $\mathbb{I}_w$ be the unique monoid anti-homomorphism on $\mathrm{dom}(\mathbb{I}_w) = \widehat{\mathcal{W}}^* * A * B$ (the submonoid generated by $\widehat{\mathcal{W}} \cup \iota_A(A) \cup \iota_B(B)$) such that

$$\mathbb{I}_w(a) = a^{-1} \text{ for } a \in A \tag{69}$$
$$\mathbb{I}_w(b) = b^{-1} \text{ for } b \in B \tag{70}$$
$$\mathbb{I}_w(V, \epsilon, \boldsymbol{\theta}, r, \varphi) = (V, -\epsilon, \mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}_{\mathsf{Q}}(r), \varphi^{-1}). \tag{71}$$

The mapping $\gamma_w : \mathcal{W}^* * A * B \to 2^{\mathcal{T}}$ is defined on the generators of $\mathcal{W}^* * A * B$ as follows:

$$\gamma_w(a) = \{(A, T, 0, A, T), (A, H, 0, A, H)\} \text{ for every } a \in A \setminus \{1\} \tag{72}$$
$$\gamma_w(b) = \{(B, T, 0, B, T), (B, H, 0, B, H)\} \text{ for every } b \in B \setminus \{1\} \tag{73}$$
$$\gamma_w(V, \epsilon, \boldsymbol{\theta}, r, \varphi) = \{\boldsymbol{\theta}\} \tag{74}$$

Now let $s \in \mathcal{W}^* * A * B$. Then s can be uniquely written as

$$s = g_1 \cdots g_n$$

with $n \geq 0$ and $g_1, \ldots, g_n \in \mathcal{W} \cup (A \setminus 1) \cup (B \setminus 1)$ such that for all $1 \leq i < n$ we have neither $g_i, g_{i+1} \in A \setminus 1$ nor $g_i, g_{i+1} \in B \setminus 1$. Such a string $g_1 \cdots g_n$ is called (A, B)-*reduced* in the following. Then, let

$$\gamma_w(g_1 g_2 \cdots g_n) = \begin{cases} \{(\theta, 0, \theta) \mid \theta \in \mathcal{T}_6\} \text{ if } n = 0 \\ \gamma_w(g_1)\gamma_w(g_2) \cdots \gamma_w(g_n) \text{ if } n > 0. \end{cases} \tag{75}$$

Note that $\gamma_w(1)$ is the identity of the monoid $2^{\mathcal{T}}$. Before we continue with the definition of the mappings μ_w and δ_w, let us first state some properties of γ_w:

Lemma 18. *The following holds:*

(a) For all $s \in \mathcal{W}^* * A * B$ we have $|\gamma_w(s)| \in \{0, 1, 2, 6\}$.
(b) If $\|s\| \geq 1$, then $|\gamma_w(s)| \in \{0, 1\}$.

(c) If $g_1g_2\cdots g_n$ is (A,B)-reduced and $\boldsymbol{\theta} \in \gamma_w(g_1g_2\cdots g_n)$ then there exist unique path types $\boldsymbol{\theta}_1 \in \gamma_w(g_1),\dots,\boldsymbol{\theta}_n \in \gamma_w(g_n)$ with $\boldsymbol{\theta} = \boldsymbol{\theta}_1\boldsymbol{\theta}_2\cdots\boldsymbol{\theta}_n$.
(d) If the (A,B)-reduced string s contains a factor from $(A\setminus 1)(B\setminus 1) \cup (B\setminus 1)(A\setminus 1)$, then $\gamma_w(s) = \emptyset$.
(e) For all $u,v \in \mathcal{W}^* * A * B$: $\gamma_w(u)\gamma_w(v) \subseteq \gamma_w(uv)$.
(f) For all $u,v \in \mathcal{W}^* * A * B$, if $\|u\| \geq 1$ and $\gamma_w(u) \neq \emptyset$ then $\gamma_w(u)\gamma_w(v) = \gamma_w(uv)$.
(g) For all $u,v \in \mathcal{W}^* * A * B$, if $\|v\| \geq 1$ and $\gamma_w(v) \neq \emptyset$ then $\gamma_w(u)\gamma_w(v) = \gamma_w(uv)$.
(h) For all $u,v,v' \in \mathcal{W}^* * A * B$, if $\|u\| \geq 1$ and $\gamma_w(u) \neq \emptyset$ then $\gamma_w(v)\gamma_w(u)\gamma_w(v') = \gamma_w(vuv')$.

Proof. We show only statements (e)–(h), the other statements are easy to prove.

Statement (e): If $\gamma_w(u) = \emptyset$ or $\gamma_w(v) = \emptyset$ then we have $\gamma_w(u)\gamma_w(v) = \emptyset \subseteq \gamma_w(uv)$. So assume that $\gamma_w(u) \neq \emptyset \neq \gamma_w(v)$. Let $u = g_1\cdots g_n$ and $v = h_1\cdots h_m$ be (A,B)-reduced strings. If $g_1\cdots g_nh_1\cdots h_m$ is again (A,B)-reduced then we have $\gamma_w(u)\gamma_w(v) = \gamma_w(uv)$ by (75). Hence, assume that $g_1\cdots g_nh_1\cdots h_m$ is not (A,B)-reduced, e.g. $g_n, h_1 \in A\setminus\{1\}$. Since $\gamma_w(u) \neq \emptyset \neq \gamma_w(v)$, the letters g_{n-1} and h_2 (if they exist) cannot be from $B\setminus\{1\}$, and they cannot be from $A\setminus\{1\}$ since $g_1\cdots g_n$ and $h_1\cdots h_m$ are (A,B)-reduced. Hence, either $uv = g_1\cdots g_{n-1}h_2\cdots h_m$ is reduced (if $g_nh_1 = 1$ in A), or $uv = g_1\cdots g_{n-1}(g_nh_1)h_2\cdots h_m$ is reduced. In the first case, we get $\gamma_w(u)\gamma_w(v) \subseteq \gamma_w(uv)$, in the second case we even have $\gamma_w(u)\gamma_w(v) = \gamma_w(uv)$.

Statement (f): Let $u = g_1\cdots g_n$ and $v = h_1\cdots h_m$ be the unique (A,B)-reduced representatives for u and v. Since $\|u\| \geq 1$ and $\gamma_w(u) \neq \emptyset$ we have $g_n \in \mathcal{W}$ or $g_n \in (A\setminus 1)\cup(B\setminus 1)$ and $g_{n-1} \in \mathcal{W}$.

Case 1. $uv = g_1\cdots g_nh_1\cdots h_m$ is (A,B)-reduced.
Applying (75) on the three (A,B)-reduced strings $g_1\cdots g_n$, $h_1\cdots h_m$ and $g_1\cdots g_nh_1\cdots h_m$ we obtain that $\gamma_w(u)\gamma_w(v) = \gamma_w(uv)$.

Case 2. $g_1\cdots g_nh_1\cdots h_m$ is not (A,B)-reduced.
This implies that either $g_n, h_1 \in A\setminus 1$ or $g_n, h_1 \in B\setminus 1$. W.l.o.g. we assume that $g_n, h_1 \in A\setminus 1$. Since $\|u\| \geq 1$ we have $n \geq 2$ and $g_{n-1} \in \mathcal{W}$. Since $\gamma_w(u) \neq \emptyset$ we have $\gamma_w(g_{n-1}) = \{\boldsymbol{\theta}\}$ for a path-type $\boldsymbol{\theta}$ with $\tau\mathrm{e}(\boldsymbol{\theta}) \in \{(A,T),(A,H)\}$.

Case 2.1. $g_nh_1 \in A\setminus 1$.
Then $uv = g_1\cdots g_{n-1}(g_nh_1)h_2\cdots h_m$ is (A,B)-reduced. By (75) we have

$$\gamma_w(uv) = \gamma_w(g_1)\cdots\gamma_w(g_{n-1})\gamma_w(g_nh_1)\gamma_w(h_2)\cdots\gamma_w(h_m).$$

Moreover, $\gamma_w(g_n) = \gamma_w(h_1) = \gamma_w(g_nh_1) = \{(A,T,0,A,T),(A,H,0,A,H)\}$ and these two path types are idempotent. Hence $\gamma_w(g_nh_1) = \gamma_w(g_n)\gamma_w(h_1)$ and we get

$$\begin{aligned}\gamma_w(uv) &= \gamma_w(g_1)\cdots\gamma_w(g_{n-1})\gamma_w(g_nh_1)\gamma_w(h_2)\cdots\gamma_w(h_m)\\ &= \gamma_w(g_1)\cdots\gamma_w(g_{n-1})\gamma_w(g_n)\gamma_w(h_1)\gamma_w(h_2)\cdots\gamma_w(h_m)\\ &= \gamma_w(u)\gamma_w(v).\end{aligned}$$

Case 2.2. $g_nh_1 = 1$.
In this case $\gamma_w(g_n) = \gamma_w(h_1) = \{(A,T,0,A,T),(A,H,0,A,H)\}$ is an idempotent that stabilizes $\gamma_w(g_{n-1}) = \{\boldsymbol{\theta}\}$ on the right. We get

$$\gamma_w(g_{n-1}) = \gamma_w(g_{n-1})\gamma_w(g_n)\gamma_w(h_1) \tag{76}$$

and hence

$$\begin{aligned}\gamma_w(uv) &= \gamma_w(g_1)\cdots\gamma_w(g_{n-1})\gamma_w(h_2)\cdots\gamma_w(h_m)\\ &\overset{(76)}{=} \gamma_w(g_1)\cdots\gamma_w(g_{n-1})\gamma_w(g_n)\gamma_w(h_1)\gamma_w(h_2)\cdots\gamma_w(h_m)\\ &= \gamma_w(u)\gamma_w(v).\end{aligned}$$

This concludes the proof of (f).

Statement (g): symetrical with (f).

Statement (h): Let $u = g_1 \cdots g_n$, $v = h_1 \cdots h_m$ and $v' = h'_1 \cdots h'_{m'}$ be (A, B)-reduced strings, with $\|u\| \geq 1$ and $\gamma_w(u) \neq \emptyset$.
The (A, B)-reduced string describing vuv' has the form

$$h_1 \cdots h_{m-1} L g_2 \cdots g_{n-1} R h'_2 \cdots h'_{m'} \tag{77}$$

where

$$L \in \{h_m \cdot g_1, (h_m g_1), \varepsilon\}, \quad R \in \{g_n \cdot h'_1, (g_n h'_1), \varepsilon\}.$$

Here $h_m \cdot g_1$ denotes a string of length 2 over $\mathcal{W} \cup (A \setminus 1) \cup (B \setminus 1)$, $(h_m g_1)$ denotes a string of length 1 over $(A \setminus 1) \cup (B \setminus 1)$ and ε is the string of length 0 and similarly for $g_n \cdot h'_1, (g_n h'_1)$. The value of L (resp. R) depends on the fact that h_m, g_1 (resp. g_n, h'_1) belong to the same subgroup A or B and on whether their product is 1. Since $\|u\| \geq 1$ and $\gamma_w(u) \neq \emptyset$, we are sure that, in case $L \neq h_m \cdot g_1$ (resp. $R \neq g_n \cdot h'_1$), $g_2 \in \mathcal{W}$ (resp. $g_{n-1} \in \mathcal{W}$), so that the string (77) is (A, B)-reduced. By the arguments used for proving statement (f), we get that:

$$\gamma_w(L)\gamma_w(g_2) = \gamma_w(h_m)\gamma_w(g_1)\gamma_w(g_2) \tag{78}$$

$$\gamma_w(g_{n-1})\gamma_w(R) = \gamma_w(g_{n-1})\gamma_w(g_n)\gamma_w(h'_1) \tag{79}$$

By (75) we have

$$\gamma_w(vuv') = \gamma_w(h_1)\cdots\gamma_w(h_{m-1})\gamma_w(L)\gamma_w(g_2)\cdots\gamma_w(g_{n-1})\gamma_w(R)\gamma_w(h'_2)\cdots\gamma_w(h'_{m'}).$$

Applying equations (78) and (79) on the right-hand sides of the above equality we obtain:

$$\gamma_w(vuv') = (\gamma_w(h_1)\cdots\gamma_w(h_m))(\gamma_w(g_1)\cdots\gamma_w(g_n))(\gamma_w(h'_1)\cdots\gamma_w(h'_{m'}))$$

thus

$$\gamma_w(vuv') = \gamma_w(v)\gamma_w(u)\gamma_w(v').$$

□

Definition 7. *Let $\equiv_\gamma$ be the equivalence relation over $\mathbb{W}$ defined by: for every $u, u' \in \mathbb{W}$,*

$$u \equiv_\gamma u' \iff (\forall v, v' \in \mathbb{W} : \gamma_w(vuv') = \gamma_w(vu'v')). \tag{80}$$

It is clear from the definition that $\equiv_\gamma$ is a monoid congruence.

Lemma 19. *Let $u, u' \in \mathbb{W}$. If $\|u\| \geq 1, \|u'\| \geq 1$ and $\gamma_w(u) = \gamma_w(u') \neq \emptyset$ then $u \equiv_\gamma u'$.*

Proof. Let $u, u' \in \mathbb{W}$ fulfilling $\|u\| \geq 1, \|u'\| \geq 1, \gamma_w(u) = \gamma_w(u') \neq \emptyset$. Let $v, v' \in \mathbb{W}$. Using (h) from Lemma 18 together with the hypothesis that $\gamma_w(u) = \gamma_w(u')$ we obtain:

$$\begin{aligned}\gamma_w(vuv') &= \gamma_w(v)\gamma_w(u)\gamma_w(v') \\ &= \gamma_w(v)\gamma_w(u')\gamma_w(v') \\ &= \gamma_w(vu'v')\end{aligned}$$

□

Let us now continue with the definition of the partial mappings $\mu_w : \mathcal{T} \times \mathcal{W}^* * A * B \to \mathsf{B}^2(\mathsf{Q})$ and $\delta_w : \mathcal{T} \times \mathcal{W}^* * A * B \to \mathrm{PGI}\{A, B\}$. Also these mappings are first defined only for generators, where $c \in (A \setminus 1) \cup (B \setminus 1)$:

$$\begin{aligned}\mu_w(\boldsymbol{\theta}, c) &= \mu_t(\boldsymbol{\theta}, c) \text{ for } \boldsymbol{\theta} \in \gamma_w(c) \\ \delta_w(\boldsymbol{\theta}, c) &= \delta_t(\boldsymbol{\theta}, c) \text{ for } \boldsymbol{\theta} \in \gamma_w(c) \\ \mu_w(\boldsymbol{\theta}, (V, \epsilon, \boldsymbol{\theta}, r, \varphi)) &= r \\ \delta_w(\boldsymbol{\theta}, (V, \epsilon, \boldsymbol{\theta}, r, \varphi)) &= \varphi\end{aligned}$$

Finally, let $s = g_1 \cdots g_n$ ($g_1, \ldots, g_n \in \mathcal{W} \cup (A \setminus 1) \cup (B \setminus 1)$) be an (A, B)-reduced string and $\boldsymbol{\theta} \in \gamma_w(s)$. If $n = 0$, i.e. $s = 1$, then $\boldsymbol{\theta} = (\theta, 0, \theta)$ for some vertex type θ, and we set

$$\begin{aligned}&\mu_w(\boldsymbol{\theta}, 1) = \mu_t(\boldsymbol{\theta}, 1) = \langle \{(q, q) \mid \tau(q) = \theta\},\ \{(p, p) \mid \tau(p) = \mathbb{I}_{\mathcal{R}}(\theta)\}\rangle \\ &\delta_w(\boldsymbol{\theta}, 1) = \mathrm{Id}_{p_1(\theta)}.\end{aligned}$$

If $n \geq 1$ then by Lemma 18, there exist unique path types $\boldsymbol{\theta}_1 \in \gamma_w(g_1), \ldots, \boldsymbol{\theta}_n \in \gamma_w(g_n)$ with $\boldsymbol{\theta} = \boldsymbol{\theta}_1 \cdots \boldsymbol{\theta}_n$. Then,

$$\mu_w(\boldsymbol{\theta}, g_1 g_2 \cdots g_n) = \mu_w(\boldsymbol{\theta}_1, g_1) \circ \mu_w(\boldsymbol{\theta}_2, g_2) \circ \cdots \circ \mu_w(\boldsymbol{\theta}_n, g_n) \tag{81}$$

$$\delta_w(\boldsymbol{\theta}, g_1 g_2 \cdots g_n) = \delta_w(\boldsymbol{\theta}_1, g_1) \circ \delta_w(\boldsymbol{\theta}_2, g_2) \circ \cdots \circ \delta_w(\boldsymbol{\theta}_n, g_n). \tag{82}$$

Proposition 4. *$\langle \mathcal{W}^* * A * B, \iota_A, \iota_B, \mathbb{I}_w, \mu_w, \gamma_w, \delta_w \rangle$ is an AB-algebra.*

Proof. Properties (AB1)–(AB4) are obvious.

(AB5): For $s \in \mathcal{W}^* * A * B$, we have $s \in \mathrm{dom}(\mathbb{I}_w) = \widehat{\mathcal{W}}^* * A * B$ if and only if all $\mathcal{W}$-symbols in s belong to $\widehat{\mathcal{W}}$. Since $\mathcal{W}$-symbols cannot cancel in a product $s_1 s_2$ it follows that $s_1 s_2 \in \mathrm{dom}(\mathbb{I}_w)$ if and only if $s_1, s_2 \in \mathrm{dom}(\mathbb{I}_w)$.

(AB6): This is stated in Lemma 18, point (e).

(AB7): Follows immediately from the definition of δ_w.

(AB8): Assume that $\boldsymbol{\theta}_1 \in \gamma_w(s_1)$ and $\boldsymbol{\theta}_2 \in \gamma_w(s_2)$ such that $\boldsymbol{\theta}_1 \boldsymbol{\theta}_2$ is defined. We prove by induction over $|s_1| + |s_2|$ that $\mu_w(\boldsymbol{\theta}_1 \boldsymbol{\theta}_2, s_1 s_2) = \mu_w(\boldsymbol{\theta}_1, s_1) \circ \mu_w(\boldsymbol{\theta}_2, s_2)$.

Case 1. $s_1 = 1$. Then $\boldsymbol{\theta}_1 = (\theta, 0, \theta)$ for some vertex type θ and

$$\mu_w(\boldsymbol{\theta}_1, s_1) = \mu_w(\boldsymbol{\theta}, 1) = \langle \{(q, q) \mid \tau(q) = \theta\},\ \{(p, p) \mid \tau(p) = \mathbb{I}_{\mathcal{R}}(\theta)\}\rangle.$$

Since $\boldsymbol{\theta}_1 \boldsymbol{\theta}_2$ is defined, we must have $\mathrm{Gi}(\boldsymbol{\theta}_2) = \theta$. Hence,

$$\mu_w(\boldsymbol{\theta}_1, s_1) \circ \mu_w(\boldsymbol{\theta}_2, s_2) = \mu_w(\boldsymbol{\theta}_2, s_2) = \mu_w(\boldsymbol{\theta}_1 \boldsymbol{\theta}_2, s_1 s_2).$$

Case 2. $s_2 = 1$. Can be dealt analogously.

Case 3. $s_1 \neq 1 \neq s_2$ and $s_1 s_2$ is (A, B)-reduced. This case can be dealt directly, using the definition of μ_w in (81).

Case 4. $s_1 \neq 1 \neq s_2$ and $s_1 s_2$ is not (A, B)-reduced. W.l.o.g. assume that $s_1 = u_1 a_1$ and $s_2 = a_2 u_2$ for $a_1, a_2 \in A \setminus 1$. We must have $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_1(\theta, 0, \theta)$ and $\boldsymbol{\theta}_2 = (\theta, 0, \theta)\boldsymbol{\theta}_2$, $\boldsymbol{\theta}_1 \in \gamma_w(u_1)$, $\boldsymbol{\theta}_2 \in \gamma_w(u_2)$, and $\theta \in \{(A, T), (A, H)\}$ by (72).

If $a_2 \neq a_1^{-1}$, then $u_1(a_1 a_2)u_2$ is (A, B)-reduced. Since the fta $\mathcal{A}$ satisfies property (AB8), we obtain:

$$\begin{aligned}
\mu_w(\boldsymbol{\theta}_1, s_1) \circ \mu_w(\boldsymbol{\theta}_2, s_2) &= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_w((\theta, 0, \theta), a_1) \circ \mu_w((\theta, 0, \theta), a_2) \circ \mu_w(\boldsymbol{\theta}_2, u_2) \\
&= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_t((\theta, 0, \theta), a_1) \circ \mu_t((\theta, 0, \theta), a_2) \circ \mu_w(\boldsymbol{\theta}_2, u_2) \\
&= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_t((\theta, 0, \theta), a_1 a_2) \circ \mu_w(\boldsymbol{\theta}_2, u_2) \\
&= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_w((\theta, 0, \theta), a_1 a_2) \circ \mu_w(\boldsymbol{\theta}_2, u_2) \\
&= \mu_w(\boldsymbol{\theta}_1 \boldsymbol{\theta_2}, u_1(a_1 a_2)u_2)
\end{aligned}$$

Finally, if $a_2 = a_1^{-1}$, then, since the fta $\mathcal{A}$ is unitary, we get:

$$\begin{aligned}
\mu_w(\boldsymbol{\theta}_1, s_1) \circ \mu_w(\boldsymbol{\theta}_2, s_2) &= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_w((\theta, 0, \theta), a_1) \circ \mu_w((\theta, 0, \theta), a_1^{-1}) \circ \mu_w(\boldsymbol{\theta}_2, u_2) \\
&= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_t((\theta, 0, \theta), a_1) \circ \mu_t((\theta, 0, \theta), a_1^{-1}) \circ \mu_w(\boldsymbol{\theta}_2, u_2) \\
&= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_t((\theta, 0, \theta), 1) \circ \mu_w(\boldsymbol{\theta}_2, u_2) \\
&= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \langle \{(q, q) \mid \tau(q) = \theta\},\ \{(p, p) \mid \tau(p) = \mathbb{I}_{\mathcal{R}}(\theta)\} \rangle \circ \mu_w(\boldsymbol{\theta}_2, u_2) \\
&= \mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_w(\boldsymbol{\theta}_2, u_2)
\end{aligned}$$

By induction over the length of strings, we obtain finally

$$\mu_w(\boldsymbol{\theta}_1, u_1) \circ \mu_w(\boldsymbol{\theta}_2, u_2) = \mu_w(\boldsymbol{\theta}_1 \boldsymbol{\theta}_2, u_1 u_2) = \mu_w(\boldsymbol{\theta}_1 \boldsymbol{\theta}_2, s_1 s_2).$$

(AB9): Can be shown with a similar calculation as (AB8).

(AB10): Assume that $s = g_1 g_2 \cdots g_n$ is (A, B)-reduced with $g_1, \ldots, g_n \in \mathcal{W} \cup (A \setminus 1) \cup (B \setminus 1)$. Then, also $\mathbb{I}_w(g_n) \cdots \mathbb{I}_w(g_1)$ is (A, B)-reduced. We have to show that $\gamma_w(\mathbb{I}_w(s)) = \mathbb{I}_{\mathcal{T}}(\gamma_w(s))$. If $n = 0$, then

$$\gamma_w(\mathbb{I}_w(1)) = \gamma_w(1) = \{(\theta, 0, \theta) \mid \theta \in \mathcal{T}_6\} = \mathbb{I}_{\mathcal{T}}(\{(\theta, 0, \theta) \mid \theta \in \mathcal{T}_6\}) = \mathbb{I}_{\mathcal{T}}(\gamma_w(1)).$$

The case $n = 1$, i.e. s is a single generator is clear by (69)–(74). Finally, for $n > 1$, we have

$$\begin{aligned}
\gamma_w(\mathbb{I}_w(g_1 \cdots g_n)) &= \gamma_w(\mathbb{I}_w(g_n) \cdots \mathbb{I}_w(g_1)) \\
&= \gamma_w(\mathbb{I}_w(g_n)) \cdots \gamma_w(\mathbb{I}_w(g_1)) \\
&= \mathbb{I}_{\mathcal{T}}(\gamma_w(g_n)) \cdots \mathbb{I}_{\mathcal{T}}(\gamma_w(g_1)) \\
&= \mathbb{I}_{\mathcal{T}}(\gamma_w(g_1) \cdots \gamma_w(g_n)) \\
&= \mathbb{I}_{\mathcal{T}}(\gamma_w(g_1 \cdots g_n))
\end{aligned}$$

(AB11): Let us first consider the case of a generator of $\mathcal{W}^* * A * B$. For $W = (V, \epsilon, \boldsymbol{\theta}, r, \varphi)$ we have indeed

$$\mathbb{I}_{\mathsf{Q}}(\mu_w(\boldsymbol{\theta}, W)) = \mathbb{I}_{\mathsf{Q}}(r) = \mu_w(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}), \mathbb{I}_w(W)).$$

For elements from $A \cup B$, we obtain (AB11) directly from (AB11) for the AB-algebra $\mathbb{H}_t$. This settles the case of generators. Now assume that $s = g_1 g_2 \cdots g_n$ is (A,B)-reduced with $g_1, \ldots, g_n \in \mathcal{W} \cup (A \setminus 1) \cup (B \setminus 1)$, $n \geq 2$, and $\gamma_w(s) \neq \emptyset$. If $\boldsymbol{\theta} \in \gamma_w(s)$ then there exist unique path types $\boldsymbol{\theta}_1 \in \gamma_w(g_1), \ldots, \boldsymbol{\theta}_n \in \gamma_w(g_n)$ with $\boldsymbol{\theta} = \boldsymbol{\theta}_1 \cdots \boldsymbol{\theta}_n$. We obtain

$$\begin{aligned}
\mu_w(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}),\ \mathbb{I}_w(s)) &= \mu_w(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1 \cdots \boldsymbol{\theta}_n),\ \mathbb{I}_w(g_1 \cdots g_n)) \\
&= \mu_w(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_n) \cdots \mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1),\ \mathbb{I}_w(g_n) \cdots \mathbb{I}_w(g_1)) \\
&\overset{\text{(AB8)}}{=} \mu_w(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_n), \mathbb{I}_w(g_n)) \circ \cdots \circ \mu_w(\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_1), \mathbb{I}_w(g_1)) \\
&= \mathbb{I}_{\mathsf{Q}}(\mu_w(\boldsymbol{\theta}_n, g_n)) \circ \cdots \circ \mathbb{I}_{\mathsf{Q}}(\mu_w(\boldsymbol{\theta}_1, g_1)) \\
&= \mathbb{I}_{\mathsf{Q}}(\mu_w(\boldsymbol{\theta}_1, g_1) \circ \cdots \circ \mu_w(\boldsymbol{\theta}_n, g_n)) \\
&\overset{\text{(AB8)}}{=} \mathbb{I}_{\mathsf{Q}}(\mu_w(\boldsymbol{\theta}_1 \cdots \boldsymbol{\theta}_n,\ g_1 \cdots g_n)) \\
&= \mathbb{I}_{\mathsf{Q}}(\mu_w(\boldsymbol{\theta}, g)).
\end{aligned}$$

(AB12): can be shown with a similar calculation. $\square$

Recall that $\|s\| \geq 1$ implies that $\gamma_w(s)$ is either empty or a singleton set $\{\boldsymbol{\theta}\}$, see (b) in Lemma 18. In the latter case, we also use the (abusive) notations $\tau\mathrm{i}(s)$, $\tau\mathrm{e}(s)$, $\mathrm{Gi}(s)$, $\mathrm{Ge}(s)$, $\mu_w(s)$, and $\delta_w(s)$ for what should be denoted, in full rigor, by $\tau\mathrm{i}(\boldsymbol{\theta})$, $\tau\mathrm{e}(\boldsymbol{\theta})$, $\mathrm{Gi}(\boldsymbol{\theta})$, $\mathrm{Ge}(\boldsymbol{\theta})$, $\mu_w(\boldsymbol{\theta}, s)$, and $\delta_w(\boldsymbol{\theta}, s)$. Similarly, we write $\delta_w(a)$ for $\delta_w(\boldsymbol{\theta}, a)$, where $\boldsymbol{\theta} \in \gamma_w(a)$. Recall that the mapping $\delta_w(\boldsymbol{\theta}, a)$ is $c \mapsto a^{-1}ca$ $(c \in A)$ independently of which of the two path types $\boldsymbol{\theta} \in \gamma_w(a)$ we choose. For $s \in \mathcal{W}^* * A * B$ let

$$\chi_{AB}(s) = \begin{cases} 1 & \text{if } s \in A \cup B \\ 0 & \text{otherwise.} \end{cases}$$

$$\chi_H(s) = \begin{cases} 0 & \text{if } |\gamma_w(s)| \neq 1 \text{ or } \gamma_w(s) \text{ has the form } \{(*, T, 1, *, H)\} \\ 1 & \text{if } \gamma_w(s) \text{ has the form } \{(*, *, 0, *, *)\} \text{ or } \{(*, T, 1, *, T)\} \\ & \qquad\qquad \text{or } \{(*, H, 1, *, H)\} \text{ or } \{(*, T, 1, 1, 1)\} \\ 2 & \text{if } \gamma_w(s) \text{ has the form } \{(*, H, 1, *, T)\} \text{ or } \{(*, H, 1, 1, 1)\} \end{cases}$$

Note that the three cases in this definition cover all possible values for $\gamma_w(s)$. The intuition behind this definition is the following: Let $s \in (\mathcal{W} \cup A \cup B)^*$ such that $|\gamma_w(s)| = 1$. Then:

- $\chi_H(s) = 0$ if s has the type of a t-sequence which begins with either t or t^{-1} and ends with either t or t^{-1}.
- $\chi_H(s) = 1$ if s has the type of a t-sequence which either belongs to $\mathbb{H}$ or begins with some $\mathbb{H}$-element or ends with some $\mathbb{H}$-element, but the other extremity is not from $\mathbb{H}$.
- $\chi_H(s) = 2$ if w has the type of a t-sequence which begins with some $\mathbb{H}$-element and ends with some $\mathbb{H}$-element and is not reduced to an $\mathbb{H}$-element.

Given $p, s, p', s' \in \mathcal{W}^* * A * B$ and $\psi_t \in \mathrm{Hom}_{AB}(\mathcal{W}^* * A * B, \mathbb{H}_t)$ we define

$$\varDelta(p, s, p', s', \psi_t) = 1 - \frac{1}{2}\Big(\chi_{AB}(p) + \chi_{AB}(p')\Big) + \chi_H(s) + \chi_H(s') + 2\|\psi_t(s)\| + 2\|\psi_t(s')\| \quad (83)$$

This notion will serve as a "progression measure" in the inductive proof of Lemma 32, which is a key point of this paper.

3.7 The AB-algebra $\mathbb{W}$

Let us consider the monoid congruence $\equiv$ over $\mathcal{W}^* * A * B$ generated by the set of pairs

$$\{(cW, Wd) \mid W \in \mathcal{W}, (c,d) \in \delta_w(W)\} \tag{84}$$

We define the quotient monoid $\mathbb{W} = \mathcal{W}^* * A * B/_{\equiv}$.

Lemma 20. *For all $s \in \mathcal{W}^* * A * B$, all $\boldsymbol{\theta} \in \gamma_w(s)$, and all $(c,d) \in \delta_w(\boldsymbol{\theta}, s)$ we have $cs \equiv sd$.*

Proof. Assume that $s = g_1 g_2 \cdots g_n$ is (A,B)-reduced. We prove the lemma by induction on n. The cases $n = 0$ and $n = 1$ follow easily from the definition of δ_w and $\equiv$. Now assume that $n \geq 2$ and let $\boldsymbol{\theta} \in \gamma_w(s)$. Since $\gamma_w(s) \neq \emptyset$, s has to contain a symbol from $\mathcal{W}$. Hence, Lemma 18 implies that $\gamma_w(s) = \{\boldsymbol{\theta}\}$. Let $\boldsymbol{\theta} = (\theta, b, \theta')$.

Case 1. $g_1 = a \in A \setminus 1$, i.e. $s = as'$. We have $\delta_w(s) = \delta_w((\theta,0,\theta),a) \circ \delta_w(s')$ and $\mathrm{Gi}(\boldsymbol{\theta}) = p_1(\theta) = A$. With $(c,d) \in \delta_w(\boldsymbol{\theta}, s)$ we obtain $c \in \mathrm{Gi}(\boldsymbol{\theta}) = A$ and $(c,a') \in \delta_w((\theta,0,\theta),a)$, $(a',d) \in \delta_w(s')$ for some $a' \in A$. Thus, $aa' = ca$ in A and $s'd \equiv a's'$ by induction. We get $sd = as'd \equiv aa's' = cas' = cs$. The case $g_1 \in B \setminus 1$ can be dealt analogously.

Case 2. $g_1 = W \in \mathcal{W}$, i.e. $s = Ws'$. We have $\delta_w(s) = \delta_w(W) \circ \delta_w(\boldsymbol{\theta}_2, s')$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_1\boldsymbol{\theta}_2$ for some path types $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ with $\gamma_w(W) = \boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2 \in \gamma_w(s')$. Moreover, there exists some $c' \in \mathrm{Ge}(\boldsymbol{\theta}_1)$ such that $(c,c') \in \delta_w(W)$ and $(c',d) \in \delta_w(\boldsymbol{\theta}_2, s')$. Hence $cW \equiv Wc'$ and $c's' \equiv s'd$ by induction. We obtain $cs = cWs' = Wc's' \equiv Ws'd = sd$. □

Lemma 21. *The congruence $\equiv$ on $\mathcal{W}^* * A * B$ is compatible (see Definition 6) with the AB-algebra $\langle \mathcal{W}^* * A * B, \iota_A, \iota_B, \mathbb{I}_w, \gamma_w, \mu_w, \delta_w \rangle$.*

Proof. Let $u, u' \in \mathcal{W}^* * A * B$ with $u \equiv u'$, $a \in A$, and $b \in B$. We have to check the following statements:

(a) $u \in \mathrm{dom}(\mathbb{I}_w) \Leftrightarrow u' \in \mathrm{dom}(\mathbb{I}_w)$ and if $u, u' \in \mathrm{dom}(\mathbb{I}_w)$ then $\mathbb{I}_w(u) \equiv \mathbb{I}_w(u')$,
(b) For all $a, a' \in A$ and $b, b' \in B$: $a \equiv a' \Rightarrow a = a'$ and $b \equiv b' \Rightarrow b = b'$
(c) $\gamma_w(u) = \gamma_w(u')$
(d) For all $\boldsymbol{\theta} \in \gamma_w(u)$: $\mu_w(\boldsymbol{\theta}, u) = \mu_w(\boldsymbol{\theta}, u')$ and $\delta_w(\boldsymbol{\theta}, u) = \delta_w(\boldsymbol{\theta}, u')$.

Point (b) is obvious. For (a), (c), and (d), it suffices to consider only the case that $u = cW$ and $u' = Wd$, where $W \in \mathcal{W}$ and $(c,d) \in \delta_w(W)$. Let $W = (V, \epsilon, \boldsymbol{\theta}, r, \varphi)$. Thus, $(c,d) \in \varphi$.

(a) We have $cW \in \mathrm{dom}(\mathbb{I}_w) \iff W \in \mathrm{dom}(\mathbb{I}_w) \iff Wd \in \mathrm{dom}(\mathbb{I}_w)$. Moreover, if $W \in \mathrm{dom}(\mathbb{I}_w)$ then, since $(c,d) \in \delta_w(W) = \varphi$ we have $(d,c) \in \delta_w(\mathbb{I}_w(W)) = \varphi^{-1}$. Thus, $d\,\mathbb{I}_w(W) \equiv \mathbb{I}_w(W)c$, i.e., $\mathbb{I}_w(cW) = \mathbb{I}_w(W)c^{-1} \equiv d^{-1}\mathbb{I}_w(W) = \mathbb{I}_w(Wd)$.

(c) Since $c \in \mathrm{Gi}(\boldsymbol{\theta})$ and $d \in \mathrm{Ge}(\boldsymbol{\theta})$ we have

$$\gamma_w(cW) = \gamma_w(c)\gamma_w(W) = \{\boldsymbol{\theta}\} = \gamma_w(W)\gamma_w(d) = \gamma_w(Wd).$$

(d) The type $\boldsymbol{\theta}$ splits as $\boldsymbol{\theta} = (\theta,0,\theta)\boldsymbol{\theta} = \boldsymbol{\theta}(\theta',0,\theta')$, where $\theta = \tau\mathrm{i}(\boldsymbol{\theta})$ and $\theta' = \tau\mathrm{e}(\boldsymbol{\theta})$. We get:

$$\begin{aligned}
\mu_w(\boldsymbol{\theta}, cW) &= \mu_w((\theta,0,\theta),c) \circ \mu_w(\boldsymbol{\theta}, W) \\
&= \mu_t((\theta,0,\theta),c) \circ r \\
&= r \circ \mu_t((\theta',0,\theta'),d) \\
&= \mu_w(\boldsymbol{\theta}, W) \circ \mu_w((\theta',0,\theta'),d) \\
&= \mu_w(\boldsymbol{\theta}, Wd)
\end{aligned}$$

For the mapping δ_w, we have

$$\delta_w(\boldsymbol{\theta}, cW) = \delta_w((\theta,0,\theta),c) \circ \delta_w(W) = \delta_w((\theta,0,\theta),c) \circ \varphi$$

and similarly $\delta_w(\boldsymbol{\theta}, Wd) = \varphi \circ \delta_w((\theta',0,\theta'),d)$. Hence, we have to show that

$$\delta_w((\theta,0,\theta),c) \circ \varphi = \varphi \circ \delta_w((\theta',0,\theta'),d),$$

where $(c,d) \in \varphi$. Let us first check that the domains of the left and right hand side are equal. Since $c \in \mathrm{dom}(\varphi)$ and $\mathrm{dom}(\varphi)$ is a subgroup of $\mathrm{Gi}(\boldsymbol{\theta})$, we have

$$\begin{aligned} x \in \mathrm{dom}(\delta_w((\theta,0,\theta),c) \circ \varphi) &\iff c^{-1}xc \in \mathrm{dom}(\varphi) \\ &\iff x \in \mathrm{dom}(\varphi) \\ &\iff x \in \mathrm{dom}(\varphi \circ \delta_w((\theta',0,\theta'),d)). \end{aligned}$$

Now let $x \in \mathrm{dom}(\delta_w((\theta,0,\theta),c) \circ \varphi) = \mathrm{dom}(\varphi)$. Then, we have:

$$\begin{aligned} (\delta_w((\theta,0,\theta),c) \circ \varphi)(x) &= \varphi(c^{-1}xc) \\ &= d^{-1}\varphi(x)d \\ &= (\varphi \circ \delta_w((\theta',0,\theta'),d))(x) \end{aligned}$$

This proves the lemma. □

By the previous lemma, we can endow $\mathbb{W}$ with the structure of an AB-algebra:

$$\langle \mathbb{W}, \iota_A, \iota_B, \mathbb{I}_w, \mu_w, \gamma_w, \delta_w \rangle \tag{85}$$

In addition $u \equiv v$ implies $\|u\| = \|v\|$ for all $u, v \in \mathcal{W}^* * A * B$, so that the notion of norm remains well-defined in the quotient $\mathbb{W}$.

3.8 The AB-algebras $\mathbb{W}_{\boldsymbol{t}}$, $\mathbb{W}_{\mathbb{H}}$, and $\widehat{\mathbb{W}}$

Let us consider the set $\mathcal{W}_t$ consisting of all the letters $W \in \mathcal{W}$ fulfilling

$$\exists s \in \mathbb{H} * \{t, \bar{t}\}^* \begin{cases} W \in \mathrm{dom}(\mathbb{I}_w) \iff s \in \mathrm{dom}(\mathbb{I}_t),\ \gamma_w(W) \subseteq \gamma_t(s) \\ \forall \boldsymbol{\theta} \in \gamma_w(W) : \mu_w(W) = \mu_t(\boldsymbol{\theta}, s),\ \delta_w(W) = \delta_t(\boldsymbol{\theta}, s). \end{cases} \tag{86}$$

Intuitively, it is the set of all symbols from $\mathcal{W}$ that can be realized by some concrete t-sequence. We define the subset

$$\mathcal{W}_{\mathbb{H}} = \{ W \in \mathcal{W}_t \mid \gamma_w(W) \text{ is an } H\text{-type } \}.$$

Note that the sets $\mathcal{W}_t$ and $\mathcal{W}_{\mathbb{H}}$ are closed under the involution $\mathbb{I}_w$. We then define the quotients

$$\mathbb{W}_t = \mathcal{W}_t^* * A * B/_{\equiv}, \quad \mathbb{W}_{\mathbb{H}} = \mathcal{W}_{\mathbb{H}}^* * A * B/_{\equiv}, \quad \widehat{\mathbb{W}} = \widehat{\mathcal{W}}^* * A * B/_{\equiv}$$

and consider the AB-structures induced by $\mathbb{W}$ on $\mathbb{W}_t$, $\mathbb{W}_{\mathbb{H}}$, and $\widehat{\mathbb{W}}$, respectively. One can easily check that the inclusions $\mathbb{W}_{\mathbb{H}} \hookrightarrow \mathbb{W}_t$, $\mathbb{W}_t \hookrightarrow \mathbb{W}$, and $\widehat{\mathbb{W}} \hookrightarrow \mathbb{W}$ are AB-homomorphisms.

3.9 Involutive automorphisms

We consider here special AB-automorphisms of $\mathbb{W}$ that occur naturally in the process of reducing equations in $\mathbb{G}$ to equations in $\mathbb{W}$. Consider a partition

$$\widehat{\mathcal{W}} = \widehat{\mathcal{W}}_0 \cup \{W_1, \ldots, W_p\} \cup \{\overline{W}_1, \ldots, \overline{W}_p\}, \tag{87}$$

where $\overline{W}_k = \mathbb{I}_w(W_k)$ for $1 \le k \le p$ and $\widehat{\mathcal{W}}_0$ is closed under the involution $\mathbb{I}_w$. Moreover, let $(a_1, b_1, \ldots, a_p, b_p)$ be a tuple with

$$\mathrm{Gi}(W_k) = \mathrm{Ge}(W_k) \text{ for } 1 \le k \le p \tag{88}$$

Lemma 22. *Let us consider a partition of $\widehat{\mathcal{W}}$ and a tuple of group elements $(a_1, b_1, \ldots, a_p, b_p)$ as above.*
1- There exists some AB-automorphism of $\Phi : \mathbb{W} \to \mathbb{W}$ satisfying

$$\Phi(c) = c \text{ for } c \in A \cup B \qquad \Phi(W) = W \qquad \text{for all } W \in (\mathcal{W} \setminus \widehat{\mathcal{W}}) \cup \widehat{\mathcal{W}}_0 \tag{89}$$

$$\Phi(\overline{W}_k) = a_k W_k b_k \qquad \Phi(W_k) = a_k^{-1} \overline{W}_k b_k^{-1} \text{ for } 1 \le k \le p. \tag{90}$$

if and only if, for every $1 \le k \le p$ we have

$$\delta_w(W_k)^{-1} = \delta_w(a_k) \circ \delta_w(W_k) \circ \delta_w(b_k). \tag{91}$$

$$(b_k^{-1} a_k, a_k b_k^{-1}) \in \delta_w(W_k) \tag{92}$$

$$\gamma_w(W_k) = \mathbb{I}_{\mathcal{T}}(\gamma_w(W_k)) \tag{93}$$

$$\mu_w(a_k W_k b_k) = \mathbb{I}_{\mathsf{Q}}(\mu_w(W_k)). \tag{94}$$

2- Every AB-homomorphism $\Phi : \mathbb{W} \to \mathbb{W}$ satisfying (89) and (90) is an involutive AB-automorphism.

Proof. 1- Suppose that $\Phi : \mathbb{W} \to \mathbb{W}$ is an AB-homomorphism satisfying formula (89)(90).
1.1- Let $1 \le k \le p$ and $(c, d) \in \mathrm{Gi}(W_k) \times \mathrm{Ge}(W_k)$. Recall that $\delta_w(a_k)$ is the mapping $x \mapsto a_k^{-1} x a_k$ (for $x \in \mathrm{Gi}(W_k)$) and $\delta_w(b_k)$ is the mapping $x \mapsto b_k^{-1} x b_k$ (for $x \in \mathrm{Ge}(W_k)$). The following calculation shows (91):

$$\begin{aligned}
(c, d) \in \delta_w(a_k) \circ \delta_w(W_k) \circ \delta_w(b_k) &\iff (a_k^{-1} c a_k, d) \in \delta_w(W_k) \circ \delta_w(b_k) \\
&\iff (a_k^{-1} c a_k, b_k d b_k^{-1}) \in \delta_w(W_k) \\
&\iff a_k^{-1} c a_k W_k \equiv W_k b_k d b_k^{-1} \\
&\iff c a_k W_k b_k \equiv a_k W_k b_k d \\
&\iff c\Phi(\overline{W}_k) = \Phi(\overline{W}_k) d \\
&\iff c\overline{W}_k \equiv \overline{W}_k d \\
&\iff (c, d) \in \delta_w(\overline{W}_k) \\
&\iff (c, d) \in \delta_w(W_k)^{-1}.
\end{aligned}$$

1.2- Let $1 \le k \le p$. Since Φ preserves the standard-involution $\mathbb{I}_w$ (axiom (Hom4)) we get successively:

$$\begin{aligned}
\Phi(\mathbb{I}_w(W_k)) = \mathbb{I}_w(\Phi(W_k)) &\Longrightarrow a_k W_k b_k \equiv b_k W_k a_k \\
&\Longrightarrow b_k^{-1} a_k W_k \equiv W_k a_k b_k^{-1} \\
&\Longrightarrow (b_k^{-1} a_k, a_k b_k^{-1}) \in \delta_w(W_k)
\end{aligned}$$

which establishes (92).
1.3- Since $\Phi(W_k) = a_k^{-1}\overline{W}_k b_k^{-1}$ and Φ preserves γ (axiom (Hom5), we get

$$\begin{aligned}\gamma_w(W_k) \subseteq &\quad \gamma_w(a_k^{-1}\overline{W}_k b_k^{-1}) \\ = &\ \gamma_w(a_k^{-1})\gamma_w(\overline{W}_k)\gamma_w(b_k^{-1}) && \text{by definition (75)} \\ \subseteq &\quad \gamma_w(\overline{W}_k) && \text{by axiom (AB6)} \\ = &\quad \mathbb{I}_{\mathcal{T}}(\gamma_w(W_k)).\end{aligned}$$

Since the two extreme terms of this sequence of inclusions are singletons, they must be equal, which establishes (93).
1.4- Since $a_k W_k b_k = \Phi(\overline{W}_k)$ and Φ preserves μ, (by axiom (Hom6), we get

$$\begin{aligned}\mu_w(a_k W_k b_k) &= \mu_w(\overline{W}_k) \\ &= \mathbb{I}_{\mathsf{Q}}(\mu_w(W_k)) \quad \text{by axiom (AB11)}\ ,\end{aligned}$$

which establishes (94).

□

1'- Suppose that the letters W_k and the tuple $(a_1, b_1, \ldots, a_k, b_k)$ are fulfilling conditions (91)(92)(93)(94)
By the universal property of the free-product, there exists a unique monoid homorphism $\tilde{\Phi} : \hat{\mathcal{W}}^* * A * B \to \hat{\mathcal{W}}^* * A * B$ such that

$$\tilde{\Phi}(c) = c \text{ for } c \in A \cup B \qquad \tilde{\Phi}(W) = W \qquad \text{for all } W \in (\mathcal{W} \setminus \widehat{\mathcal{W}}) \cup \widehat{\mathcal{W}_0} \tag{95}$$

$$\tilde{\Phi}(\overline{W}_k) = a_k W_k b_k \qquad \tilde{\Phi}(W_k) = a_k^{-1}\overline{W}_k b_k^{-1} \text{ for } 1 \leq k \leq p. \tag{96}$$

Let $1 \leq k \leq p$ and $(c, d) \in \delta_w(W_k)$

$$\tilde{\Phi}(cW_k) = ca_k^{-1}\overline{W}_k b_k^{-1}, \quad \tilde{\Phi}(W_k d) = a_k^{-1}\overline{W}_k b_k^{-1} d.$$

Thus

$$\begin{aligned}\tilde{\Phi}(cW_k) \equiv \tilde{\Phi}(W_k d) &\Leftrightarrow ca_k^{-1}\overline{W}_k b_k^{-1} \equiv a_k^{-1}\overline{W}_k b_k^{-1} d \\ &\Leftrightarrow a_k c a_k^{-1}\overline{W}_k \equiv \overline{W}_k b_k^{-1} d b_k \\ &\Leftrightarrow (a_k c a_k^{-1}, b_k^{-1} d b_k) \in \delta_w(W_k)^{-1} \\ &\Leftrightarrow (c, d) \in \delta_w(a_k)^{-1} \circ \delta_w(W_k)^{-1} \circ \delta_w(b_k)^{-1}.\end{aligned}$$

But (91) ensures that $\delta_w(a_k)^{-1} \circ \delta_w(W_k)^{-1} \circ \delta_w(b_k)^{-1} = \delta_w(W_k)$ and, by hypothesis, $(c, d) \in \delta_w(W_k)$. We have thus established that

$$\tilde{\Phi}(cW_k) \equiv \tilde{\Phi}(W_k d). \tag{97}$$

Using $(d, c) \in \delta_w(\overline{W}_k)$, a similar argument shows

$$\tilde{\Phi}(d\overline{W}_k) \equiv \tilde{\Phi}(\overline{W}_k c). \tag{98}$$

It follows from the equalities (97) and (98) that $\tilde{\Phi} : \hat{\mathcal{W}}^* * A * B \to \hat{\mathcal{W}}^* * A * B$ induces a monoid homomorphism $\Phi : \mathbb{W} \to \mathbb{W}$. This monoid homomorphism satisfies (89) (90).
The AB-algebra $\mathcal{W}^* * A * B$ fulfills hypotheses (A),(B) and (C) from Lemma 12 for the set of

generators $\Gamma := \mathcal{W} \cup A \cup B$: (B) clearly holds and the decomposition for $s \in \mathbb{W}$ that has to exist according to (C) is obtained by writting s as an (A, B)-reduced product, see (75). Thus, in order to show that Φ is an AB-homomorphism, we just have to ckeck that it meets the hypotheses of Lemma 12:

(a) is clear
(b) is clear
(c) amounts to the property that

$$\forall k \in [1, p], b_k W_k a_k \equiv a_k W_k b_k \text{ and } b_k^{-1} W_k a_k^{-1} \equiv a_k^{-1} W_k b_k^{-1}$$

These properties follow from (92).

(d) asserts that $\forall g \in \Gamma, \gamma_w(g) \subseteq \gamma_w(\Phi(g))$. This is clear for $g \notin \{W_k \mid 1 \leq k \leq p\} \cup \{\overline{W}_k \mid 1 \leq k \leq p\}$. For $g = W_k$, we have to check that

$$\gamma_w(W_k) = \gamma_w(a_k W_k b_k)$$

which reduces to

$$a_k \in \mathrm{Gi}(W_k), b_k \in \mathrm{Ge}(W_k) \tag{99}$$

which is true by hypothesis. For $g = \overline{W}_k$, we have to check that

$$\gamma_w(\overline{W}_k) = \gamma_w(a_k^{-1} \overline{W}_k b_k^{-1})$$

which also follows also from (99).

(e) is ensured by (94).
(f) is ensured by (91).

2- Let us prove Point (2) of the lemma.
Let $\Phi : \mathbb{W} \to \mathbb{W}$ be an AB-homomorphism satisfying (89) and (90). For every $k \in [1, p]$

$$\Phi(\Phi(W_k)) = \Phi(a_k^{-1} \overline{W}_k b_k^{-1}) = a_k^{-1} a_k W_k b_k b_k^{-1} = W_k$$

and, similarly

$$\Phi(\Phi(\overline{W}_k)) = \overline{W}_k.$$

For all other generators $g \in \mathcal{W} \cup A \cup B$, we have $\Phi(g) = g$ and thus $\Phi(\Phi(g)) = g$. Hence the map Φ is involutive i.e. $\Phi \circ \Phi = \mathrm{I}_{\mathbb{W}}$. This shows that Φ has an inverse (namely Φ), which is itself an AB-homomorphism. Thus Φ is an AB-automorphism.

Note that condition (93) implies for all $1 \leq k \leq p$

$$\gamma_w(W_k) \in \{(A, T, 1, A, H), (B, T, 1, B, H), (A, H, 0, A, T), (B, H, 0, B, T), (1, H, 0, 1, 1)\}, \tag{100}$$

since these are the only T-types and H-types with $\mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}) = \boldsymbol{\theta}$. Moreover, by (91), a_k and b_k belong to the same group $\mathrm{Gi}(W_k) = \mathrm{Ge}(W_k) = \mathrm{Gi}(\overline{W}_k) = \mathrm{Ge}(\overline{W}_k)$.

Definition 8. *We denote by* HInv *(for* $\mathbb{H}$*-involutive automorphisms) the set of all involutive AB-automorphisms* Φ *of the form (89),(90) satisfying the four conditions (91),(92),(93),(94) and such that,*

$$\forall 1 \leq k \leq p, \gamma_w(W_k) \textit{ is an H-type }. \tag{101}$$

For $\Phi \in \mathsf{HInv}$ we denote with $\mathbb{W}/\Phi$ the AB-algebra defined as the quotient $\mathbb{W}/\simeq_\Phi$ where $\simeq_\Phi$ is the monoid-congruence over $\mathbb{W}$ generated by the set $\{(W, \Phi(W) \mid W \in \mathcal{W}\}$ (one can easily check that this monoid-congruence is compatible with the AB-algebra structure). Alternatively, we can define $\mathbb{W}/\Phi$ as the quotient $\mathcal{W}^* * A * B/\equiv_\Phi$, where $\equiv_\Phi$ is the monoid-congruence over $\mathbb{W}$ generated by the set $\{(cW, Wd) \mid W \in \mathcal{W}, (c,d) \in \delta_w(W)\} \cup \{(W, \Phi(W) \mid W \in \mathcal{W}\}$ (this monoid-congruence is compatible with the AB-algebra structure).

As noticed in the course of the above proof, the AB-algebra $\mathcal{W}^* * A * B$ fulfills hypotheses (A),(B) and (C) from Lemma 12 for the set of generators $\Gamma := \mathcal{W} \cup A \cup B$. By Lemma 14 the AB-algebras $\mathbb{W}$ and $\mathbb{W}/\Phi$ also satisfy these conditions (A)(B)(C).

3.10 AB-homomorphisms on $\mathcal{W}^* * A * B$ or $\mathbb{W}$

Let us show here some properties of AB-homomorphisms on $\mathcal{W}^* * A * B$ or $\mathbb{W}$. Let $\mathcal{W}_0 \subseteq \mathcal{W}$ be some subset which is closed under $\mathbb{I}_w$, let $\mathbb{W}_0$ be the submonoid of $\mathbb{W}$ generated by $\mathcal{W}_0 \cup A \cup B$. This submonoid induces an AB-subalgebra of $\mathbb{W}$ (as defined in Section 3.2), which we denote again by $\mathbb{W}_0$.

Lemma 23. *Let $\psi : \mathbb{W} \to \mathbb{W}$ be an AB-homomorphism and let $s \in \mathbb{W}$ with $\|s\| \geq 1$. Then also $\|\psi(s)\| \geq 1$.*

Proof. It suffices to show that $\|\psi(W)\| \geq 1$ for all $W \in \mathcal{W}$. Assume that $\psi(W) \in A * B$. By (Hom5) we have $\emptyset \neq \gamma_w(W) \subseteq \gamma_w(\psi(W))$. Hence, $\gamma_w(\psi(W))$ contains an H-type or a T-type. By point (d) from Lemma 18, $\psi(W)$ cannot contain a factor from $(A \setminus 1)(B \setminus 1)$ or $(B \setminus 1)(A \setminus 1)$. Hence, we have $\psi(W) \in A \cup B$. But also the cases $\psi(W) = 1$, $\psi(W) \in A \setminus 1$, and $\psi(W) \in B \setminus 1$ can be excluded, since in these cases $\psi(W)$ does not posess an H-type or a T-type, see (72), (73), and (75). □

Lemma 24. *Let $\psi : \mathbb{W}_t \to \mathbb{H}_t$ be some AB-homomorphism. Let $P, S, P', S' \in \mathbb{W}_t$ such that the following holds:*

- $\gamma_w(P) = \gamma_w(P') \neq \emptyset$,
- $\gamma_w(S) \neq \emptyset \neq \gamma_w(S')$,
- $\gamma_w(PS) = \gamma_w(P'S') \neq \emptyset$
- $\psi(P) = \psi(P')$ *and* $\psi(PS) = \psi(P'S')$.

Then $\gamma_w(S) = \gamma_w(S')$.

Proof. Let P, S, P', S' fulfill the hypothesis of the lemma. Since $\mathbb{H}_t$ is cancellative by Lemma 3, $\psi(PS) = \psi(P'S')$ and $\psi(P) = \psi(P')$ imply $\psi(S) = \psi(S')$.

We distinguish several cases according to whether the norm of P, P', S, S' is zero. Let us notice that, since $\gamma_w(P) = \gamma_w(P') \neq \emptyset$, either $\|P\| = \|P'\| = 0$ or $\|P\|, \|P'\| \geq 1$. We also notice that (P, S) and (P', S') play the same role in the above lemma. Therefore we only have to treat the following six cases.

Case 1: $\|P\| \geq 1, \|P'\| \geq 1, \|S\| \geq 1, \|S'\| \geq 1$.
Hence, by Lemma 18, $\gamma_w(P)$, $\gamma_w(P')$, $\gamma_w(S)$, and $\gamma_w(S')$ are all singleton sets. Since $\gamma_w(PS) = \gamma_w(P'S') \neq \emptyset$ and $\gamma_w(P) = \gamma_w(P')$, we must have

$$\tau\mathrm{i}(S) = \tau\mathrm{e}(P) = \tau\mathrm{e}(P') = \tau\mathrm{i}(S') \text{ and } \tau\mathrm{e}(S) = \tau\mathrm{e}(S').$$

Moreover, since $\psi(S) = \psi(S')$, we have $\gamma_w(S) \in \gamma_t(\psi(S)) = \gamma_t(\psi(S')) \ni \gamma_w(S')$. This implies that the boolean components of $\gamma_w(S)$ and $\gamma_w(S')$ are equal, hence $\gamma_w(S) = \gamma_w(S')$.

Case 2: $\|P\| \geq 1, \|P'\| \geq 1, \|S\| = \|S'\| = 0$.
Since $\gamma_w(S) \neq \emptyset \neq \gamma_w(S')$, we must have $S, S' \in A \cup B$. Moreover, the type $\gamma_w(P) = \gamma_w(P')$ is a singleton. Since $\gamma_w(PS) \neq \emptyset \neq \gamma_w(P'S')$, we must have $S \in \mathrm{Ge}(P) = \mathrm{Ge}(P') \ni S'$. Thus, $S, S' \in A$ or $S, S' \in B$. Since ψ must be injective on A and B (by (Hom2)) we have either $S, S' \in A \setminus 1$ or $S, S' \in B \setminus 1$, or $S = S' = 1$. Thus, we get $\gamma_w(S) = \gamma_w(S')$.

Case 3: $\|P\| \geq 1, \|P'\| \geq 1, \|S\| \geq 1, \|S'\| = 0$.
Thus, $\gamma_w(P)$, $\gamma_w(P')$, and $\gamma_w(S)$ are singeltons and $S' \in A \cup B$. Since $\psi(S) = \psi(S') \in A \cup B$, the boolean component of $\gamma_w(S) \in \gamma_t(\psi(S))$ is 0. Now, an inspection of the graph $\mathcal{B}_6$ from Figure 1 shows that there does not exist a non-empty product of H-types that starts and ends in the same vertex type. Thus, $\|S\| \geq 1$ and the fact that the boolean component of $\gamma_w(S)$ is 0 imply $\tau\mathrm{e}(S) \neq \tau\mathrm{i}(S) = \tau\mathrm{e}(P)$. Thus,

$$\tau\mathrm{e}(S) \neq \tau\mathrm{e}(P). \tag{102}$$

Since $S' \in A \cup B$, we have $\tau\mathrm{i}(\boldsymbol{\theta}) = \tau\mathrm{e}(\boldsymbol{\theta})$ for every $\boldsymbol{\theta} \in \gamma_w(S')$. Hence,

$$\tau\mathrm{e}(P'S') = \tau\mathrm{e}(P'). \tag{103}$$

We also have $\gamma_w(PS) = \gamma_w(P'S') \neq \emptyset$ which implies that $\tau\mathrm{e}(S) = \tau\mathrm{e}(PS) = \tau\mathrm{e}(P'S')$ hence, taking into account (103), we get

$$\tau\mathrm{e}(S) = \tau\mathrm{e}(P'). \tag{104}$$

But equations (102) and (104) entail that $\tau\mathrm{e}(P) \neq \tau\mathrm{e}(P')$ contradicting the hypothesis that $\gamma_w(P) = \gamma_w(P') \neq \emptyset$. This case is thus impossible.

Case 4: $\|P\| = \|P'\| = 0, \|S\| \geq 1, \|S'\| \geq 1$.
In this case $P, P' \in A \cup B$. The fact that $\gamma_w(P) = \gamma_w(P')$ implies that

$$P, P' \in A \setminus \{1\} \text{ or } P, P' \in B \setminus \{1\} \text{ or } P = P' = 1.$$

Here $\gamma_w(S) = \gamma_w(PS) = \gamma_w(P'S') = \gamma_w(S')$, i.e. the conclusion of the lemma holds.

Case 5: $\|P\| = \|P'\| = 0, \|S\| \geq 1, \|S'\| = 0$.
Then $|\gamma_w(PS)| = 1$ while $|\gamma_w(P'S')| \in \{2, 6\}$. This contradicts the hypothesis $\gamma_w(PS) = \gamma_w(P'S')$. This case is thus impossible.

Case 6: $\|P\| = \|P'\| = 0, \|S\| = \|S'\| = 0$.
Then $P, P', S, S' \in A \cup B$. Since $\gamma_w(P) = \gamma_w(P')$, we have

$$P, P' \in A \setminus \{1\} \text{ or } P, P' \in B \setminus \{1\} \text{ or } P = P' = 1. \tag{105}$$

Case 6.1: $S \in A \setminus 1$ and $S' \in B \setminus 1$. With (105) this implies either $\gamma_w(P'S') = \emptyset$ or $\gamma_w(PS) = \emptyset$ or $\gamma_w(PS) \neq \gamma_w(P'S')$, which is a contradiction.

Case 6.2: $S, S' \in A$. Since $\psi(S) = \psi(S')$, we have $S = 1 \Leftrightarrow S' = 1$. We get $\gamma_w(S) = \gamma_w(S')$. All other subcases can be dealt analogously to Case 6.1 or 6.2. $\square$

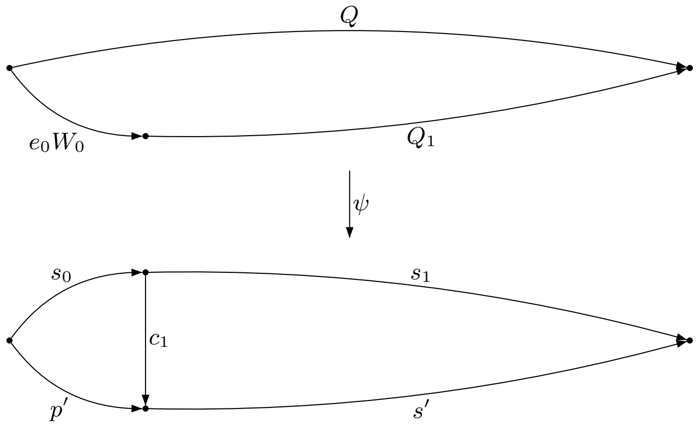


Fig. 4. Proof of Lemma 25

Lemma 25. *Let $\psi : \mathbb{W}_t \to \mathbb{H}_t$ be some AB-homomorphism. Let $Q \in \mathbb{W}_t, P', S' \in \mathbb{H}_t, \boldsymbol{\theta} \in \gamma_t(P'), \boldsymbol{\rho} \in \gamma_t(S')$ such that $\boldsymbol{\theta}$ is an H-type, $\boldsymbol{\theta\rho}$ is defined, $\gamma_w(Q) = \{\boldsymbol{\theta\rho}\}$ and $\psi(Q) = P'S'$. Then, there exist $P, S \in \mathbb{W}_t$ such that:*

$$Q = PS,\ \ \psi(P) = P',\ \ \gamma_w(P) = \{\boldsymbol{\theta}\},\ \ \psi(S) = S',\ \ \boldsymbol{\rho} \in \gamma_w(S).$$

Proof. Let us consider $\psi, Q, P', S', \boldsymbol{\theta}, \boldsymbol{\rho}$ fulfilling the hypothesis of the lemma. Since $\gamma_w(Q) = \{\boldsymbol{\theta\rho}\}$ is a singleton set, $Q \in \mathbb{W}_t$ must contain a letter from $\mathcal{W}_t$, i.e. we can factorize Q as

$$Q = e_0W_0Q_1 \tag{106}$$

where $W_0 \in \mathcal{W}_t$ and $e_0 \in \mathrm{Gi}(W_0)$. We also have $Q_1 \in \mathbb{W}_t$. Since $\gamma_w(Q) \neq \emptyset$, (Hom5) implies that $\gamma_t(\psi(Q)) \neq \emptyset$. Recall that a t-sequence s is reduced if and only if $\gamma_t(s) \neq \emptyset$. It follows that $\psi(Q) = P'Q'$ viewed as a $\sim$-equivalence class of t-sequences contains only reduced t-sequences. Since $P'Q' = \psi(e_0W_0)\psi(Q_1)$, we can choose reduced t-sequences $s_0, s_1, p', s' \in \mathbb{H} * \{t, \bar{t}\}^*$ such that

$$\psi(e_0W_0) = [s_0]_\sim,\ \ \psi(Q_1) = [s_1]_\sim,\ \ P' = [p']_\sim,\ \ S' = [s']_\sim, \tag{107}$$

and s_0s_1 as well as $p's'$ are reduced, see Figure 4. Let $\gamma_w(W_0) = \{\boldsymbol{\theta_1}\}$. Thus, $\boldsymbol{\theta_1}$ is either an H-type or a T-type. We have

$$\{\boldsymbol{\theta\rho}\} = \gamma_w(Q) = \gamma_w(e_0W_0Q_1) = \gamma_w(W_0Q_1) = \{\boldsymbol{\theta_1}\}\gamma_w(Q_1). \tag{108}$$

Hence, we have $\tau\mathrm{i}(\boldsymbol{\theta}_1) = \tau\mathrm{i}(\boldsymbol{\theta})$. But since $\boldsymbol{\theta}$ is an H-type, also $\boldsymbol{\theta_1}$ must be an H-type. Moreover, $\boldsymbol{\theta} \in \gamma_t(P')$ and $\boldsymbol{\theta_1} \in \gamma_w(W_0) = \gamma_w(e_0W_0) \subseteq \gamma_t(\psi(e_0W_0)) = \gamma_t([s_0]_\sim)$ imply

$$p', s_0 \in \mathbb{H}. \tag{109}$$

Since $\boldsymbol{\theta}$ is an H-type and $\boldsymbol{\theta\rho}$ is defined, we can distinguish the following four cases for $\boldsymbol{\rho}$:

Case 1: $\boldsymbol{\rho}$ has the form $(*, T, 1, *, *)$.
Hence, (108) implies that the boolean component of the unique type in $\{\boldsymbol{\theta}_1\}\gamma_w(Q_1)$ must be 1. Since $\boldsymbol{\theta}_1$ is an H-type, i.e. its boolean component is 0, it follows that $\gamma_w(Q_1)$ contains a

type with boolean component 1. But this is only possible if Q_1 contains a symbol from $\mathcal{W}_t$. Hence, $\gamma_w(Q_1)$ is a singleton set, i.e., $\gamma_w(Q_1) = \{\boldsymbol{\rho_1}\}$. It follows

$$\boldsymbol{\theta}_1\boldsymbol{\rho}_1 = \boldsymbol{\theta}\boldsymbol{\rho}. \tag{110}$$

Since $\boldsymbol{\theta}_1$ is an H-type, $\boldsymbol{\rho}_1$ must be of the form $(*, T, 1, *, *)$.

Since $\boldsymbol{\rho} \in \gamma_t(s')$ and $\boldsymbol{\rho}_1 \in \gamma_w(Q_1) \subseteq \gamma_t(s_1)$ (by (Hom5)) and both types are of the form $(*, T, 1, *, *)$, we can choose the representatives s_1 and s' in such a way that they begin with the letter t or t^{-1}. This choice ensures that no $\mathbb{H}$-product takes place in the products s_0s_1 and $p's'$ in $\mathbb{H} * \{t, \bar{t}\}^*$. We know that $P'S' = \psi(Q) = \psi(e_0W_0Q_1)$, i.e. $p's' \sim s_0s_1$. Since all t-sequences are reduced and $s_0, p' \in \mathbb{H}$ by (109), there exists a connecting element c_1 such that a van Kampen diagram as in Figure 4 holds. In particular,

$$c_1 \in \mathrm{Ge}(\boldsymbol{\theta}), \quad s_0c_1 \sim p', \quad c_1s' \sim s_1. \tag{111}$$

Let us define

$$P = e_0W_0c_1 \quad \text{and} \quad S = c_1^{-1}Q_1. \tag{112}$$

We obtain from (106) and (112) that $Q = PS$. From (107), (111), and (112) we get $\psi(P) = \psi(e_0W_0c_1) = [s_0]_\sim c_1 = [p']_\sim = P'$ and $\psi(S) = \psi(c_1^{-1}Q_1) = c_1^{-1}[s_1]_\sim = [s']_\sim = S'$.

It remains to determine the types of P and Q. Since $\boldsymbol{\rho} \in \gamma_t(s')$ and $\boldsymbol{\rho}_1 \in \gamma_t(s_1)$ and s' and s_1 both start with either t or t^{-1}, we must have $\tau\mathrm{i}(s') = \tau(s_1)$ (which is either (A, T) or (B, T)). Moreover, the boolean component of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}_1$ is 1. Finally, by (110), $\tau\mathrm{e}(\boldsymbol{\rho}) = \tau\mathrm{e}(\boldsymbol{\theta}\boldsymbol{\rho}) = \tau\mathrm{e}(\boldsymbol{\rho}_1)$. This establishes that $\boldsymbol{\rho}_1 = \boldsymbol{\rho}$. Using (110) we know that $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}$ have same initial vertex type. Moreover, $\boldsymbol{\rho}_1 = \boldsymbol{\rho}$ implies that also the ending vertex types of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}$ coincide. Finally, since they are H-types, they also have the same boolean component, namely 0. Hence $\boldsymbol{\theta}_1 = \boldsymbol{\theta}$. We get $\gamma_w(P) = \gamma_w(e_0W_0c_1) = \gamma_w(W_0) = \{\boldsymbol{\theta}_1\} = \{\boldsymbol{\theta}\}$ and $\gamma_w(S) = \gamma_w(c_1^{-1}Q_1) = \gamma_w(Q_1) = \{\boldsymbol{\rho_1}\} = \{\boldsymbol{\rho}\}$.

Case 2: $\boldsymbol{\rho} = (A, T, 0, A, T)$.
Since $\boldsymbol{\rho} = (A, T, 0, A, T) \in \gamma_t(S')$, we have $S' = a \in A$. Hence, we can set $S = a$ and $P = Qa^{-1}$. Thus, $\psi(S) = a = S'$ and $\psi(P)a = \psi(P)\psi(S) = \psi(PS) = \psi(Q) = P'S' = P'a$ implies $\psi(P) = P'$. Since $\boldsymbol{\theta}\boldsymbol{\rho}$ is defined, the H-type $\boldsymbol{\theta}$ must have the form $(*, *, 0, A, T)$. Hence $\gamma_w(P) = \gamma_w(Qa^{-1}) = \gamma(Q) = \{\boldsymbol{\theta}\boldsymbol{\rho}\} = \{\boldsymbol{\theta}\}$ and $\boldsymbol{\rho} = (A, T, 0, A, T) \in \gamma_w(a) = \gamma_w(S)$.

Case 3: $\boldsymbol{\rho} = (B, T, 0, B, T)$.
This case can be treated in the same way as Case 2.

Case 4: $\boldsymbol{\rho} = (1, 1, 0, 1, 1)$.
Hence $S' = 1$. The choice $S = 1$ and $P = Q$ fulfills the conclusion of the lemma. □

In the following, $\pi_\equiv : \mathcal{W}^* * A * B \to \mathbb{W}$ denotes the canonical homomorphism that maps $s \in \mathcal{W}^* * A * B$ to its equivalence class w.r.t. $\equiv$.

Lemma 26 (factorization). *Let $\psi' : \mathcal{W}^* * A * B \to \mathbb{W}$ be some AB-homomorphism. Then, there exists a unique AB-homomorphism $\psi : \mathbb{W} \to \mathbb{W}$ such that $\pi_\equiv \circ \psi = \psi'$.*

Proof. Since ψ' is an AB-homomorphism, for every $W \in \mathcal{W}$ we have $\delta_w(W) = \delta_w(\psi'(W))$. We claim that the congruence $\equiv$ is contained in the kernel of the homomorphism ψ': Consider a pair $(c, d) \in \delta_w(W)$ (i.e, $cW \equiv Wd$). Thus $(c, d) \in \delta_w(\psi'(W))$, i.e., $\psi'(cW) = c\psi'(W) \equiv \psi'(W)d = \psi'(Wd)$. Therefore (cW, Wd) belongs to the kernel of ψ'. It follows that there exists a monoid homomorphism $\psi : \mathbb{W} \to \mathbb{W}$ with $\pi_\equiv \circ \psi = \psi'$. By Lemma 13, ψ is in fact an AB-homomorphism. Unicity of ψ follows directly from the requirement that $\psi([s]_\equiv) = \psi'(s)$ for all $s \in \mathcal{W}^* * A * B$. □

Lemma 27 (quotient). *Let $\sigma' : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B$ be some AB-homomorphism. Then, there exists a unique AB-homomorphism $\sigma : \mathbb{W} \to \mathbb{W}$ such that $\pi_\equiv \circ \sigma = \sigma' \circ \pi_\equiv$.*

Proof. Applying Lemma 26 to the AB-homomorphism $\psi' = \sigma' \circ \pi_\equiv$ we obtain this lemma. □

Lemma 28 (lifting). *Let $\psi' : \mathcal{W}^* * A * B \to \mathbb{W}$ be some AB-homomorphism. Then, there exists an AB-homomorphism $\psi : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B$ such that $\psi' = \psi \circ \pi_\equiv$.*

Proof. Choose a partition $\widehat{\mathcal{W}} = \widehat{\mathcal{W}_1} \cup \widehat{\mathcal{W}_2}$ such that $\widehat{\mathcal{W}_1} \cap \widehat{\mathcal{W}_2} = \emptyset$ and $\{\mathbb{I}_w(W) \mid W \in \widehat{\mathcal{W}_1}\} = \widehat{\mathcal{W}_2}$. Since $\mathbb{I}_w(W) \neq W$ for all $W \in \widehat{\mathcal{W}}$, such a partition exists. Let us consider some map $\psi : (\mathcal{W} \cup A \cup B) \to \mathcal{W}^* * A * B$ such that for all $a \in A$, $b \in B$, and $W \in (\mathcal{W} \setminus \widehat{\mathcal{W}}) \cup \widehat{\mathcal{W}_1}$:

$$\psi(a) = a, \quad \psi(b) = b, \quad \psi(W) \in \pi_\equiv^{-1}(\psi'(W)). \tag{113}$$

Moreover, for $W \in \widehat{\mathcal{W}_2}$ we set

$$\psi(W) = \mathbb{I}_w(\psi(\mathbb{I}_w(W))). \tag{114}$$

Note that this implies

$$\pi_\equiv(\psi(W)) = \pi_\equiv(\mathbb{I}_w(\psi(\mathbb{I}_w(W)))) = \mathbb{I}_w(\pi_\equiv(\psi(\mathbb{I}_w(W)))) = \mathbb{I}_w(\psi'(\mathbb{I}_w(W))) = \psi'(W)$$

for all $W \in \widehat{\mathcal{W}_2}$. Hence, together with (113), we get

$$\pi_\equiv(\psi(W)) = \psi'(W) \text{ for all } W \in \mathcal{W}. \tag{115}$$

Also notice that (114) easily implies

$$\mathbb{I}_w(\psi(W)) = \psi(\mathbb{I}_w(W)) \text{ for all } W \in \widehat{\mathcal{W}}. \tag{116}$$

By the universal property of the free product, ψ can be extended into a monoid homomorphism $\psi : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B$. Since $\pi_\equiv(\psi(g)) = \psi(g)$ for all $g \in \mathcal{W} \cup A \cup B$, we have $\psi' = \psi \circ \pi_\equiv$.

It remains to show that ψ is in fact an AB-homomorphism. For this, we check conditions (a)–(f) from Lemma 12. Property (a) is obvious. For (b)–(f), assume that $g \in \mathcal{W} \cup A \cup B$. For $g \in A \cup B$, we have $\psi(g) = g$ and (b)–(f) are obvious. Thus, assume that $g = W \in \mathcal{W}$. For (b), we get: We have:

$$\begin{aligned} W \in \mathrm{dom}(\mathbb{I}_w) &\overset{\text{(Hom3)}}{\iff} \psi'(W) \in \mathrm{dom}(\mathbb{I}_w) \\ &\iff \pi_\equiv(\psi(W)) \in \mathrm{dom}(\mathbb{I}_w) \\ &\overset{\text{(Hom3)}}{\iff} \psi(W) \in \mathrm{dom}(\mathbb{I}_w). \end{aligned}$$

For (c), assume that $W \in \widehat{\mathcal{W}}$. Then (c) is stated in (116). Property (d) follows from

$$\begin{aligned} \gamma_w(W) &\overset{\text{(Hom5)}}{\subseteq} \gamma_w(\psi'(W)) \\ &= \gamma_w(\pi_\equiv(\psi(W))) \\ &\overset{(61)}{=} \gamma_w(\psi(W)). \end{aligned}$$

Finally, (e) and (f) follows directly from the preservation of the μ- and δ-mappings by ψ' and $\pi_\equiv$. Hence, ψ is indeed an AB-homomorphism. □

$$\begin{tikzcd} \mathcal{W}^* * A * B \arrow[d, "\pi_\equiv"'] \arrow[dr, "\psi'"] & \\ \mathbb{W} \arrow[r, dashed, "\psi"] & \mathbb{W} \end{tikzcd}$$

(factorization)

$$\begin{tikzcd} \mathcal{W}^* * A * B \arrow[r, "\sigma'"] \arrow[d, "\pi_\equiv"'] & \mathcal{W}^* * A * B \arrow[d, "\pi_\equiv"] \\ \mathbb{W} \arrow[r, dashed, "\sigma"] & \mathbb{W} \end{tikzcd}$$

(quotient)

$$\begin{tikzcd} \mathcal{W}^* * A * B \arrow[r, dashed, "\psi"] \arrow[dr, "\psi'"'] & \mathcal{W}^* * A * B \arrow[d, "\pi_\equiv"] \\ & \mathbb{W} \end{tikzcd}$$

(lifting)

$$\begin{tikzcd} \mathcal{W}^* * A * B \arrow[r, dashed, "\sigma'"] \arrow[d, "\pi_\equiv"'] & \mathcal{W}^* * A * B \arrow[d, "\pi_\equiv"] \\ \mathbb{W} \arrow[r, "\sigma"] & \mathbb{W} \end{tikzcd}$$

(inverse image)

Fig. 5. AB-homomorphisms

Lemma 29 (inverse image). *Let $\sigma : \mathbb{W} \to \mathbb{W}$ be some AB-homomorphism. Then, there exists an AB-homomorphism $\sigma' : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B$ such that $\pi_\equiv \circ \sigma = \sigma' \circ \pi_\equiv$.*

Proof. Applying Lemma 28 to the AB-homomorphism $\psi' = \pi_\equiv \circ \sigma$, we obtain this lemma. □

4 Equations and disequations with rational constraints over $\mathbb{G}$

Let us consider a system $(\mathcal{S}, \mathsf{C})$ of equations and disequations over the HNN-extension $\mathbb{G}$ with variables from $\mathcal{U}$ and rational constraints, see Section 2.6. By Proposition 2 we can assume that $\mathcal{S}$ is quadratic. Recall that $\mathsf{C} : \mathcal{U} \to \mathcal{B}(Rat(\mathbb{G}))$. By Proposition 2, for every variable $U \in \mathcal{U}$ one can construct a strict normal partitioned fta $\mathcal{A}_U$ over the labelling set $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$ such that

$$\mathsf{C}(U) = \pi_{\mathbb{G}}(\mathrm{L}(\mathcal{A}_U)).$$

Let $\mathcal{A}$ be the direct product (see [LS08]) of the fta $\mathcal{A}_U$ for $U \in \mathcal{U}$ and let Q be the set of states of $\mathcal{A}$. This fta $\mathcal{A}$ is strict, normal, partitioned and, for every $U \in \mathcal{U}$, there exists $\mathsf{I}_U \subseteq \mathsf{Q}$ and $\mathsf{T}_U \subseteq \mathsf{Q}$ such that

$$\mathsf{C}(U) = \{g \in \mathbb{G} \mid (\mathsf{I}_U \times \mathsf{T}_U) \cap \mu_{\mathcal{A},\mathbb{G}}(g) \neq \emptyset\}$$

(recall the definition of the mapping $\mu_{\mathcal{A},\mathbb{G}}$ from (44)). Let us define, for every $U \in \mathcal{U}$, the set of binary relations $\mathsf{B}(U) \subseteq \mathsf{B}(\mathsf{Q})$ by:

$$\mathsf{B}(U) = \{\rho \in \mathsf{B}(\mathsf{Q}) \mid (I_U \times T_U) \cap \rho \neq \emptyset\}.$$

Note that $\mathsf{B}(U)$ is *upwards-closed* i.e., if $\rho \subseteq \rho'$ and $\rho \in \mathsf{B}(U)$, then $\rho' \in \mathsf{B}(U)$. It is then clear that $\mathcal{A}$ recognizes the constraint C in the sense that, for every $U \in \mathcal{U}$,

$$\mathsf{C}(U) = \mu_{\mathcal{A},\mathbb{G}}^{-1}(\mathsf{B}(U)).$$

Let us define

$$M(\mathsf{C}) = \{\mu : \mathcal{U} \to \mathsf{B}(\mathsf{Q}) \mid \forall U \in \mathcal{U} : \mu(U) \in \mathsf{B}(U)\}.$$

Since $\mathsf{B}(U)$ is upwards-closed, we have:

$$(\mu \in M(\mathsf{C}) \text{ and } \forall U \in \mathcal{U} : \mu(U) \subseteq \mu'(U)) \quad \Longrightarrow \quad \mu' \in M(\mathsf{C}). \tag{117}$$

We can now express the initial system $(\mathcal{S}, \mathsf{C})$ of equations and disequations with rational constraint as a finite disjunction

$$\bigvee_{\mu \in M(\mathsf{C})} (\mathcal{S}, \mu_{\mathcal{A},\mathbb{G}}, \mu). \tag{118}$$

A solution of the system $(\mathcal{S}, \mu_{\mathcal{A},\mathbb{G}}, \mu)$ is now any monoid homomorphism $\sigma : \mathcal{U}^* \to \mathbb{G}$ such that the following holds:

$$\forall (u = u') \in \mathcal{S} : \ \sigma(u) = \sigma(u') \tag{119}$$

$$\forall (u \neq u') \in \mathcal{S} : \ \sigma(u) \neq \sigma(u') \tag{120}$$

$$\forall U \in \mathcal{U} : \ \mu_{\mathcal{A},\mathbb{G}}(\sigma(U)) = \mu(U). \tag{121}$$

Any solution of a system $(\mathcal{S}, \mu_{\mathcal{A},\mathbb{G}}, \mu)$ for some $\mu \in M(\mathsf{C})$ is a solution of the disjunction (118). An *over*-solution of the system $(\mathcal{S}, \mu_{\mathcal{A},\mathbb{G}}, \mu)$ is any monoid homomorphism $\sigma : \mathcal{U}^* \to \mathbb{G}$ fulfilling the above conditions (119) and (120) together with the condition

$$\forall U \in \mathcal{U} : \mu_{\mathcal{A},\mathbb{G}}(\sigma(U)) \supseteq \mu(U). \tag{122}$$

By condition (117), any over-solution of a system $(\mathcal{S}, \mu_{\mathcal{A},\mathbb{G}}, \mu)$ for some $\mu \in M(\mathsf{C})$ is a solution of the disjunction (118). We have thus proved the following proposition.

Proposition 5. *Given a system $(\mathcal{S}, \mathsf{C})$ of equations and disequations over $\mathbb{G}$ with variables $\mathcal{U}$ and rational constraints, one can compute a finite family of systems $(\mathcal{S}_j, \mu_{\mathcal{A},\mathbb{G}}, \mu_j)$ $(j \in J)$ such that the following holds:*

(a) Every $\mathcal{S}_j$ is a quadratic system of equations and disequations with variables $\mathcal{V} \supseteq \mathcal{U}$.
(b) $\mu_{\mathcal{A},\mathbb{G}}$ is the map associated with a strict normal partitioned fta $\mathcal{A}$ over the labelling set $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$ (see 44)).
(c) μ_j is a map $\mu_j : \mathcal{V} \to \mathsf{B}(\mathsf{Q}_\mathcal{A})$
(d) Every solution $\sigma : \mathcal{U}^* \to \mathbb{G}$ of $(\mathcal{S}, \mathsf{C})$ extends to a solution $\sigma' : \mathcal{V}^* \to \mathbb{G}$ of $\bigvee_{j \in J}(\mathcal{S}_j, \mu_{\mathcal{A},\mathbb{G}}, \mu_j)$.
(e) For every solution $\sigma' : \mathcal{V}^* \to \mathbb{G}$ of $\bigvee_{j \in J}(\mathcal{S}_j, \mu_{\mathcal{A},\mathbb{G}}, \mu_j)$, the restriction of σ' to $\mathcal{U}^*$ is a solution of $\mathcal{S}$.
(f) Every over-solution of $\bigvee_{j \in J}(\mathcal{S}_j, \mu_{\mathcal{A},\mathbb{G}}, \mu_j)$ is also a solution of $\bigvee_{j \in J}(\mathcal{S}_j, \mu_{\mathcal{A},\mathbb{G}}, \mu_j)$.

Moreover, if $\mathcal{S}$ is a system of equations (without disequations), then every $\mathcal{S}_j$ is a system of equations (without disequations).

A disjunction of systems $\bigvee_{j \in J}(\mathcal{S}_j, \mu_{\mathcal{A},\mathbb{G}}, \mu_j)$ that satisfies (a), (b), and (c) from Proposition 5 is said to be in *quadratic normal form.* If the disjunction moreover satisfies (f), then it is said to be in *closed quadratic normal form.*

5 Equations over $\mathbb{H}_t$

5.1 t-equations

A *system of t-equations* is a set

$$\mathcal{S} \subseteq \mathbb{W} \times \mathbb{W} \tag{123}$$

such that for all $(w, w') \in \mathcal{S}$ we have $\gamma_w(w) = \gamma_w(w') \neq \emptyset$. Such a system can be only *solvable* if $\mathcal{S} \subseteq \mathbb{W}_t \times \mathbb{W}_t$. In this case, a *solution* of $\mathcal{S}$ is any AB-homomorphism $\psi : \mathbb{W}_t \to \mathbb{H}_t$ such that:

$$\forall (w, w') \in \mathcal{S} : \psi(w) = \psi(w'). \tag{124}$$

Notice that here, the rational constraints are replaced by the even more restrictive conditions that define the notion of AB-homomorphism: besides preservation of the μ-map, the homomorphism ψ must also preserve $\mathbb{I}$, γ and δ.

5.2 From $\mathbb{G}$-equations to t-equations

Let us start with a system

$$\mathcal{S} = (\{(u_i = u'_i) \mid 1 \leq i \leq n\},\ \mu_{\mathcal{A},\mathbb{G}},\ \mu_{\mathcal{U}}) \tag{125}$$

of equations over $\mathbb{G}$ with variables $\mathcal{U}$ and rational constraints, which is in quadratic normal form (see Proposition 5). Assume that $u_i = U_{i,1}$ and $u'_i = U_{i,2}U_{i,3}$ with $U_{i,1}, U_{i,2}, U_{i,3} \in \mathcal{U}$ for $1 \leq i \leq n$. In the following we denote the interval $\{1, \ldots, m\}$ with $[1, m]$.

We define a reduction of the satisfiability problem for such systems to the satisfiability problem for systems of t-equations. The leading idea is simply that, since $\pi_{\mathbb{G}} : \mathrm{Red}(\mathbb{H}, t)/_\sim \to \mathbb{G}$ is a bijection, every solution in $\mathbb{G}$ corresponds to a map into $\mathbb{H}_t$. Nevertheless the product in $\mathbb{G}$ corresponds to a somewhat complicated operation over $\mathrm{Red}(\mathbb{H}, t)/_\sim$ that we must deal with.

Let us consider the alphabets

$$\mathcal{V}_0 = [1, n] \times [1, 3] \times [1, 7] \times [0, N_0] \tag{126}$$

and the alphabet $\mathcal{W}$ constructed from this set $\mathcal{V}_0$ in Section 3.6. We choose the integer N_0 in such a way that

$$\mathrm{Card}(\{W \in \mathcal{W} \mid \exists i, j, k : p_1(W) = (i, j, k, 0)\}) < \frac{1}{2}\mathrm{Card}(\mathcal{V}_0) \tag{127}$$

One can take, for example, $N_0 = 2\mathrm{Card}(\{-1, 0, 1\} \times \mathcal{T}_{HT} \times \mathsf{B}^2(\mathsf{Q}) \times \mathrm{PGI}\{A, B\}) + 1$ in order to achieve this inequality. We consider all tuples $(W_{i,j,k})_{(i,j,k) \in \mathcal{V}_0}$ with $W_{i,j,k} \in \mathcal{W} \cup \{1\}$ and all triple $(e_{i,1,2}, e_{i,2,3}, e_{i,3,1}) \in (A \cup B)^3$ such that the conditions in Figure 6 hold. A vector $(\boldsymbol{W}, \boldsymbol{e})$ fulfilling all the properties (128)-(139) is called an *admissible* vector. For every

$$W_{i,j,4} \;\in\; \mathcal{W} \;\wedge\; \gamma_w(W_{i,j,4}) \text{ is an H-type} \tag{128}$$

$$p_1(W_{i,j,k}) \;=\; (i,j,k,0) \in \mathcal{V}_0 \text{ if } W_{i,j,k} \neq 1 \tag{129}$$

$$\gamma_w(\prod_{k=1}^{7} W_{i,j,k}) \;\in\; \{(1,H,b,1,1) \mid b \in \{0,1\}\} \tag{130}$$

$$\gamma_w(\prod_{k=1}^{7} W_{i,j,k}) = \gamma_w(\prod_{k=1}^{7} W_{i',j',k}) \text{ if } U_{i,j} = U_{i',j'} \tag{131}$$

$$\gamma_w(\prod_{k=1}^{3} W_{i,1,k}) \;=\; \gamma_w(\prod_{k=1}^{3} W_{i,2,k}) \tag{132}$$

$$\gamma_w(\prod_{k=5}^{7} W_{i,1,k}) \;=\; \gamma_w(\prod_{k=5}^{7} W_{i,3,k}) \tag{133}$$

$$\gamma_w(\prod_{k=5}^{7} W_{i,2,k}) \;=\; \gamma_w(\prod_{k=1}^{3} \mathbb{I}_w(W_{i,3,4-k})) \tag{134}$$

$$e_{i,1,2} \;\in\; \mathrm{Gi}(W_{i,1,4}) \;=\; \mathrm{Gi}(W_{i,2,4}) \tag{135}$$

$$e_{i,2,3} \;\in\; \mathrm{Ge}(W_{i,2,4}) \;=\; \mathrm{Gi}(W_{i,3,4}) \tag{136}$$

$$e_{i,3,1} \;\in\; \mathrm{Ge}(W_{i,3,4}) \;=\; \mathrm{Ge}(W_{i,1,4}) \tag{137}$$

$$p_1(\mu_w(\prod_{k=1}^{7} W_{i,j,k})) \;=\; \mu_{\mathcal{U}}(U_{i,j}) \tag{138}$$

$$W_{i,2,k} \;\in\; \widehat{\mathcal{W}} \cup \{1\} \;\ni\; W_{i,3,8-k} \text{ for } 5 \leq k \leq 7 \tag{139}$$

Fig. 6. Conditions for an admissible vector

admissible vector $(\boldsymbol{W}, \boldsymbol{e})$ we define for all $1 \leq i, i' \leq n, 1 \leq j, j' \leq 3$ the following equations:

$$\prod_{k=1}^{7} W_{i,j,k} = \prod_{k=1}^{7} W_{i',j',k} \text{ if } U_{i,j} = U_{i',j'} \tag{140}$$

$$W_{i,1,1}W_{i,1,2}W_{i,1,3}e_{i,1,2} = W_{i,2,1}W_{i,2,2}W_{i,2,3} \tag{141}$$

$$W_{i,2,5}W_{i,2,6}W_{i,2,7} = e_{i,2,3}\mathbb{I}_w(W_{i,3,3})\mathbb{I}_w(W_{i,3,2})\mathbb{I}_w(W_{i,3,1}) \tag{142}$$

$$e_{i,3,1}W_{i,1,5}W_{i,1,6}W_{i,1,7} = W_{i,3,5}W_{i,3,6}W_{i,3,7} \tag{143}$$

$$W_{i,1,4} = e_{i,1,2}W_{i,2,4}e_{i,2,3}W_{i,3,4}e_{i,3,1} \tag{144}$$

Equations (141)–(144) correspond to a decomposition of the planar diagram associated with the $\mathbb{G}$-relation $U_{i,1} = U_{i,2}U_{i,3}$ into four pieces, as indicated in Figure 7. Equation (140) expresses the fact that some variables from $\mathcal{U}$ are common to several equations of $\mathcal{S}$.

We denote by $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ the set of equations (140)-(143) and by $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ the set of equations (144). For every $(i,j) \in [1,n] \times [1,3]$ we denote by $\overline{i,j}$ the lexicographically smallest

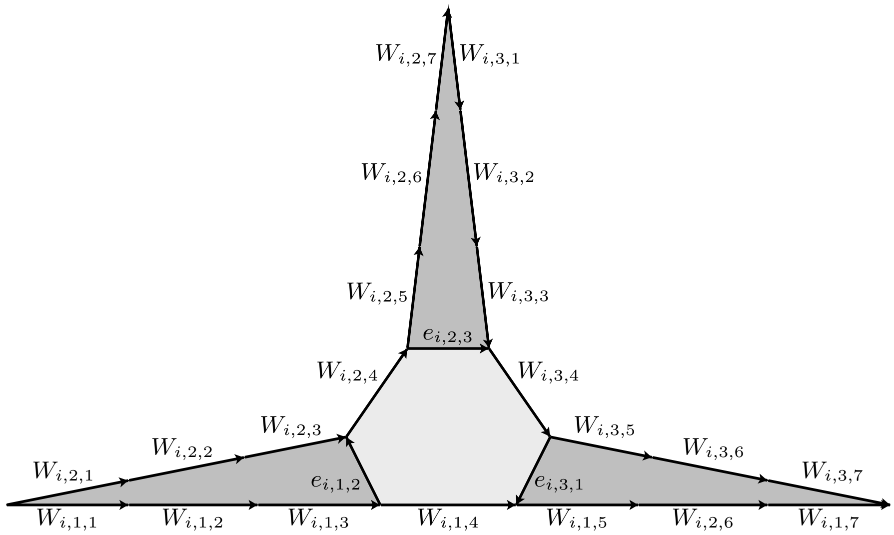


Fig. 7. Equation cut into 4 pieces

pair such that $U_{i,j} = U_{\overline{i,j}}$. By $\sigma_{\boldsymbol{W},\boldsymbol{e}} : \mathcal{U}^* \to \mathbb{W}$ we denote the unique monoid homomorphism such that

$$\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,j}) = \prod_{k=1}^{7} W_{\overline{i,j},k}. \tag{145}$$

Notice that, by the conditions imposed through the notion of an admissible vector, the system of equations $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ is really a system of t-equations, while some of the right-hand sides of the equations from $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ might have an empty image by γ_w.

Lemma 30. *A monoid homomorphism*

$$\sigma : \mathcal{U}^* \to \mathbb{G}$$

is a solution of the system (125) if and only if there exists an admissible vector $(\boldsymbol{W}, \boldsymbol{e})$ and an AB-homomorphism $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ such that:

- *All components of $(\boldsymbol{W}, \boldsymbol{e})$ belong to $\mathcal{W}_t \cup A \cup B$.*
- *σ_t solves both systems $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$.*
- *$\sigma = \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \sigma_t \circ \pi_{\mathbb{G}}$*

Here, we denote by $\pi_{\mathbb{G}} : \mathbb{H}_t \to \mathbb{G}$ the canonical projection, see Figure 8. We prove this lemma in the following two subsections.

From $\mathbb{G}$-solutions to t-solutions Let $\sigma : \mathcal{U}^* \to \mathbb{G}$ be a monoid homomorphism solving the system $(\{(U_{i,1} = U_{i,2}U_{i,3}) \mid 1 \le i \le n\},\ \mu_{\mathcal{A},\mathbb{G}},\ \mu_{\mathcal{U}})$ of equations over $\mathbb{G}$ with variables $\mathcal{U}$ and rational constraints ($U_{i,1}, U_{i,2}, U_{i,3} \in \mathcal{U}$ for $1 \le i \le n$). Let us fix some integer $1 \le i \le n$ and the corresponding equation $U_{i,1} = U_{i,2}U_{i,3}$. We construct the vectors $(W_{i,*,*}), (e_{i,*,*})$ corresponding to this equation. Let us choose for every $j \in [1,3]$ some $s_{i,j} \in \mathrm{Red}(\mathbb{H}, t)$ such that $\sigma(U_{i,j}) = \pi_{\mathbb{G}}(s_{i,j})$. This ensures that

$$s_{i,1} \approx s_{i,2}s_{i,3} \text{ for all } 1 \le i \le n$$

$$p_1(\mu_t((1, H, \|s_{i,j}\|_b, 1, 1), s_{i,j})) \overset{(63)}{=} \mu_{\mathcal{A},1}((1, H, \|s_{i,j}\|_b, 1, 1), s_{i,j}) \overset{(44)}{=} \mu_{\mathcal{A},\mathbb{G}}(\sigma(U_{i,j})) \overset{(122)}{=} \mu_{\mathcal{U}}(U_{i,j})$$

$$\begin{tikzcd}
\mathcal{U}^* \arrow[r, "\sigma_{\boldsymbol{W},\boldsymbol{e}}"] \arrow[d, "\sigma"'] & \mathbb{W}_t \arrow[d, "\sigma_t"] \\
\mathbb{G} & \mathbb{H}_t \arrow[l, "\pi_{\mathbb{G}}"]
\end{tikzcd}$$

Fig. 8. Lemma 30

Let us consider decompositions of the form (5) for $s_{i,2}$ and $s_{i,3}$:

$$s_{i,2} = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_\lambda} h_\lambda \cdots t^{\alpha_\ell} h_\ell, \tag{146}$$
$$s_{i,3} = k_0 t^{\beta_1} k_1 \cdots t^{\beta_\rho} k_\rho \cdots t^{\beta_m} k_m. \tag{147}$$

There exist integers $\lambda \in [0, \ell]$ and $\rho \in [0, m]$ such that $\rho = \ell - \lambda + 1$

$$\alpha_\lambda + \beta_\rho = 0, \tag{148}$$

$$t^{\alpha_\lambda} h_\lambda \cdots t^{\alpha_\ell} h_\ell k_0 t^{\beta_1} \cdots k_{\rho-1} t^{\beta_\rho} \approx e_{i,2,3} \in A(\alpha_\lambda) = B(\beta_\rho), \tag{149}$$

$$h_0 t^{\alpha_1} \cdots h_{\lambda-2} t^{\alpha_{\lambda-1}} (h_{\lambda-1} e_{i,2,3} k_\rho) t^{\beta_{\rho+1}} k_{\rho+1} \cdots t^{\beta_m} k_m \in \mathrm{Red}(\mathbb{H}, t). \tag{150}$$

Note that $h_\lambda, \ldots, h_\ell, k_0, \ldots, k_{\rho-1}$ are invertible in the monoid $\mathbb{H}$. We include in the above notation the following "degenerated" cases:

Left-degenerated case $\lambda = 1$: Then $h_0 t^{\alpha_1} \cdots h_{\lambda-2} t^{\alpha_{\lambda-1}}$ must be understood as 1.
Right-degenerated case $\rho = m$: Then $t^{\beta_{\rho+1}} k_{\rho+1} \cdots t^{\beta_m} k_m$ must be understood as 1.
Middle-degenerated case $\alpha_\ell + \beta_1 \neq 0$ or ($\alpha_\ell + \beta_1 = 0$ and $h_\ell k_0 \notin A(\beta_1)$): In this case we set $\lambda = \ell + 1$, $\rho = 0$, and $e_{i,2,3} = 1$. Equality (148) disappears, (149) becomes the trivial equation $1 = 1$, while (150) remains valid.
LM-degenerated case $\ell = 0$, i.e., $s_{i,2} = h_0$: We set $\lambda = 1$, $\rho = 0$ and $e_{i,2,3} = 1$. Equality (148) disappears, (149) becomes the trivial equation $1 = 1$. The lefthand-side of assertion (150) consists just of $h_0 s_{i,3}$.
MR-degenerated case $m = 0$: Analogous to the previous case. The left-hand side of (150) consists just of $s_{i,2} k_0$.

Notice that these cases are not disjoint; in particular, when $\ell = m = 0$ the three kinds of degeneracy occur simultaneously. Each kind of degeneracy can be visuallized in Figure 9 as one of the three triangular pieces consisting of a trivial relation.

Let us consider the following factors of the reduced sequences $s_{i,2}$ and $s_{i,3}$:

$$\begin{array}{lll}
L_{i,2} = h_0 t^{\alpha_1} \cdots h_{\lambda-2} t^{\alpha_{\lambda-1}} & M_{i,2} = h_{\lambda-1} & R_{i,2} = t^{\alpha_\lambda} h_\lambda \cdots t^{\alpha_\ell} h_\ell \\
L_{i,3} = k_0 t^{\beta_1} \cdots k_{\rho-1} t^{\beta_\rho} & M_{i,3} = k_\rho & R_{i,3} = t^{\beta_{\rho+1}} k_{\rho+1} \cdots t^{\beta_m} k_m
\end{array}$$

Note that $R_{i,2}, L_{i,3} \in \mathrm{dom}(\mathbb{I}_t)$. Moreover, since $s_{i,1} \approx s_{i,2} s_{i,3} \approx L_{i,2}(M_{i,2} e_{i,2,3} M_{i,3}) R_{i,3}$ and $L_{i,2}(M_{i,2} e_{i,2,3} M_{i,3}) R_{i,3}$ as well as $s_{i,1}$ are reduced sequences, by Lemma 2 we get

$$s_{i,1} \sim L_{i,2}(M_{i,2} e_{i,2,3} M_{i,3}) R_{i,3}. \tag{151}$$

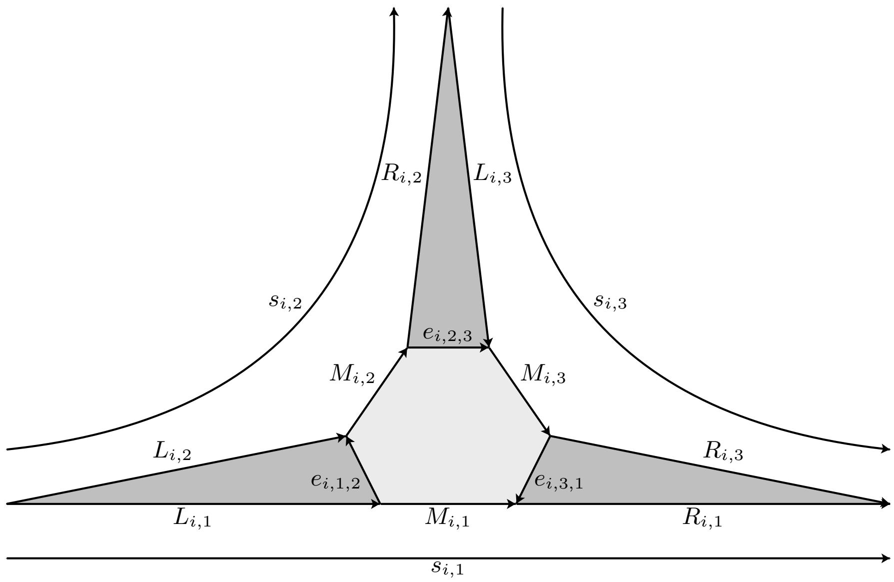


Fig. 9. Cutting the solution into three factors

Case 1 (standard case) $\lambda \in [2, \ell]$ and $\rho \in [1, m-1]$.

Then $L_{i,2}$ (resp. $R_{i,3}$) ends (resp. starts) with $t^{\alpha_{\lambda-1}}$ (resp. $t^{\beta_{\rho+1}}$). There must exist a factorization

$$s_{i,1} = L_{i,1} M_{i,1} R_{i,1}$$

with $L_{i,1}, R_{i,1} \in \mathrm{Red}(\mathbb{H}, t)$, $M_{i,1} \in \mathbb{H}$, and connecting elements $e_{i,1,2} \in B(\alpha_{\lambda-1})$, $e_{i,2,1} \in A(\beta_{\rho+1})$ such that:

$$L_{i,1} e_{i,1,2} \sim L_{i,2}, \tag{152}$$

$$e_{i,1,2} M_{i,2} e_{i,2,3} M_{i,3} e_{i,3,1} = M_{i,1} \text{ in } \mathbb{H} \tag{153}$$

$$e_{i,3,1} R_{i,1} \sim R_{i,3}, \tag{154}$$

see also the following diagram:

$$\begin{tikzcd}
& \bullet \arrow[rr, "M_{i,2}e_{i,2,3}M_{i,3}"] & & \bullet \arrow[dd, "e_{i,3,1}"] \arrow[dr, bend left, "R_{i,3}"] & \\
\bullet \arrow[ur, bend left, "L_{i,2}"] \arrow[dr, bend right, "L_{i,1}"'] & & & & \bullet \\
& \bullet \arrow[uu, "e_{i,1,2}"] \arrow[rr, "M_{i,1}"'] & & \bullet \arrow[ur, bend right, "R_{i,1}"'] &
\end{tikzcd}$$

The relation (149) can be rewritten as $R_{i,2} L_{i,3} \approx e_{i,2,3}$, or, since $R_{i,2}$ and $L_{i,3}$ are reduced, as

$$R_{i,2} \sim e_{i,2,3} \mathbb{I}_t(L_{i,3}), \tag{155}$$

see Figure 9. Let $\pi_t : \mathbb{H} * \{t, \bar{t}\}^* \to \{t, t^{-1}\}^*$ be the natural projection. By (152), (154), and (155) we know that

$$\pi_t(L_{i,1}) = \pi_t(L_{i,2}) \neq \varepsilon, \quad \pi_t(R_{i,1}) = \pi_t(R_{i,3}) \neq \varepsilon, \quad \pi_t(R_{i,2}) = \pi_t(\mathbb{I}_t(L_{i,3})) \neq \varepsilon.$$

Hence the fta $\mathcal{R}_6$ (see Figure 3) has computations of the following forms:

$$\begin{aligned}
(1,H) &\xrightarrow{L_{i,1}} \quad q_{i,1} \quad \xrightarrow{M_{i,1}} r_{i,1} \xrightarrow{R_{i,1}} (1,1)\\
(1,H) &\xrightarrow{L_{i,2}} \quad q_{i,1} \quad \xrightarrow{M_{i,2}} r_{i,2} \xrightarrow{R_{i,2}} (1,1)\\
(1,H) &\xrightarrow{L_{i,3}} \mathbb{I}_{\mathcal{R}}(r_{i,2}) \xrightarrow{M_{i,3}} r_{i,1} \xrightarrow{R_{i,3}} (1,1).
\end{aligned}$$

Note that $(q_{i,1},0,r_{i,1})$, $(q_{i,1},0,r_{i,2})$, and $(\mathbb{I}_{\mathcal{R}}(r_{i,2}),0,r_{i,1})$ are H-types.

An inspection of the fta $\mathcal{R}_6$ shows that the computation $(1,H) \xrightarrow{L_{i,1}} q_{i,1}$ can be factored into three subcomputations

$$(1,H) = q_{i,1,0} \xrightarrow{v_{i,1,1}} q_{i,1,1} \xrightarrow{v_{i,1,2}} q_{i,1,2} \xrightarrow{v_{i,1,3}} q_{i,1,3} = q_{i,1} \text{ with } L_{i,1} = v_{i,1,1}v_{i,1,2}v_{i,1,3} \tag{156}$$

such that each triple $(q_{i,1,k-1},\ \|v_{i,1,k}\|_b,\ q_{i,1,k})$ is either a T-type or an H-type or satisfies $q_{i,1,k-1} = q_{i,1,k}$ and $v_{i,1,k} = 1$. Any decomposition of the computation $(1,H) \xrightarrow{L_{i,1}} q_{i,1}$ having this property will be called $\mathcal{R}_6$-compatible. Similarly the computations $r_{i,1} \xrightarrow{R_{i,1}} (1,1)$ and $r_{i,2} \xrightarrow{R_{i,2}} (1,1)$ have $\mathcal{R}_6$-compatible decompositions:

$$q_{i,1,4} = r_{i,1} \xrightarrow{v_{i,1,5}} q_{i,1,5} \xrightarrow{v_{i,1,6}} q_{i,1,6} \xrightarrow{v_{i,1,7}} q_{i,1,7} = (1,1) \text{ with } R_{i,1} = v_{i,1,5}v_{i,1,6}v_{i,1,7} \tag{157}$$

$$q_{i,2,4} = r_{i,2} \xrightarrow{v_{i,2,5}} q_{i,2,5} \xrightarrow{v_{i,2,6}} q_{i,2,6} \xrightarrow{v_{i,2,7}} q_{i,2,7} = (1,1) \text{ with } R_{i,2} = v_{i,2,5}v_{i,2,6}v_{i,2,7}. \tag{158}$$

Combining decomposition (156) with equation (152), (157) with (154), and (158) with (155), we get the following three $\mathcal{R}_6$-compatible decompositions:

$$\begin{aligned}
(1,H) = q_{i,2,0} = q_{i,1,0} \xrightarrow{v_{i,2,1}} q_{i,2,1} = q_{i,1,1} \xrightarrow{v_{i,2,2}} q_{i,2,2} = \\
q_{i,1,2} \xrightarrow{v_{i,2,3}} q_{i,2,3} = q_{i,1,3} = q_{i,1} \text{ with } L_{i,2} = v_{i,2,1}v_{i,2,2}v_{i,2,3}
\end{aligned} \tag{159}$$

$$\begin{aligned}
q_{i,3,4} = q_{i,1,4} = r_{i,1} \xrightarrow{v_{i,3,5}} q_{i,3,5} = q_{i,1,5} \xrightarrow{v_{i,3,6}} q_{i,3,6} = \\
q_{i,1,6} \xrightarrow{v_{i,3,7}} q_{i,3,7} = q_{i,1,7} = (1,1) \text{ with } R_{i,3} = v_{i,3,5}v_{i,3,6}v_{i,3,7}
\end{aligned} \tag{160}$$

$$\begin{aligned}
(1,H) = \mathbb{I}_{\mathcal{R}}(q_{i,2,7}) = q_{i,3,0} \xrightarrow{v_{i,3,1}} \mathbb{I}_{\mathcal{R}}(q_{i,2,6}) = q_{i,3,1} \xrightarrow{v_{i,3,2}} \mathbb{I}_{\mathcal{R}}(q_{i,2,5}) = \\
q_{i,3,2} \xrightarrow{v_{i,3,3}} \mathbb{I}_{\mathcal{R}}(q_{i,2,4}) = q_{i,3,3} \text{ with } L_{i,3} = v_{i,3,1}v_{i,3,2}v_{i,3,3}
\end{aligned} \tag{161}$$

Finally, we define $v_{i,j,4} = M_{i,j} \in \mathbb{H}$ $(1 \le j \le 3)$. Note that $v_{i,2,5}, v_{i,2,6}, v_{i,2,7}, v_{i,3,1}, v_{i,3,2}, v_{i,3,3} \in \mathrm{dom}(\mathbb{I}_t)$. We summarize in Figure 10 the above decompositions and relations.

Let us extract from these the vector $(\boldsymbol{W}, \boldsymbol{e})$ and the AB-homomorphism σ_t: Let $\boldsymbol{\theta}_{i,j,k} = (q_{i,j,k-1}, \|v_{i,j,k}\|_b, q_{i,j,k})$. Note that $\boldsymbol{\theta}_{i,j,k} \in \gamma_t(v_{i,j,k})$ and that $\boldsymbol{\theta}_{i,j,4}$ is an H-type. If $v_{i,j,k} = 1$, $q_{i,j,k-1} = q_{i,j,k}$, and $k \neq 4$, then we set $W_{i,j,k} = 1$, otherwise, we choose for $W_{i,j,k}$ a letter from $\mathcal{W}$ with the following properties:

- $p_1(W_{i,j,k}) = (i,j,k,0)$, this ensures property (129).
- $\gamma_w(W_{i,j,k}) = \boldsymbol{\theta}_{i,j,k}$, this ensures property (130)–(137).

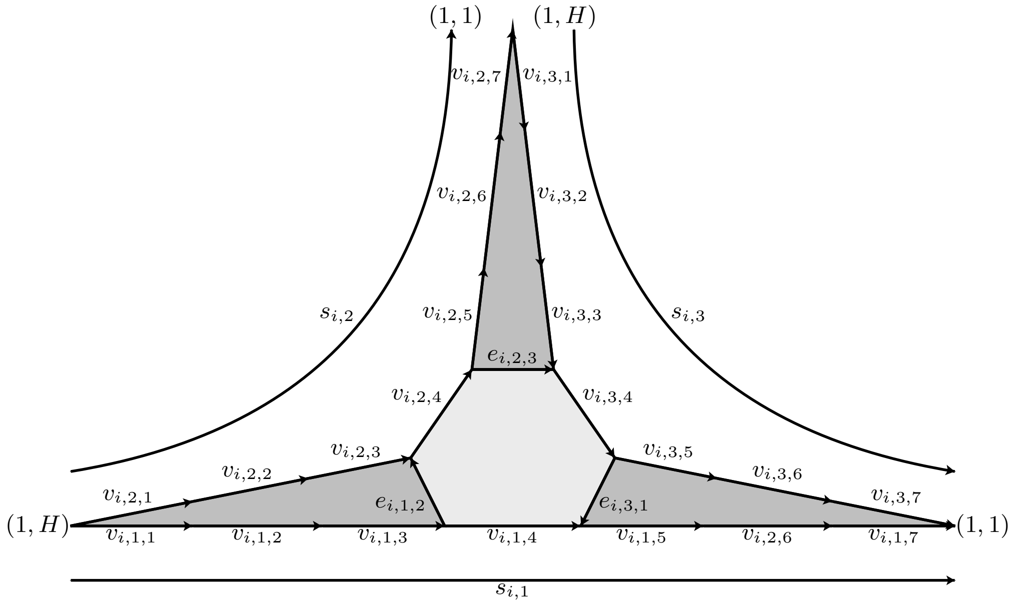


Fig. 10. Cutting the solution into seven factors

- $\mu_w(W_{i,j,k}) = \mu_t(\boldsymbol{\theta}_{i,j,k}, v_{i,j,k})$, this ensures property (138).
- $\delta_w(W_{i,j,k}) = \delta_t(\boldsymbol{\theta}_{i,j,k}, v_{i,j,k})$
- $W_{i,j,k} \in \widehat{\mathcal{W}} \Longleftrightarrow v_{i,j,k} \in \mathrm{dom}(\mathbb{I}_t)$, this ensures property (139).

Hence, $(\boldsymbol{W}, \boldsymbol{e})$ is an admissible vector. Note that $W_{i,j,k} \in \mathcal{W}_t$: one can choose $s = v_{i,j,k}$ in (86). Moreover, if $W_{i,j,k} \in \widehat{\mathcal{W}}$, then $\mathbb{I}_w(W_{i,j,k})$ does not occur among the letters $W_{i',j',k'}$ for $(i',j',k') \in \mathcal{V}_0$ (because $(i,j,k) = p_1(W_{i,j,k}) = p_1(\mathbb{I}_w(W_{i,j,k}))$ and $\mathbb{I}_w(W_{i,j,k}) \neq W_{i,j,k}$). This allows us to define a mapping σ_t by $\sigma_t(W_{i,j,k}) = [v_{i,j,k}]_\sim$ and $\sigma_t(\mathbb{I}_w(W_{i,j,k})) = [\mathbb{I}_t(v_{i,j,k})]_\sim$ in case $W_{i,j,k} \in \widehat{\mathcal{W}}$ (i.e. $v_{i,j,k} \in \mathrm{dom}(\mathbb{I}_t)$). By Lemma 12, this mapping σ_t induces an AB-homomorphism $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$. One can check that σ_t solves the systems of equations $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ and that

$$\sigma = \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \sigma_t \circ \pi_{\mathbb{G}}.$$

We indicate now how these arguments must be adapted to the degenerated cases.

Case 2 (left-degenerated case): We have $\lambda = 1$. Let us set $L_{i,2} = L_{i,1} = e_{i,1,2} = 1$ and, accordingly, for all $1 \leq k \leq 3$:

$$v_{i,2,k} = v_{i,1,k} = 1, \;\; W_{i,2,k} = W_{i,1,k} = 1.$$

Case 3 (right-degenerated case): We have $\rho = m$. Let us set $R_{i,3} = R_{i,1} = e_{i,3,1} = 1$ and, accordingly, for all $5 \leq k \leq 7$:

$$v_{i,3,k} = v_{i,1,k} = 1, \;\; W_{i,3,k} = W_{i,1,k} = 1.$$

Case 4 (middle-degenerated case): We have $\alpha_\ell + \beta_1 \neq 0$ or $(\alpha_\ell + \beta_1 = 0$ and $h_\ell k_0 \notin A(\beta_1))$. Let us set $R_{i,2} = L_{i,3} = e_{i,2,3} = 1$ and, accordingly, for all $5 \leq k \leq 7$:

$$v_{i,2,k} = v_{i,3,8-k} = 1, \;\; W_{i,2,k} = W_{i,3,8-k} = 1.$$

Case 5 (LM-degenerated case): We have $\ell = 0$. We choose all the special values chosen in Case 2 and Case 4, i.e. $L_{i,2} = L_{i,1} = R_{i,2} = L_{i,3} = e_{i,1,2} = e_{i,2,3} = 1$ and the resulting choices for $W_{i,*,*}$.

Case 6 (MR-degenerated case): We have $m = 0$. We choose all the special values chosen in Case 3 and Case 4.

From t-solutions to $\mathbb{G}$-solutions Let $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ be an AB-homomorphism solving both systems $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$, where $(\boldsymbol{W}, \boldsymbol{e})$ is some admissible vector with all components in $\mathcal{W}_t \cup A \cup B$. We have to show that $\sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \sigma_t \circ \pi_{\mathbb{G}}$ solves the system (125).

First, we show that for every $i \in [1, n]$,

$$\sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,1})) \approx \sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,2}U_{i,3})). \tag{162}$$

Let us fix such an integer $i \in [1, n]$. Since σ_t solves (141)–(144), we have

$$\sigma_t(\prod_{k=1}^{7} W_{i,1,k}) \approx \sigma_t(\prod_{k=1}^{7} W_{i,2,k} \prod_{k=1}^{7} W_{i,3,k}). \tag{163}$$

Figure 7 gives a decomposition of the Van-Kampen diagram corresponding to the above equivalence into four diagrams corresponding to (141)–(144). Hence,

$$\begin{aligned}
\sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,1})) &\overset{(145)}{=} \sigma_t(\prod_{k=1}^{7} W_{\overline{i,1},k}) \\
&\overset{(140)}{=} \sigma_t(\prod_{k=1}^{7} W_{i,1,k}) \\
&\overset{(163)}{\approx} \sigma_t(\prod_{k=1}^{7} W_{i,2,k} \prod_{k=1}^{7} W_{i,3,k}) \\
&\overset{(140)}{=} \sigma_t(\prod_{k=1}^{7} W_{\overline{i,2},k} \prod_{k=1}^{7} W_{\overline{i,3},k}) \\
&\overset{(145)}{=} \sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,2}U_{i,3})),
\end{aligned}$$

which is (162).

It remains to show that for every $i \in [1, n]$,

$$\mu_{\mathcal{A},\mathbb{G}}(\pi_{\mathbb{G}}(\sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,j})))) = \mu_{\mathcal{U}}(U_{i,j}).$$

By definition (44), this is equivalent to

$$\mu_{\mathcal{A},1}((1, H, b, 1, 1), \sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,j}))) = \mu_{\mathcal{U}}(U_{i,j}), \text{ where} \tag{164}$$

$$b = \|\sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,j}))\|_b \tag{165}$$

We first remark that

$$\sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,j})) \overset{(145)}{=} \sigma_t(\prod_{k=1}^{7} W_{\overline{i,j},k}) \overset{(140)}{=} \sigma_t(\prod_{k=1}^{7} W_{i,j,k}). \tag{166}$$

By (130), $\gamma_w(\prod_{k=1}^{7} W_{i,j,k}) = \{(1,H,b',1,1)\}$ for some $b' \in \{0,1\}$. Since σ_t is an AB-homomorphism, the type $(1,H,b',1,1)$ also belongs to $\gamma_t(\sigma_t(\prod_{k=1}^{7} W_{i,j,k})) = \gamma_t(\sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,j})))$, which by (165) implies that $b' = b$. Thus

$$\gamma_w(\prod_{k=1}^{7} W_{i,j,k}) = \{(1,H,b,1,1)\}. \tag{167}$$

As σ_t is an AB-homomorphism,

$$\mu_t((1,H,b,1,1), \sigma_t(\prod_{k=1}^{7} W_{i,j,k})) = \mu_w((1,H,b,1,1), \prod_{k=1}^{7} W_{i,j,k}). \tag{168}$$

Now, the following calculation yields (164):

$$\begin{aligned}
\mu_{\mathcal{A},1}((1,H,b,1,1), \sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,j}))) &\overset{(166)}{=} \mu_{\mathcal{A},1}((1,H,b,1,1), \sigma_t(\prod_{k=1}^{7} W_{i,j,k})) \\
&\overset{(63)}{=} p_1(\mu_t((1,H,b,1,1), \sigma_t(\prod_{k=1}^{7} W_{i,j,k}))) \\
&\overset{(168)}{=} p_1(\mu_w((1,H,b,1,1), \prod_{k=1}^{7} W_{i,j,k})) \\
&\overset{(167)}{=} p_1(\mu_w(\gamma_w(\prod_{k=1}^{7} W_{i,j,k}), \prod_{k=1}^{7} W_{i,j,k})) \\
&= p_1(\mu_w(\prod_{k=1}^{7} W_{i,j,k})) \\
&\overset{(138)}{=} \mu_{\mathcal{U}}(U_{i,j})
\end{aligned}$$

6 Equations over $\mathbb{W}$

We suppose here that a system of equations with rational constraints over $\mathbb{G}$ is fixed. Thus the AB-algebras $\mathbb{W}$ and $\mathbb{H}_t$ are completely defined (from the variable alphabet of the system and the fta expressing the constraints). Given an $\mathbb{H}$-involution Φ (this notion is defined at the end of Section 3.9) we abbreviate with $\mathbb{W}/\Phi$ the quotient of the AB-algebra $\mathbb{W}$ by the congruence $\simeq_\Phi$ generated by the set of relations $\{(W, \Phi(W)) \mid W \in \mathcal{W}\}$ (see Lemma 22). Recall that HInv denotes the set of all involutive $\mathbb{H}$-automorphisms of $\mathbb{W}$ (see definition 8).

6.1 $\mathbb{W}$-equations

A system of $\mathbb{W}$-equations is a pair

$$(\mathcal{S}, \Phi) \tag{169}$$

such that $\mathcal{S} \subseteq \mathbb{W} \times \mathbb{W}$ is finite, $\gamma_w(s) = \gamma_w(s') \neq \emptyset$ for all $(s,s') \in \mathbb{W}$ and $\Phi \in \mathsf{HInv}$.

This system only has a solution if $\mathcal{S} \subseteq \mathbb{W}_t \times \mathbb{W}_t$. In this case, a *solution* of $(\mathcal{S}, \Phi)$ is any AB-homomorphism $\sigma_{\mathbb{W}} : \mathbb{W}_t \to \mathbb{W}_t/\Phi$ such that for every $(s,s') \in \mathcal{S}$:

$$\sigma_{\mathbb{W}}(s) = \sigma_{\mathbb{W}}(s'). \tag{170}$$

$$\begin{tikzcd}
\mathcal{U}^* \arrow[r, "\sigma_{\boldsymbol{W},\boldsymbol{e}}"] \arrow[d, "\sigma"] & \mathbb{W}_t \arrow[d, "\sigma_t"] \arrow[dr, "\sigma_{\mathbb{W}}"] & \\
\mathbb{G} & \mathbb{H}_t \arrow[l, "\pi_{\mathbb{G}}"'] & \mathbb{W}_t/\varPhi \arrow[l, "\psi_t"']
\end{tikzcd}$$

Fig. 11. Lemma 31

6.2 From t-equations to $\mathbb{W}$-equations

From t-solutions to $\mathbb{W}$-solutions For every $w \in \mathbb{W}$ we use the notation

$$\mathrm{Alph}(w) = \{W \in \mathcal{W} \setminus \widehat{\mathcal{W}} \mid W \text{ occurs in } w\} \cup \{W \in \widehat{\mathcal{W}} \mid W \text{ or } \overline{W} \text{ occurs in } w\}$$
$$\mathrm{A}(w) = \mathrm{Card}\{W \in \mathcal{W} \setminus \widehat{\mathcal{W}} \mid W \text{ occurs in } w\} + \frac{1}{2}\mathrm{Card}\{W \in \widehat{\mathcal{W}} \mid W \text{ or } \overline{W} \text{ occurs in } w\}.$$

Given a system of t-equations $\mathcal{S} = \{(w_i, w_i') \mid 1 \leq i \leq n\} \subseteq \mathbb{W} \times \mathbb{W}$, let:

$$\mathrm{Alph}(\mathcal{S}) = \mathrm{Alph}(\prod_{i=1}^{n} w_i w_i').$$

Lemma 31 (factorization of t-solutions). *Let $\mathcal{S} \subseteq \mathbb{W} \times \mathbb{W}$ be a system of t-equations (see (123)). Let us suppose that $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ is an AB-homomorphism solving the system $\mathcal{S}$. Then there exists an $\mathbb{H}$-involution $\varPhi \in \mathsf{HInv}$ and AB-homomorphisms*

$$\sigma_{\mathbb{W}} : \mathbb{W}_t \to \mathbb{W}_t/\varPhi \quad \text{and} \quad \psi_t : \mathbb{W}_t/\varPhi \to \mathbb{H}_t$$

such that, for every $W \in \mathrm{Alph}(\mathcal{S})$, $\sigma_t(W) = \psi_t(\sigma_{\mathbb{W}}(W))$ and for every $(s, s') \in \mathcal{S}$, $\sigma_{\mathbb{W}}(s) = \sigma_{\mathbb{W}}(s')$.

In other words: every solution in $\mathbb{H}_t$ of a system of t-equations factorizes, over the variables of the equation, through a solution in $\mathbb{W}_t/\varPhi$ of the same system of equations, where $\varPhi$ is an involution which belongs to HInv.

Remark 1. In the case where the system $\mathcal{S}$ is one of the systems $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ introduced by Lemma 30, every variable W occurring in the system $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ occurs in the system $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ too (it must occur at least in one trivial equation (140)). Thus the factorization property asserted by Lemma 31 will hold for every common solution of $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$.

This factorization lemma is obtained via the following more technical lemma:

Lemma 32. *Let $\mathcal{S} = \{(w_i, w_i') \mid 1 \leq i \leq n + m\} \subseteq \mathbb{W}_t \times \mathbb{W}_t$ be a system of t-equations such that $\gamma_w(w_i) = \gamma_w(w_i') \neq \emptyset$ for all $1 \leq i \leq m + n$. Let us suppose that $\lambda_i : \mathbb{W}_t \to \mathbb{W}_t$ $(1 \leq i \leq n + m)$ and $\theta_t : \mathbb{W}_t \to \mathbb{H}_t$ are AB-homomorphisms such that*

$$\theta_t(\lambda_i(w_i)) = \theta_t(\lambda_i(w_i')) \text{ for all } 1 \leq i \leq n + m, \tag{171}$$
$$\lambda_i = \lambda_j \text{ for all } 1 \leq i, j \leq n, \tag{172}$$
$$\mathrm{A}(\prod_{i=1}^{n+m} \lambda_i(w_i w_i')) < \frac{1}{2}\mathrm{Card}(\mathcal{V}_0).$$

Then there exist an $\mathbb{H}$-involution $\Phi \in \mathsf{HInv}$ and AB-homomorphisms

$$\lambda'_i : \mathbb{W}_t \to \mathbb{W}_t \ (1 \le i \le n+m) \quad \textit{and} \quad \theta'_t : \mathbb{W}_t \to \mathbb{H}_t$$

such that

$$\theta_t(\lambda_i(W)) = \theta'_t(\lambda'_i(W)) \textit{ for all } 1 \le i \le n+m, W \in \mathrm{Alph}(\mathcal{S}) \tag{173}$$
$$\theta'_t(W) = \theta'_t(\Phi(W)) \textit{ for all } W \in \mathcal{W} \tag{174}$$
$$\lambda'_i(w_i) \equiv_\Phi \lambda'_i(w'_i) \textit{ for all } 1 \le i \le n+m, \tag{175}$$
$$\lambda'_i = \lambda'_j \textit{ for all } 1 \le i,j \le n. \tag{176}$$

Proof. Let $\mathcal{S} = \{(w_i, w'_i) \mid 1 \le i \le n+m\} \subseteq \mathbb{W}_t \times \mathbb{W}_t$ such that $\gamma_w(w_i) = \gamma_w(w'_i) \neq \emptyset$ and let $\boldsymbol{\lambda} = (\lambda_i)_{1 \le i \le n+m}$ be a sequence of AB-homomorphisms $\lambda_i : \mathbb{W}_t \to \mathbb{W}_t$.

Claim 1. For every $1 \le i \le n+m$ we either have $w_i = w'_i \in A \cup B$ (and hence $\lambda_i(w_i) = \lambda_i(w'_i) \in A \cup B$) or $\|w_i\|, \|w'_i\| \ge 1$ and $\gamma_w(\lambda_i(w_i)) = \gamma_w(\lambda_i(w'_i))$. In particular, $\gamma_w(\lambda_i(w_i)) = \gamma_w(\lambda_i(w'_i))$ for all $1 \le i \le n+m$.

Since $\gamma_w(w_i) = \gamma_w(w'_i) \neq \emptyset$, we can distinguish the following cases:

Case 1. $w_i = 1$: We get $\gamma_w(w'_i) = \gamma_w(w_i) = \{(\theta, 0, \theta) \mid \theta \in \mathcal{T}_6\}$. But this implies $w'_i = 1$.

Case 2. $w_i \in A \setminus 1$: We get $\gamma_w(w'_i) = \gamma_w(w_i) \overset{(72)}{=} \{(A, T, 0, A, T), (A, H, 0, A, H)\}$, which implies $w'_i \in A \setminus 1$. Since λ_i and θ_t are AB-homomorphisms and therefore injective on A, we get

$$w_i = \theta_t(\lambda_i(w_i)) \overset{(171)}{=} \theta_t(\lambda_i(w'_i)) = w'_i.$$

The cases $w_i \in B \setminus 1$ and $w'_i \in A \cup B$ are symmetric.

Case 3. $\|w_i\|, \|w'_i\| \ge 1$: By Lemma 18 we have $|\gamma_w(w_i)| = 1 = |\gamma_w(w'_i)|$. Moreover, $\|w_i\|, \|w'_i\| \ge 1$ implies $\|\lambda_i(w_i)\|, \|\lambda_i(w'_i)\| \ge 1$ by Lemma 23. Hence, $|\gamma_w(\lambda_i(w_i))| = 1 = |\gamma_w(\lambda_i(w'_i))|$. With (Hom5) we finally get $\gamma_w(\lambda_i(w_i)) = \gamma_w(w_i) = \gamma_w(w'_i) = \gamma_w(\lambda_i(w'_i))$. This concludes the proof of Claim 1.

For every $i \in [1, n+m]$ we define $\equiv_i$ as the least monoid congruence over $\mathbb{W}$ containing $\{(\lambda_j(w_j), \lambda_j(w'_j)) \mid i+1 \le j \le n+m\}$. For every $i \in [1, n+m]$ let us consider some decompositions

$$\lambda_i(w_i) = P_i S_i \quad \text{and} \quad \lambda_i(w'_i) = P'_i S'_i \tag{177}$$

such that $P_i \equiv_i P'_i$, $\gamma_w(P_i) \neq \emptyset$, $\gamma_w(S_i) \neq \emptyset$, $\gamma_w(P'_i) \neq \emptyset$, $\gamma_w(S'_i) \neq \emptyset$, and this choice (177) minimizes the integer

$$\Delta(P_i, S_i, P'_i, S'_i, \theta_t)$$

(this value was defined in (83)). Such a (P_i, P'_i) is called a *distinguishing pair* for $(\lambda_i(w_i), \lambda_i(w'_i))$ and we denote by

$$\Delta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)$$

the corresponding value of $\Delta(P_i, S_i, P'_i, S'_i, \theta_t)$.

The *size* of the triple $(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)$ is defined as the multiset of natural numbers

$$\|(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)\| = \{\{\Delta_1(\mathcal{S}, \boldsymbol{\lambda}, \theta_t), \ldots, \Delta_{n+m}(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)\}\}. \tag{178}$$

For every $w \in \mathbb{W}$ we use the notation

$$\mathrm{Alph}(\mathcal{S},\boldsymbol{\lambda}) = \mathrm{Alph}(\prod_{i=1}^{n+m} \lambda_i(w_i w_i')),$$
$$\mathrm{A}(\mathcal{S},\boldsymbol{\lambda}) = \mathrm{A}(\prod_{i=1}^{n+m} \lambda_i(w_i w_i')).$$

Let us prove Lemma 32 by induction over $\|(\mathcal{S},\boldsymbol{\lambda},\theta_t)\|$ with respect to the partial ordering over multisets of integers induced by the natural ordering over $\mathbb{N}$ (it is known that this ordering is well-founded). Let $(\mathcal{S},\boldsymbol{\lambda},\theta_t)$ fulfill the hypothesis of the lemma.

Case 1: For every $i \in [1, n+m]$, one of the following two situations occurs:

$$\lambda_i(w_i) \equiv_i \lambda_i(w_i') \tag{179}$$
$$\exists W \in \widehat{\mathcal{W}}, (e_i, f_i), (d_i, c_i) \in \mathrm{Gi}(W) \times \mathrm{Ge}(W) : \lambda_i(w_i) = e_i W f_i, \quad \lambda_i(w_i') = c_i^{-1}\overline{W} d_i^{-1} \tag{180}$$

In the following, a pair of variables $(W, \overline{W})$ occuring in some equation (180) is called a *twisted pair*. By Claim 1 we get

$$\gamma_w(W) = \gamma_w(e_i W f_i) = \gamma_w(\lambda_i(w_i)) = \gamma_w(\lambda_i(w_i')) = \gamma_w(c_i^{-1}\overline{W} d_i^{-1}) = \gamma_w(\overline{W}) \tag{181}$$

for every twisted pair $(W, \overline{W})$. Hence, $\mathrm{Gi}(W) = \mathrm{Ge}(W)$ and thus $\mathrm{Gi}(W) = \mathrm{Ge}(W) = \mathrm{Gi}(\overline{W}) = \mathrm{Ge}(\overline{W})$.

Let us consider the partition

$$\widehat{\mathcal{W}} = \widehat{\mathcal{W}}_0 \cup \mathcal{W}_1' \cup \overline{\mathcal{W}_1'} \cup \mathcal{W}_1'' \cup \overline{\mathcal{W}_1''},$$

where $\mathcal{W}_1' \cup \overline{\mathcal{W}_1'}$ (resp. $\mathcal{W}_1'' \cup \overline{\mathcal{W}_1''}$) is exactly the set of symbols with an H-type (resp. with a T-type) which are members of some twisted pair. Let us write

$$\mathcal{W}_1' = \{W_1, \dots, W_p\}, \qquad \overline{\mathcal{W}_1'} = \{\overline{W} \mid W \in \mathcal{W}_1'\},$$
$$\mathcal{W}_1'' = \{W_{p+1}, \dots, W_{p+q}\}, \qquad \overline{\mathcal{W}_1''} = \{\overline{W} \mid W \in \mathcal{W}_1''\}.$$

We can modify the system $\mathcal{S}$ in such a way that $e_i = f_i = 1$ in every equation of type (180) with preservation of the hypothesis of the lemma (with the same morphisms) and also of the size of $(\mathcal{S},\boldsymbol{\lambda},\theta_t)$. We can also suppose that $\mathcal{W}_1' \cup \mathcal{W}_1''$ is exactly the set of variables appearing in the righthand-sides of the first equation of (180). For every $k \in [1, p+q]$, let $(W_k, c_{\nu(k)}^{-1}\overline{W}_k d_{\nu(k)}^{-1})$ be the equation of type (180) with smallest index $\nu(k) \in [1, n+m]$ such that $\lambda_{\nu(k)}(w_{\nu(k)}) = W_k$. Since θ_t is an AB-homomorphism and

$$\theta_t(W_k) = \theta_t(\lambda_{\nu(k)}(w_{\nu(k)})) \overset{(171)}{=} \theta_t(\lambda_{\nu(k)}(w'_{\nu(k)})) = \theta_t(c_{\nu(k)}^{-1}\overline{W}_k d_{\nu(k)}^{-1})$$

we know that

$$\theta_t(\overline{W}_k) = \theta_t(c_{\nu(k)} W_k d_{\nu(k)}). \tag{182}$$

Let us "cut" the variables W_k, which have a T-type, into three parts in such a way that we transform the twisted pair $(W_k, \overline{W}_k)$ into another twisted pair but with an H-type. For every $k \in [p+1, p+q]$, let s_k be a reduced t-sequence such that

$$\theta_t(W_k) = [s_k]_\sim \tag{183}$$

Since W_k has a T-type, s_k has to contain at least one occurrence of t or t^{-1}. Since $\mathbb{I}_t(s_k) \sim c_{\nu(k)} s_k d_{\nu(k)}$ by (182), the t-sequence s_k can be decomposed as

$$s_k = u_k\, v\, u \text{ where } u_k = u_k' t^{\alpha}, u = t^{-\alpha} u' \tag{184}$$

for some $v \in \mathbb{H}$, reduced t-sequences u_k, u, and $\alpha \in \{1, -1\}$, such that $\pi_t(u) = \mathbb{I}_t(\pi_t(u_k))$. We obtain a van Kampen diagram of the following form:

$$\begin{tikzcd}
\bullet \arrow[r, "u_k"'] \arrow[rrr, bend left, "s_k"] & \bullet \arrow[r, "v"'] & \bullet \arrow[r, "u"'] \arrow[d, "b"] & \bullet \arrow[d, "d_{\nu(k)}"] \\
\bullet \arrow[u, "c_{\nu(k)}"] \arrow[r, "\mathbb{I}_t(u)"] \arrow[rrr, bend right, "\mathbb{I}_t(s_k)"'] & \bullet \arrow[u, "a"] \arrow[r, "v^{-1}"'] & \bullet \arrow[r, "\mathbb{I}_t(u_k)"] & \bullet
\end{tikzcd}$$

Define $v_k = vb \in \mathbb{H}$. We obtain

$$s_k \sim u_k v_k \mathbb{I}_t(u_k) d_{\nu(k)}^{-1} \text{ and } \mathbb{I}_t(s_k) \sim c_{\nu(k)} u_k v_k \mathbb{I}_t(u_k). \tag{185}$$

Thus $c_{\nu(k)} u_k v_k \mathbb{I}_t(u_k) \sim \mathbb{I}_t(s_k) \sim d_{\nu(k)} u_k v_k^{-1} \mathbb{I}_t(u_k)$. Since $\mathbb{H}_t$ is cancellative by Lemma 3, we get

$$d_{\nu(k)} u_k v_k^{-1} \sim c_{\nu(k)} u_k v_k. \tag{186}$$

Since $\gamma_w(W_k) = \gamma_w(\overline{W}_k)$ (see (181)) is a T-type, we must have $\gamma_w(W_k) = (C, T, 1, C, H)$ for either $C = A$ or $C = B$; see the list of T-types in (22). We have $(C, T, 1, C, H) \in \gamma_t(u_k v_k \mathbb{I}_t(u_k) d_{\nu(k)}^{-1})$. We can decompose the type $\gamma_w(W_k)$ as

$$\gamma_w(W_k) = (C, T, 1, B(\alpha), H)(B(\alpha), H, 0, B(\alpha), T)(B(\alpha), T, 1, C, H)$$

(here α is from (184)) with $(C, T, 1, B(\alpha), H) \in \gamma_t(u_k)$ and $(B(\alpha), H, 0, B(\alpha), T) \in \gamma_t(v_k)$. Let $\boldsymbol{\theta}_k' = (C, T, 1, B(\alpha), H)$ (which is a T-type) and $\boldsymbol{\theta}_k = (B(\alpha), H, 0, B(\alpha), T)$ (which is an H-type). We get

$$\gamma_w(\overline{W}_k) = \gamma_w(W_k) = \{\boldsymbol{\theta}_k' \boldsymbol{\theta}_k \mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_k')\} \text{ with } \boldsymbol{\theta}_k = \mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_k). \tag{187}$$

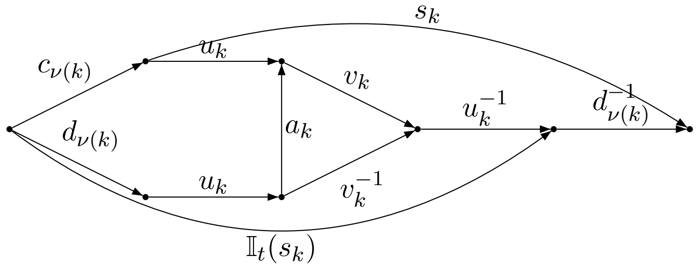
Moreover, by (186), there exists $a_k \in \mathrm{Gi}(\boldsymbol{\theta}_k) = \mathrm{Ge}(\boldsymbol{\theta}_k')$ with

$$c_{\nu(k)} u_k a_k^{-1} \sim d_{\nu(k)} u_k \quad \text{and} \quad a_k v_k = v_k^{-1} \text{ in } \mathbb{H}. \tag{188}$$

Note that the last equation implies

$$a_k v_k = v_k a_k \text{ in } \mathbb{H}. \tag{189}$$

Omitting the words a, b, u, v and inserting the new words v_k, a_k in the preceding van Kampen diagram, we obtain the following van Kampen diagram:



Since the number of letters from $\mathcal{W}$ occuring in $\prod_{i=1}^{n+m} \lambda_i(w_i w_i')$ is strictly less than $\frac{1}{2}\mathrm{Card}(\mathcal{V}_0)$, we are sure that $q < \frac{1}{2}\mathrm{Card}(\mathcal{V}_0)$, so that there are at least q symbols $\alpha_{p+1}, \ldots, \alpha_{p+q} \in \mathcal{V}_0$ that do not occur as first component of a symbol in $\mathrm{Alph}(\prod_{i=1}^{n+m} \lambda_i(w_i w_i'))$. Let us define for every $p+1 \leq k \leq p+q$ variables $U_k, V_k, \overline{U}_k, \overline{V}_k \in \mathcal{W}_t \cap \widehat{\mathcal{W}}$ with first component α_k and such that

$$\begin{aligned}
\gamma_w(U_k) &= \{\boldsymbol{\theta}_k'\}, & \gamma_w(V_k) &= \{\boldsymbol{\theta}_k\}, && (190)\\
\delta_w(\boldsymbol{\theta}_k', U_k) &= \delta_t(\boldsymbol{\theta}_k', u_k), & \delta_w(\boldsymbol{\theta}_k, V_k) &= \delta_t(\boldsymbol{\theta}_k, v_k), && (191)\\
\mu_w(\boldsymbol{\theta}_k', U_k) &= \mu_t(\boldsymbol{\theta}_k', u_k), & \mu_w(\boldsymbol{\theta}_k, V_k) &= \mu_t(\boldsymbol{\theta}_k, v_k). && (192)
\end{aligned}$$

Note that (191) implies with (188) and (189):

$$c_{\nu(k)} U_k a_k^{-1} = d_{\nu(k)} U_k \qquad \text{and} \qquad a_k V_k = V_k a_k \tag{193}$$

Let $\lambda' : \mathcal{W}^* * A * B \to \mathcal{W}^* * A * B$ be the unique monoid homomorphism fulfilling:

$$\begin{aligned}
\lambda'(e) &= e && \text{for } e \in A \cup B && (194)\\
\lambda'(W_k) &= U_k V_k \overline{U}_k d_{\nu(k)}^{-1} && \text{for } p+1 \leq k \leq p+q && (195)\\
\lambda'(\overline{W}_k) &= d_{\nu(k)} U_k \overline{V}_k \overline{U}_k && \text{for } p+1 \leq k \leq p+q. && (196)\\
\lambda'(W) &= W && \text{for all other } W \in \mathcal{W} && (197)
\end{aligned}$$

Claim 2. λ' induces a monoid homomorphism $\lambda' : \mathbb{W} \to \mathbb{W}$.

Let $W \in \mathcal{W}$ and $(c, d) \in \delta_w(W)$. We have to prove that

$$\lambda'(cW) = c\lambda'(W) \equiv \lambda'(W)d = \lambda'(Wd). \tag{198}$$

By Lemma 20 it suffices to show $(c, d) \in \delta_w(\lambda'(W))$, i.e.

$$\delta_w(W) = \delta_w(\lambda'(W)) \tag{199}$$

Case 1. $\lambda'(W) = W$: Then (199) trivially holds.

Case 2. $W = W_k$ for some $p+1 \leq k \leq p+q$: We have

$$\begin{aligned}
\delta_w(W_k) &\overset{(\mathrm{Hom7})}{=} \delta_t(\gamma_w(W_k), \theta_t(W_k))\\
&\overset{(183)}{=} \delta_t(\gamma_w(W_k), s_k)\\
&\overset{(185)}{=} \delta_t(\gamma_w(W_k), u_k v_k \mathbb{I}_t(u_k) d_{\nu(k)}^{-1})\\
&\overset{(187)}{=} \delta_t(\boldsymbol{\theta}_k' \boldsymbol{\theta}_k \mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_k'), u_k v_k \mathbb{I}_t(u_k) d_{\nu(k)}^{-1})\\
&\overset{(191)}{=} \delta_w(\boldsymbol{\theta}_k' \boldsymbol{\theta}_k \mathbb{I}_{\mathcal{T}}(\boldsymbol{\theta}_k'), U_k V_k \overline{U}_k d_{\nu(k)}^{-1})\\
&\overset{(195)}{=} \delta_w(\gamma_w(W_k), \lambda'(W_k)),
\end{aligned}$$

i.e., (199) holds.

Case 3. $W = \overline{W}_k$ for some $p+1 \leq k \leq p+q$: One can check on the formulas (195)–(197) that, $\forall W \in \mathcal{W}, \lambda'(\mathbb{I}_w(W)) = \mathbb{I}_w(\lambda'(W))$. Thus Case 3 reduces to Case 2.

Claim 2 is thus established.

Claim 3. $\lambda' : \mathbb{W} \to \mathbb{W}$ is an AB-homomorphism.

We will apply Lemma 12. Hence,we have to check properties (a)–(f) from Lemma 12. By (194), λ' preserves ι_A and ι_B, i.e., (a) is satisfied. Also (b) follows immediately from the definition of λ'.

Condition (c) from Lemma 12: We have to show that $\lambda'(\mathbb{I}_w(W)) = \mathbb{I}_w(\lambda'(W))$ for all $W \in \widehat{\mathcal{W}}$. This property can be checked directly on the equations (194)–(197) defining λ'.

Condition (d) from Lemma 12: The definition of λ' together with (187)(190) implies in fact $\gamma_w(W) = \gamma_w(\lambda'(W))$ for all $W \in \mathcal{W}$.

Condition (f) from Lemma 12: This was shown in (199).

Conditions (e) from Lemma 12: The same arguments as for (f) work. We have thus established Claim 3.

For $1 \leq i \leq n+m$ let us define

$$\lambda'_i = \lambda_i \circ \lambda'. \tag{200}$$

As every λ'_i is the composition of two AB-homomorphisms, λ'_i is an AB-homomorphism from $\mathbb{W}_t$ to $\mathbb{W}_t$. Let $\theta'_t : \mathcal{W}_t^* * A * B \to \mathbb{H}_t$ be the unique monoid homomorphism satisfying:

$$\begin{aligned}
\theta'_t(e) &= e && \text{for all } e \in A \cup B && (201)\\
\theta'_t(U_k) &= u_k && \text{for all } p+1 \leq k \leq p+q && (202)\\
\theta'_t(\overline{U}_k) &= \mathbb{I}_t(u_k) && \text{for all } p+1 \leq k \leq p+q && (203)\\
\theta'_t(V_k) &= v_k && \text{for all } p+1 \leq k \leq p+q && (204)\\
\theta'_t(\overline{V}_k) &= v_k^{-1} && \text{for all } p+1 \leq k \leq p+q && (205)\\
\theta'_t(W) &= \theta_t(W) && \text{for all other } W \in \mathcal{W}_t && (206)
\end{aligned}$$

Claim 4. θ'_t induces a monoid homomorphism $\theta'_t : \mathbb{W}_t \to \mathbb{H}_t$.

It suffices to prove that $c\theta'_t(W) = \theta'_t(W)d$ in $\mathbb{H}_t$ for every $W \in \mathcal{W}_t$ and $(c,d) \in \delta_w(W)$. For this, it suffices to show that

$$\delta_w(W) = \delta_t(\gamma_w(W), \theta'_t(W)). \tag{207}$$

If $\theta'_t(W) = \theta_t(W)$ then (207) follows from the fact that θ_t is an AB-homomorphism. In all other cases, i.e., (202)–(205), we get (207) from (190) and (191).

Claim 5. $\theta'_t : \mathbb{W}_t \to \mathbb{H}_t$ is an AB-homomorphism.

$$
\begin{tikzcd}
\mathbb{W}_t \arrow[r, "\lambda_i"] \arrow[dr, "\lambda_i'"'] & \mathbb{W}_t \arrow[r, "\theta_t"] \arrow[d, "\lambda'"] & \mathbb{H}_t \\
& \mathbb{W}_t \arrow[ur, "\theta_t'"'] &
\end{tikzcd}
$$

Fig. 12. Lemma 32, Case 1, Claim 6

We will apply Lemma 12. Hence, we have to check properties (a)-(f) from Lemma 12. By (201), θ_t' preserves ι_A and ι_B, i.e., (a) is satisfied. Condition (b) for the variables U_k, V_k, $\overline{U}_k$, and $\overline{V}_k$ (which all belong to $\mathrm{dom}(\mathbb{I}modif) = \mathrm{dom}(\mathbb{I}_w)$) follows from the fact that $u_k, v_k \in \mathrm{dom}(\mathbb{I}_t)$. Condition (b) for other variables $W \in \mathcal{W}_t$ follows from (206) and the fact that $\theta_t : \mathbb{W}_t \to \mathbb{H}_t$ is an AB-homomorphism.

Condition (c) from Lemma 12: We have to show that $\theta_t'(\mathbb{I}_w(W)) = \mathbb{I}_t(\theta_t'(W))$ for all $W \in \widehat{\mathcal{W}_t}$.

Let us first consider a variable $W \in \mathcal{W}_t$ with $\theta_t'(W) = \theta_t(W)$. Then, since $\theta_t : \mathbb{W}_t \to \mathbb{H}_t$ is an AB-homomorphism, we have

$$\mathbb{I}_t(\theta_t'(W)) = \mathbb{I}_t(\theta_t(W)) = \theta_t(\mathbb{I}_w(W)) = \theta_t'(\mathbb{I}_w(W)).$$

For a variable U_k we have

$$\mathbb{I}_t(\theta_t'(U_k)) \overset{(202)}{=} \mathbb{I}_t(u_k) \overset{(203)}{=} \theta_t'(\mathbb{I}_w(U_k))$$

and similarly for a variable $\overline{U}_k$. Finally, for a variable V_k we have

$$\mathbb{I}_t(\theta_t'(V_k)) \overset{(204)}{=} \mathbb{I}_t(v_k) = v_k^{-1} \overset{(205)}{=} \theta_t'(\mathbb{I}_w(V_k))$$

`verifier equa ci-dessus` and similarly for a variable $\overline{V}_k$.

Conditions (d)–(f) from Lemma 12: For variables $W \in \mathcal{W}_t$ with $\theta_t'(W) = \theta_t(W)$ we get these conditions directly from the fact that θ_t is an AB-homomorphism. For variables from $\{U_k, \overline{U}_k, V_k, \overline{V}_k \mid p+1 \le k \le p+q\}$ we get (d)–(f) from (190)–(192).

Claim 6. λ_i' and θ_t' fulfill conclusion (173) of the lemma.

We have to show $\theta_t(\lambda_i(W)) = \theta_t'(\lambda_i'(W))$ for all $1 \le i \le n+m$ and all $W \in \mathrm{Alph}(\mathcal{S})$. Recall that $\lambda_i' = \lambda_i \circ \lambda'$ by (200). Hence, it suffices to show that $\theta_t = \lambda' \circ \theta_t'$ over $\mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda})$ i.e. that

$$\theta_t(W) = \theta_t'(\lambda'(W)) \tag{208}$$

for every $W \in \mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda})$.

Case 1. $W \in \mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda})$ is such that $\lambda'(W) = W$:

$$\theta_t'(\lambda'(W)) = \theta_t'(W) = \theta_t(W),$$

which establishes (208).

Case 2. $W = W_k$ for some $p+1 \le k \le p+q$: We have

$$\begin{aligned}
\theta'_t(\lambda'(\overline{W}_k)) &\overset{(195)}{=} \theta'_t(U_k V_k \overline{U}_k d_{\nu(k)}^{-1}) \\
&\overset{(202)\text{–}(204)}{=} u_k v_k \mathbb{I}_t(u_k) d_{\nu(k)}^{-1} \\
&\overset{(185)}{=} s_k \\
&\overset{(183)}{=} \theta_t(W_k).
\end{aligned}$$

Case 3. $W = \overline{W}_k$ for some $p+1 \le k \le p+q$: We have

$$\begin{aligned}
\theta'_t(\lambda'(\overline{W}_k)) &\overset{(196)}{=} \theta'_t(d_{\nu(k)} U_k \overline{V}_k \overline{U}_k) \\
&\overset{(202)\text{–}(205)}{=} d_{\nu(k)} u_k \mathbb{I}_t(v_k) \mathbb{I}_t(u_k) \\
&\overset{(185)}{=} \mathbb{I}_t(s_k) \\
&\overset{(183)}{=} \theta_t(\overline{W}_k).
\end{aligned}$$

Note that the symbols $U_k, \overline{U}_k, V_k, \overline{V}_k$ for $p+1 \le k \le p+q$ were chosen in such a way that they do *not* belong to $\mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda})$.

We next define a mapping $\varPhi : \widehat{\mathbb{W}} \to \widehat{\mathbb{W}}$ as follows:

$$\begin{aligned}
\varPhi(e) &= e && \text{for } e \in A \cup B && (209)\\
\varPhi(\overline{W}_k) &= c_{\nu(k)} W_k d_{\nu(k)} && \text{for } 1 \le k \le p && (210)\\
\varPhi(W_k) &= c_{\nu(k)}^{-1} \overline{W}_k d_{\nu(k)}^{-1} && \text{for } 1 \le k \le p && (211)\\
\varPhi(\overline{V}_k) &= a_k V_k && \text{for } p+1 \le k \le p+q && (212)\\
\varPhi(V_k) &= a_k^{-1} \overline{V}_k && \text{for } p+1 \le k \le p+q && (213)\\
\varPhi(W) &= W && \text{for all other } W \in \widehat{\mathcal{W}} && (214)
\end{aligned}$$

Claim 7. $\varPhi$ belongs to HInv.

We have to check that $\varPhi$ fulfills conditions (91)–(94) from Lemma 22 and also the H-type condition (101).

Condition (91): For $1 \le k \le p$ we have to show that $\delta_w(\overline{W}_k) = \delta_w(c_{\nu(k)} W_k d_{\nu(k)})$.

$$\begin{aligned}
\delta_w(\overline{W}_k) &\overset{(\mathrm{Hom7})}{=} \delta_t(\gamma_w(\overline{W}_k), \theta_t(\overline{W}_k)) \\
&\overset{(182)}{=} \delta_t(\gamma_w(\overline{W}_k), \theta_t(c_{\nu(k)} W_k d_{\nu(k)})) \\
&= \delta_t(\gamma_w(c_{\nu(k)} W_k d_{\nu(k)}), \theta_t(c_{\nu(k)} W_k d_{\nu(k)})) \\
&\overset{(\mathrm{Hom7})}{=} \delta_w(c_{\nu(k)} W_k d_{\nu(k)})
\end{aligned}$$

For $p+1 \le k \le q$ we have to show that $\delta_w(\overline{V}_k) = \delta_w(a_k V_k)$.

$$
\begin{aligned}
\delta_w(\overline{V}_k) &= \delta_w(\boldsymbol{\theta}_k, \overline{V}_k) \\
&\overset{(AB12)}{=} \delta_w(\boldsymbol{\theta}_k, V_k)^{-1} \\
&\overset{(191)}{=} \delta_t(\boldsymbol{\theta}_k, v_k)^{-1} \\
&\overset{(AB12)}{=} \delta_t(\boldsymbol{\theta}_k, v_k^{-1}) \\
&\overset{(188)}{=} \delta_t(\boldsymbol{\theta}_k, a_k v_k) \\
&\overset{(191)}{=} \delta_w(a_k V_k)
\end{aligned}
$$

Condition (92): For $1 \leq k \leq p$ we have to show that

$$
(d_{\nu(k)}^{-1} c_{\nu(k)}, c_{\nu(k)} d_{\nu(k)}^{-1}) \in \delta_w(W_k).
$$

Since $s_k \sim c_{\nu(k)}^{-1} \mathbb{I}_t(s_k) d_{\nu(k)}^{-1}$ we have $\mathbb{I}_t(s_k) \sim d_{\nu(k)} s_k c_{\nu(k)}$ and hence $s_k \sim c_{\nu(k)}^{-1} d_{\nu(k)} s_k c_{\nu(k)} d_{\nu(k)}^{-1}$. Thus, $s_k \sim d_{\nu(k)}^{-1} c_{\nu(k)} s_k d_{\nu(k)} c_{\nu(k)}^{-1}$, i.e., $(d_{\nu(k)}^{-1} c_{\nu(k)}, d_{\nu(k)} c_{\nu(k)}^{-1}) \in \delta_t(\gamma_w(W_k), s_k) = \delta_w(W_k)$. For $p+1 \leq k \leq p+q$, condition (92) is equivalent to $(a_k, a_k) \in \delta_w(V_k)$, which is true, by (193).

Condition (93):this follows from (181), (187), and (190).

Condition (94): First assume that $1 \leq k \leq p$. Since θ_t is an AB-homomorphism, we get:

$$
\begin{aligned}
\mathbb{I}_{\mathsf{Q}}(\mu_w(W_k)) &= \mathbb{I}_{\mathsf{Q}}(\mu_t(\gamma_w(W_k), \theta_t(W_k))) \\
&= \mu_t(\gamma_w(W_k), \mathbb{I}_t(\theta_t(W_k))) \\
&= \mu_t(\gamma_w(W_k), c_{\nu(k)} \theta_t(W_k) d_{\nu(k)}) \\
&= \mu_w(c_{\nu(k)} W_k d_{\nu(k)})
\end{aligned}
$$

For $p+1 \leq k \leq p+q$ we get:

$$
\begin{aligned}
\mathbb{I}_{\mathsf{Q}}(\mu_w(V_k)) &\overset{(192)}{=} \mathbb{I}_{\mathsf{Q}}(\mu_t(\gamma_w(V_k), v_k)) \\
&\overset{(AB11)}{=} \mu_t(\gamma_w(V_k), \mathbb{I}_t(v_k)) \\
&= \mu_t(\gamma_w(V_k), v_k^{-1}) \\
&\overset{(188)}{=} \mu_t(\gamma_w(V_k), a_k v_k) \\
&\overset{(192)}{=} \mu_w(\gamma_w(V_k), a_k V_k)
\end{aligned}
$$

Condition (101): Recall that $\gamma_w(W_k)$ is an H-type for $1 \leq k \leq p$, and that $\boldsymbol{\theta}_k$ (see the assertion before (187)) is an H-type for $p+1 \leq k \leq p+q$. Hence Φ meets the requirements of (101). Finally, Φ belongs to the set HInv of $\mathbb{H}$-involutions.

Claim 8. λ_i' and Φ fulfill conclusion (174) of the lemma, i.e. $\theta_t'(W) = \theta_t'(\Phi(W))$ for all $W \in \mathcal{W}$.

Let $W \in \mathcal{W}$.
Case 1. $W = W_k$ where $1 \leq k \leq p$.

$$
\theta_t'(W) \overset{(206)}{=} \theta_t(W_k)
$$

while

$$\theta'_t(\Phi(W)) \overset{(211)}{=} \theta'_t(c^{-1}_{\nu(k)}\overline{W}_k d^{-1}_{\nu(k)}) \overset{(206)}{=} \theta_t(c^{-1}_{\nu(k)}\overline{W}_k d^{-1}_{\nu(k)}) \overset{(171)}{=} \theta_t(W_k)$$

hence $\theta'_t(W) = \theta'_t(\Phi(W))$.
Case 1'. $W = \overline{W}_k$ where $1 \le k \le p$.
This case reduces to Case 1 since θ'_t and Φ are AB-homomorphisms, hence commute with the suitable standard involutions $\mathbb{I}_w, \mathbb{I}_t$.
Case 2. $W = V_k$ where $p+1 \le k \le p+q$.

$$\theta'_t(W) \overset{(204)}{=} v_k$$

while

$$\theta'_t(\Phi(W)) \overset{(213)}{=} \theta'_t(a_k^{-1}\overline{V}_k) \overset{(205)}{=} a_k^{-1}v_k^{-1} \overset{(188)}{=} v_k$$

hence $\theta'_t(W) = \theta'_t(\Phi(W))$.
Case 2'. $W = \overline{V}_k$ where $p+1 \le k \le p+q$.
Reduces to Case 2 (same argument as for Case 1').
Case 3. $W \in \mathcal{W}$ and W does not fulfill any of the above cases.

$$\theta'_t(\Phi(W)) \overset{(214)}{=} \theta'_t(W).$$

Claim 8 has been established in all cases.

Claim 9. λ'_i and Φ fulfill conclusion (175) of the lemma i.e. $\lambda'_i(w_i) \equiv_\Phi \lambda'_i(w'_i)$ for all $1 \le i \le n+m$.

Let $1 \le i \le n+m$. Suppose first that (w_i, w'_i) is one of the pairs of the form (180) (recall that we reduced to the case where $e_i = f_i = 1$). There exists some $k \in [1, p+q]$ such that

$$\lambda_i(w_i) = W_k \text{ and } \lambda_i(w'_i) = c_i^{-1}\overline{W}_k d_i^{-1}. \tag{215}$$

Let us first suppose that $k \in [1,p]$. Then

$$\lambda'_i(w_i) \overset{(200)}{=} \lambda'(\lambda_i(w_i)) \overset{(197)}{=} W_k \text{ and } \lambda'_i(w'_i) \overset{(200)}{=} \lambda'(\lambda_i(w'_i)) \overset{(197)}{=} c_i^{-1}\overline{W}_k d_i^{-1}. \tag{216}$$

Applying the automorphism Φ on W_k we obtain:

$$\Phi(W_k) \overset{(211)}{=} c^{-1}_{\nu(k)}\overline{W}_k d^{-1}_{\nu(k)} \tag{217}$$

The equality

$$\theta_t(c_i^{-1}\overline{W}_k d_i^{-1}) \overset{(171)}{=} \theta_t(c^{-1}_{\nu(k)}\overline{W}_k d^{-1}_{\nu(k)})$$

shows that $(c_{\nu(k)}c_i^{-1}, d^{-1}_{\nu(k)}d_i) \in \delta_t(\gamma_w(\overline{W}_k), \theta_t(\overline{W}_k))$. Since θ_t is an AB-homomorphism, this implies that $(c_{\nu(k)}c_i^{-1}, d^{-1}_{\nu(k)}d_i) \in \delta_w(\overline{W}_k)$, i.e.,

$$c_{\nu(k)}c_i^{-1}\overline{W}_k = \overline{W}_k d^{-1}_{\nu(k)} d_i \tag{218}$$

We get:

$$\begin{aligned}
\lambda_i'(w_i) &\overset{(216)}{=} W_k \\
&\overset{(217)}{\equiv_\Phi} c_{\nu(k)}^{-1}\overline{W}_k d_{\nu(k)}^{-1} \\
&\overset{(218)}{=} c_i^{-1}\overline{W}_k d_i^{-1} \\
&\overset{(216)}{=} \lambda_i'(w_i')
\end{aligned}$$

Let us now suppose that $k \in [p+1, p+q]$. Then

$$\lambda_i'(w_i) \overset{(200)}{=} \lambda'(\lambda_i(w_i)) \overset{(197)}{=} U_k V_k \overline{U}_k d_{\nu(k)}^{-1} \text{ and } \lambda_i'(w_i') \overset{(200)}{=} \lambda'(\lambda_i(w_i')) \overset{(197)}{=} c_i^{-1} d_{\nu(k)} U_k \overline{V}_k \overline{U}_k d_i^{-1}. \tag{219}$$

Using the definition of Φ, we calculate

$$\Phi(U_k V_k \overline{U}_k d_{\nu(k)}^{-1}) = U_k a_k^{-1} \overline{V}_k \overline{U}_k d_{\nu(k)}^{-1} \tag{220}$$

The particular commutation (193) around U_k can be equivalently formulated as

$$U_k a_k^{-1} = c_{\nu(k)}^{-1} d_{\nu(k)} U_k \tag{221}$$

By (219)(220)(221) we get

$$\begin{aligned}
\lambda_i'(w_i) &\equiv_\Phi U_k a_k^{-1} \overline{V}_k \overline{U}_k d_{\nu(k)}^{-1} \\
&= c_{\nu(k)}^{-1}(d_{\nu(k)} U_k \overline{V}_k \overline{U}_k) d_{\nu(k)}^{-1}
\end{aligned} \tag{222}$$

By similar arguments as for the case where $1 \leq k \leq p$, we know that

$$(c_i c_{\nu(k)}^{-1}, d_{\nu(k)}^{-1} d_i) \in \delta_w(d_{\nu(k)} U_k \overline{V}_k \overline{U}_k)$$

hence,

$$c_{\nu(k)}^{-1}(d_{\nu(k)} U_k \overline{V}_k \overline{U}_k) d_{\nu(k)}^{-1} \equiv c_i^{-1}(d_{\nu(k)} U_k \overline{V}_k \overline{U}_k) d_i^{-1} \tag{223}$$

Finally, chaining (222) and (223), we obtain

$$\lambda_i'(w_i) \equiv_\Phi \lambda_i'(w_i')$$

We treat now the pairs (w_i, w_i') of the form (179) by descending induction over i. So, assume that $\lambda_i(w_i) \equiv_i \lambda_i(w_i')$ and $\lambda_j'(w_j) = \lambda_j'(w_j')$ for all $j \in [i+1, n+m]$. Hence, $\lambda'(\lambda_j(w_j)) \equiv_\Phi \lambda'(\lambda_j(w_j'))$ for all $j \in [i+1, n+m]$. Recall that the congruence $\equiv_i$ is generated by the set of pairs $P = \{(\lambda_j(w_j), \lambda_j(w_j')) \mid j \in [i+1, n+m]\}$, i.e., $\equiv_i = \equiv_P$. Hence, $\lambda'(P)$ is included in the congruence $\equiv_\Phi$ on $\mathbb{W}$. Therefore, $\lambda_i(w_i) \equiv_P \lambda_i(w_i')$ together with Lemma 1 implies $\lambda_i'(w_i) = \lambda'(\lambda_i(w_i)) \equiv_\Phi \lambda'(\lambda_i(w_i')) = \lambda_i'(w_i')$. We have thus established (175) in all cases.

Claim 10. λ_i' fulfill conclusion (176) of the lemma.

$$
\begin{tikzcd}
\mathbb{W}_t \arrow[rr, "\lambda_i"] \arrow[dr, "\lambda'_i"] \arrow[ddrr, dashed, swap, "w_i"] \arrow[ddrr, dashed, "w'_i"] & & \mathbb{W}_t \arrow[rr, "\theta_t"] \arrow[d, "\lambda'"] & & \mathbb{H}_t \\
& & \mathbb{W}_t \arrow[urr, "\theta'_t"] \arrow[d, "\pi_\Phi"] & & \\
& & \mathbb{W}_t/\Phi \arrow[uurr, dashed] & &
\end{tikzcd}
$$

Fig. 13. Lemma 32, Case 1, Claims 8,9

For all $1 \leq i, j \leq n$ we get

$$\lambda'_i \overset{(200)}{=} \lambda_i \circ \lambda' \overset{(172)}{=} \lambda_j \circ \lambda' \overset{(200)}{=} \lambda'_j.$$

Case 2: There exists some $i \in [1, n+m]$, such that neither condition (179) nor (180) holds.

Let $i \in [1, n+m]$ be the minimal integer for which neither (179) nor (180) holds. Let (P_i, P'_i) be a distinguishing pair for $(\lambda_i(w_i), \lambda_i(w'_i))$. The generators of the congruence $\equiv_i$ are, by definition, all pairs $(\lambda_j(w_j), \lambda_j(w'_j))$ for $j \in [i+1, n+m]$. Hence, by Claim 1, $\equiv_i$ is also generated by the subset

$$\mathcal{P}_i = \{(\lambda_j(w_j), \lambda_j(w'_j)) \mid j \in [i+1, n+m], \|\lambda_j(w_j)\| \geq 1, \|\lambda_j(w'_j)\| \geq 1\}.$$

By Lemma 19, every pair from $\mathcal{P}_i$ belongs to the congruence $\equiv_\gamma$ from (80). Hence $\equiv_i \subseteq \equiv_\gamma$, which implies

$$\gamma_w(P_i) = \gamma_w(P'_i).$$

Moreover, by (171) every element of $\mathcal{P}_i$ belongs to the kernel of θ_t, hence

$$\theta_t(P_i) = \theta_t(P'_i). \tag{224}$$

(Hom2) for θ_t implies

$$P_i \in A \iff P'_i \in A \quad \text{and} \quad P_i \in B \iff P'_i \in B. \tag{225}$$

Lemma 24 applied on $P_i, S_i, P'_i, S'_i \in \mathbb{W}$ asserts that $\gamma_w(S_i) = \gamma_w(S'_i)$. By Lemma 3, $\mathbb{H}_t$ is left-cancellative. Hence, $\theta_t(P_i) = \theta_t(P'_i)$ and $\theta_t(P_i S_i) = \theta_t(\lambda_i(w_i)) = \theta_t(\lambda_i(w'_i)) = \theta_t(P'_i S'_i)$ imply

$$\theta_t(S_i) = \theta_t(S'_i). \tag{226}$$

As θ_t restricted to A and B is injective by (Hom2), the value $\|S_i\| = 0$ or $\|S'_i\| = 0$ would lead to $S_i = S'_i$ and hence to $\lambda_i(w_i) = P_i S_i \equiv_i P'_i S'_i = \lambda_i(w'_i)$, i.e., to (179), which contradicts the choice of i. Hence, we have

$$\|S_i\| \geq 1,\ \|S'_i\| \geq 1 \text{ and } \gamma_w(S_i) = \gamma_w(S'_i). \tag{227}$$

Recall the definition of $\Delta(P_i, S_i, P'_i, S'_i, \theta_t)$ from (83). Equations (225), (226), and (227) imply

$$\Delta(P_i, S_i, P'_i, S'_i, \theta_t) = 1 - \chi_{AB}(P_i) + 2\chi_H(S_i) + 4\|\theta_t(S_i)\| \tag{228}$$

Since $\|S_i\| \geq 1$ and $\|S'_i\| \geq 1$, we can write S_i and S'_i as

$$S_i = cWL_i \text{ and } S'_i = c'W'L'_i, \tag{229}$$

where $W, W' \in \mathcal{W}$ and $c, c' \in \mathrm{Gi}(W) = \mathrm{Gi}(W')$. Since $\tau\mathrm{i}(W) = \tau\mathrm{i}(W')$, either $\gamma_w(W)$ and $\gamma_w(W')$ are both T-types or $\gamma_w(W)$ and $\gamma_w(W')$ are both H-types.

Case 2.1: $\gamma_w(W)$ and $\gamma_w(W')$ are both H-types.

Claim 11. $\gamma_w(W) = \gamma_w(W')$

We know that $\tau\mathrm{i}(W) = \tau\mathrm{i}(S_i) \overset{(227)}{=} \tau\mathrm{i}(S_i') = \tau\mathrm{i}(W')$. Moreover, the boolean component of $\gamma_w(W)$ and $\gamma_w(W')$ is 0. Hence, it remains to show that $\tau\mathrm{e}(W) = \tau\mathrm{e}(W')$. Assume that $\tau\mathrm{e}(W) \neq \tau\mathrm{e}(W')$. There are several cases according to the possible H-type, which can be all dealt in the same way. Let us consider the case $\tau\mathrm{e}(W) = (A, T)$ and $\tau\mathrm{e}(W') = (B, T)$. Since $\theta_t(cWL_i) = \theta_t(c'W'L_i')$ by (226) and (229) and $\theta_t(cW), \theta_t(c'W') \in \mathbb{H}$ we get $\pi_t(\theta_t(L_i)) = \pi_t(\theta_t(L_i'))$. Moreover, by (227) there exists a vertex type θ such that $\gamma_t(\theta_t(L_i))$ contains a path type of the form $(A, T, *, \theta)$ and $\gamma_t(\theta_t(L_i'))$ contains a path type of the form $(B, T, *, \theta)$. But this contradicts $\pi_t(\theta_t(L_i)) = \pi_t(\theta_t(L_i'))$. This proves Claim 11.

Let $\gamma_w(W) = \gamma_w(W') = \{\boldsymbol{\theta}\}$ in the following.

Claim 12. There exists a path type $\boldsymbol{\rho} \in \gamma_w(L_i) \cap \gamma_w(L_i') \neq \emptyset$ such that $\boldsymbol{\theta\rho}$ is defined.

There exist $\boldsymbol{\rho} \in \gamma_w(L_i)$ and $\boldsymbol{\rho'} \in \gamma_w(L_i')$ such that $\{\boldsymbol{\theta\rho}\} = \gamma_w(S_i) = \gamma_w(S_i') = \{\boldsymbol{\theta\rho'}\}$. Hence, $\tau\mathrm{i}(\boldsymbol{\rho}) = \tau\mathrm{e}(\boldsymbol{\theta}) = \tau\mathrm{i}(\boldsymbol{\rho'})$ and $\tau\mathrm{e}(\boldsymbol{\rho}) = \tau\mathrm{e}(\boldsymbol{\rho'})$. Moreover, the boolean component of $\boldsymbol{\rho}$ is 1 if and only if the boolean component of $\gamma_w(S_i) = \gamma_w(S_i')$ is 1 if and only if the boolean component of $\boldsymbol{\rho'}$ is 1. Hence $\boldsymbol{\rho} = \boldsymbol{\rho'}$. This proves the claim.

Since θ_t is an AB-homomorphism, we have $\boldsymbol{\theta} \in \gamma_t(\theta_t(cW)) \cap \gamma_t(\theta_t(c'W'))$ and $\boldsymbol{\rho} \in \gamma_t(\theta_t(L_i)) \cap \gamma_t(\theta_t(L_i'))$. Lemma 16 applied to the identity $\theta_t(cW)\theta_t(L_i) = \theta_t(S_i) = \theta_t(S_i') = \theta_t(c'W')\theta_t(L_i')$ yields $d \in \mathrm{Ge}(\boldsymbol{\theta})$ such that

$$\theta_t(cWd) = \theta_t(c'W') \text{ and } \theta_t(L_i) = \theta_t(dL_i'). \tag{230}$$

Since $\theta_t(W), \theta_t(W') \in \mathbb{H}$, we have

$$\|\theta_t(L_i)\| = \|\theta_t(dL_i')\| = \|\theta_t(S_i)\|. \tag{231}$$

Claim 13. $\Delta(P_icWd, d^{-1}L_i, P_i'c'W', L_i', \theta_t) < \Delta(P_i, S_i, P_i', S_i', \theta_t)$

We have

$$\begin{aligned}
\Delta(P_icWd, d^{-1}L_i, P_i'c'W', L_i', \theta_t) &= 1 - \frac{1}{2}\Big(\chi_{AB}(P_icWd) + \chi_{AB}(P_i'c'W')\Big) + \\
&\qquad \chi_H(d^{-1}L_i) + \chi_H(L_i') + 2\|\theta_t(d^{-1}L_i)\| + 2\|\theta_t(L_i')\| \\
&\overset{(231)}{=} 1 + \chi_H(L_i) + \chi_H(L_i') + 4\|\theta_t(S_i)\| \\
\Delta(P_i, S_i, P_i', S_i', \theta_t) &\overset{(228)}{=} 1 - \chi_{AB}(P_i) + 2\chi_H(S_i) + 4\|\theta_t(S_i)\| \\
&\geq 2\chi_H(S_i) + 4\|\theta_t(S_i)\|.
\end{aligned}$$

We claim that $\chi_H(L_i) = \chi_H(L_i') = \chi_H(S_i) - 1$, which implies the claim. We distinguish several cases according to the H-type $\boldsymbol{\theta}$ of W and W'.

Case A. $\boldsymbol{\theta}$ has the form $(*, H, 0, 1, 1)$: Then $\gamma_w(S_i)$ has the form $\{(*, H, *, *)\}$. Moreover, a path type of the form $(1, 1, *, *, *)$ belongs to both $\gamma_w(L_i)$ and $\gamma_w(L_i')$. But this already implies $L_i = L_i' = 1$. Hence, $|\gamma_w(L_i)| = |\gamma_w(L_i')| > 1$. We get $\chi_H(L_i) = \chi_H(L_i') = 0$. Moreover, $\gamma_w(S_i)$ has the form $\{(*, H, 0, 1, 1)\}$. This implies $\chi_H(S_i) = 1$.

Case B. $\boldsymbol{\theta}$ has the form $(*, H, 0, *, T)$: Again, $\gamma_w(S_i)$ has the form $\{(*, H, *, *, *)\}$. Moreover, $\gamma_w(L_i)$ and $\gamma_w(L_i')$ contain a path type of the form $(*, T, *, *, *, *)$. We distinguish the following three subcases:

Case B.1 $\gamma_w(S_i)$ has the form $(*, H, 0, *, *)$: We get $\chi_H(S_i) = 1$. Moreover, $\gamma_w(L_i)$ and $\gamma_w(L_i')$ contain a path type of the form $(*, T, 0, *, *)$. Note that there is no proper product of H-types and T-type of this form. Hence, $|\gamma_w(L_i)|, |\gamma_w(L_i)| > 1$ and $\chi_H(L_i) = \chi_H(L_i') = 0$.

Case B.2 $\gamma_w(S_i)$ has the form $\{(*, H, 1, *, T)\}$: We get $\chi_H(S_i) = 2$. Moreover, $\gamma_w(L_i)$ and $\gamma_w(L_i')$ contain a path type of the form $(*, T, 1, *, T)$. But this implies $\gamma_w(L_i) = \gamma_w(L_i') = \{(*, T, 1, *, T)\}$ and $\chi_H(L_i) = \chi_H(L_i') = 1$.

Case B.3 $\gamma_w(S_i)$ has the form $\{(*, H, 1, *, H)\}$: We get $\chi_H(S_i) = \chi_H(S_i') = 1$. Moreover, $\gamma_w(L_i)$ and $\gamma_w(L_i')$ contain a path type of the form $(*, T, 1, *, H)$. But this implies $\gamma_w(L_i) = \gamma_w(L_i') = \{(*, T, 1, *, H)\}$ and $\chi_H(L_i) = \chi_H(L_i') = 0$.

Case B.4 $\gamma_w(S_i)$ has the form $\{(*, H, 1, 1, 1)\}$: We get $\chi_H(S_i) = 2$. Moreover, $\gamma_w(L_i)$ and $\gamma_w(L_i')$ contain a path type of the form $(*, T, 1, 1, 1)$. But this implies $\gamma_w(L_i) = \gamma_w(L_i') = \{(*, T, 1, 1, 1)\}$ and $\chi_H(L_i) = \chi_H(L_i') = 1$.

Case 2.1.1. $W' = W$: As the AB-homomorphism θ_t is δ-preserving by (Hom7), $\theta_t(cWd) = \theta_t(c'W)$ implies $cWd = c'W$. Hence, we have $P_i cWd \equiv_i P_i' c'W$. Together with Claim 13, this violates the hypothesis of minimality in the choice of decomposition (177). This case is thus impossible.

Case 2.1.2. $W' = \overline{W}$: Hence, $\gamma_w(W) = \{\boldsymbol{\theta}\} = \gamma_w(\overline{W})$ and we get

$$\boldsymbol{\theta} \in \{(A, H, 0, A, T), (B, H, 0, B, T), (1, H, 0, 1, 1)\}. \tag{232}$$

Hence, $\gamma_w(S_i)$ has the form $\{(*, H, *, *, *)\}$ and we get

$$\chi_H(S_i) \geq 1 \tag{233}$$

Moreover, we either have $P_i, P_i' \notin A \cup B$ or $L_i, L_i' \notin A \cup B$, because otherwise the index i would satisfy (180), which we exclude in Case 2.

Let us define a new equation (w_{n+m+1}, w_{n+m+1}') with

$$w_{n+m+1} = cWd \quad \text{and} \quad w_{n+m+1}' = c'\overline{W} \tag{234}$$

and the AB-morphisms

$$\lambda_i' = \lambda_i \text{ for all } 1 \leq i \leq n+m \quad \text{and} \quad \lambda_{n+m+1}' = \mathrm{Id}_{\mathbb{W}_t}. \tag{235}$$

Hence, $\gamma_w(w_{n+m+1}) = \gamma_w(w_{n+m+1}')$. Let

$$\mathcal{S}' = \{(w_i, w_i') \mid 1 \leq i \leq n+m+1\} \quad \text{and} \quad \boldsymbol{\lambda}' = (\lambda_i)_{1 \leq i \leq n+m+1}. \tag{236}$$

The triple $(\mathcal{S}', \boldsymbol{\lambda}', \theta_t)$ fulfills the hypothesis of Lemma 32. Moreover, w.r.t. this triple, we have

$$P_i cWd \equiv_i P_i' c'\overline{W}.$$

The definition of Δ_i and Claim 13 imply

$$\Delta_i(\mathcal{S}', \boldsymbol{\lambda}', \theta_t) < \Delta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t). \tag{237}$$

Let us now bound the size of $\Delta(P_i, S_i, P'_i, S'_i, \theta_t)$:

$$\begin{aligned}\Delta(P_i, S_i, P'_i, S'_i, \theta_t) &\overset{(228)}{=} 1 - \chi_{AB}(P_i) + 2\chi_H(S_i) + 4\|\theta_t(S_i)\| \\ &\overset{(233)}{\geq} 3 - \chi_{AB}(P_i) + 4\|\theta_t(S_i)\|\end{aligned}$$

In case $P_i, P'_i \notin A \cup B$, we have $\chi_{AB}(P_i) = 0$ and hence $\Delta(P_i, S_i, P'_i, S'_i, \theta_t) \geq 3$. In case $P_i, P'_i \in A \cup B$ we have $L_i, L'_i \notin A \cup B$ (as argued above) and $\chi_{AB}(P_i) = 1$. Hence, $\Delta(P_i, S_i, P'_i, S'_i, \theta_t) \geq 2 + 4\|\theta_t(S_i)\|$. Since $L_i \notin A \cup B$ and $\gamma_w(L_i) \neq \emptyset$, we can write L_i as $L_i = eUL'_i$ for some $U \in \mathcal{W}$ and $e \in \mathrm{Gi}(U)$. Moreover, (232) and $\tau\mathrm{i}(U) = \tau\mathrm{e}(W)$ imply that $\gamma_w(U)$ has the form $\{(*, T, *, *, *)\}$ or $\{(1, 1, *, *, *)\}$. The latter is not possible (it is not an H-type or T-type), hence $\gamma_w(U)$ has the form $\{(*, T, *, *, *)\}$, i.e., it is a T-type. This implies $\|\theta_t(S_i)\| = \|\theta_t(L_i)\| \geq \|\theta_t(U)\| \geq 1$. This implies $\Delta(P_i, S_i, P'_i, S'_i, \theta_t) \geq 2 + 4\|\theta_t(S_i)\| \geq 6$. In both cases, we have shown

$$\Delta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t) = \Delta(P_i, S_i, P'_i, S'_i, \theta_t) \geq 3. \tag{238}$$

Let us choose

$$P_{n+m+1} = \varepsilon, \quad P'_{n+m+1} = \varepsilon, \quad S_{n+m+1} = cWd, \quad S'_{n+m+1} = c'\overline{W}.$$

We compute

$$\begin{aligned}\Delta(P_{n+m+1}, S_{n+m+1}, P'_{n+m+1}, S'_{n+m+1}, \theta_t) &= 1 - \chi_{AB}(\varepsilon) + \chi_H(W) + \chi_H(\overline{W}) + \\ &\qquad 2\|\theta_t(W)\| + 2\|\theta_t(\overline{W})\| \\ &= 1 - 1 + 2 + 0 \\ &= 2.\end{aligned} \tag{239}$$

Finally, using (238) and (239), we obtain:

$$\begin{aligned}\Delta_{n+m+1}(\mathcal{S}', \boldsymbol{\lambda}', \theta_t) &\leq \Delta(P_{n+m+1}, S_{n+m+1}, P'_{n+m+1}, S'_{n+m+1}, \theta_t) \\ &= 2 \\ &< 3 \\ &\leq \Delta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t).\end{aligned} \tag{240}$$

The above inequalities (237) and (240) prove that

$$\{\{\Delta_i(\mathcal{S}', \boldsymbol{\lambda}', \theta_t), \Delta_{n+m+1}(\mathcal{S}', \boldsymbol{\lambda}', \theta'_t)\}\} < \{\{\Delta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)\}\}.$$

Hence

$$\|(\mathcal{S}', \boldsymbol{\lambda}', \theta_t)\| < \|(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)\|.$$

By induction hypothesis, the conclusion of the lemma holds for $(\mathcal{S}', \boldsymbol{\lambda}', \theta_t)$. This proves that it holds for $(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)$ too.

Case 2.1.3. $W' \notin \{W, \overline{W}\}$: Let us consider the monoid homomorphism $\lambda' : \mathbb{W} \to \mathbb{W}$ fulfilling:

$$\lambda'(e) = e \text{ for all } e \in A \cup B \tag{241}$$

$$\lambda'(W) = c^{-1}c'W'd^{-1} \tag{242}$$

$$\lambda'(\overline{W}) = d\overline{W'}c'^{-1}c \tag{243}$$

$$\lambda'(W'') = W'' \text{ for all } W'' \in \mathcal{W} \setminus \{W, \overline{W}\} \tag{244}$$

This definition is written for the case where $W \in \widehat{\mathcal{W}}$, in the case where $W \notin \widehat{\mathcal{W}}$, line (243) of this definition must be cancelled. Such a homomorphism exists because (230) ensures that

$$\delta_w(W) = \delta_t(\theta_t(W)) \overset{(230)}{=} \delta_t(\theta_t(c^{-1}c'W'd^{-1})) = \delta_w(c^{-1}c'W'd^{-1}) = \delta_w(\lambda'(W))$$

(and analogously $\delta_w(\overline{W}) = \delta_w(\lambda'(\overline{W}))$ in case $W \in \widehat{\mathcal{W}}$). The same argument shows that λ' also preserves μ_w. Moreover, λ' preserves γ_w, since $\gamma_w(W) = \gamma_w(W')$. Finally, λ' preserves the partial involution $\mathbb{I}_w$ because $\lambda'(\overline{W}) = d\overline{W'}c'^{-1}c = \mathbb{I}_w(c^{-1}c'W'd^{-1}) = \mathbb{I}_w(\lambda'(W))$. Hence, Lemma 12 implies that λ' is an AB-homomorphism. Let us define $\boldsymbol{\lambda}' = (\lambda_i \circ \lambda')_{1 \le i \le n+m}$. Since $\mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda}') = \mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda}) \setminus \{W, \overline{W}\}$, the inequality $\mathrm{A}(\mathcal{S}, \boldsymbol{\lambda}') < \frac{1}{2}\mathrm{Card}(\mathcal{V}_0)$ still holds. Moreover, we have

$$\theta_t(\lambda'(W)) = \theta_t(c^{-1}c'W'd^{-1}) \overset{(230)}{=} \theta_t(W)$$

and hence also $\theta_t(\lambda'(\overline{W})) = \theta_t(\overline{W})$. Since moreover $\theta_t(\lambda'(W'')) = \theta_t(W'')$ for all $W'' \in \mathcal{W} \setminus \{W, \overline{W}\}$, we get $\theta_t = \lambda' \circ \theta_t$. This implies

$$\theta_t(\lambda'(\lambda_i(w_i))) = \theta_t(\lambda_i(w_i)) = \theta_t(\lambda_i(w_i')) = \theta_t(\lambda'(\lambda_i(w_i'))).$$

It follows that the triple $(\mathcal{S}, \boldsymbol{\lambda}', \theta_t)$ fulfills the hypothesis of Lemma 32. W.r.t. to this triple we have

$$\lambda'(P_i cWd) \equiv_i \lambda'(P_i'c'W'). \tag{245}$$

Also note that for all $w \in \mathbb{W}$ we have $\chi_{AB}(w) = \chi_{AB}(\lambda'(w))$ and $\chi_H(w) = \chi_H(\lambda'(w))$, since w and $\lambda'(w)$ have the same type which uniquely determines the value under χ_H. Furthermore, also $\|\theta_t(w)\| = \|\theta_t(\lambda'(w))\|$. This implies that

$$\begin{aligned}
\Delta_i(\mathcal{S}, \boldsymbol{\lambda}', \theta_t) &\overset{(245)}{\le} \Delta(\lambda'(P_i cWd), \lambda'(d^{-1}L_i), \lambda'(P_i'c'W'), \lambda'(L_i'), \theta_t) \\
&= \Delta(P_i cWd, d^{-1}L_i, P_i'c'W', L_i', \theta_t) \\
&\overset{\text{Claim 13}}{<} \Delta(P_i, S_i, P_i', S_i', \theta_t) \\
&= \Delta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t).
\end{aligned}$$

Hence, we have $\|(\mathcal{S}, \boldsymbol{\lambda}', \theta_t)\| < \|(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)\|$, i.e., the triple $(\mathcal{S}, \boldsymbol{\lambda}', \theta_t)$ fulfills the hypothesis of Lemma 32 and has smaller size. By induction hypothesis the conclusion of the lemma holds for $(\mathcal{S}, \boldsymbol{\lambda}', \theta_t)$. This proves that it holds for $(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)$ too.

Case 2.2: $\gamma_w(W)$ and $\gamma_w(W')$ are T-types and $W' \in \{W, \overline{W}\}$. This implies $\|\theta_t(cW)\| = \|\theta_t(c'W')\|$. In view of this equality, and equations (226) and (229), point (1) of Lemma 17 applies. Hence, there exists $d \in \mathrm{Ge}(W)$ such that

$$\theta_t(cWd) = \theta_t(c'W) \quad \text{and} \quad \theta_t(d^{-1}L_i) = \theta_t(L_i').$$

Claim 14. $\Delta(P_i cWd, d^{-1}L_i, P_i'c'W', L_i', \theta_t) < \Delta(P_i, L_i, P_i', L_i', \theta_t)$.

First note that

$$\|\theta_t(d^{-1}L_i)\| = \|\theta_t(L_i)\| = \|\theta_t(S_i)\| - \|\theta_t(W)\| \le \|\theta_t(S_i)\| - 1 \text{ and}$$
$$\|\theta_t(L_i')\| = \|\theta_t(S_i')\| - \|\theta_t(W')\| \le \|\theta_t(S_i')\| - 1 = \|\theta_t(S_i)\| - 1,$$

because W and W' have a T-type and therefore $\|\theta_t(W)\|, \|\theta_t(W')\| \geq 1$. Next, since $\tau\mathrm{i}(W) = \tau\mathrm{i}(S_i)$ and W has a T-type, $\gamma_w(S_i)$ has the form $\{(*, T, 1, *, *)\}$ which implies $\chi_H(S_i) \leq 1$. Hence, we have

$$\chi_H(d^{-1}L_i) = \chi_H(L_i) \leq \chi_H(S_i) + 1 \quad \text{and} \quad \chi_H(L'_i) \leq \chi_H(S_i) + 1.$$

Hence, we get

$$\begin{aligned}\varDelta(P_i cWd, d^{-1}L_i, P'_i c'W', L'_i, \theta_t) = 1 - \frac{1}{2}\Big(\chi_{AB}(P_i cWd) + \chi_{AB}(P'_i c'W')\Big) + \\ \chi_H(d^{-1}L_i) + \chi_H(L'_i) + 2\|\theta_t(d^{-1}L_i)\| + 2\|\theta_t(L'_i)\| \\ &\leq 1 + 2\chi_H(S_i) + 2 + 4\|\theta_t(S_i)\| - 4 \\ &= 2\chi_H(S_i) + 4\|\theta_t(S_i)\| - 1 \\ &< \varDelta(P_i, L_i, P'_i, L'_i, \theta_t).\end{aligned}$$

This proves Claim 14.

Case 2.2.1. $W' = W$: Then, as in Case 2.1.1, $\theta_t(cWd) = \theta_t(c'W)$ implies $cWd = c'W$ and hence $P_i cWd \equiv_i P'_i c'W$. With Claim 14, this contradicts the minimality of $\varDelta(P_i, L_i, P'_i, L'_i, \theta_t)$.

Case 2.2.1. $W' = \overline{W}$: Then $\gamma_w(W) = \{(C, T, 1, C, H)\} = \gamma_w(\overline{W})$ for either $C = A$ or $C = B$. A new system $\mathcal{S}'$ and tuple $\boldsymbol{\lambda}'$ of AB-homomorphisms can be defined as in (234)–(236) in Case 2.1.2. W.r.t. the triple $(\mathcal{S}', \boldsymbol{\lambda}', \theta_t)$ we have

$$P_i cWd \equiv_i P'_i c'\overline{W}.$$

The definition of $\varDelta_i$ and Claim 14 imply

$$\varDelta_i(\mathcal{S}', \boldsymbol{\lambda}', \theta_t) < \varDelta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t). \tag{246}$$

Let us define

$$P_{n+m+1} = \varepsilon, \quad P'_{n+m+1} = \varepsilon, \quad S_{n+m+1} = cWd, \quad S'_{n+m+1} = c'\overline{W}.$$

Note that $\chi_H(W) = \chi_H(\overline{W}) = 0$. Hence, we get

$$\begin{aligned}\varDelta(P_{n+m+1}, S_{n+m+1}, P'_{n+m+1}, S'_{n+m+1}, \theta_t) &= 1 - \chi_{AB}(\varepsilon) + \chi_H(W) + \chi_H(\overline{W}) + \\ &\quad 2\|\theta_t(W)\| + 2\|\theta_t(\overline{W})\| \\ &= 4\|\theta_t(W)\|\end{aligned} \tag{247}$$

Next, let us estimate the size of

$$\varDelta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t) = \varDelta(P_i, S_i, P'_i, S'_i, \theta_t) = 1 - \chi_{AB}(P_i) + 2\chi_H(S_i) + 4\|\theta_t(S_i)\|.$$

We claim that

$$1 - \chi_{AB}(P_i) + 2\chi_H(S_i) + 4\|\theta_t(S_i)\| > 4\|\theta_t(W)\| \tag{248}$$

which implies

$$\begin{aligned}\varDelta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t) &> 4\|\theta_t(W)\| \\ &\overset{(247)}{=} \varDelta(P_{n+m+1}, S_{n+m+1}, P'_{n+m+1}, S'_{n+m+1}, \theta_t) \\ &\geq \varDelta_{n+m+1}(\mathcal{S}', \boldsymbol{\lambda}', \theta_t).\end{aligned} \tag{249}$$

So, let us prove (248). As in Case 2.1.2, we have either $P_i, P'_i \notin A \cup B$ or $L_i, L'_i \notin A \cup B$ (otherwise the index i would satisfy (180), which we exclude in Case 2). If $P_i, P'_i \notin A \cup B$, then $\chi_{AB}(P_i) = 0$ and hence

$$1 - \chi_{AB}(P_i) + 2\chi_H(S_i) + 4\|\theta_t(S_i)\| = 1 + 2\chi_H(S_i) + 4\|\theta_t(S_i)\| > 4\|\theta_t(S_i)\| \geq 4\|\theta_t(W)\|.$$

On the other hand, if $P_i, P'_i \in A \cup B$ and $L_i, L'_i \notin A \cup B$, then

$$1 - \chi_{AB}(P_i) + 2\chi_H(S_i) + 4\|\theta_t(S_i)\| = 2\chi_H(S_i) + 4\|\theta_t(S_i)\|.$$

If moreover $\|\theta_t(L_i)\| \geq 1$ then $\|\theta_t(S_i)\| = \|\theta_t(W)\| + \|\theta_t(L_i)\| > \|\theta_t(W)\|$ and hence $2\chi_H(S_i) + 4\|\theta_t(S_i)\| > 4\|\theta_t(W)\|$. If $\|\theta_t(L_i)\| = 0$, then $\|\theta_t(S_i)\| = \|\theta_t(W)\|$ and all symbols from $\mathcal{W}$ that occur in L_i have H-types. Since $\|L_i\| \geq 1$, there is at least one such symbol. It follows that $\gamma_w(S_i)$ is either of the form $\{(*, T, 1, *, T)\}$ or of the form $\{(*, T, 1, 1, 1)\}$. In both cases we get $\chi_H(S_i) \geq 1$ and hence again $2\chi_H(S_i) + 4\|\theta_t(S_i)\| \geq 2 + 4\|\theta_t(W)\| > \|\theta_t(W)\|$. This proves (248) and hence (249). Together with (246), we get

$$\|(\mathcal{S}', \boldsymbol{\lambda}', \theta_t)\| < \|(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)\|.$$

By induction hypothesis, the conclusion of the lemma holds for $(\mathcal{S}', \boldsymbol{\lambda}', \theta_t)$. This proves that it holds for $(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)$ too.

Case 2.3. $W' \notin \{W, \overline{W}\}$ and $\gamma_w(W)$, $\gamma_w(W')$ are T-types: By (226), $\theta_t(cWL_i) = \theta_t(c'W'L'_i)$. Using that $\gamma_w(W)$ and $\gamma_w(W')$ are T-types, we can apply Lemma 17 with $P = \theta_t(cW)$, $P' = \theta_t(cW')$, $S = \theta_t(L_i)$, and $S' = \theta_t(L'_i)$. We distinguish 3 subcases according to which point of Lemma 17 occurs.

Case 2.3.1. $\|\theta_t(cW)\| = \|\theta_t(c'W')\|$ (and hence $\|\theta_t(W)\| = \|\theta_t(W')\|$): This corresponds to point (1) of Lemma 17: There exists some $d \in \mathrm{Ge}(W)$ such that,

$$\theta_t(cWd) = \theta_t(c'W') \quad \text{and} \quad \theta_t(L_i) = \theta_t(dL'_i). \tag{250}$$

Moreover, $\gamma_w(W) = \gamma_w(W')$. We can end this case as for Case 2.1.3: Using (250) and $\gamma_w(W) = \gamma_w(W')$, we can define an AB-homomorphism λ' as in (241)–(243). Then, we define $\boldsymbol{\lambda}' = (\lambda_i \circ \lambda')_{1 \leq i \leq n+m}$. We get again (245). The definition of λ' implies that for all $w \in \mathbb{W}$: $\chi_{AB}(w) = \chi_{AB}(\lambda'(w))$, $\chi_H(w) = \chi_H(\lambda'(w))$ (since w and $\lambda'(w)$ have the same type), and $\|\theta_t(w)\| = \|\theta_t(\lambda'(w))\|$ (since $\|\theta_t(W)\| = \|\theta_t(W')\|$). This allows to derive $\Delta_i(\mathcal{S}, \boldsymbol{\lambda}', \theta_t) < \Delta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)$ in the same way as in Case 2.1.3 (using Claim 14 instead of Claim 13).

Case 2.3.2. $\|\theta_t(cW)\| < \|\theta_t(c'W')\|$: This corresponds to point (2) of Lemma 17: Let $\gamma_w(W) = \{\boldsymbol{\theta}\}$, $\gamma_w(W') = \{\boldsymbol{\theta}'\}$, $\boldsymbol{\rho} \in \gamma_w(L_i)$ and $\boldsymbol{\rho}' \in \gamma_w(L'_i)$ such that $\boldsymbol{\theta}\boldsymbol{\rho} = \boldsymbol{\theta}'\boldsymbol{\rho}'$. By point (2) of Lemma 17, there exist $d \in \mathrm{Ge}(W)$ and $P'_1, P'_2, P'_3 \in \mathbb{H}_t$ such that P'_1 has a T-type $\boldsymbol{\theta}'_1$, P'_3 has a T-type $\boldsymbol{\theta}'_3$, P'_2 has an H-type $\boldsymbol{\theta}'_2$, and

$$\theta_t(cW) = P'_1 d,\ \theta_t(c'W') = P'_1 P'_2 P'_3,\ \theta_t(dL_i) = P'_2 P'_3 \theta_t(L'_i),\ \boldsymbol{\theta}' = \boldsymbol{\theta}'_1 \boldsymbol{\theta}'_2 \boldsymbol{\theta}'_3,\ \boldsymbol{\rho} = \theta'_2 \theta'_3 \boldsymbol{\rho}'. \tag{251}$$

Note that $\|L_i\| \geq 1$, because otherwise we have $\theta_t(L_i) \in \mathbb{H}$, which contradicts the fact that P'_3 has a T-type. Hence, $\gamma_w(dL_i) = \{\boldsymbol{\rho}\} = \{\theta'_2 \theta'_3 \boldsymbol{\rho}'\}$. Let us apply Lemma 25 to the AB-homomorphism θ_t, the equality $\theta_t(dL_i) = P'_2(P'_3 \theta_t(L'_i))$, the H-type $\boldsymbol{\theta}'_2 \in \gamma_t(P'_2)$, and the type $\theta'_3 \boldsymbol{\rho}' \in \gamma_t(P'_3 \theta_t(L'_i))$ (note that $\gamma_w(dL_i) = \{\theta'_2 \theta'_3 \boldsymbol{\rho}'\}$ as required in Lemma 25). We obtain $P_2, S \in \mathbb{W}_t$ with

$$dL_i = P_2 S, \quad \theta_t(P_2) = P'_2, \quad \gamma_w(P_2) = \{\boldsymbol{\theta}'_2\}, \quad \theta_t(S) = P'_3 \theta_t(L'_i), \quad \boldsymbol{\theta}'_3 \boldsymbol{\rho}' \in \gamma_w(S). \tag{252}$$

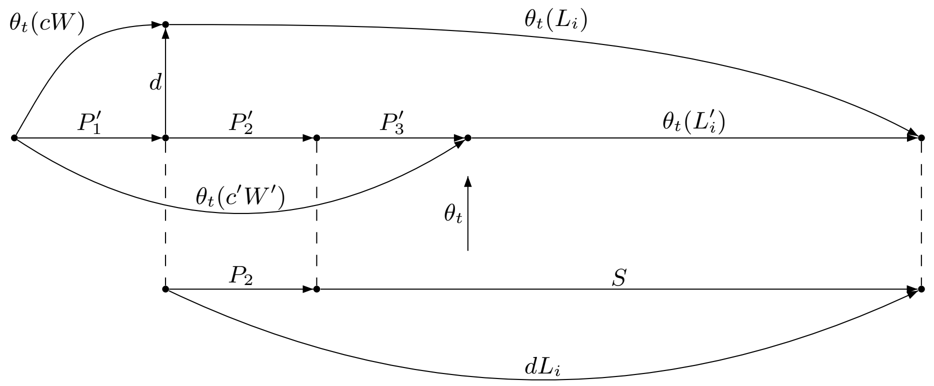


Fig. 14. Case 2.3.2

see Figure 14.

Let us first assume that $W' \in \widehat{\mathcal{W}}$. By axiom (AB3), $c' \in \widehat{\mathbb{W}}$ and by axiom (AB1), $c'W' \in \widehat{\mathbb{W}}$. Axiom (Hom3) on AB-homomorphisms implies that $P'_1P'_2P'_3 \in \mathrm{dom}(\mathbb{I}_t)$. Axiom (AB5) implies that $P'_1, P'_2, P'_3 \in \mathrm{dom}(\mathbb{I}_t)$ and axiom (Hom3) implies that

$$W, P_2 \in \widehat{\mathbb{W}}. \tag{253}$$

We saw above that $\gamma_t(P'_3)$ contains the T-type $\boldsymbol{\theta}'_3$. Moreover, by hypothesis, $\mathrm{A}(\mathcal{S}, \boldsymbol{\lambda}) < \frac{1}{2}\mathrm{Card}(\mathcal{V}_0)$.
Hence, we can choose a letter $W_3 \in \mathcal{W}_t$ such that

$$W_3 \notin \mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda}), \quad \gamma_w(W_3) = \boldsymbol{\theta}'_3 \in \gamma_t(P'_3), \quad \mu_w(W_3) = \mu_t(\boldsymbol{\theta}'_3, P'_3), \quad \delta_w(W_3) = \delta_t(\boldsymbol{\theta}'_3, P'_3).$$

We define a monoid homomorphism $\lambda' : \mathbb{W} \to \mathbb{W}$ by:

$$\begin{aligned}
\lambda'(e) &= e \text{ for } e \in A \cup B \\
\lambda'(W') &= c'^{-1}cWd^{-1}P_2W_3 \qquad (254)\\
\lambda'(\overline{W'}) &= \mathbb{I}_w(c'^{-1}cWd^{-1}P_2W_3) \qquad (255)\\
\lambda'(W'') &= W'' \text{ for } W'' \in \mathcal{W} \setminus \{W', \overline{W'}\}
\end{aligned}$$

Note that

$$\begin{aligned}
\delta_w(W') &= \delta_t(\boldsymbol{\theta}', \theta_t(W')) \\
&\overset{(251)}{=} \delta_t(\boldsymbol{\theta}', c'^{-1}P'_1P'_2P'_3) \\
&\overset{(251),(252)}{=} \delta_t(\boldsymbol{\theta}', \theta_t(c'^{-1}cWd^{-1}P_2)P'_3) \\
&= \delta_w(c'^{-1}cWd^{-1}P_2W_3) = \delta_w(\lambda'(W'))
\end{aligned}$$

and similarly $\delta_w(\overline{W'}) = \delta_w(\lambda'(\overline{W'}))$. Hence, λ' is indeed a monoid homomorphism. Next, since $\delta_t(\boldsymbol{\theta}'_3, P'_3) = \delta_w(W_3)$, we can define a monoid homomorphism $\theta'_t : \mathbb{W}_t \to \mathbb{H}_t$ by

$$\begin{aligned}
\theta'_t(e) &= e \text{ for } c \in A \cup B \\
\theta'_t(W_3) &= P'_3 \qquad (256)\\
\theta'_t(\overline{W}_3) &= \mathbb{I}_t(P'_3). \qquad (257)\\
\theta'_t(W'') &= \theta_t(W'') \text{ for } W'' \in \mathcal{W} \setminus \{W_3, \overline{W}_3\}
\end{aligned}$$

As in the above cases we can check that λ' and θ'_t are AB-homomorphisms. Moreover, we claim that

$$\forall W'' \in \mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda}) : \theta'_t(\lambda'(W'')) = \theta_t(W''). \tag{258}$$

Since $W_3, \overline{W}_3 \notin \mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda})$, we have $\theta'_t(\lambda'_t(W'')) = \theta'_t(W'') = \theta_t(W'')$ for every $W'' \in \mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda}) \setminus \{W', \overline{W'}\}$. Moreover

$$\begin{aligned}
\theta'_t(\lambda'(W')) &\overset{(254)}{=} \theta'_t(c'^{-1}cWd^{-1}P_2W_3) \\
&\overset{(256)}{=} c'^{-1}\theta_t(cW)d^{-1}\theta'_t(P_2)P'_3 \\
&\overset{W_3 \notin \mathrm{Alph}(P_2)}{=} c'^{-1}\theta_t(cW)d^{-1}\theta_t(P_2)P'_3 \\
&\overset{(252)}{=} c'^{-1}\theta_t(cW)d^{-1}P'_2P'_3 \\
&\overset{(251)}{=} c'^{-1}P'_1dd^{-1}P'_2P'_3 \\
&\overset{(251)}{=} \theta_t(W').
\end{aligned}$$

Finally, since λ' preserves $\mathbb{I}_w$ and θ'_t, θ_t preserve $\mathbb{I}_t$, we also get $\theta'_t(\lambda'(\overline{W'})) = \theta_t(\overline{W'})$. Equation (258) implies that

$$\theta'_t(\lambda'(\lambda_i(w_i))) = \theta_t(\lambda_i(w_i)) = \theta_t(\lambda_i(w'_i)) = \theta'_t(\lambda'(\lambda_i(w'_i))) \tag{259}$$

for all $1 \leq i \leq n+m$. Let us define

$$\lambda'_i = \lambda_i \circ \lambda' \text{ for all } 1 \leq i \leq n+m.$$

Recall that P_2 has the H-type $\boldsymbol{\theta}'_2$. Since $\gamma_w(W')$ is a T-type, this shows that

$$W', \overline{W'} \notin \mathrm{Alph}(P_2). \tag{260}$$

Hence $\mathrm{Alph}(\mathcal{S}', \boldsymbol{\lambda}') = \mathrm{Alph}(\mathcal{S}, \boldsymbol{\lambda}) \cup \{W_3, \overline{W}_3\} \setminus \{W', \overline{W'}\}$. We thus obtain

$$\mathrm{A}(\mathcal{S}', \boldsymbol{\lambda}') < \frac{1}{2}\mathrm{Card}(\mathcal{V}_0). \tag{261}$$

Equality (259) and inequality (261) ensure that the new triple $(\mathcal{S}, \boldsymbol{\lambda'}, \theta')$ fulfills the hypothesis of Lemma 32. Let us evaluate now the size of this new triple. For the i-th equation (w_i, w'_i) we have:

$$\begin{aligned}
(\lambda'_i(w_i), \lambda'_i(w'_i)) &= (\lambda'(P_icWL_i),\ \lambda'(P'_ic'W'L'_i)) \\
&\overset{(252),\ (254)}{=} (\lambda'(P_i)cW\lambda'(d^{-1}P_2S),\ \lambda'(P'_i)c'(c'^{-1}cWd^{-1}P_2W_3)\lambda'(L'_i)) \\
&\overset{(260)}{=} (\lambda'(P_i)cWd^{-1}P_2\lambda'(S)),\ \lambda'(P'_i)cWd^{-1}P_2W_3\lambda'(L'_i))
\end{aligned}$$

Let us set

$$Q_i = \lambda'(P_i)cWd^{-1}P_2,\ \ T_i = \lambda'(S),\ \ Q'_i = \lambda'(P'_i)cWd^{-1}P_2,\ \ T'_i = W_3\lambda'(L'_i).$$

We next want to bound the size of

$$\mathit{\Delta}(Q_i, T_i, Q'_i, T'_i, \theta'_t) = 1 - \frac{1}{2}(\chi_{AB}(Q_i) + \chi_{AB}(Q'_i)) + \chi_H(T_i) + \chi_H(T'_i) + 2\|\theta'_t(T_i)\| + 2\|\theta'_t(T'_i)\|.$$

First note that $\chi_{AB}(Q_i) = \chi_{AB}(Q'_i) = 0$ since Q_i and Q'_i contain the symbol W. Moreover

$$\theta'_t(T_i) = \theta'_t(\lambda'(S)) = \theta_t(S) = P'_3\theta_t(L'_i) = P'_3\theta'_t(\lambda'(L'_i)) = \theta'_t(W_3\lambda'(L'_i)) = \theta'_t(T'_i). \qquad (262)$$

Since $T'_i = W_3\lambda'(L'_i)\}$ and W_3 has a T-type, it follows that $\gamma_w(T'_i)$ is of the form $\{(*, T, 1, *, *)\}$. Hence, $\chi_H(T'_i) \leq 1$. For $T_i = \lambda'(S)$ we either have $\|\gamma_w(T_i)\| > 0$ and hence $\chi_H(T_i) = 0$ or $\gamma_w(T_i) = \gamma_w(S) = \{\boldsymbol{\theta}'_3\boldsymbol{\rho}'\}$. Since $\boldsymbol{\theta}'_3$ is a T-type, it follows again $\chi_H(T'_i) \leq 1$. We therefore obtain:

$$\begin{aligned}
\Delta(Q_i, T_i, Q'_i, T'_i, \theta'_t) &= 1 - \frac{1}{2}(\chi_{AB}(Q_i) + \chi_{AB}(Q'_i)) + \chi_H(T_i) + \chi_H(T'_i) + 2\|\theta'_t(T_i)\| + 2\|\theta'_t(T'_i)\| \\
&\leq 3 + 4\|\theta'_t(T'_i)\| \\
&\overset{(262)}{=} 3 + 4\|P'_3\| + 4\|\theta_t(L'_i)\| \\
&\overset{\|P'_1\|>0}{=} 4(\|P'_1\| + \|P'_2\| + \|P'_3\| + \|\theta_t(L'_i)\|) \\
&= 4\|\theta_t(c'W'L'_i)\| \\
&= 4\|\theta_t(S'_i)\| \\
&\leq \Delta(P_i, S_i, P'_i, S'_i, \theta_t).
\end{aligned}$$

By Lemma 1, the hypothesis that P_i and P'_i are related by the monoid congruence generated by the set of pairs $\{(\lambda_j(w_j), \lambda_j(w'_j)) \mid i+1 \leq j \leq n+m\}$ implies that $\lambda'(P_i)$ and $\lambda'(P'_i)$ (and hence Q_i and Q'_i) are related by the monoid congruence generated by the set of pairs $\{(\lambda'(\lambda_j(w_j)), \lambda'(\lambda_j(w'_j))) \mid i+1 \leq j \leq n+m\}$. It follows that

$$\Delta_i(\mathcal{S}, \boldsymbol{\lambda}', \theta'_t) \leq \Delta(Q_i, T_i, Q'_i, T'_i, \theta'_t) < \Delta(P_i, S_i, P'_i, S'_i, \theta_t) = \Delta_i(\mathcal{S}, \boldsymbol{\lambda}, \theta_t)$$

and thus $\|(\mathcal{S}', \boldsymbol{\lambda}', \theta')\| < \|(\mathcal{S}, \boldsymbol{\lambda}, \theta)\|$. By induction hypothesis, conclusions (173)–(176) are true for $(\mathcal{S}', \boldsymbol{\lambda}')$, which also implies that they hold for $(\mathcal{S}, \boldsymbol{\lambda})$. This concludes the case that $W' \in \widehat{\mathcal{W}}$. In case $W' \notin \widehat{\mathcal{W}}$ we just cancel the last line of (255) and can conclude as before.

Case 2.3.3. $\|\theta_t(cW)\| > \|\theta_t(c'W')\|$: Symmetric to Case 2.3.2.

This concludes the proof of Lemma 32. □

Let us now prove Lemma 31.

Proof of Lemma 31. Let us suppose that $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ is an AB-homomorphism solving the system of t-equations $\mathcal{S} = \{(w_i, w'_i) \mid 1 \leq i \leq n\}$. Let us define $m = 0$, $\lambda_i = \mathrm{Id}$ for $1 \leq i \leq n$, and $\theta_t = \sigma_t$. Since, by construction of $\mathcal{V}_0$,

$$\mathrm{A}(\prod_{i=1}^{n} w_i w'_i) < \frac{1}{2}\mathrm{Card}(\mathcal{V}_0)$$

(see condition (129) and inequality (127)), the system $\mathcal{S}$, the integers m, n, and the maps $(\lambda_i)_{1\leq i\leq n+m}$, θ_t are fulfiling the hypothesis of Lemma 32. Let us consider the maps Φ, λ'_i and θ'_t given by the conclusion of Lemma 32, the natural projection $\pi_{\approx_\Phi} : \mathbb{W}_t \to \mathbb{W}_t/\simeq_\Phi$ and choose

$$\sigma_{\mathbb{W}} = \lambda'_1 \circ \pi_{\simeq_\Phi} : \mathbb{W}_t \to \mathbb{W}_t/\simeq_\Phi$$

and let

$$\psi_t : \mathbb{W}_t/\simeq_\Phi \to \mathbb{H}_t$$

$$\begin{tikzcd}
\mathbb{W}_t \arrow[r, "\lambda_1"] \arrow[dr, "\lambda'_1"] \arrow[ddr, "\sigma_{\mathbb{W}}"'] & \mathbb{W}_t \arrow[r, "\theta_t"] \arrow[d, "\lambda'"] & \mathbb{H}_t \\
& \mathbb{W}_t \arrow[ur, "\theta'_t"] \arrow[d, "\pi_{\simeq_{\Phi}}"] & \\
& \mathbb{W}_t/\simeq_{\Phi} \arrow[uur, "\psi_t"'] &
\end{tikzcd}$$

Fig. 15. Proof of Lemma 31

be the AB-homomorphism defined by

$$\forall W \in \mathcal{W}_t, \psi_t([W]_{\simeq_{\Phi}}) = \theta'_t(W).$$

By point (174) of Lemma 32, $\simeq_{\Phi} \subseteq \mathrm{Ker}(\theta'_t)$ This implies the existence and unicity of the monoid homomorphism ψ_t. Since the map Φ is an AB-automorphism, the projection $u \mapsto [u]_{\Phi}$ preserves invertibility, non-invertibility and preserves the three maps γ, μ, δ. It follows that ψ_t is an AB-homomorphism.
The objects $\sigma_{\mathbb{W}}, \psi_t$ are fulfiling the properties asserted in Lemma 31, as required. (See figure 15). □

From $\mathbb{W}$-solutions to t-solutions Conversely to Lemma 31, every $\mathbb{W}$-solution of a given system $\mathcal{S}$ of the form (123) provides a t-solution of the same system. Let us state this formally.

Lemma 33 ($\mathbb{W}$-solutions provide t-solutions). *Let $\mathcal{S}_t = \{(w_i, w'_i) \mid 1 \leq i \leq n\}$ be a system of t-equations of the form (123) and $\mathcal{S}_{\mathbb{H}} = \{(v_i, v'_i) \mid 1 \leq i \leq m\}$ a system of equations over the monoid $\mathbb{H}$.*
Let us suppose that there exists an $\mathbb{H}$-involution $\Phi \in \mathsf{HInv}$, an AB-homomorphism $\sigma_{\mathbb{W}} : \mathbb{W}_t \to \mathbb{W}_t/\Phi$ such that

$$\sigma_{\mathbb{W}}(w_i) = \sigma_{\mathbb{W}}(w'_i) \text{ for all } 1 \leq i \leq n.$$

and an AB-homomorphism

$$\psi_{H,t} : \mathbb{W}_{\mathbb{H}}/\Phi \to \mathbb{H}_t$$

such that

$$\sigma_{H,t}(\sigma_{\mathbb{W}}(v_i)) = \sigma_{\mathbb{W}}(\sigma_{\mathbb{W}}(v'_i)) \text{ for all } 1 \leq i \leq m.$$

Then, there is an AB-homomorphism $\psi_t : \mathbb{W}_t/\Phi \to \mathbb{H}_t$, which extends $\psi_{H,t}$ such that $\sigma_{\mathbb{W}} \circ \psi_t$ solves the system $\mathcal{S}_t \wedge \mathcal{S}_{\mathbb{H}}$.

Proof. Let $\Phi, \sigma_{\mathbb{W}}, \psi_{H,t}$ fulfill the hypothesis of the lemma. Since $\Phi \in \mathsf{HInv}$, there exists a partition $\widehat{\mathcal{W}} = \widehat{\mathcal{W}_0} \cup \{W_1, \ldots, W_p\} \cup \{\overline{W}_1, \ldots, \overline{W}_p\}$ (see (87)), where $\widehat{\mathcal{W}_0}$ is closed under the involution $\mathbb{I}_w$. Moreover, there exists a tuple $(a_1, b_1, \ldots, a_p, b_p)$ with $(a_k, b_k) \in \mathrm{Gi}(W_k) \times \mathrm{Ge}(W_k)$ $(1 \leq k \leq p)$ such that (89) and (90) hold. We define a monoid homomorphism $\psi_t : \mathcal{W}_t^* * A * B \to \mathbb{H}_t$ in the following way:

- $\psi_t(c) = c$ for $c \in A \cup B$
- $\psi_t(W) = \psi_{H,t}([W]_{\equiv_\Phi})$ for $W \in \mathcal{W}_{\mathbb{H}}$
- Let $W \in \mathcal{W}_t \setminus \mathcal{W}_{\mathbb{H}}$. By (86), we can choose a t-sequence $s \in \mathbb{H} * \{t, \bar{t}\}^*$ which realizes W in the sense that

$$s \in \mathrm{dom}(\mathbb{I}_t) \Leftrightarrow W \in \mathrm{dom}(\mathbb{I}_w), \quad \gamma_w(W) \subseteq \gamma_t(s),$$

$$\forall \boldsymbol{\theta} \in \gamma_w(W) : \mu_w(\boldsymbol{\theta}, W) = \mu_t(\boldsymbol{\theta}, s),\ \delta_w(\boldsymbol{\theta}, W) = \delta_t(\boldsymbol{\theta}, s),$$

and we set $\psi_t(W) = [s]_\sim$.

Since, for every $W \in \mathcal{W}_t$ and every $(c, d) \in \delta_w(W)$, $c\psi_t(W) \sim \psi_t(W)d$, the map ψ_t induces a monoid homomorphism $\psi_t : \mathbb{W}_t \to \mathbb{H}_t$. This monoid homomorphism clearly fulfills conditions (a), (b), (c) and (d)–(f) from Lemma 12. Thus ψ_t is an AB-homomorphism from $\mathbb{W}_t \to \mathbb{H}_t$. By hypothesis on $\psi_{H,t}$, for every $w, w' \in \mathbb{W}_{\mathbb{H}}$, $w \simeq_\Phi w' \Rightarrow \psi_t(w) = \psi_t(w')$ and, since the congruence $\simeq_\Phi$ is generated by its restriction over $\mathbb{W}_{\mathbb{H}}$ (all the $W_k, \bar{W}_k$ belong to $\mathcal{W}_{\mathbb{H}}$),

$$\forall w, w' \in \mathbb{W}_t, w \simeq_\Phi w' \Rightarrow \psi_t(w) = \psi_t(w').$$

It follws that ψ_t induces an AB-homomorphism $\mathbb{W}_t/\Phi \to \mathbb{H}_t$. Since $\sigma_{\mathbb{W}}$ solves $\mathcal{S}_t$, $\sigma_{\mathbb{W}} \circ \psi_t$ solves $\mathcal{S}_t$ too. Since $\sigma_{\mathbb{W}} \circ \psi_{H,t}$ solves $\mathcal{S}_{\mathbb{H}}$, $\sigma_{\mathbb{W}} \circ \psi_t$ solves $\mathcal{S}_t$ too. □

7 The groups $\mathbb{U}$ and $\mathbb{E}$

We define in this section a group $\mathbb{U}$, in which the partial monoid consisting of the elements of $\mathbb{W}$ with a non-empty type, is embedded. Well-typed equations over $\mathbb{W}$ thus translate into equations, with rational constraints, over $\mathbb{U}$. But $\mathbb{W}$-equations involve, beside well-typed equations, an involutive $\mathbb{H}$-automorphism Φ, which induces an involutive automorphism $\Phi_{\mathbb{U}}$ of $\mathbb{U}$. We define a quotient $\mathbb{E}$ of $\mathbb{U}$, by making equal every element $u \in \mathbb{U}$ with its image $\Phi_{\mathbb{U}}(u)$. The notion of solution $\mathbb{W}_t \to \mathbb{W}_t/\simeq_\Phi$ of a t-equation over $\mathbb{W}_t$ can thus be translated into the notion of a solution $\mathcal{W}^* * A * B \to \mathbb{E}$ of an equation, with rational constraints, in $\mathbb{E}$.

7.1 The group $\mathbb{U}$

Let us adjoin for every non-invertible symbol $W \in \mathcal{W} \setminus \widehat{\mathcal{W}}$ a formal inverse $\overline{W}$. The extended alphabet $\mathcal{W}' = \mathcal{W} \cup \{\overline{W} \mid W \in \mathcal{W} \setminus \widehat{\mathcal{W}}\}$ is now endowed with a total involution, which extends the partial involution $\mathbb{I}_w$ by the rules $\mathbb{I}_w(W) = \overline{W}$ and $\mathbb{I}_w(\overline{W}) = W$ for $W \in \mathcal{W} \setminus \widehat{\mathcal{W}}$. The maps γ_w, μ_w, and δ_w are extended to $\mathcal{W}'$ in such a way that the axioms (AB10), (AB11), and (AB12) for AB-algebras are fulfilled by $\mathcal{W}'^* * A * B$ endowed with this involution $\mathbb{I}_w$. We denote by $\equiv'$ the monoid-congruence over $\mathcal{W}'^* * A * B$ generated by the set of pairs

$$\{(cW, Wd) \mid W \in \mathcal{W}', (c, d) \in \delta_w(W)\} \tag{263}$$

and by $\mathbb{W}'$ the AB-algebra obtained from $\mathcal{W}'^* * A * B$ by quotienting it by the equivalence $\equiv'$. We define the group $\mathbb{U}$ by the following monoid-presentation:

$$\mathbb{U} = \langle A * B, \mathcal{W}';\ \mathbb{I}_w(W)cW = d\ (W \in \mathcal{W}', (c, d) \in \delta_w(W))\rangle. \tag{264}$$

Note that the above relations include $\mathbb{I}_w(W)W = \varepsilon$ which guaranty that the monoid $\mathbb{U}$ is a group. When working in the group $\mathbb{U}$, we will also write W^{-1} instead of $\mathbb{I}_w(W)$.

The group $\mathbb{U}$ is an HNN-extension of the free product $A*B$: the elements from $\mathcal{W}'$ are the stable letters and the mappings $\delta_w(W')$ ($W \in \mathcal{W}'$) are the partial isomorphisms. We identify ι_A (resp. ι_B) with the natural embedding of A (resp. B) into $A * B$. We denote by $\equiv_{\mathbb{U}}$ the monoid congruence over $\mathcal{W}'^* * A * B$ such that $u \equiv_{\mathbb{U}} v$ if and only if u and v represent the same element of the group $\mathbb{U}$. We denote by

$$\pi_{\mathbb{U}} : \mathcal{W}'^* * A * B \to \mathbb{U} \tag{265}$$

the canonical monoid homomorphism with $\pi_{\mathbb{U}}(z) = [z]_{\equiv_{\mathbb{U}}}$. Clearly, all the pairs in (84) also belong to $\equiv_{\mathbb{U}}$, hence $\equiv \subseteq \equiv_{\mathbb{U}}$. Thus, there exists a unique monoid homomorphism $\overline{\pi}_{\mathbb{U}} : \mathbb{W} \to \mathbb{U}$ such that

$$\pi_{\mathbb{U}}\restriction_{\mathcal{W}^* * A * B} = \pi_{\equiv} \circ \overline{\pi}_{\mathbb{U}}.$$

An element $z \in \mathcal{W}'^* * A * B$ is said to be a $\mathbb{U}$-*reduced sequence* if its corresponding (A, B)-reduced string does not contain any factor of the form $\mathbb{I}_w(W)cW$ with $W \in \mathcal{W}', c \in \mathrm{dom}(\delta_w(W))$. We denote by $\mathrm{Red}_{\mathbb{U}}(A*B, \mathcal{W}')$ the subset of $\mathcal{W}'^* * A * B$ consisting of all $\mathbb{U}$-reduced sequences.

Lemma 34. *Let s, s' be some reduced sequences in $\mathcal{W}'^* * A * B$. Then $s \equiv_{\mathbb{U}} s'$ if and only if $s \equiv' s'$.*

This lemma is an analogue of Lemma 2 for HNN-extensions with a set $\mathcal{W}'$ of stable letters (instead of just a single stable letter t). This analogue can be obtained from Lemma 2 by induction over the number of stable letters.

Lemma 35. *Let $s \in \mathcal{W}^* * A * B$ with $\gamma_w(s) \neq \emptyset$. Then s is $\mathbb{U}$-reduced.*

Proof. It suffices to prove that $\gamma_w(\mathbb{I}_w(W)cW) = \emptyset$ for all $W \in \widehat{\mathcal{W}}$ and $c \in \mathrm{Gi}(W) = \mathrm{Ge}(\mathbb{I}_w(W))$: If $\gamma_w(W) = \{(\theta, b, \rho)\}$ for $\theta, \rho \in \mathcal{T}_6$, $b \in \{0, 1\}$, then we obtain

$$\gamma_w(\mathbb{I}_w(W)cW) = \{(\mathbb{I}_{\mathcal{R}}(\rho), b, \mathbb{I}_{\mathcal{R}}(\theta))\}\{(\theta, b, \rho)\} = \emptyset,$$

since $\mathbb{I}_{\mathcal{R}}(\theta) \neq \theta$ (the mapping $\mathbb{I}_{\mathcal{R}}$ defined in (13)–(15) has no fixpoints).

7.2 The group $\mathbb{E}$

Let us recall that an involutive H-automorphism $\Phi \in \mathsf{HInv}$ is defined through p letters W_k ($1 \leq k \leq p$), their inverses $\overline{W}_k$, and a tuple $(a_1, b_1, \ldots, a_p, b_p)$, as specified in Section 3.9 (the tuple fulfills (88)). Let us fix such an involution and assume that ((91)–(94)) hold. We extend Φ to an involutive AB-automorphism of $\mathbb{W}'$ by setting $\Phi(W) = W$ for $W \in \mathcal{W}' \setminus \mathcal{W}$. We define the group $\mathbb{E}$ by the group-presentation:

$$\mathbb{E} := \langle \mathbb{U};\ W_k = a_k^{-1} W_k^{-1} b_k^{-1}, \quad \text{for } 1 \leq k \leq p \rangle. \tag{266}$$

where we know that

$$W_k^{-1} a_k^{-1} b_k W_k \equiv_{\mathbb{U}} b_k a_k^{-1} \text{ for } 1 \leq k \leq p \tag{267}$$

by (92), and

$$(W_k b_k W_k a_k)^{-1} x (W_k b_k W_k a_k) \equiv_{\mathbb{U}} x \text{ for } 1 \leq k \leq p \text{ and } x \in \mathrm{dom}(\delta_w(W_k)) \tag{268}$$

by (91). We denote by $\equiv_{\mathbb{E}}$ the monoid-congruence over $\mathcal{W}'^* * A * B$ such that $u \equiv_{\mathbb{E}} v$ if and only if u and v represent the same element of the group $\mathbb{E}$. We denote by

$$\pi_{\mathbb{E}} : \mathcal{W}'^* * A * B \to \mathbb{E}$$

the canonical monoid homomorphism with $\pi_{\mathbb{E}}(z) = [z]_{\equiv_{\mathbb{E}}}$.

Definition 9. *For a given finite group A, we denote by* $\mathrm{PHNN}(A)$ *the set of all presentations consisting of a set of relations of the form* (264) *and* (266)*, where the letters $W_k, \overline{W}_k$ ($1 \leq k \leq p$) and the tuple $(a_1, b_1, \dots, a_p, b_p)$ are fulfilling conditions* (88)*,* (91)*,* (92)*,* (93)*and* (94)*.*

Note that the results demonstrated in the rest of this section hold for any group $\mathbb{E}$, provided it has a presentation in the set $\mathrm{PHNN}(A)$ for some finite group A.

Lemma 36. *The monoid homorphism $\Phi : \mathbb{W}' \to \mathbb{W}'$ (from Lemma 22) induces an involutive group automorphism of $\mathbb{U}$.*

Proof. By Point 2 from Lemma 22, adapted to the larger alphabet $\mathcal{W}'$, we know that the monoid homomorphism $\Phi : \mathbb{W}' \to \mathbb{W}'$ is an involutive AB-automorphism. Let $W \in \mathcal{W}'$. By (Hom7), we have

$$\delta_w(\Phi(W)) = \delta_w(W). \tag{269}$$

Let $e \in \mathrm{dom}(\delta_w(W)) = \mathrm{dom}(\delta_w(\Phi(W)))$. Since Φ commutes with $\mathbb{I}_w$ and $\Phi(c) = c$ for all $c \in A \cup B$, we get

$$\Phi(\mathbb{I}_w(W)eW) \equiv' \mathbb{I}_w(\Phi(W))e\Phi(W) \equiv_{\mathbb{U}} \delta_w(\Phi(W))(e) \overset{(269)}{=} \delta_w(W)(e) = \Phi(\delta_w(W)(e)).$$

Hence, Φ is compatible with the defining relations of $\mathbb{U}$. It follows that, for all $m, m' \in \mathbb{W}'$, $m \equiv_{\mathbb{U}} m'$ implies $\Phi(m) \equiv_{\mathbb{U}} \Phi(m')$. Thus $\Phi : \mathbb{W}' \to \mathbb{W}'$ induces a group homomorphism $\Phi_{\mathbb{U}} : \mathbb{U} \to \mathbb{U}$ and since $\Phi_{\mathbb{U}} \circ \Phi_{\mathbb{U}} = \mathrm{Id}_{\mathbb{U}}$, $\Phi_{\mathbb{U}}$ is an involutive group automorphism of $\mathbb{U}$. □

Note that the presentation (266) can be rewritten as

$$\mathbb{E} = \langle \mathbb{U}; w = \Phi_{\mathbb{U}}(w) \text{ for } w \in \mathbb{U} \rangle. \tag{270}$$

Let us define a finite semi-Thue system $S_{\mathbb{E}}$ over the alphabet $\mathcal{G}_{\mathbb{E}} := \mathcal{W}' \cup (A \setminus \{1\}) \cup (B \setminus \{1\})$ which will provide a *complete* monoid presentation for the group $\mathbb{E}$. Let us choose, for every $W \in \mathcal{W}'$ a transversal R_W of the left-congruence modulo the subgroup $\mathrm{dom}(\delta_w(W))$ over the group $\mathrm{Gi}(W)$ i.e.

$$\forall x \in \mathrm{Gi}(W), \exists! r \in R_W, x \in r \cdot \mathrm{dom}(\delta_w(W)).$$

For every $x, y \in A$ we denote by $\underline{x \cdot y}$ the element of $(A \setminus \{1\}) \cup \{\varepsilon\}$ that represents the result of the product $x \cdot y$, performed in the group A. (Note that, in the rules below, $x \cdot y$ can denote a word of length 2 over the alphabet $A \setminus \{1\}$ while $\underline{x \cdot y}$ denotes either an element of $A \setminus \{1\}$ or the empty word). The same notation is used for B, as well.

For every $W \in \mathcal{W}'$, we also denote by φ_W the partial automorphism $\delta_w(W)$. In the case $W = W_k$ (for $1 \leq k \leq p$) we also denote it by φ_k.

For every $W \in \mathcal{W}'$, we choose $1 \in R_W$ and for every $k \in [1, p]$, we choose $b_k \in R_W$. These two assumptions are made compatible after , possibly, a preliminary change of the defining tuple $(a_1, b_1, \dots, a_k, b_k, \dots, a_p, b_p)$ into another tuple $(a'_1, b'_1, \dots, a'_k, b'_k, \dots, a'_p, b'_p)$ as follows:
- if $b_k \notin \mathrm{dom}(\delta_w(W_k))$, we can choose $1 \in R_W$
- if $b_k \in \mathrm{dom}(\delta_w(W_k))$, since (by (91)), the same automorphism Φ is determined by , instead of (a_k, b_k), the pair (b_k, a_k) , thus the pair $(1, \varphi_k(b_k)a_k)$ and finally $(\varphi_k(b_k)a_k, 1))$, we choose this last pair $(a'_k, b'_k) := (\varphi_k(b_k)a_k, 1))$ of cefficients as k-th component of the defining tuple and choose $1 = b'_k \in R_W$.

The semi-Thue system $S_{\mathbb{E}}$ consists of the set of rules:

$$\begin{aligned}
a_1 \cdot a_2 \rightarrow &\quad \underline{a_1 \cdot a_2}, && \text{for } a_1 \in A \setminus \{1\}, a_2 \in A \setminus \{1\} && (271)\\
b_1 \cdot b_2 \rightarrow &\quad \underline{b_1 \cdot b_2}, && \text{for } b_1 \in B \setminus \{1\}, b_2 \in B \setminus \{1\} && (272)\\
\underline{r \cdot x}W \rightarrow &\; rW\varphi_W(x), && \text{for } W \in \mathcal{W}', r \in R_W, x \in \mathrm{dom}(\delta_w(W)) \setminus \{1\} && (273)\\
\overline{W}xW \rightarrow &\quad \varphi_W(x), && \text{for } W \in \mathcal{W}' \setminus \{W_k, \bar{W}_k\} \mid 1 \leq k \leq p\}, x \in \mathrm{dom}(\delta_w(W)) && (274)\\
\overline{W}_k \rightarrow &\quad a_k W_k b_k, && \text{for } 1 \leq k \leq p && (275)\\
W_k b_k W_k \rightarrow &\quad a_k^{-1}, && \text{for } 1 \leq k \leq p. && (276)
\end{aligned}$$

Lemma 37.
1- The monoid $\mathcal{G}_{\mathbb{E}}^*/ \overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}}$ is isomorphic to $\mathbb{E}$.
2- The semi-Thue system $S_{\mathbb{E}}$ is confluent and Noetherian.

(some recalls on semi-Thue systems are given in [LS08, Section 2]).

Proof. 1.1- It is clear that, for every rule $u \rightarrow v$ of $S_{\mathbb{E}}$, $u \equiv_{\mathbb{E}} v$. Hence, $\overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}} \subseteq \equiv_{\mathbb{E}}$.
1.2- Let us show that $\equiv_{\mathbb{U}} \subseteq \overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}}$.
The rules defining the product-tables for A (resp. B) are also rules of $S_{\mathbb{E}}$ (see rules (271),(272).
Let us consider a rule of the presentation of $\mathbb{U}$ of the form:

$$(\mathbb{I}_w(W)cW, d) \quad \text{for some } W \in \mathcal{W}' \text{ and } (c, d) \in \delta_w(W).$$

If $W \notin \{W_k, \bar{W}_k, k \in [1, p]\}$:

$$\mathbb{I}_w(W)cW \rightarrow_{S_{\mathbb{E}}} d$$

If $W = W_k$ for some $k \in [1, p]$:

$$\begin{aligned}
\bar{W}_k c W_k &\rightarrow_{S_{\mathbb{E}}} a_k W_k b_k c W_k\\
&\rightarrow_{S_{\mathbb{E}}} a_k W_k b_k W_k d\\
&\rightarrow_{S_{\mathbb{E}}} a_k a_k^{-1} d\\
&\rightarrow_{S_{\mathbb{E}}} d
\end{aligned} \tag{277}$$

If $W = \bar{W}_k$ for some $k \in [1, p]$ and $c = d = 1$:

$$\begin{aligned}
W_k \bar{W}_k &\rightarrow_{S_{\mathbb{E}}} W_k a_k W_k b_k\\
&= W_k \underline{b_k (b_k^{-1} a_k)} W_k b_k\\
&\rightarrow_{S_{\mathbb{E}}} W_k b_k W_k \varphi_k(b_k^{-1} a_k) b_k\\
&\rightarrow_{S_{\mathbb{E}}} a_k^{-1} \varphi_k(b_k^{-1} a_k) b_k\\
&= a_k^{-1}(a_k b_k^{-1}) b_k\\
&\rightarrow_{S_{\mathbb{E}}}^* \varepsilon
\end{aligned} \tag{278}$$

(Here we used the fact that, by (91), $\varphi_k(b_k^{-1}a_k) = a_k b_k^{-1}$).
Note that, by (277) and (278), $W_k, \bar{W}_k$ are inverses in the monoid presented by $S_{\mathbb{E}}$.
If $W = \bar{W}_k$ for some $k \in [1,p]$ and $(c,d) \in \delta_w(\bar{W}_k)$:
then $(d,c) \in \delta_w(W_k)$, hence by (277)

$$\bar{W}_k d W_k \rightarrow_{S_{\mathbb{E}}} c$$

hence, since $W_k, \bar{W}_k$ are inverses in the monoid $\mathcal{G}_{\mathbb{E}}^*/\overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}}$,

$$d \overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}} W_k c \bar{W}_k.$$

We have checked that $\equiv_{\mathbb{U}} \subseteq \overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}}$.
1.3- Let consider now an additional rule from the presentation (270) defining $\mathbb{E}$:

$$(g, \Phi(g)) \text{ for some } g \in \mathcal{G}_{\mathbb{E}}.$$

If $g \in \mathcal{G}_{\mathbb{E}} \setminus \{W_k, \bar{W}_k \mid 1 \le k \le p\}$, then $\Phi(g) = g$, so that $g \overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}} \Phi(g)$.
If $g = \bar{W}_k$ for some $k \in [1,p]$:

$$\bar{W}_k \rightarrow_{S_{\mathbb{E}}} a_k W_k b_k = \Phi(\bar{W}_k).$$

If $g = W_k$ for some $k \in [1,p]$:

$$W_k b_k W_k \rightarrow_{S_{\mathbb{E}}} a_k^{-1}.$$

Hence, using the fact that $(\bar{W}_k, W_k)$, (b_k, b_k^{-1}) are inverses in the monoid $\mathcal{G}_{\mathbb{E}}^*/\overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}}$,

$$W_k \overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}} a_k^{-1} \bar{W}_k b_k^{-1} = \Phi(W_k).$$

Points 1.1,1.2,1.3 ensure that $\equiv_{\mathbb{E}} = \overset{*}{\longleftrightarrow}_{S_{\mathbb{E}}}$.
2.1- All the rules of $S_{\mathbb{E}}$, excepted those of type (273) are strictly weight-decreasing for the function assigning the weight 4 to every letter $\bar{W}_k$ (for $1 \le k \le p$) and the weight 1 to every other letter. Thus, any infinite derivation for $S_{\mathbb{E}}$ would consist, ultimately, of an infinite derivation made of rules of type (273). But any application of a rule of type (273) either strictly decreases the number of redexes for these rules, or replaces a given redex by another redex which is strictly more on the right. Hence, an infinite derivation made of rules of type (273) cannot exist. We have established that $S_{\mathbb{E}}$ is Noetherian.
2.2- In view of point 2.1 above, in order to prove point 2 of the lemma, it suffices to show that every critical pair of the system $S_{\mathbb{E}}$ is resolved.
The critical pairs of $S_{\mathbb{E}}$ consist of the following:

$$r \cdot \bar{W}\varphi_W^{-1}(x)yW \leftarrow \underline{r \cdot x}\bar{W}yW \rightarrow \underline{r \cdot x}\varphi_W(y) \quad \text{for } r \in R_{\bar{W}}, x, y \in \mathrm{dom}(\delta_w(W)) \tag{279}$$

$$\varphi_W(y) \leftarrow \bar{W}yW \rightarrow \bar{W}W\varphi(y) \quad \text{for } y \in \mathrm{dom}(\delta_w(W)) \tag{280}$$

$$\underline{xry}W \leftarrow x \cdot \underline{r \cdot y}W \rightarrow x \cdot rW\varphi_W(y) \quad \text{for } r \in R_W, x \in \mathrm{Gi}(W), y \in \mathrm{dom}(\delta_w(W)) \tag{281}$$

$$rW_k\varphi_k(x)b_kW_k \leftarrow \underline{r \cdot x}W_kb_kW_k \rightarrow \underline{r \cdot x}a_k^{-1} \quad \text{for } r \in R_{W_k}, x \in \mathrm{dom}(\delta_w(W_k)) \tag{282}$$

$$a_k^{-1}b_kW_k \leftarrow W_kb_kW_kb_kW_k \rightarrow W_kb_ka_k^{-1} \tag{283}$$

$$r\bar{W}_k\varphi_k^{-1}(x) \leftarrow \underline{r \cdot x}\bar{W}_k \rightarrow \underline{r \cdot x}a_kW_kb_k \quad \text{for } r \in R_{\bar{W}_k}, x \in \mathrm{im}(\delta_w(W_k)) \tag{284}$$

Let us give, for each of them, a common descendant modulo($S_{\mathbb{E}}$).
Pair of type (279):

$$r \cdot \bar{W}\varphi_W^{-1}(x)yW \to_{S_{\mathbb{E}}} \bar{W}\underline{\varphi_W^{-1}(x)y}W \to_{S_{\mathbb{E}}} r\underline{x\varphi_W(y)} \to_{S_{\mathbb{E}}} \underline{rx\varphi_W(y)}.$$

$$\underline{r \cdot x}\varphi_W(y) \to_{S_{\mathbb{E}}} \underline{rx\varphi_W(y)}.$$

Pair of type (280):

$$\bar{W}W\varphi(y) \to_{S_{\mathbb{E}}} \varphi(y).$$

Pair of type (281):
Let us choose a representative $s \in R_W$ and an element $y' \in \mathrm{dom}(\delta_w(W))$ such that $xry = sy'$.

$$\underline{xry}W \to_{S_{\mathbb{E}}} = \underline{sy'}W \to_{S_{\mathbb{E}}} s\varphi_W(y').$$

$$\begin{aligned} x \cdot rW\varphi_W(y) &\to_{S_{\mathbb{E}}} \underline{xr}W\varphi_W(y) = \underline{sy'y^{-1}}W\varphi_W(y) \to_{S_{\mathbb{E}}} sW\varphi_W(y'y^{-1})\varphi_W(y) \to_{S_{\mathbb{E}}} sW\underline{\varphi_W(y'y^{-1})\varphi_W(y)} \\ &= s\varphi_W(y'). \end{aligned}$$

Pair of type (282):

$$\begin{aligned} rW_k\varphi_k(x)b_kW_k &\to_{S_{\mathbb{E}}} rW_k\underline{b_kb_k^{-1}\varphi_k(x)b_k}W_k \to_{S_{\mathbb{E}}} rW_kb_kW_k\varphi_k(b_k^{-1}\varphi_k(x)b_k) \to_{S_{\mathbb{E}}} ra_k^{-1}\varphi_k(b_k^{-1}\varphi_k(x)b_k) \\ &\to_{S_{\mathbb{E}}}^* \underline{ra_k^{-1}\varphi_k(b_k^{-1}\varphi_k(x)b_k)} \end{aligned}$$

$$\underline{r \cdot x}a_k^{-1} \to_{S_{\mathbb{E}}} \underline{rxa_k^{-1}}$$

We used here the fact that , for every $x \in \mathrm{dom}(\delta_w(W_k))$, $b_k^{-1}\varphi_k(x)b_k$ belongs to $\mathrm{dom}(\delta_w(W_k))$: this follows from equality (91). We also deduce from equality (91) that

$$\delta_w(W_k) \circ \delta_w(b_k) \circ \delta_w(W_k) \circ \delta_w(a_k) = \mathrm{Id}_{\mathrm{dom}(\delta_w(W_k))}$$

which implies successively

$$\begin{aligned} \varphi_k \circ \delta_w(b_k) \circ \varphi_k &= \delta_w(a_k^{-1}) \\ \varphi_k(b_k^{-1}\varphi_k(x)b_k) &= a_kxa_k^{-1} \\ a_k^{-1}\varphi_k(b_k^{-1}\varphi_k(x)b_k) &= xa_k^{-1} \\ ra_k^{-1}\varphi_k(b_k^{-1}\varphi_k(x)b_k) &= rxa_k^{-1} \end{aligned}$$

showing that the critical pair is resolved.
Pair of type (283):

$$a_k^{-1}b_kW_k \to_{S_{\mathbb{E}}} W_kb_ka_k^{-1}$$

since, by (92), $(a_k^{-1}b_k, b_ka_k^{-1}) \in \delta_w(W_k)$.
Pair of type (284):
let us choose a representative $s \in R_{W_k}$ and an element $x' \in \mathrm{dom}(\delta_w(W_k))$ such that $ra_k = sx'$.

$$r\bar{W}_k\varphi_k^{-1}(x) \to_{S_{\mathbb{E}}} ra_kW_kb_k\varphi_k^{-1}(x) \to_{S_{\mathbb{E}}} \underline{ra_k}W_kb_k\varphi_k^{-1}(x) = \underline{sx'}W_kb_k\varphi_k^{-1}(x) \to_{S_{\mathbb{E}}}^* sW_k\underline{\varphi_k(x')b_k\varphi_k^{-1}(x)}$$

$$\underline{r \cdot x}a_kW_kb_k \rightarrow_{S_{\mathbb{E}}} \underline{rxa_k}W_kb_k = \underline{ra_k(a_k^{-1}xa_k)}W_kb_k = \underline{sx'(a_k^{-1}xa_k)}W_kb_k \rightarrow^*_{S_{\mathbb{E}}} sW_k\underline{\varphi_k(x'a_k^{-1}xa_k)b_k}$$

We used here the fact that , for every $x \in \mathrm{im}(\delta_w(W_k))$, $a_k^{-1}xa_k$ belongs to $\mathrm{dom}(\delta_w(W_k))$: this follows from equality (91). We also deduce from equality (91) that

$$\mathrm{Id}_{\mathrm{dom}(\delta_w(W_k))} = \delta_w(W_k) \circ \delta_w(b_k) \circ \delta_w(W_k) \circ \delta_w(a_k)$$

which implies successively

$$\begin{aligned}
\mathrm{Id}_{\mathrm{dom}(\delta_w(W_k))} &= \varphi_k \circ \delta_w(b_k) \circ \varphi_k \circ \delta_w(a_k)\\
\mathrm{Id}_{\mathrm{im}(\delta_w(W_k))} &= \delta_w(a_k) \circ \varphi_k \circ \delta_w(b_k) \circ \varphi_k\\
\varphi_k^{-1} &= \delta_w(a_k) \circ \varphi_k \circ \delta_w(b_k)\\
\varphi_k^{-1}(x) &= b_k^{-1}\varphi_k(a_k^{-1}xa_k)b_k\\
\varphi_k(x')b_k\varphi_k^{-1}(x) &= \varphi_k(x')\varphi_k(a_k^{-1}xa_k)b_k
\end{aligned}$$

showing that the critical pair is resolved.

An element $s \in \mathcal{W}'^* * A * B$ is said to be a $\mathbb{E}$-*reduced sequence* if its corresponding (A, B)-reduced string does not contain any factor of the form $\mathbb{I}_w(W)cW$ with $W \in \mathcal{W}', c \in \mathrm{dom}(\delta_w(W))$ nor of the form $W_kb_kW_k$ for $1 \leq k \leq p$. We denote by $\mathrm{Red}_{\mathbb{E}}(A*B, \mathcal{W}')$ the subset of $\mathcal{W}'^* * A * B$ consisting of all $\mathbb{E}$-reduced sequences.

Lemma 38. *Let $s \in \mathcal{W}^* * A * B$ with $\gamma_w(s) \neq \emptyset$. Then s is $\mathbb{E}$-reduced.*

Proof. Let $s \in \mathcal{W}^* * A * B$ with $\gamma_w(s) \neq \emptyset$. We already know, by Lemma 35, that s is $\mathbb{U}$-reduced. Every W_k has a H-type, which implies that $\gamma(W_kb_kW_k) = \emptyset$, hence s cannot have any factor of the form $W_kb_kW_k$.

Let us consider the monoid congruence $\equiv'_\Phi$ over $\mathcal{W}'^* * A * B$ generated by the set of pairs

$$\{(cW, Wd) \mid W \in \mathcal{W}', (c, d) \in \delta_w(W)\} \cup \{(W, \Phi(W) \mid W \in \mathcal{W}'\} \tag{285}$$

Note that the congruence $\equiv_\Phi$ over $\mathcal{W}^* * A * B$. which was defined at the end of subsection 3.9, is generated by the set of pairs

$$\{(cW, Wd) \mid W \in \mathcal{W}, (c, d) \in \delta_w(W)\} \cup \{(W, \Phi(W) \mid W \in \mathcal{W}\} \tag{286}$$

The congruence $\equiv$ over $\mathcal{W}^* * A * B$ was defined by (84), the congruence $\equiv'$ over $\mathcal{W}'^* * A * B$ was defined by (263) and the congruences $\equiv_{\mathbb{U}}, \equiv_{\mathbb{E}}$ over $\mathcal{W}'^* * A * B$ are defined just above.

Leaning on the semi-Thue system $S_{\mathbb{E}}$ we are now able to prove the following lemma which is a key-argument for reducing equations over $\mathbb{W}/\Phi$ to equations over the group $\mathbb{E}$.

Lemma 39. *1- Let $s, s' \in \mathcal{W}'^* * A * B$ such that $\gamma(s) \neq \emptyset$ and $\gamma(s') \neq \emptyset$.*
Then $s \equiv_{\mathbb{E}} s'$ if and only if $s \equiv'_\Phi s'$.
2- Let $s, s' \in \mathcal{W}^* * A * B$ such that $\gamma(s) \neq \emptyset$ and $\gamma(s') \neq \emptyset$.
Then $s \equiv_{\mathbb{E}} s'$ if and only if $s \equiv_\Phi s'$.
3- Let $s, s' \in \mathcal{W}'^* * A * B$ such that $\gamma(s) \neq \emptyset$ and $\gamma(s') \neq \emptyset$. Then $s \equiv_{\mathbb{U}} s'$ if and only if $s \equiv' s'$.
4- Let $s, s' \in \mathcal{W}^* * A * B$ such that $\gamma(s) \neq \emptyset$ and $\gamma(s') \neq \emptyset$.
Then $s \equiv_{\mathbb{U}} s'$ if and only if $s \equiv s'$.

Proof. Let us give a proof of point 1.
Let $s, s' \in \mathcal{W}'^* * A * B$ such that $\gamma(s) \neq \emptyset, \gamma(s') \neq \emptyset$ and $s \equiv_{\mathbb{E}} s'$.
Let us denote by z (resp. z') the unique (A, B)-reduced string representing z (resp. z'). In this proof, for every string $w \in \mathcal{G}_{\mathbb{E}}^*$, we denote by $[w]_{AB}$ its image in the free product $\mathcal{W}'^* * A * B$.
Let us consider the reduction of z (resp. z') towards its normal form w.r.t. the system $S_{\mathbb{E}}$:

$$z \rightarrow^*_{S_{\mathbb{E}}} \rho_{S_{\mathbb{E}}}(z), \quad z' \rightarrow^*_{S_{\mathbb{E}}} \rho_{S_{\mathbb{E}}}(z'). \tag{287}$$

By Lemma 37 point 1, $z \xleftrightarrow{*}_{S_{\mathbb{E}}} z'$ and by Lemma 37 point 2, $\rho_{S_{\mathbb{E}}}(z) = \rho_{S_{\mathbb{E}}}(z')$. Let us denote by $\gamma : \mathcal{G}_{\mathbb{E}}^* \to 2^{\mathcal{T}}$ the unique homomorphism such that, for every $g \in \mathcal{G}_{\mathbb{E}}, \gamma(g) = \gamma_w([g]_{AB})$. Note that every rule of type (271–273) or (275) increases (for the inclusion ordering) the value of γ; note also that, if $\gamma(w) \neq \emptyset$, then by Lemma 38, the sequence $[w]_{AB}$ is $\mathbb{E}$-reduced which implies that the only rules of $S_{\mathbb{E}}$ that might have a redex in w are those of type (271–273) or (275). Thus, for every $w, w' \in \mathcal{G}_{\mathbb{E}}^*$

$$(\gamma(w) \neq \emptyset \text{ and } w \rightarrow_{S_{\mathbb{E}}} w') \Rightarrow \gamma(w') \neq \emptyset.$$

By induction, using the above property, every intermediate step $w \rightarrow_{S_{\mathbb{E}}} w$ such that

$$z \rightarrow^*_{S_{\mathbb{E}}} w \rightarrow_{S_{\mathbb{E}}} w' \rightarrow^*_{S_{\mathbb{E}}} \rho_{S_{\mathbb{E}}}(z)$$

(or similarly starting from z') must fulfill $\gamma(w) \neq \emptyset$. Hence, every reduction step in both reductions (287) uses a rule of type (271–273) or (275). Since, every rule (u, v) of these types fulfills $[u]_{AB} \equiv'_{\Phi} [v]_{AB}$, we conclude that:

$$[z]_{AB} \equiv'_{\Phi} [\rho_{S_{\mathbb{E}}}(z)]_{AB} = [\rho_{S_{\mathbb{E}}}(z')]_{AB} \equiv'_{\Phi} [z']_{AB}.$$

Conversely, the congruence $\equiv'_{\Phi}$ is generated by all the defining relations of $\mathbb{W}'$ union the set of rules $\{(W, \Phi(W)) \mid W \in \mathcal{W}'\}$ which all belong to the presentation (270) of $\mathbb{E}$. Hence $\equiv'_{\Phi} \subseteq \equiv_{\mathbb{E}}$.
Point 2:
Let $T_{\mathbb{E}}$ be the set of all rules of type (271)(272)(275)(276) together with those rules of type (273) where $W \in \mathcal{W}$ and those rules of type (274) where $W \in \hat{\mathcal{W}}$. One can check that $T_{\mathbb{E}}$ is complete too (because, in the above proof of Lemma 37, the resolutions given for the critical pairs of $T_{\mathbb{E}}$ use only rules of $T_{\mathbb{E}}$ itself). Moreover, every rule (u, v) of $T_{\mathbb{E}}$ of type (271–273) or (275) fulfills that $[u]_{AB} \equiv_{\Phi} [v]_{AB}$. Hence, by similar arguments, if $s, s' \in \mathcal{W}^* * A * B$ fulfill the hypothesis of point 2 then $s \equiv_{\Phi} s'$.
Point 3:
Point 3 of the lemma follows immediately from Lemma 34 together with Lemma 35. It can also be proved by an adaptation of point 1, where instead of the semi-Thue system $S_{\mathbb{E}}$, we use the semi-Thue system $S_{\mathbb{U}}$ consisting of all rules of type (271)(272)(273) and (274).
Point 4: Point 4 can be proved also by an adaptation of point 1, where instead of the semi-Thue system $S_{\mathbb{E}}$, we use the semi-Thue system $T_{\mathbb{U}}$ consisting of all rules of type (271)(272) together with those rules of type (273) where $W \in \mathcal{W}$ and those rules of type (274) where $W \in \hat{\mathcal{W}}$.
Every rule (u, v) of $T_{\mathbb{U}}$ of type (271)(272)or (273) with $W \in \mathcal{W}$, fulfills that $[u]_{AB} \equiv [v]_{AB}$. Hence, by similar arguments, if $s, s' \in \mathcal{W}^* * A * B$ fulfill the hypothesis of point 4 then $s \equiv s'$.

7.3 Extensions of degree 2

The structure of $\mathbb{E}$ turns out to be based on quadratic extensions of A that are combined together by free product with amalgamation over A. We thus state some basic facts about quadratic extensions of groups. Let K be a group and E be the group presented by

$$E = \langle K, u;\ u^2 = c,\ u^{-1}xu = \psi(x)\,(x \in K)\rangle, \tag{288}$$

where u is a new letter, c is some element from K and $\psi : K \to K$ is a group automorphism fulfilling

$$\psi(c) = c \quad \text{and} \quad \forall x \in K : \psi(\psi(x)) = c^{-1}xc. \tag{289}$$

One can easily check that, under hypothesis (289) on c and ψ, the group E is an extension of K with $[E : K] = 2$. Conversely, every extension E of K with $[E : K] = 2$ must have the form (288) for some $c \in K$ and some group automorphism $\psi : K \to K$ fulfilling (289).

7.4 The structure of $\mathbb{E}$

We show here, by applying successive Tietze transformations, that $\mathbb{E}$ can be obtained from a certain group $\mathbb{K}$ by a finite number of HNN-extensions over some strict subgroups of A. In Section 7.5, we will show that equations with rational constraints in $\mathbb{K}$ are solvable by a reduction to Theorem 1 of Section 2 (treating the case of free products).

Remark 2. Note that $\mathbb{U}$ is, by definition, the fundamental group of a finite graph of groups where the vertex groups are A and B. Thus $\mathbb{U}$ is virtually free. The decomposition of $\mathbb{E}$ that we exhibit in this subsection can be seen as a finite graph of groups whose fundamental group is $\mathbb{E}$, showing that $\mathbb{E}$ is virtually-free, too.

First transformation: Let us choose some symbol from $\mathcal{W}$ such that the corresponding δ_w-value is a total isomorphism from A to B. We call this symbol t in the following, since any generic symbol from $\mathcal{W}$ that represents the stable letter t may serve. Hence $\delta_w(t) = \varphi$.

By applying the Tietze transformation

$$b \longrightarrow t^{-1}\varphi^{-1}(b)t \text{ for all } b \in B \setminus \{1\}$$

to the presentation (266), we obtain a group presentation with the set of generators $A \cup \mathcal{W}'$ and the following set of relations:

$$\begin{gathered}
W^{-1}aW = \delta_w(W)(a) \text{ if } \mathrm{Gi}(W) = \mathrm{Ge}(W) = A, a \in \mathrm{dom}(\delta_w(W)) \\
W^{-1}aW = t^{-1}\varphi^{-1}(\delta_w(W)(a))t \text{ if } \mathrm{Gi}(W) = A, \mathrm{Ge}(W) = B, W \neq t, a \in \mathrm{dom}(\delta_w(W)) \\
W^{-1}t^{-1}\varphi^{-1}(b)tW = t^{-1}\varphi^{-1}(\delta_w(W)(b))t \text{ for } , \mathrm{Gi}(W) = \mathrm{Ge}(W) = B, b \in \mathrm{dom}(\delta_w(W)) \\
a_1a_2 = a_3 \text{ for } a_1, a_2, a_3 \in A \text{ with } a_1a_2 = a_3 \text{ in } A \\
W_k = a_k^{-1}W_k^{-1}b_k^{-1} \text{ for } , 1 \leq k \leq p
\end{gathered}$$

where we know that (267) and (268) hold.

Second transformation: Let us define

$$\mathcal{V} = \{t\} \cup \{W \in \mathcal{W}' \mid \mathrm{Gi}(W) = \mathrm{Ge}(W) = A\} \ \cup \ \{Wt^{-1} \mid \mathrm{Gi}(W) = A, \mathrm{Ge}(W) = B, W \neq t\}$$
$$\cup \ \{tWt^{-1} \mid \mathrm{Gi}(W) = \mathrm{Ge}(W) = B\}.$$

We take the new set of group generators $A \cup \mathcal{V}$ and, for every $1 \leq k \leq p$, we define:

$$\begin{array}{lll} a'_k = a_k, & b'_k = b_k & \text{if } \mathrm{Gi}(W_k) = \mathrm{Ge}(W_k) = A \\ a'_k = \varphi^{-1}(a_k), & b'_k = \varphi^{-1}(b_k) & \text{if } \mathrm{Gi}(W_k) = \mathrm{Ge}(W_k) = B \end{array}$$

Hence, we have

$$a'_k, b'_k \in A \text{ for } 1 \leq k \leq p.$$

Moreover, for every $V \in \mathcal{V} \setminus \{t\}$ we define the partial automorphism $\delta_w(V) : A \to A$ in the natural way. E.g. if $V = tWt^{-1}$, where $W \in \mathcal{W}'$ with $\mathrm{Gi}(W) = \mathrm{Ge}(W) = B$, then we set $\delta_w(V)(a) = \varphi \circ \delta_w(W) \circ \varphi^{-1}$. For $V = t$, since all the relations involving t and $A \cup B$ have now disappeared, $\delta_w(t)$ is defined as the trivial partial isomorphism i.e. $\mathrm{dom}(\delta_w(t)) = \{1\}$.

We obtain the following finite set of relations:

$$V^{-1}aV = \delta_w(V)(a) \text{ for } V \in \mathcal{V}, a \in \mathrm{dom}(\delta_w(V)) \tag{290}$$
$$a_1a_2 = a_3 \text{ for } a_1, a_2, a_3 \in A \text{ with } a_1a_2 = a_3 \text{ in } A \tag{291}$$
$$V_k = (a'_k)^{-1}V_k^{-1}(b'_k)^{-1} \text{ for } 1 \leq k \leq p \tag{292}$$

In order to get (292), it is important to note that $\mathrm{Gi}(W_k) = \mathrm{Ge}(W_k)$ for $1 \leq k \leq p$. I If, for instance, $\mathrm{Gi}(W_k) = \mathrm{Ge}(W_k) = B$, then we have $V_k = tW_kt^{-1}$, i.e., $W_k = t^{-1}V_kt$. Hence, the relation

$$W_k = t^{-1}\varphi^{-1}(a_k^{-1})tW_k^{-1}t^{-1}\varphi^{-1}(b_k^{-1})t$$

becomes

$$t^{-1}V_kt = t^{-1}\varphi^{-1}(a_k^{-1})tt^{-1}V_k^{-1}tt^{-1}\varphi^{-1}(b_k^{-1})t = t^{-1}\varphi^{-1}(a_k^{-1})V_k^{-1}\varphi^{-1}(b_k^{-1})t.$$

which is equivalent to

$$V_k = \varphi^{-1}(a_k^{-1})V_k^{-1}\varphi^{-1}(b_k^{-1}) = (a'_k)^{-1}V_k^{-1}(b'_k)^{-1},$$

i.e., (292). Moreover, by translating the relations (267) and (268) over W_k and W_k^{-1} into relations over V_k and V_k^{-1}, we obtain:

$$V_k^{-1}(a'_k)^{-1}b'_kV_k \equiv_{\mathbb{U}} b'_k(a'_k)^{-1} \text{ for } 1 \leq k \leq p \tag{293}$$
$$(V_kb'_kV_ka'_k)^{-1}x(V_kb'_kV_ka'_k) \equiv_{\mathbb{U}} x \text{ for } 1 \leq k \leq p, x \in \mathrm{dom}(\delta_w(V_k)) \tag{294}$$

Third transformation: Let us define

$$\mathcal{U} = (\mathcal{V} \setminus \{V_k, V_k^{-1} \mid 1 \leq k \leq p\}) \cup \{U_k, U_k^{-1} \mid 1 \leq k \leq p\},$$

where $U_k = V_kb'_k$ and $U_k^{-1} = (b'_k)^{-1}V_k^{-1}$ and take as set of generators $A \cup \mathcal{U}$. Let us define $c_k = (a'_k)^{-1}b'_k \in A$. Moreover, let $\delta_w(U_k) = \delta_w(T_k) \circ \delta_w(b'_k)$. We obtain a set of relations of the form:

$$
\begin{gathered}
U^{-1}aU = \delta_w(U)(a) \text{ for } U \in \mathcal{U}, a \in \mathrm{dom}(\delta_w(U)) \\
a_1a_2 = a_3 \text{ for } a_1, a_2, a_3 \in A \text{ with } a_1a_2 = a_3 \text{ in } A \\
U_k^2 = c_k \text{ for } 1 \leq k \leq p
\end{gathered}
$$

Here, the last equation is obtained as follows: We have $a'_kT_kb'_k = T_k^{-1}$, i.e., $a'_kT_kb'_kT_k = 1$, i.e., $a'_kT_kb'_kT_kb'_k = b'_k$. Hence, $a'_kU_k^2 = b'_k$, i.e., $U_k^2 = (a'_k)^{-1}b'_k = c_k$. Moreover, translating the relations (293) and (294) over V_k into relations over the U_k we obtain:

$$U_k^{-1}c_kU_k \equiv_{\mathbb{U}} c_k \text{ for } 1 \leq k \leq p \tag{295}$$

$$U_k^{-2}xU_k^2 \equiv_{\mathbb{U}} c_k^{-1}xc_k \text{ for } 1 \leq k \leq p, x \in \mathrm{dom}(\delta_w(U_k)) \tag{296}$$

Fourth transformation: Let us set $\varphi_U = \delta_w(U)$ for $U \in \mathcal{U} \setminus \{U_k, U_k^{-1} \mid 1 \leq k \leq p\})$ and $\varphi_k = \delta_w(U_k)$ for $1 \leq k \leq p$; these are all partial automorphisms of A. The last set of relations can be rewritten as:

$$
\begin{gathered}
U^{-1}aU = \varphi_U(a) \text{ for } U \in \mathcal{U} \setminus \{U_k, U_k^{-1} \mid 1 \leq k \leq p\}), a \in \mathrm{dom}(\varphi_U) \\
U_k^{-1}aU_k = \varphi_k(a) \text{ for } 1 \leq k \leq p, a \in \mathrm{dom}(\varphi_k) \\
a_1a_2 = a_3 \text{ for } a_1, a_2, a_3 \in A \text{ with } a_1a_2 = a_3 \text{ in } A \\
U_k^2 = c_k \text{ for } 1 \leq k \leq p
\end{gathered}
$$

Moreover, by (295) and (296), we know that:

$$\varphi_k(c_k) \equiv_{\mathbb{U}} c_k \text{ for } 1 \leq k \leq p \tag{297}$$

$$(\varphi_k \circ \varphi_k)(x) \equiv_{\mathbb{U}} c_k^{-1}xc_k \text{ for } x \in \mathrm{dom}(\varphi_k) \tag{298}$$

Remark 3. But $\mathbb{U}$ is an HNN-extension of A, hence A is embedded in $\mathbb{U}$ thus (297), (298) hold in A too.

Decomposition: Let

$$\mathcal{U}_A = \{U \in \mathcal{U} \setminus \{U_k, U_k^{-1} \mid 1 \leq k \leq p\}, \mathrm{dom}(\varphi_U) = A\} \cup \{U_k \mid 1 \leq k \leq p, \mathrm{dom}(\varphi_k) = A\} \text{ and}$$

$$\mathbb{K} = \langle \mathcal{U}_A \cup A \rangle,$$

i.e. $\mathbb{K}$ is the subgroup of $\mathbb{E}$ generated by $\mathcal{U}_A \cup A$. It can be seen on the last presentation given above for the group $\mathbb{E}$ that it is obtained from $\mathbb{K}$ by a finite number of operations, each of which is either an HNN-extension operation with associated subgroups of cardinality $< |A|$ or a free product with amalgamation over subgroups of cardinality $< |A|$.

7.5 The subgroup $\mathbb{K}$

The structure of $\mathbb{K}$ Up to a permutation of the indices $1, 2, \ldots, p$, we can assume that $\mathrm{dom}(\varphi_k) = A$ for $k \in [1, p_A]$, while $\mathrm{dom}(\varphi_k) \subsetneq A$ for $k \in [p_A + 1, p]$. The following group presentation of $\mathbb{K}$ can be extracted from the above presentation :

$$a_1a_2 = a_3 \text{ for } a_1, a_2, a_3 \in A \text{ with } a_1a_2 = a_3 \text{ in } A \tag{299}$$

$$U^{-1}aU = \varphi_U(a) \text{ for } a \in A, U \in \mathcal{U}_A \setminus \{U_k, U_k^{-1} \mid 1 \leq k \leq p_A\}, \tag{300}$$

$$U_k^{-1}aU_k = \varphi_k(a) \text{ for } a \in A, 1 \leq k \leq p_A \tag{301}$$

$$U_k^2 = c_k \text{ for } 1 \leq k \leq p_A \tag{302}$$

Let us set $\mathcal{U}_{A,2} = \{U_k, U_k^{-1} \mid 1 \leq k \leq p_A\}$ and $\mathcal{U}_{A,\infty} = \mathcal{U}_A \setminus \mathcal{U}_{A,2}$. We define

$$C_2 = \langle u; u^2 = \varepsilon\rangle$$

$$B_k = \langle A, U_k; U_k^{-1}aU_k = \varphi_k(a), U_k^2 = c_k\rangle \text{ for } 1 \leq k \leq p_A$$

By Remark 3 and the paragraph on quadratic extensions, each B_k is a quadratic extension of A while,

$$\mathbb{K} = \langle B_1 *_A \cdots *_A B_{p_A}, \mathcal{U}_{A,\infty}; U^{-1}aU = \varphi_u(a), \text{ for } U \in \mathcal{U}_{A,\infty}, a \in A\rangle$$

i.e. $\mathbb{K}$ is a multiple HNN-extension (over A) of a free product with amalgamation (over A) of p_A quadratic extensions of A.

Equations over $\mathbb{K}$ Let us show that equations with rational constraints are solvable in $\mathbb{K}$. Let us define the groups:

$$H_0 := \langle \mathcal{U}_{A,2}; U_k^2 = \varepsilon \text{ for } 1 \leq k \leq p_A\rangle \approx (C_2)^{*,p_A},$$

$$K_0 := \langle \mathcal{U}_{A,\infty}\rangle$$

i.e. H_0 is a free product of p_A copies of the group C_2 and K_0 is the free group generated by $\mathcal{U}_{A,\infty}$.

Lemma 40. *Equations with rational constrainst in $H_0 * K_0$ are solvable.*

Proof. By the main theorem of [DHG05], equations with rational constraints in K_0 are solvable. Applying Theorem 1 (about free-products), solvability of equations with rational constraints in the group C_2 implies successively that equations with rational constraints in H_0, and then in the free product $H_0 * K_0$ are solvable.

All the remaining of this section is devoted to a reduction of equations with rational constraints in $\mathbb{K}$ to equations with rational constraints in $H_0 * K_0$.

Let us consider the alphabet:

$$\mathcal{G}_{\mathbb{K}} := \mathcal{U}_{A,\infty} \cup \{U_k \mid 1 \leq k \leq p\} \cup (A \setminus \{1\}).$$

Since $U_k^{-1} = U_kc_k^{-1}$ in $\mathbb{K}$, $\mathcal{G}_{\mathbb{K}}$ is a set of monoid-generators of $\mathbb{K}$. Let us define the sets of words over $\mathcal{G}_{\mathbb{K}}^*$:

$$\text{FORB} = \{U^{-1} \cdot U \mid U \in \mathcal{U}_{A,\infty}\} \cup \{U_kU_k \mid 1 \leq k \leq p_A\} \cup (A \setminus \{1\})$$

$$R := \mathcal{G}_{\mathbb{K}}^* \setminus \ (\mathcal{G}_{\mathbb{K}}^* \cdot \text{FORB} \cdot \mathcal{G}_{\mathbb{K}}^*).$$

Let us define a finite semi-Thue system S over the alphabet $(\mathcal{G}_{\mathbb{K}} \setminus A)$ by the set of rules:

$$U^{-1}U \to \varepsilon \quad \text{for } U \in \mathcal{U}_{A,\infty}, \quad U_kU_k \to \varepsilon \quad \text{for } 1 \leq k \leq p_A.$$

Lemma 41.
1- The monoid $(\mathcal{G}_{\mathbb{K}} \setminus A)^*/ \overset{*}{\longleftrightarrow}_S$ *is isomorphic to* $H_0 * K_0$.
2- The semi-Thue system S *is confluent and Noetherian.*
3- The set of words $R \subseteq (\mathcal{G}_{\mathbb{K}} \setminus A)^*$ *is a transvversal for* $\overset{*}{\longleftrightarrow}_S$.

Sketch of proof:
1- Point 1 is straightforward.
2- Since every rule of S decreases (strictly) the length, S is Noetherian. One can check that every critical pair of S is resolved, which, knowing that S is Noetherian, proves that S is confluent.
3- In fact R is exactly the set of irreducible words with respect to the semi-Thue system S. Hence, by point 2, R is a transversal for $\overset{*}{\longleftrightarrow}_S$ i.e.

$$\forall u \in (\mathcal{G}_{\mathbb{K}} \setminus A)^*, \exists! v \in R, u \overset{*}{\longleftrightarrow}_S v.$$

□

Lemma 42. *There exists a unique group homomorphism* $\psi : \mathbb{K} \to H_0 * K_0$ *such that,*

$$\forall g \in \mathcal{G}_{\mathbb{K}} \setminus A : \psi([g]_{\mathbb{K}}) = [g]_{H_0 * K_0} \text{ and } \forall a \in A : \psi([a]_{\mathbb{K}}) = [\varepsilon]_{H_0 * K_0}$$

Proof. Let us first prove existence. Let us consider the monoid homomorphism $\theta : \mathcal{G}_{\mathbb{K}}^* \to H_0 * K_0$ defined by:

$$\forall g \in \mathcal{G}_{\mathbb{K}} \setminus A, \theta(g) = [g]_{H_0 * K_0}, \ \ \forall a \in A, \theta(a) = \varepsilon.$$

For every relation $u = v$ of (299–302), $\theta(u) = \theta(v)$. Hence there exists an homomorphism $\psi : \mathbb{K} \to H_0 * K_0$ such that:

$$\forall w \in \mathcal{G}_{\mathbb{K}}^* : \theta(w) = \psi([w]_{\mathbb{K}})$$

hence such that

$$\forall g \in \mathcal{G}_{\mathbb{K}} : \psi([g]_{\mathbb{K}}) = [g]_{H_0 * K_0}.$$

To show unicity, note that $\mathcal{G}_{\mathbb{K}}$ is a set of monoid generators of $\mathbb{K}$, hence the values of ψ on these generators completely determine ψ. □

Lemma 43. *The homomorphism* ψ *introduced by Lemma 42 is such that:*

(1) $\mathrm{Ker}(\psi) = A$
(2) $[R]_{\mathbb{K}}$ *is a transversal of* $\mathbb{K}$ *for the congruence* $(modA)$

Proof. 0- Let us first prove that every element of $\mathbb{K}$ has the form $[w \cdot a]_{\mathbb{K}}$ for some $w \in R, a \in A$. Let $v \in \mathcal{G}_{\mathbb{K}}^*$. Let us consider the following set of rules, $S_{\mathbb{K}}$, obtained (essentially) by orientating the relations ((299)–(302)):

$$a_1 a_2 \to a_3 \text{ for } a_1, a_2, a_3 \in A \text{ with } a_1 a_2 = a_3 \text{ in } A \quad (303)$$
$$aU \to U\varphi_U(a) \text{ for } a \in A, U \in \mathcal{U}_{A,\infty} \quad (304)$$
$$aU_k \to U_k\varphi_k(a) \text{ for } a \in A, 1 \leq k \leq p_A \quad (305)$$
$$U_k^2 \to c_k \text{ for } 1 \leq k \leq p_A \quad (306)$$

Applying iteratively these rules, until no rule is applicable, one can find a word $v' \in \mathcal{G}_{\mathbb{K}}^*$ and some $a \in A$ such that

$$v \equiv_{\mathbb{K}} v' \cdot a.$$

Since $v' \cdot a$ has no occurrence of a lefthand-side of rule of $S_{\mathbb{K}}$, $v' \in R$.

1- Now, we can show point (1) from the lemma. Let $w \in R, a \in A$ such that $\psi([w \cdot a]_{\mathbb{K}}) = 1$. Hence

$$w \equiv_{H_0 * K_0} \varepsilon. \tag{307}$$

Since R is the set of irreducible words for S , which presents $H_0 * K_0$, we get that $w = \varepsilon$, hence $[w \cdot a]_{\mathbb{K}} \in A$. This establishes that $\mathrm{Ker}(\psi) = A$.

2- For point (2) from the lemma, let $w, w' \in R, a, a' \in A$ such that $[w \cdot a]_{\mathbb{K}} = [w' \cdot a']_{\mathbb{K}}$. Then, $\psi([w \cdot a]_{\mathbb{K}}) = \psi([w' \cdot a']_{\mathbb{K}})$ i.e., since $A \subseteq \mathrm{Ker}(\psi)$, $\psi([w]_{\mathbb{K}}) = \psi([w']_{\mathbb{K}})$ i.e. $[w]_{H_0 * K_0} = [w']_{H_0 * K_0}$, and by point (1)(3) of Lemma 41, $w = w'$ which proves that $[w]_{\mathbb{K}} = [w']_{\mathbb{K}}$. Points (0) and (2) of this proof establish point (2) of the lemma. □

In other words,

$$1 \to A \to \mathbb{K} \xrightarrow{\psi} H_0 * K_0 \to 1$$

is an exact sequence and the restriction $\psi_{|[R]_{\mathbb{K}}} : [R]_{\mathbb{K}} \to H_0 * K_0$ is bijective.

Lemma 44. *For every regular language L of $H_0 * K_0$, $\psi^{-1}(L)$ is a regular language of $\mathbb{K}$.*

Proof. Let L be a regular subset of $H_0 * K_0$. Let M be a regular subset of $(\mathcal{G}_{\mathbb{K}} \setminus A)^*$ such that $L = [M]_{H_0 * K_0}$. Let us use again the finite semi-Thue system S defined above: This system is finite and monadic, which implies that the set of descendants $M' := M \to_S^*$ is regular too. Defining $M'' := M' \cap R$ we get: $L = [M'']_{H_0 * K_0}$. Finally, $\psi^{-1}(L) = [M'' \cdot A]_{\mathbb{K}}$, hence is regular. □

Lemma 45. *Let L be a subset of $\mathbb{K}$. L is a regular subset of $\mathbb{K}$, if and only if, for every $a \in A$, $\psi(L \cap [R \cdot a]_{\mathbb{K}})$ is a regular subset of $H_0 * K_0$.*

Proof. First, suppose that L is a regular subset of $\mathbb{K}$. The subsets R and $\{a\}$ are regular subsets of $\mathcal{G}_{\mathbb{K}}^*$, thus $R \cdot a$ is a regular subset of $\mathcal{G}_{\mathbb{K}}^*$, which implies that its homomorphic image in $\mathbb{K}$, $[R \cdot a]_{\mathbb{K}}$ is a regular subset of $\mathbb{K}$. It is well-known that regular subsets of a virtually-free group are closed under intersection (see, e.g. [Sén96,LS08]), hence $L \cap [R \cdot a]_{\mathbb{K}}$ is regular. Since ψ is a monoid-homomorphism, we get that $\psi(L \cap [R \cdot a]_{\mathbb{K}})$ is a regular subset of $H_0 * K_0$.

Now, suppose that for every $a \in A$, $\psi(L \cap [R \cdot a]_{\mathbb{K}})$ is a regular subset of $H_0 * K_0$. One can check that

$$L = \bigcup_{a \in A} (\psi^{-1}(\psi(L \cap [R \cdot a]_{\mathbb{K}})) \cap [R]_{\mathbb{K}}) \cdot a$$

Using closure by intersection of the set of regular subsets of $\mathbb{K}$ and Lemma 44, each term $\psi^{-1}(\psi(L \cap [R \cdot a]_{\mathbb{K}}))$ is regular. Using again closure by intersection, by product by singletons and by finite union, we get that L is regular. □

We define the map $\rho : \mathbb{K} \to (H_0 * K_0) \times A$ by

$$\rho(x) = ([w]_{H_0 * K_0}, a), \quad \text{where} \quad w \in R, a \in A, x = [w]_{\mathbb{K}} \cdot a.$$

By Lemma 43 ρ is a bijection.

Let us introduce now a right-action $a \mapsto a^w$ and a right-action $a \mapsto a \odot w$ of the free monoid $(\mathcal{G}_{\mathbb{K}} \setminus A)^*$ over the set A in order to express some equalities in $\mathbb{K}$ by some rational constraints in $H_0 * K_0$. For every $a \in A$

$$\begin{aligned} a^U &:= \varphi_U(a) \quad \text{if } U \in \mathcal{U}_{A,\infty} \\ a^{U_k} &:= \varphi_k(a) \quad \text{if } 1 \leq k \leq p_A \\ a \odot U &:= \varphi_U(a) \quad \text{if } U \in \mathcal{U}_{A,\infty} \\ a \odot U_k &:= c_k \cdot \varphi_k(a) \quad \text{if } 1 \leq k \leq p_A \end{aligned}$$

Remark 4. The action $a \mapsto a^w$ does *not* induce an action of the *group* $H_0 * K_0$ over the set A since, for example, $a^{U_k U_k} = c_k^{-1} a c_k$ which might be different from $a^{\varepsilon} = a$, while $U_k U_k \equiv_{H_0 * K_0} \varepsilon$. The same remark applies on the action $\odot$, with the counter-example $a \odot U_k U_k = c_k \varphi_k(c_k) c_k^{-1} a c_k$
For every $w \in (\mathcal{G}_{\mathbb{K}} \setminus A)^*$, $a \mapsto a^w$ is a group-isomorphism. But, for some $w \in (\mathcal{G}_{\mathbb{K}} \setminus A)^*$, $a \mapsto a \odot w$ might *not* be a group-*homomorphism*: for example, $(a_1 \cdot a_2) \odot U_k = c_k \varphi_k(a_1) \varphi_k(a_2)$ while $(a_1 \odot U_k) \cdot (a_2 \odot U_k) = c_k \varphi_k(a_1) c_k \varphi_k(a_2)$, which is a different element of A, as soon as $c_k \neq 1$.

Lemma 46. *Let $w, w' \in R$ such that $w \equiv_{H_0 * K_0} w'$. Then, for every $a \in A$, $a^w = a^{w'}$ and $a \odot w = a \odot w'$.*

This statement is straighforward: R is a transversal for $\xleftrightarrow{*}_S = \equiv_{H_0 * K_0}$, hence $w \equiv_{H_0 * K_0} w'$ implies $w = w'$.
In view of this lemma, we can use the notation $a^{[w]_{H_0 * K_0}}$, $a \odot [w]_{H_0 * K_0}$ for every $a \in A, w \in R$.

Lemma 47. *For every $w \in (\mathcal{G}_{\mathbb{K}} \setminus A)^*$, $a \cdot w \equiv_{\mathbb{K}} w \cdot a^w$*

Let us denote by $\mathbb{J} : (\mathcal{G}_{\mathbb{K}} \setminus A)^* \to (\mathcal{G}_{\mathbb{K}} \setminus A)^*$ the anti-monoid-homomorphism such that: $\mathbb{J}(U) = U^{-1}, \mathbb{J}(U^{-1}) = U$ if $U \in \mathcal{U}_{A,\infty}$, and $\mathbb{J}(U_k) = U_k$ if $1 \leq k \leq p_A$.

Lemma 48. *For every $w \in (\mathcal{G}_{\mathbb{K}} \setminus A)^*$, $\mathbb{J}(w) \cdot a \cdot w \equiv_{\mathbb{K}} a \odot w$.*

In order to describe the product in the group $\mathbb{K}$, via the bijection ρ, as an operation over the group $(H_0 * K_0) \times A$, we introduce rational sets of $(\mathcal{G}_{\mathbb{K}} \setminus A)^*$, that correspond to specifying the beginning (resp. ending) symbol of a word of R.

$$\forall g \in \mathcal{G}_{\mathbb{K}} \setminus A, \;\; B_g := g(\mathcal{G}_{\mathbb{K}} \setminus A)^* \cap R, \;\; E_g := (\mathcal{G}_{\mathbb{K}} \setminus A)^* g \cap R,$$

and

$$B_{\varepsilon} := \{\varepsilon\}, \;\; E_{\varepsilon} := \{\varepsilon\}.$$

The following lemma expresses the product in $\mathbb{K}$ by means of the product in $H_0 * K_0$ and some rational conditions. Let us use the abreviated notation $[z]_0$ for the class $[z]_{H_0 * K_0}$

Lemma 49. *For every $w_1, w_2, w \in R, a_1, a_2, a \in A$,*

$$\rho(\rho^{-1}([w_1]_0, a_1) \cdot \rho^{-1}([w_2]_0, a_2)) = ([w]_0, a)$$

if and only if, there exist $a_0 \in A, g_1, g_2, h_1, h_2 \in (\mathcal{G}_{\mathbb{K}} \setminus A) \cup \{\varepsilon\}, \alpha_1, \beta_1, \beta_2, \alpha_2 \in R$*, such that,*

$$\forall i, j \in \{1,2\},\ \textit{such that } i \leq j, g_i h_j \notin \mathrm{FORB} \tag{308}$$

$$\alpha_1 \in E_{g_1}, \beta_1 \in B_{h_1}, \beta_2 \in E_{g_2}, \alpha_2 \in B_{h_2} \tag{309}$$

$$a_1 \odot [\beta_2]_0 = a_0 \tag{310}$$

$$a_0^{[\alpha_2]_0} \cdot a_2 = a \tag{311}$$

$$[w_1]_0 = [\alpha_1]_0 \cdot [\beta_1]_0 \tag{312}$$

$$[w_2]_0 = [\beta_2]_0 \cdot [\alpha_2]_0 \tag{313}$$

$$[w]_0 = [\alpha_1]_0 \cdot [\alpha_2]_0 \tag{314}$$

$$1 = [\beta_1]_0 \cdot [\beta_2]_0 \tag{315}$$

Sketch of proof:
1- Suppose that $w_1, w_2, w \in R, a_1, a_2, a \in A$ and

$$\rho(\rho^{-1}([w_1]_0, a_1) \cdot \rho^{-1}([w_2]_0, a_2)) = ([w]_0, a)$$

This means that

$$w_1 \cdot a_1 \cdot w_2 \cdot a_2 \equiv_{\mathbb{K}} w \cdot a \tag{316}$$

Let β_2 be the longuest common prefix of w_2 and $\mathbb{J}(w_1)$ (the mirror image of w_1). There exist some words $\alpha_1, \alpha_2 \in (\mathcal{G}_{\mathbb{K}} \setminus A)^*$ such that

$$w_1 = \alpha_1 \cdot \beta_1, \quad w_2 = \beta_2 \cdot \alpha_2$$

Since α_1, α_2 are factors of R, they belong to R. Let us denote by g_1, g_2, h_1, h_2 the elements of $(\mathcal{G}_{\mathbb{K}} \setminus A) \cup \{\varepsilon\}$ such that $\alpha_1 \in E_{g_1}, \beta_1 \in B_{h_1}, \beta_2 \in E_{g_2}, \alpha_2 \in B_{h_2}$. The hypothesis that $\forall i \in \{1,2\}, \alpha_i \cdot \beta_i \in R$ implies that $\forall i, j \in \{1,2\}, g_i h_i \notin$ FORB and, by maximality of β_2, $g_1 h_2 \notin$ FORB.
By Lemma 48,

$$\mathbb{J}(\beta_2) \cdot a_1 \cdot \beta_2 \equiv_{\mathbb{K}} a_1 \odot \beta_2$$

Let us define

$$a_0 := a_1 \odot \beta_2.$$

The initial equality (316) can be now rephrased as:

$$\alpha_1 \cdot a_0 \cdot \alpha_2 \cdot a_2 \equiv_{\mathbb{K}} w \cdot a$$

By Lemma 47,

$$a_0 \cdot \alpha_2 \equiv_{\mathbb{K}} \alpha_2 \cdot a^{[\alpha_2]_0}$$

hence

$$(\alpha_1 \cdot \alpha_2) \cdot (a^{[\alpha_2]_0} a_2) \equiv_{\mathbb{K}} w \cdot a$$

and using the bijection ρ

$$[w]_0 = [\alpha_1 \cdot \alpha_2]_0 \text{ and } a^{[\alpha_2]_0} \cdot a_2 = a.$$

2- Suppose now that there exist $a_0 \in A, g_1, g_2, h_1, h_2 \in (\mathcal{G}_{\mathbb{K}} \setminus A) \cup \{\varepsilon\}, \alpha_1, \beta_1, \beta_2, \alpha_2 \in R$ fulfilling conditions (308)–(315). Translating the right-actions by appropriate products (by means of Lemma 48 and Lemma 47) one can recover the equality (316). □

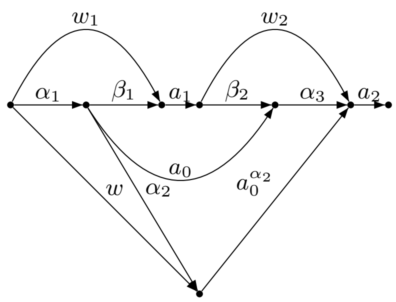


Fig. 16. Proof of Lemma 49

Proposition 6. *The satisfiability problem for systems of equations with rational constraints in $\mathbb{K}$ is decidable.*

Proof. By Lemma 49, the product in $\mathbb{K}$ can be expressed by a finite boolean combination of equalities in $H_0 * K_0$ together with some rational constraints and the additional conditions (310)(311) using the operations $\odot$ and $a \mapsto a^u$.
Condition (310) can be expressed by the rational constraint

$$[\beta_2]_0 \in \{[w]_0 \mid w \in R, a_1 \odot [w]_0 = a_0\}$$

Similarly, condition (311) can be expressed by the rational constraint:

$$a_0^{[\alpha_2]_0} = a_2^{-1} a.$$

Every equation over $\mathbb{K}$ can thus be translated into a finite disjunction of systems of equations with rational constraints over $H_0 * K_0$: the disjunction enumerates the finite list of all the possible values of a_0, a_1, a_2, a as well as the finite list of possible g_i, h_i fulfilling (308). Moreover, by Lemma 45, every rational constraint over $\mathbb{K}$ can be translated as a rational constraint over $H_0 * K_0$.

8 Equations over $\mathbb{E}$

8.1 From $\mathbb{W}$-equations to $\mathbb{E}$-equations

We use here the notion of system of equations with rational constraints over $\mathbb{E}$, as defined in Section 2.6 for any monoid. Recall from Section 3.8 that $\mathbb{W}_{\mathbb{H}}$ is the AB-subalgebra of $\mathbb{W}_t$ generated by those $W \in \mathcal{W}_t$ which have an H-type. Let us consider a system of $\mathbb{W}$-equations of the form described in (169) together with an AB-homomorphism:

$$\mathcal{S}_{\mathbb{W}} = (\{(w_i, w_i') \mid 1 \leq i \leq n\}, \Phi), \quad \sigma_{\mathbb{H}} \in \mathrm{Hom}_{AB}(\mathbb{W}_{\mathbb{H}}, \mathbb{W}_{\mathbb{H}}/\Phi).$$

Assume that $\Phi \in \mathsf{HInv}$. It is thus given by formulas of the form (87–90)with teh additional conditions (91–94); for every $1 \leq k \leq p$, $\gamma_w(W_k)$ must be an H-type and $\mathrm{Gi}(W_k) = \mathrm{Ge}(W_k)$ (see 88).

Let $z_i, z_i' \in \mathcal{W}_t^* * A * B$ be some representatives, modulo $\equiv$, of w_i, w_i'. We can also choose a monoid-homomorphism $\widetilde{\sigma}_{\mathbb{H}} : \mathcal{W}_{\mathbb{H}}^* * A * B \to \mathcal{W}_{\mathbb{H}}^* * A * B$ such that $\pi_{\equiv} \circ \sigma_{\mathbb{H}} = \widetilde{\sigma}_{\mathbb{H}} \circ \pi_{\equiv_\Phi}$. For every $W \in \mathcal{W}_t$ we consider the following rational subsets of $\mathcal{W}_t^* * A * B$:

$$\begin{aligned}
R_{\mathbb{I},W} &= \{z \in \mathcal{W}_t^* * A * B \mid \gamma_w(z) = \gamma_w(W) \ \wedge \ z \in \widehat{\mathcal{W}_t^*} * A * B \Leftrightarrow W \in \widehat{\mathcal{W}_t}\} \\
R_{\mu,W} &= \{z \in \mathcal{W}_t^* * A * B \mid \gamma_w(z) = \gamma_w(W) \ \wedge \ \mu_w(z) = \mu_w(W)\} \\
R_{\delta,W} &= \{z \in \mathcal{W}_t^* * A * B \mid \gamma_w(z) = \gamma_w(W) \ \wedge \ \delta_w(z) = \delta_w(W)\} \\
R_{\mathbb{H},W} &= \begin{cases} \{\widetilde{\sigma}_{\mathbb{H}}(W)\} & \text{if } W \in \mathcal{W}_{\mathbb{H}} \\ \mathcal{W}_t^* * A * B & \text{if } W \in \mathcal{W}_t \setminus \mathcal{W}_{\mathbb{H}} \end{cases}
\end{aligned}$$

We next want to define a system of equations over $\mathbb{E}$ (in the sense of Section 2.6), which comprises, as equations, the pairs (z_i, z_i') together with equations expressing the compatibility with $\mathbb{I}_w$ (i.e. the fact that images of inverses for $\mathbb{I}_w$ must be inverses in the group $\mathbb{E}$). In order to be in accordance with Section 2.6, we have to identify z_i, z_i' with their (A, B)-reduced representatives in $(\mathcal{W} \cup A \cup B)^*$ and to consider elements from $(A \cup B) \setminus \{1\}$ (which may occur in these representatives) as variables, since the z_i and z_i' are only allowed to contain variables. With these conventions, we introduce the following rational constraint $\mathsf{C} : \mathcal{W}_t \cup (A \cup B) \setminus \{1\} \to \mathcal{B}(\mathrm{Rat}(\mathbb{E}))$:

$$\begin{gathered}
\forall W \in \mathcal{W}_t : \mathsf{C}(W) = \pi_{\mathbb{E}}(R_{\mathbb{I},W}) \cap \pi_{\mathbb{E}}(R_{\mu,W}) \cap \pi_{\mathbb{E}}(R_{\delta,W}) \cap \pi_{\mathbb{E}}(R_{H,W}) \\
\forall e \in (A \cup B) \setminus \{1\} : \mathsf{C}(e) = \pi_{\mathbb{E}}(\{e\})
\end{gathered}$$

where $\pi_{\mathbb{E}} : \mathcal{W}_t^* * A * B \to \mathbb{E}$ is the natural projection onto $\mathbb{E}$. Let us now define a system of equations over $\mathbb{E}$ with rational constraints:

$$\mathcal{S}_{\mathbb{E}}(\sigma_{\mathbb{H}}) := (\{(z_i, z_i') \mid 1 \le i \le n\}, \mathsf{C}) \cup \{(W \mathbb{I}_w(W), \varepsilon) \mid W \in \hat{\mathcal{W}}_t \setminus \mathcal{W}_{\mathbb{H}}\}$$

Note that $\mathcal{S}_{\mathbb{E}}(\sigma_{\mathbb{H}})$ does really depend on $\mathcal{S}_{\mathbb{W}}$ and $\sigma_{\mathbb{H}}$, but not on the choice of $\widetilde{\sigma}_{\mathbb{H}}$. Let us denote by $\overline{\pi}_{\mathbb{E}} : \mathbb{W}_t/\Phi \to \mathbb{E}$ the map induced by $\pi_{\mathbb{E}}$ (i.e. $\overline{\pi}_{\mathbb{E}}([w]_{\equiv_\Phi}) := \pi_{\mathbb{E}}(w)$).

Lemma 50. *The map $\Psi : \mathrm{Hom}_{AB}(\mathbb{W}_t, \mathbb{W}_t/\Phi) \to \mathrm{Hom}(\mathcal{W}_t^* * A * B, \mathbb{E})$ defined by:*

$$\Psi(\sigma_{\mathbb{W}}) = \pi_{\equiv} \circ \sigma_{\mathbb{W}} \circ \overline{\pi}_{\mathbb{E}}$$

induces a bijection from the set of solutions of $\mathcal{S}_{\mathbb{W}}$, which extend $\sigma_{\mathbb{H}}$, into the set of solutions of $\mathcal{S}_{\mathbb{E}}(\sigma_{\mathbb{H}})$.

We prove this lemma in the subsequent three subsections, see Figure 17 for the general context and Figure 18 for the details of the proof. Note that the diagram in Figure 17 contains the diagram from Figure 11.

8.2 From $\mathbb{W}$-solutions to $\mathbb{E}$-solutions

Let $\sigma_{\mathbb{W}} : \mathbb{W}_t \to \mathbb{W}_t/\Phi$ be a solution of $\mathcal{S}_{\mathbb{W}}$, extending $\sigma_{\mathbb{H}}$. Let $\sigma_{\mathbb{E}} = \Psi(\sigma_{\mathbb{W}})$. From the definitions of z_i and z_i' we get

$$\sigma_{\mathbb{E}}(z_i) = \overline{\pi}_{\mathbb{E}}(\sigma_{\mathbb{W}}(w_i)) = \overline{\pi}_{\mathbb{E}}(\sigma_{\mathbb{W}}(w_i')) = \sigma_{\mathbb{E}}(z_i'). \tag{317}$$

By hypothesis, $\sigma_{\mathbb{W}}$ is an AB-homomorphism, hence it commutes with the involutions given by the AB-structures i.e.

$$\forall W \in \hat{\mathcal{W}}_t, \sigma_{\mathbb{W}}(\mathbb{I}_w(W)) = \mathbb{I}_w(\sigma_{\mathbb{W}}(W)). \tag{318}$$

$$\begin{tikzcd}
& \mathcal{W}_t^* * A * B \arrow[d, "\pi_{\equiv}"] \arrow[dddrr, "\sigma_{\mathbb{E}}"] & & \\
\mathcal{U}^* \arrow[r, "\sigma_{\boldsymbol{W},\boldsymbol{e}}"] \arrow[dd, "\sigma"'] & \mathbb{W}_t \arrow[dd, "\sigma_t"'] \arrow[ddr, "\sigma_{\mathbb{W}}"] & \mathbb{W}_{\mathbb{H}} \arrow[l] \arrow[d, "\sigma_{\mathbb{H}}"] & \\
& & \mathbb{W}_{\mathbb{H}}/\Phi \arrow[d] & \\
\mathbb{G} & \mathbb{H}_t \arrow[l, "\overline{\pi}_{\mathbb{G}}"] & \mathbb{W}_t/\Phi \arrow[l, "\psi_t"] \arrow[r, "\overline{\pi}_{\mathbb{E}}"] & \mathbb{E}
\end{tikzcd}$$

Fig. 17. Lemma 50: the context

The defining relations of $\mathbb{E}$ ensure that, for every $w \in \mathbb{W}$.

$$\overline{\pi}_{\mathbb{E}}(\sigma_{\mathbb{W}}(w) \cdot \mathbb{I}_w(\sigma_{\mathbb{W}}(W))) = 1$$

hence, the relation (318) implies that

$$\forall W \in \hat{\mathcal{W}}_t, \overline{\pi}_{\mathbb{E}}(\sigma_{\mathbb{W}}(W)\sigma_{\mathbb{W}}(\mathbb{I}_w(W))) = 1. \tag{319}$$

The map $\pi_{\equiv} \circ \sigma_{\mathbb{W}} : \mathcal{W}^* * A * B \to \mathbb{W}_t/\Phi$ is an AB-homomorphism, hence it preserves γ_w, μ_w, and δ_w. It follows that, for every $W \in \mathcal{W}_t$:

$$\sigma_{\mathbb{W}}(\pi_{\equiv}(W)) \in \pi_{\equiv_\Phi}(R_{\mathbb{I},W}) \cap \pi_{\equiv_\Phi}(R_{\mu,W}) \cap \pi_{\equiv_\Phi}(R_{\delta,W}).$$

As $\sigma_{\mathbb{W}}$ extends $\sigma_{\mathbb{H}}$ we get, for every $W \in \mathcal{W}_t$: $\sigma_{\mathbb{W}}(\pi_{\equiv}(W)) \in \pi_{\equiv_\Phi}(R_{\mathbb{H},W})$. Applying $\overline{\pi}_{\mathbb{E}}$ on both sides of this membership relation we obtain

$$\forall W \in \mathcal{W}_t : \sigma_{\mathbb{E}}(W) \in \mathsf{C}(W). \tag{320}$$

Moreover the three maps $\pi_{\equiv}, \sigma_{\mathbb{W}}, \overline{\pi}_{\mathbb{E}}$ are fixing every element of $A \cup B$ thus,

$$\forall e \in (A \cup B) \setminus \{1\} : \sigma_{\mathbb{E}}(e) \in \mathsf{C}(e). \tag{321}$$

By (317), (319), (320) and (321), $\sigma_{\mathbb{E}}$ is a solution of $\mathcal{S}_{\mathbb{E}}(\sigma_{\mathbb{H}})$.

8.3 From $\mathbb{E}$-solutions to $\mathbb{W}$-solutions

Let $\sigma_{\mathbb{E}}$ be a solution of $\mathcal{S}_{\mathbb{E}}(\sigma_{\mathbb{H}})$. Since, for every $W \in \mathcal{W}_t$, $\sigma_{\mathbb{E}}(W) \in \mathsf{C}(W) \subseteq \pi_{\mathbb{E}}(R_{\mu,W})$, there is a choice map $\widetilde{\sigma}_{\mathbb{E}} : \mathcal{W}_t \to \mathcal{W}_t^* * A * B$ fulfilling

$$\forall e \in A \cup B : \widetilde{\sigma}_{\mathbb{E}}(e) = e. \tag{322}$$

$$\forall W \in \mathcal{W}_t : \widetilde{\sigma}_{\mathbb{E}}(W) \in R_{\mu,W}, \text{ and } \pi_{\mathbb{E}}(\widetilde{\sigma}_{\mathbb{E}}(W)) = \sigma_{\mathbb{E}}(W). \tag{323}$$

From $\widetilde{\sigma}_{\mathbb{E}}(W) \in R_{\mu,W}$ we get

$$\gamma_w(\widetilde{\sigma}_{\mathbb{E}}(W)) = \gamma_w(W) \text{ and } \mu_w(\widetilde{\sigma}_{\mathbb{E}}(W)) = \mu_w(W). \tag{324}$$

$$
\begin{tikzcd}
\mathcal{W}_t^* * A * B \arrow[r, "\widetilde{\sigma}_{\mathbb{E}}"] \arrow[dd, "\pi_{\equiv}"] \arrow[rrd, bend left, "\sigma_{\mathbb{E}}"] & \mathcal{W}_t^* * A * B \arrow[dr, "\pi_{\mathbb{E}}"] \arrow[dd, "\pi_{\equiv_\Phi}"] & \\
 & & \mathbb{E} \\
\mathbb{W}_t \arrow[r, "\sigma_{\mathbb{W}}"] & \mathbb{W}_t/\Phi \arrow[ur, "\overline{\pi}_{\mathbb{E}}"] & \\
\mathbb{W}_{\mathbb{H}} \arrow[u] \arrow[r, "\sigma_{\mathbb{H}}"] & \mathbb{W}_{\mathbb{H}}/\Phi \arrow[u] & \\
\mathcal{W}_{\mathbb{H}}^* * A * B \arrow[u, "\pi_{\equiv}"] \arrow[r, "\widetilde{\sigma}_{\mathbb{H}}"] & \mathcal{W}_{\mathbb{H}}^* * A * B \arrow[u, "\pi_{\equiv_\Phi}"] &
\end{tikzcd}
$$

Fig. 18. Lemma 50: the proof

Let us denote by $\widetilde{\sigma}_{\mathbb{E}} : \mathcal{W}_t^* * A * B \to \mathcal{W}_t^* * A * B$ the unique monoid homomorphism fixing every element of $A \cup B$ and extending the above choice map. Since $\sigma_{\mathbb{E}}(W) \in \mathsf{C}(W)$, there exist $z_{\mathbb{I},W} \in R_{\mathbb{I},W}$, $z_{\delta,W} \in R_{\delta,W}$, and $z_{\mathbb{H},W} \in R_{H,W}$ fulfilling

$$\sigma_{\mathbb{E}}(W) = \pi_{\mathbb{E}}(\widetilde{\sigma}_{\mathbb{E}}(W)) = \pi_{\mathbb{E}}(z_{\mathbb{I},W}) = \pi_{\mathbb{E}}(z_{\delta,W}) = \pi_{\mathbb{E}}(z_{\mathbb{H},W}).$$

All these $z_{\mathbb{I},W}$, $z_{\delta,W}$, $z_{\mathbb{H},W}$ have a non-empty image under γ_w and are equivalent with $\widetilde{\sigma}_{\mathbb{E}}(W)$ modulo $\equiv_{\mathbb{E}}$. By Lemma 39 point 2, we have

$$\widetilde{\sigma}_{\mathbb{E}}(W) \equiv_\Phi z_{\mathbb{I},W} \equiv_\Phi z_{\delta,W} \equiv_\Phi z_{\mathbb{H},W}.$$

The equivalence $\equiv_\Phi$ is compatible with the AB-algebra $\mathcal{W}^* * A * B$ (see the end of section 3.9). Hence, for every $W \in \mathcal{W}_t$, we have

$$\widetilde{\sigma}_{\mathbb{E}}(W) \in \widehat{\mathcal{W}_t} * A * B \iff z_{\mathbb{I},W} \in \widehat{\mathcal{W}_t} * A * B \iff W \in \widehat{\mathcal{W}_t}$$

$$\delta_w(\widetilde{\sigma}_{\mathbb{E}}(W)) = \delta_w(z_{\delta,W}) = \delta_w(W) \tag{325}$$

$$\pi_{\equiv_\Phi}(\widetilde{\sigma}_{\mathbb{E}}(W)) = \pi_{\equiv_\Phi}(\widetilde{\sigma}_{\mathbb{H}}(W)) \text{ if } W \in \mathcal{W}_{\mathbb{H}}. \tag{326}$$

By (325), the map $\widetilde{\sigma}_{\mathbb{E}}$ induces a monoid-homomorphism $\sigma_{\mathbb{W}} : \mathbb{W}_t \to \mathbb{W}_t/\Phi$ fulfilling:

$$\pi_{\equiv} \circ \sigma_{\mathbb{W}} = \widetilde{\sigma}_{\mathbb{E}} \circ \pi_{\equiv_\Phi}. \tag{327}$$

Since $\sigma_{\mathbb{E}}$ solves every equation $(W\mathbb{I}_w(W), \varepsilon)$ for $W \in \hat{\mathcal{W}}_t \setminus \mathcal{W}_{\mathbb{H}}$, we are sure that:

$$\forall W \in \hat{\mathcal{W}}_t \setminus \mathcal{W}_{\mathbb{H}}, \sigma_{\mathbb{E}}(\mathbb{I}_w(W)) = \sigma_{\mathbb{E}}(W)^{-1}$$

thus, since $\pi_{\mathbb{E}}$ transforms the involution $\mathbb{I}_w$ into the inversion in $\mathbb{E}$,

$$\forall W \in \hat{\mathcal{W}}_t \setminus \mathcal{W}_{\mathbb{H}}, \sigma_{\mathbb{E}}(\mathbb{I}_w(W)) = \pi_{\mathbb{E}}(\mathbb{I}_w(\widetilde{\sigma}_{\mathbb{E}}(W)))$$

hence, by Lemma 39, point 2,

$$\forall W \in \hat{\mathcal{W}}_t \setminus \mathcal{W}_{\mathbb{H}}, \sigma_{\mathbb{W}}(\mathbb{I}_w(W)) = \mathbb{I}_w(\sigma_{\mathbb{W}}(W)). \tag{328}$$

By (326) and (327),

$$\sigma_{\mathbb{W}} \text{ extends } \sigma_{\mathbb{H}}. \tag{329}$$

This in particular ensures that

$$\forall W \in \hat{\mathcal{W}}_t \cap \mathcal{W}_{\mathbb{H}}, \sigma_{\mathbb{W}}(\mathbb{I}_w(W)) = \mathbb{I}_w(\sigma_{\mathbb{W}}(W)). \tag{330}$$

By the properties (324,325,328,330), $\sigma_{\mathbb{W}}$ preserves γ, μ, δ and the standard involutions $\mathbb{I}_w$, thus

$$\sigma_{\mathbb{W}} \in \mathrm{Hom}_{AB}(\mathbb{W}_t, \mathbb{W}_t/\Phi) \tag{331}$$

By hypothesis $\widetilde{\sigma}_{\mathbb{E}} \circ \pi_{\mathbb{E}}$ is a solution of the equations of $\mathcal{S}_{\mathbb{E}}(\sigma_{\mathbb{H}})$ that are written over $\mathcal{W}_t^* * A * B$. Hence, we have

$$\pi_{\mathbb{E}}(\widetilde{\sigma}_{\mathbb{E}}(z_i)) = \pi_{\mathbb{E}}(\widetilde{\sigma}_{\mathbb{E}}(z_i')),$$

i.e. $\widetilde{\sigma}_{\mathbb{E}}(z_i) \equiv_{\mathbb{E}} \widetilde{\sigma}_{\mathbb{E}}(z_i')$. We know that $\gamma_w(z_i) = \gamma_w(z_i') \neq \emptyset$ and that $\widetilde{\sigma}_{\mathbb{E}}$ preserves γ_w. Using Lemma 39, point 2, we conclude that $\widetilde{\sigma}_{\mathbb{E}}(z_i) \equiv_{\Phi} \widetilde{\sigma}_{\mathbb{E}}(z_i')$, hence that

$$\sigma_{\mathbb{W}}(w_i) = \sigma_{\mathbb{W}}(w_i'). \tag{332}$$

By (332) $\sigma_{\mathbb{W}}$ is a solution of $\mathcal{S}_{\mathbb{W}}$ which, by (329), extends $\sigma_{\mathbb{H}}$. Using (327), $\pi_{\equiv} \circ \sigma_{\mathbb{W}} \circ \overline{\pi}_{\mathbb{E}} = \widetilde{\sigma}_{\mathbb{E}} \circ \pi_{\equiv_{\Phi}} \circ \overline{\pi}_{\mathbb{E}} = \widetilde{\sigma}_{\mathbb{E}} \circ \pi_{\mathbb{E}} = \sigma_{\mathbb{E}}$. Thus,

$$\sigma_{\mathbb{E}} = \Psi(\sigma_{\mathbb{W}}).$$

8.4 Ψ is bijective:

The previous considerations established that Ψ is surjective. Let us check that it is injective. Suppose that $\sigma_{\mathbb{W}}, \sigma_{\mathbb{W}}' \in \mathrm{Hom}_{AB}(\mathbb{W}_t, \mathbb{W}_t/\Phi)$ fulfill $\Psi(\sigma_{\mathbb{W}}) = \Psi(\sigma_{\mathbb{W}}')$. This means that:

$$\pi_{\equiv} \circ \sigma_{\mathbb{W}} \circ \overline{\pi}_{\mathbb{E}} = \pi_{\equiv} \circ \sigma_{\mathbb{W}}' \circ \overline{\pi}_{\mathbb{E}}$$

As $\pi_{\equiv}$ is surjective we get

$$\sigma_{\mathbb{W}} \circ \overline{\pi}_{\mathbb{E}} = \sigma_{\mathbb{W}}' \circ \overline{\pi}_{\mathbb{E}}.$$

By Lemma 39, point 2, $\overline{\pi}_{\mathbb{E}}$ is injective over $\{z \in \mathbb{W}_t \mid \gamma_w(z) \neq \emptyset\}$. Hence, $\sigma_{\mathbb{W}}(g) = \sigma_{\mathbb{W}}'(g)$ for every $g \in \mathcal{W}_t \cup A \cup B$. This implies $\sigma_{\mathbb{W}} = \sigma_{\mathbb{W}}'$. By the above three paragraphs, Lemma 50 is proved. □

9 Transfer of solvability

We prove a general transfer theorem for systems of equations with rational constraints. We first treat the case of groups since it is technically simpler.

9.1 Transfer for groups

Proposition 7. *Let $\mathbb{H}$ be a group and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups A,B. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the pair of problems (Q_1, Q_2), where*

- *Q_1 is the SAT-problem for systems of equations with rational constraints in a group defined by a presentation in* $\mathrm{PHNN}(A)$

- *Q_2 is the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$.*

(the set of presentations $\mathrm{PHNN}(A)$ is from Definition 9).

Proof. Let us consider a system $\mathcal{S}_0$ of equations with rational constraints in $\mathbb{G}$. By Proposition 5 it reduces to a disjunction $\mathcal{D}S = \bigvee_{j\in J} \mathcal{S}_j$ in closed quadratic normal form where $\mathcal{S}_j = (\mathcal{S}, \mu_{\mathcal{A}}, \mu_{\mathcal{U},j})$ (i.e. the different systems are differing by their map $\mu_{\mathcal{U},j}$ only). This disjunction $\mathcal{D}S$ can thus be solved by the following
Algorithm Scheme:
1- `Compute` the subalphabet $\mathcal{W}_t$.
2- $\mathbb{W}_t := \langle \mathcal{W}_t \rangle$.
3- `Enumerate` the involutions $\Phi \in \mathsf{HInv}$, the indices $j \in J$ and the admissible vectors $\boldsymbol{W}, \boldsymbol{e}$.
4- `For` every value of $\Phi, j, \boldsymbol{W}, \boldsymbol{e}$:
5- `Find` a solution $\sigma_{\mathbb{W}} \in \mathrm{Hom}_{AB}(\mathbb{W}_t, \mathbb{W}_t/\Phi)$ to the system $\mathcal{S}_t(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$ by the procedure:
5.1- `Enumerate` all the $\sigma_{\mathbb{H}} \in \mathrm{Hom}_{AB}(\mathbb{W}_{\mathbb{H}}, \mathbb{W}_{\mathbb{H}}/\Phi)$,
5.2- `Find` a solution $\sigma_{\mathbb{E}}$ for the system $\mathcal{S}_{\mathbb{E}}(\sigma_{\mathbb{H}})$,
5.3- $\sigma_{\mathbb{W}} := \Psi^{-1}(\sigma_{\mathbb{E}})$
6- `Find` a $\psi_{H,t} \in \mathrm{Hom}_{AB}(\mathbb{W}_{\mathbb{H}}/\Phi, \mathbb{H}_t)$ such that $\sigma_{\mathbb{W}} \circ \psi_{H,t}$ solves $\mathcal{S}_{\mathbb{H}}(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$.
7- `Endfor`
8- `If` some pair $\sigma_{\mathbb{W}}, \psi_{H,t}$ is found `then` $\mathcal{D}S$ is satisfiable
9- `else` $\mathcal{D}S$ is unsatisfiable.
Let us precise how one can achieve every line of this scheme.
Line 1 consists, for every letter $W \in \widehat{\mathcal{W}}$, in deciding whether there exists an AB-homomorphism from the induced AB-subalgebra $\langle W, \mathbb{I}_w(W)\rangle$ into $\mathbb{H}_t$. By Lemma 12, this amounts to test that conditions 1-5 of this lemma are fulfilled. Let us define constraints on the variables W, that express these conditions. For every $W \in \mathcal{W}_{\mathbb{H}}$ we define the following sets:

$$\begin{aligned}
\mathsf{C}_\gamma(W) &= \{h \in \mathbb{H} \mid \gamma_w(W) \subseteq \gamma_t(h)\}\\
\mathsf{C}_\mu(W) &= \{h \in \mathbb{H} \mid \mu_w(W) = \mu_t(h)\}\\
\mathsf{C}_\delta(W) &= \{h \in \mathbb{H} \mid \forall \boldsymbol{\theta} \in \gamma_w(W) : \delta_w(\boldsymbol{\theta}, W) = \delta_t(\boldsymbol{\theta}, h)\}\\
\mathsf{C}(W) &= \mathsf{C}_\gamma(W) \cap \mathsf{C}_\mu(W) \cap \mathsf{C}_\delta(W)
\end{aligned} \tag{333}$$

Recall the notions of equational and positively definable subsets of a monoid were defined in Section 2.6. We observe that C_γ takes values in $\mathcal{B}(\{\{1\}, A, B, \mathbb{H}\})$ and C_μ takes values in $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$. The map C_δ takes values which are intersections of subsets of the form

$$\mathrm{CD}_{\mathbb{H}}(c, d) = \{h \in \mathbb{H} \mid ch =_{\mathbb{H}} hd\} \tag{334}$$

for $c, d \in A \cup B$ and of subsets of the form $\mathbb{H} \setminus \mathrm{CD}_{\mathbb{H}}(c, d)$. It is clear that $\mathrm{CD}_{\mathbb{H}}(c, d) \in \mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$ and, by Lemma 8, $\mathbb{H} \setminus \mathrm{CD}_{\mathbb{H}}(c, d)$ belongs to $\mathrm{Def}_{\exists+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$. Therefore, each of the three subsets $\mathsf{C}_\gamma(W)$, $\mathsf{C}_\mu(W)$, and $\mathsf{C}_\delta(W)$ belongs to $\mathrm{Def}_{\exists+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$, so that $\mathsf{C}(W)$ also belongs to $\mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$. Deciding whether $\mathsf{C}(W) = \emptyset$ is thus an instance of Q_2.

Let $W \in \mathcal{W} \setminus \mathcal{W}_{\mathbb{H}}$. We set now

$$\begin{aligned}
\mathsf{C}_\gamma(W) &= \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \gamma_w(W) \subseteq \gamma_t(s)\},\\
\mathsf{C}_\mu(W) &= \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \mu_1(W) = \mu_{\mathcal{A}}(s)\},\\
\mathsf{C}_\delta(W) &= \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \forall \boldsymbol{\theta} \in \gamma_w(W) : \delta_w(\boldsymbol{\theta}, W) = \delta_t(\boldsymbol{\theta}, s)\},\\
\mathsf{C}(W) &= \mathsf{C}_\gamma(W) \cap \mathsf{C}_\mu(W) \cap \mathsf{C}_\delta(W).
\end{aligned}$$

Claim 15 (γ-constraint). For every $W \in \mathcal{W} \setminus \mathcal{W}_{\mathbb{H}}$, $\mathsf{C}_\gamma(W)$ is recognized by some t-automaton with labels in $\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$.

For every type $\boldsymbol{\theta} \in \mathcal{T}_6$, the subset $\mathrm{CG}(\boldsymbol{\theta}) = \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \boldsymbol{\theta} \in \gamma(s)\}$ is recognized by a variant of the t-automaton $\mathcal{R}_6$ where we choose as initial vertex the initial vertex-type of $\boldsymbol{\theta}$, as terminal vertex the ending vertex-type of $\boldsymbol{\theta}$, and on which we apply a direct product with a finite automaton that memorizes whether at least one letter t or t^{-1} has been encountered. The subset $\mathsf{C}_\gamma(W)$ can be described as an intersection of the subsets $\mathrm{CG}(\boldsymbol{\theta})$:

$$\mathsf{C}_\gamma(W) = \bigcap_{\boldsymbol{\theta} \in \gamma_w(W)} \mathrm{CG}(\boldsymbol{\theta}).$$

Each set $\mathrm{CG}(\boldsymbol{\theta})$ is recognized by some fta with labelling set $\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$. By Lemma 3 of [LS08] (property of closure under intersection) the set $\mathsf{C}_\gamma(W)$ is thus recognized by an fta with labelling set $\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$.

Claim 16 (μ-constraint). For every $W \in \mathcal{W} \setminus \mathcal{W}_{\mathbb{H}}$, $\mathsf{C}_\mu(W)$ is recognized by some t-automaton with labels in $\mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$.

For every states $q, r \in \mathsf{Q}$ the set

$$\mathrm{CM}(q, r) = \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid (q, r) \in \mu_{\mathcal{A}}(s)\}$$

is recognized by the variant of the strict automaton $\mathcal{A}$ obtained just by taking q as initial state and r as terminal state. Similarly, the set

$$\overline{\mathrm{CM}}(q, r) = \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid (q, r) \in (\mu_{\mathcal{A}})^{-1}(\mathbb{I}_t(s))\}$$

is recognized by the variant of the fta $\mathcal{A}$ obtained by replacing the labels by their inverses, the vertex-type of every state by its image by $\mathbb{I}_{\mathcal{R}}$, taking q as initial state and r as terminal state. The subset $\mathsf{C}_\mu(W)$ can be described as:

$$\begin{aligned}
\mathsf{C}_\mu(W) = (&\bigcap_{(q,r) \in \mu_1(W)} \mathrm{CM}(q, r)) \cap (\bigcap_{(q,r) \notin \mu_1(W)} (\mathbb{H} * \{t, \bar{t}\}^* \setminus \mathrm{CM}(q, r)))\\
(&\bigcap_{(q,r) \in (\mu_1(\mathbb{I}_w(W)))^{-1}} \overline{CM}(q, r)) \cap (\bigcap_{(q,r) \notin (\mu_1(\mathbb{I}_w(W)))^{-1}} (\mathbb{H} * \{t, \bar{t}\}^* \setminus \overline{\mathrm{CM}}(q, r)))
\end{aligned}$$

By Proposition 1 and Proposition 2 of [LS08], each $\mathrm{CM}(q, r)$ is recognized by a partitioned, saturated, deterministic and complete fta with labelling set $\mathcal{B}(\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H}))))$. Hence every subset $\mathbb{H} * \{t, \bar{t}\}^* \setminus \mathrm{CM}(q, r)$ or $\mathbb{H} * \{t, \bar{t}\}^* \setminus \overline{CM}(q, r)$ is also recognized by a partitioned, saturated, deterministic and complete fta with labelling set $\mathcal{B}(\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H}))))$. By Lemma 8 this labeling set is included in $\mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$ and by Lemma 3 of [LS08], we conclude that $\mathsf{C}_\mu(W)$ is recognized by some t-automaton with labels in $\mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$.

Claim 17 (δ-constraint). For every $W \in \mathcal{W} \setminus \mathcal{W}_H$, $\mathsf{C}_\delta(W)$ is recognized by some t-automaton with labels in $\mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$.

The subset $\mathsf{C}_\delta(W)$ is a boolean combination of subsets of the form:

$$CD(c,d) = \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid cs \sim sd\} \tag{335}$$

for some $c, d \in A \cup B$. Each set $CD(c,d)$ is recognized by some t-automaton with $A \cup B$ as set of states, with c as initial state, d as terminal state, and with deterministic transitions such that, the state reached after reading the sequence $s \in \mathbb{H} * \{t, \bar{t}\}^*$, either is undefined (if $s^{-1}cs \notin A \cup B$) or is equal to $s^{-1}cs$ (if $s^{-1}cs \in A \cup B$). Such a fta has labels in $\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$. The claim follows by the same arguments as for the previous claim.
Finally, due to the Claims 15, 16, 17 and to Lemma 3 of [LS08], the subset $\mathsf{C}(W)$ is recognized by some t-automaton $\mathcal{D}$ with labels in $\mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$. The emptiness problem for $\mathsf{C}(W) = \mathrm{L}(\mathcal{D})$ reduces to the emptiness problem for elements of $\mathrm{Def}_{\exists^+}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$ (given by a system of equations with rational constraints), which are themselves instances of Q_2.
Line 3 can be achieved by an exhaustive enumeration of all possible partitions of $\mathcal{W}_\mathbb{H}$ and a test for equations (91), (92), (**??**), in the finite groups A, B.
Line 5.1 consists in enumerating all the maps $\mathcal{W}_\mathbb{H} \to \mathcal{W}_\mathbb{H}$, test whether it preserves γ, μ, δ and output only those which pass the test.
Line 5.2 is an instance of $Q1$.
Let us treat Line 6.
The involution Φ (choosen in Line 2) is given by the formulas (90) of Section 3.9. By Lemma 12, Line 6 amounts to find some tuple $(\psi_{H,t}(W))_{W \in \mathcal{W}_\mathbb{H}}$ such that conditions 1-5 of Lemma 12 are fulfilled and $(\psi_{H,t}(W))_{W \in \mathcal{W}_\mathbb{H}}$ is a solution of

$$\sigma_\mathbb{W}(\mathcal{S}_\mathbb{H}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})) \cup \{(W_k, a_k^{-1}\bar{W}_k b_k^{-1}) \mid 1 \le k \le p\} \cup \{(\bar{W}_k, a_k W_k b_k) \mid 1 \le k \le p\}.$$

Conditions 1-5 can be expressed by the constraints $\mathsf{C}(W)$ defined in (333). Hence Line 6 reduces to an instance of Q_2.

Proposition 8. *Let $\mathbb{H}$ be a group with decidable SAT-problem for systems of equations with rational constraints and let $\mathbb{G}$ be an HNN-extension of $\mathbb{H}$ with finite associated subgroups. Then the satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is decidable.*

Proof. We prove this proposition by induction over the size of the finite associated subgroups A,B used in the HNN-extension leading from $\mathbb{H}$ to $\mathbb{G}$.

Induction base: $|A| = 1$.
In this case $\mathbb{G}$ is the free product of $\mathbb{H}$ by $\mathbb{Z}$. It is known that the additive group $\mathbb{Z}$ has a decidable SAT-problem for systems of equations with rational constraints (see, for example, [ES69]). By Theorem 1, a free product of two groups having decidable SAT-problem for systems of equations with rational constraints has also a decidable SAT-problem for systems of equations with rational constraints. Hence $\mathbb{G}$ has decidable SAT-problem for systems of equations with rational constraints.

Induction step: $|A| > 1$.
By Proposition 7, the SAT-problem for systems of equations with rational constraints reduces to two other decision problems Q_1, Q_2. By hypothesis, here, problem Q_2 is decidable. Q_1 is the SAT-problem for systems of equations with rational constraints in the group $\mathbb{E}$, which

has a presentation in $\mathrm{PHNN}(A)$. By Section 7.4, $\mathbb{E}$ is obtained from the group $\mathbb{K}$ by a finite number of HNN-extension (or amalgamated product) operations, with associated subgroups (or amalgamated subgroups) of cardinality $< |A|$. By Proposition 6, $\mathbb{K}$ has a decidable satisfiability problem for equations with rational constraints. By induction hypothesis, each of the HNN-extension operations preserves this decidability property and, by the combination of this induction hypothesis with Lemma 9 and Theorem 1, each of the free products with amalgamation also preserves this decidability property. Hence $\mathbb{E}$ has a decidable satisfiability problem for equations with rational constraints. Thus problem Q_1 is decidable too, so that $\mathbb{G}$ has a decidable satisfiability problem for equations with rational constraints. □

Theorem 2. *If $\mathbb{G}$ is a virtually-free group of finite type, then the satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is decidable.*

Proof. Recall that the virtually-free groups of finite type are exactly the groups obtained from finite groups by a finite number of operations which are either HNN-extensions with finite associated subgroups or free products with amalgamation with finite amalgamated subgroups [DD90]. In a finite group, systems of equations with rational constraints are algorithmically solvable. By Proposition 8, every HNN-extension with finite associated subgroups preserves this decidability property. The combination of Proposition 8 with Lemma 9 and Theorem 1, shows that the free product with amalgamation over finite subgroups also preserves this decidability property. Hence every virtually-free group of finite type has the asserted decidability property. □

Note that another proof of this result has been exposed in [DG07].

Theorem 3. *Let $\mathbb{H}$ be a group and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$.*

Proof. By Proposition 7 the satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to a pair of problems Q_1, Q_2 (precisely defined in the proposition). But Q_1 is the SAT-problem for systems of equations with rational constraints in $\mathbb{E}$, where $\mathbb{E}$ is a virtually-free group of finite type. Since, by Theorem 2, problem Q_1 is decidable, the satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the single problem Q_2, i.e., to the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$. □

9.2 Transfer for cancellative monoids

We will prove the extension of Proposition 7 to cancellative monoids:

Proposition 9. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups A, B. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the pair of problems (Q_1, Q_2), where*
1- Q_1 is the SAT-problem for systems of equations with rational constraints in a group defined by a presentation in $\mathrm{PHNN}(A)$
2- Q_2 is the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$

A new difficulty arises here for the computation of $\mathcal{W}_t$: it requires to determine, for a given symbol $W \in \mathcal{W}$ whether there exists a t-sequence s such that: $s \in \mathbb{H}$ (if $\gamma_w(W)$ is a H-type), s is not invertible (if $\gamma_w(W)$ specifies that W does not belong to the domain of $\mathbb{I}_{\mathbb{W}}$), and $cs \neq sd$ for some $c, d \in A \cup B$ (if the value of $\delta(W)$ imposes this). The non-invertibility condition is expressed via an universally quantified formula $(\forall h', h \cdot h' \neq 1 \vee h' \cdot h \neq 1)$, and the non-commutation condition is a disequation. Since no hypothesis ensures that the satisfiability of such formulas over $\mathbb{H}$ is decidable, we give up the hope to compute $\mathcal{W}_t$. The same kind of difficulty arises in the computation of an AB-homomorphism $\psi_t : \mathbb{W}' \to \mathbb{H}_t$ (whatever variant $\mathbb{W}'$ of the AB-algebra $\mathbb{W}_t$ we use). We have to compute, for every $W \in \mathcal{W}'$, an image $\psi_t(W)$ having the same behaviour as W w.r.t. $\mathbb{I}, \gamma, \mu, \delta$, while Lemma 8 is no more ensured when $\mathbb{H}$ is not assumed to be a group.

We overcome the above difficulties by introducing a notion of *weak* AB-homomorphism. Let us call weak-AB-homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ any map $\psi : \mathbb{M}_1 \to \mathbb{M}_2$ fulfilling the seven properties (336-342) below:

$$\psi : (\mathbb{M}_1, \cdot, 1_{\mathbb{M}_1}) \to (\mathbb{M}_2, \cdot, 1_{\mathbb{M}_2}) \text{ is a monoid homomorphism} \tag{336}$$

$$\forall a \in A, \forall b \in B, \psi(\iota_{A,1}(a)) = \iota_{A,2}(a), \quad \psi(\iota_{B,1}(b)) = \iota_{B,2}(b) \tag{337}$$

$$\forall m \in \mathbb{M}_1 - \gamma_1^{-1}(\{\emptyset\}), \quad m \in \mathrm{dom}(\mathbb{I}_1) \Rightarrow \psi(m) \in \mathrm{dom}(\mathbb{I}_2) \tag{338}$$

$$\forall m \in \widehat{\mathbb{M}}_1, \quad \mathbb{I}_2(\psi(m)) = \psi(\mathbb{I}_1(m)) \tag{339}$$

$$\forall m \in \mathbb{M}_1, \gamma_2(\psi(m)) \supseteq \gamma_1(m) \tag{340}$$

$$\forall m \in \mathbb{M}_1, \forall \boldsymbol{\theta} \in \gamma_1(m), \mu_2(\boldsymbol{\theta}, \psi(m)) \supseteq \mu_1(\boldsymbol{\theta}, m), \tag{341}$$

$$\forall m \in \mathbb{M}_1, \forall \boldsymbol{\theta} \in \gamma_1(m), \delta_2(\boldsymbol{\theta}, \psi(m)) \supseteq \delta_1(\boldsymbol{\theta}, m). \tag{342}$$

We have thus replaced the axioms (Hom3),(Hom6), and (Hom7) by the weaker axioms (338), (341), and (342). It remains true that the above list of axioms can be checked on the generators only, i.e. the following analogue of Lemma 12 is true.

Lemma 51. *Let $\mathcal{M}_i = \langle \mathbb{M}_i, \cdot, 1_{\mathbb{M}_i}, \iota_{A,i}, \iota_{B,i}, \mathbb{I}_i, \gamma_i, \mu_i, \delta_i \rangle$ (for $i \in \{1,2\}$) be two AB-algebra. and assume that $\mathcal{M}_1$ satisfies the following:*

(A) The monoid $\mathbb{M}_1$ is generated by the set Γ and $A \cup B \subseteq \Gamma$
(B) $\gamma_1(g) \neq \emptyset$ for all $g \in \Gamma$
(C) For every $m \in \mathbb{M}_1$ there exists a decomposition $m = g_1 \cdots g_n$ with $g_1, \ldots, g_n \in \Gamma$ such that: $\gamma_1(m) = \gamma_1(g_1) \cdots \gamma_1(g_n)$ and $[m \in \mathrm{dom}(\mathbb{I}_1) \iff g_1, \ldots, g_n \in \mathrm{dom}(\mathbb{I}_1)]$.[4]

Let $\psi : \mathbb{M}_1 \to \mathbb{M}_2$ be some monoid homomorphism. This map ψ is a weak AB-homomorphism if and only if:

- $\iota_A \circ \psi = \iota_{A,2}, \ \ \iota_B \circ \psi = \iota_{B,2}$ *and for every $g \in \Gamma, \boldsymbol{\theta} \in \gamma_1(g)$:*
- $g \in \mathrm{dom}(\mathbb{I}_1) \Rightarrow \psi(g) \in \mathrm{dom}(\mathbb{I}_2)$
- $\mathbb{I}_2(\psi(g)) = \psi(\mathbb{I}_1(g))$
- $\gamma_2(\psi(g)) \supseteq \gamma_1(g)$
- $\mu_2(\boldsymbol{\theta}, \psi(g)) \supseteq \mu_1(\boldsymbol{\theta}, g)$
- $\delta_2(\boldsymbol{\theta}, \psi(g)) \supseteq \delta_1(\boldsymbol{\theta}, g)$.

[4] If $n = 0$, i.e., $m = 1$, then $\gamma_1(g_1) \cdots \gamma_1(g_n)$ is the neutral element $\{(\theta, 0, \theta) \mid \theta \in \mathcal{T}_6\}$ of the monoid $2^{\mathcal{T}}$.

Given two AB-algebras $\mathcal{M}_1, \mathcal{M}_2$, we denote by $\mathrm{WHom}_{AB}(\mathcal{M}_1, \mathcal{M}_2)$ the set of all weak AB-homomorphisms from $\mathcal{M}_1$ to $\mathcal{M}_2$.
Let us notice that the subalphabet $\mathcal{W}_t$ was equal to $\{W \in \mathcal{W} \mid \mathrm{Hom}_{AB}(\langle W\rangle, \mathbb{H}_t) \neq \emptyset\}$. We are thus naturally led to define

$$\mathcal{W}'_t := \{W \in \mathcal{W} \mid \mathrm{WHom}_{AB}(\langle W\rangle, \mathbb{H}_t) \neq \emptyset\}.$$

(here $\langle W\rangle$ is the smallest sub-AB-algebra containing $\{W\}$). We then define

$$\mathbb{W}'_t := \mathcal{W}'_t * A * B / \equiv$$

As well we note $\mathcal{W}'_{\mathbb{H}} := \{W \in \mathcal{W}'_t \mid \gamma_w(W) \text{ is a H-type}\}$ and $\mathbb{W}'_{\mathbb{H}} := \mathcal{W}'^*_{\mathbb{H}} * A * B/\equiv$. We can then express the solutions of the original equation over $\mathbb{G}$ via some $AB-$ (resp. weak $AB-$) homomorphisms.

Lemma 52. *Let $\mathcal{D}S = \bigvee_{j\in J} \mathcal{S}_j$ be a finite disjunction of systems of equations over $\mathbb{G}$, with rational constraint, where $\mathcal{S}_j = ((E_i)_{1\le i\le n}, \mu_{\mathcal{A}}, \mu_{\mathcal{U},j})$. Let us suppose that $\mathcal{D}S$ is in closed quadratic normal form. A monoid homomorphism*

$$\sigma : \mathcal{U}^* \to \mathbb{G}$$

is a solution of $\mathcal{D}S$ if and only if, there exists an index $j \in J$, an admissible choice $(\boldsymbol{W}, \boldsymbol{e})$ of variables of $\mathcal{W}$ (resp. elements of $A \cup B$), an involution $\Phi \in \mathsf{HInv}$, an AB-homomorphism $\sigma_{\mathbb{W}} : \mathbb{W}'_t \to \mathbb{W}'_t/\Phi$ and a weak AB-homomorphism $\psi_t : \mathbb{W}'_t/\Phi \to \mathbb{H}_t$ such that:

(S1) $\sigma_{\mathbb{W}}$ *is a solution of* $\mathcal{S}_t(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$
(S2) $\sigma_{\mathbb{W}} \circ \psi_t$ *is a solution of* $\mathcal{S}_{\mathbb{H}}(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$
(S3) $\sigma = \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \sigma_{\mathbb{W}} \circ \psi_t \circ \overline{\pi}_{\mathbb{G}}$

(Discarding the adjective "weak" in the above statement and replacing $\mathbb{W}'_t$ by $\mathbb{W}_t$ results in a straightforward synthesis of Lemma 30 with Lemma 31; the introduction of "weak" and $\mathbb{W}'_t$ make the statement usable for an *effective* characterisation of satisfiable systems of equations). These maps were summarized on Figure 11 and form the left-lower part of Figure 19.

Proof. First suppose that a monoid homomorphism

$$\sigma : \mathcal{U}^* \to \mathbb{G}$$

is a solution of $\mathcal{D}S$. By Lemma 30, there exists an admissible choice $(\boldsymbol{W}, \boldsymbol{e})$ of variables of $\mathcal{W}_t$ and an AB-homomorphism $\tilde{\sigma}_t : \mathbb{W}_t \to \mathbb{H}_t$ such that:

$$\tilde{\sigma}_t \text{ solves } \mathcal{S}_t(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e}) \wedge \mathcal{S}_{\mathbb{H}}(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e}) \tag{343}$$

$$\sigma = \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \tilde{\sigma}_t \circ \overline{\pi}_{\mathbb{G}} \tag{344}$$

By Lemma 31 $\tilde{\sigma}_t$ can be decomposed, over $\mathrm{Im}(\sigma_{\boldsymbol{W},\boldsymbol{e}})$, as

$$\tilde{\sigma}_t = \tilde{\sigma}_{\mathbb{W}} \circ \tilde{\psi}_t \tag{345}$$

where

$$\tilde{\sigma}_{\mathbb{W}} \in \mathrm{Hom}_{AB}(\mathbb{W}_t, \mathbb{W}_t/\Phi), \quad \tilde{\psi}_t \in \mathrm{Hom}_{AB}(\mathbb{W}_t/\Phi, \mathbb{H}_t) \tag{346}$$

$$\Phi \in \mathsf{HInv}. \tag{347}$$

such that

$$\tilde{\sigma}_{\mathbb{W}} \text{ solves } \mathcal{S}_t(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e}) \tag{348}$$

Let us extend the map $\tilde{\sigma}_{\mathbb{W}}, \tilde{\psi}_t$ respectively into maps $\sigma_{\mathbb{W}} \in \mathrm{Hom}_{AB}(\mathbb{W}'_t, \mathbb{W}'_t/\Phi)$, $\psi_t \in \mathrm{WHom}_{AB}(\mathbb{W}'_t/\Phi, \mathbb{H}_t)$ by setting:

$$\forall W \in \mathcal{W}'_t \setminus \mathcal{W}_t, \sigma_{\mathbb{W}}(W) := [W]_{\equiv_\Phi}, \ \ \forall W \in \mathcal{W}'_t \setminus \mathcal{W}_t, \psi_t([W]_{\equiv_\Phi}) := h_W(W),$$

where $h_W \in \mathrm{WHom}_{AB}(\langle W \rangle, \mathbb{H}_t)$ and, if $\Phi(W) = W'$, h_W is chosen to be equal to $h_{\mathbb{I}_w(W)}$. Lemma 51 ensures that such extensions exist. We then define

$$\sigma_t = \sigma_{\mathbb{W}} \circ \psi_t \tag{349}$$

By (348), the restriction of $\sigma_{\mathbb{W}}$ to $\mathbb{W}_t$ solves the system $\mathcal{S}_t(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$. Thus any extension of $\sigma_{\mathbb{W}}$ also solves this system. Hence (S1) is true.
By the same argument, (343) and (345) imply that (S2) is true.
By (344) and (345):

$$\sigma = \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \tilde{\sigma}_{\mathbb{W}} \circ \tilde{\psi}_t \circ \overline{\pi}_{\mathbb{G}}$$

But the image of $\sigma_{\boldsymbol{W},\boldsymbol{e}}$ is included in $\mathcal{W}_t$, and the image of $\tilde{\sigma}_{\mathbb{W}}$ is included in $\mathcal{W}_t/\Phi$, thus the above identity is still true after replacing the maps $\tilde{\sigma}_{\mathbb{W}}$ and $\tilde{\psi}_t$ respectively by $\sigma_{\mathbb{W}}$ and ψ_t, i.e. (S3) is true.

Conversely, suppose that $(\boldsymbol{W}, \boldsymbol{e}), \Phi \in \mathsf{HInv}, \sigma_{\mathbb{W}} \in \mathrm{Hom}_{AB}(\mathbb{W}'_t, \mathbb{W}'_t/\Phi), \psi_t \in \mathrm{WHom}_{AB}(\mathbb{W}'_t/\Phi, \mathbb{H}_t)$ are fulfiling (S1),(S2), and (S3) and let us define $\sigma_t := \sigma_{\mathbb{W}} \circ \psi_t, \sigma := \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \sigma_t \circ \overline{\pi}_{\mathbb{G}}$. By (S1) and (S2), the map σ_t is a weak-AB-homomorphism solving $\mathcal{S}_t(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e}) \wedge \mathcal{S}_{\mathbb{H}}(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$. The arguments given in Section 5.2, adapted to a *weak*-AB-homomorphism ψ_t from the AB-algebra $\mathbb{W}'_t/\Phi$ into $\mathbb{H}_t$ show that σ is an *over*-solution of $\mathcal{D}S$, which is assumed in closed quadratic normal form, thus σ is a solution of $\mathcal{D}S$. □

By Proposition 5, every system $\mathcal{S}_0$ of equations with rational constraints can be reduced to a disjunction $\mathcal{D}S = \bigvee_{j \in J} \mathcal{S}_j$ in closed quadratic normal form where $\mathcal{S}_j = (\mathcal{S}, \mu_{\mathcal{A}}, \mu_{\mathcal{U},j})$ (i.e. the different systems are differing by their map $\mu_{\mathcal{U},j}$ only). This disjunction $\mathcal{D}S$ can thus be solved by the following

Algorithm Scheme:
1- `Compute` the subalphabet $\mathcal{W}'_t$.
2- $\mathbb{W}'_t := \langle \mathcal{W}'_t \rangle$.
3- `Enumerate` the involutions $\Phi \in \mathsf{HInv}$, the indices $j \in J$ and the admissible vectors $\boldsymbol{W}, \boldsymbol{e}$.
4- `For` every value of $\Phi, j, \boldsymbol{W}, \boldsymbol{e}$:
5- `Find` a solution $\sigma_{\mathbb{W}} \in \mathrm{Hom}_{AB}(\mathbb{W}'_t, \mathbb{W}'_t/\Phi)$ to the system $\mathcal{S}_t(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$ by the procedure:
5.1- `Enumerate` all the $\sigma_{\mathbb{H}} \in \mathrm{Hom}_{AB}(\mathbb{W}'_{\mathbb{H}}, \mathbb{W}'_{\mathbb{H}}/\Phi)$,
5.2- `Find` a solution $\sigma_{\mathbb{E}}$ for the system $\mathcal{S}_{\mathbb{E}}(\sigma_{\mathbb{H}})$,
5.3- $\sigma_{\mathbb{W}} := \Psi^{-1}(\sigma_{\mathbb{E}})$
6- `Find` a $\psi_{H,t} \in \mathrm{WHom}_{AB}(\mathbb{W}'_{\mathbb{H}}/\Phi, \mathbb{H}_t)$ such that $\sigma_{\mathbb{W}} \circ \psi_{H,t}$ solves $\mathcal{S}_{\mathbb{H}}(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$.
7- `Endfor`
8- `If` some pair $\sigma_{\mathbb{W}}, \psi_{H,t}$ is found `then` $\mathcal{D}S$ is satisfiable
9- `else` $\mathcal{D}S$ is unsatisfiable.

(We summarize on Figure 19 the different maps to be found; dashed arrows correspond to weak AB-homomorphisms). Let us precise how one can achieve every line of this scheme.

$$
\begin{tikzcd}
 & \mathcal{W}'^{*} * A * B \arrow[d, "\pi_{\equiv}"] \arrow[dddrr, "\sigma_{\mathbb{E}}"] & & \\
\mathcal{U}^{*} \arrow[r, "\sigma_{\boldsymbol{W},\boldsymbol{e}}"] \arrow[dd, "\sigma"] & \mathbb{W}' \arrow[dd, dashed, "\sigma_t"] \arrow[ddr, "\sigma_{\mathbb{W}}"] & \mathbb{W}'_{\mathbb{H}} \arrow[l] \arrow[d, "\sigma_{\mathbb{H}}"] & \\
 & & \mathbb{W}'_{\mathbb{H}}/\Phi \arrow[d] & \\
\mathbb{G} & \mathbb{H}_t \arrow[l, "\bar{\pi}_{\mathbb{G}}"] & \mathbb{W}'/\Phi \arrow[l, dashed, "\psi_t"] \arrow[r, "\overline{\pi}_{\mathbb{E}}"] & \mathbb{E}
\end{tikzcd}
$$

Fig. 19. the algorithm: monoid case

Line 1 consists, for every letter $W \in \mathcal{W}$, in deciding whether there exists an AB-homomorphism from the induced AB-subalgebra $\langle W \rangle$ into $\mathbb{H}_t$. By Lemma 12, this amounts to test that conditions 1-5 of this lemma are fulfiled. Let us define constraints on the variables W, that express these conditions. For every $W \in \mathcal{W}'_{\mathbb{H}}$:

$$
\begin{aligned}
\mathsf{C}_I(W) &= \mathbb{H} \quad \text{if } W \in \mathcal{W}' - \widehat{\mathcal{W}}, && (350)\\
\mathsf{C}_I(W) &= \mathsf{I}(\mathbb{H}) \quad \text{if } W \in \widehat{\mathcal{W}}, && (351)\\
\mathsf{C}_\gamma(W) &= \{h \in \mathbb{H} \mid \gamma_w(W) \subseteq \gamma_t(h)\}, && (352)\\
\mathsf{C}_\mu(W) &= \{h \in \mathbb{H} \mid \mu_w(W) \subseteq \mu_t(h)\}, && (353)\\
\mathsf{C}_\delta(W) &= \{h \in \mathbb{H} \mid \forall \boldsymbol{\theta} \in \gamma_w(W), \delta_w(\boldsymbol{\theta}, W) \subseteq \delta_t(\boldsymbol{\theta}, h)\}. && (354)\\
\mathsf{C}(W) &= \mathsf{C}_I(W) \cap \mathsf{C}_\gamma(W) \cap \mathsf{C}_\mu(W) \cap \mathsf{C}_\delta(W). && (355)
\end{aligned}
$$

For every $W \in \mathcal{W}' \setminus \mathcal{W}'_{\mathbb{H}}$:

$$
\begin{aligned}
\mathsf{C}_I(W) &= \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid W \in \mathrm{dom}(\mathbb{I}_w) \Rightarrow s \in \mathrm{dom}(\mathbb{I}_t)\}, && (356)\\
\mathsf{C}_\gamma(W) &= \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \gamma_w(W) \subseteq \gamma_t(s)\}, && (357)\\
\mathsf{C}_\mu(W) &= \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \mu_w(W) \subseteq \mu_t(s)\}, && (358)\\
\mathsf{C}_\delta(W) &= \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \forall \boldsymbol{\theta} \in \gamma_w(W), \delta_w(\boldsymbol{\theta}, W) \subseteq \delta_t(\boldsymbol{\theta}, s)\}, && (359)\\
\mathsf{C}(W) &= \mathsf{C}_I(W) \cap \mathsf{C}_\gamma(W) \cap \mathsf{C}_\mu(W) \cap \mathsf{C}_\delta(W). && (360)
\end{aligned}
$$

The values of the $\mathsf{C}_*(W)$ have been defined in such a way that, now, every subset $\mathsf{C}(W)$ with $W \in \mathcal{W}'_{\mathbb{H}}$ belongs to $\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$ while every subset $\mathsf{C}(W)$ with $W \in \mathcal{W}' - \mathcal{W}'_H$ is recognized by some t-automaton with labels in $\mathrm{EQ}(\mathbb{H}, \mathcal{B}(\mathrm{Rat}(\mathbb{H})))$. Line 1 thus reduces to instances of Q_2.
Line 5.2 is an instance of Q_1.
Let us show that Line 6 can be achieved by a Turing-reduction to Q_2. By Lemma 51 again, Line 6 amounts to find some tuple $(\psi_{H,t}(W))_{W \in \mathcal{W}'_{\mathbb{H}}}$ solving the system of equations

$$
\sigma_{\mathbb{W}}(\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})) \cup \{(W_k, a_k^{-1} \bar{W}_k b_k^{-1}) \mid 1 \leq k \leq p\} \cup \{(\bar{W}_k, a_k W_k b_k) \mid 1 \leq k \leq p\},
$$

together with the constraint C defined for Line 1. Line 6 is thus an instance of Q_2.
We have thus proved Proposition 9.

Theorem 4. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups A,B. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$.*

Proof. Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. By Proposition 9 the satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to a certain pair of problems (Q_1, Q_2), where Q_1 is the SAT-problem for systems of equations with rational constraints in a group $\mathbb{E}$ having a presentation in PHNN(A). But, due to the structure of $\mathbb{E}$ (see §7.4) and to Theorem 3, this problem Q_1 is decidable. Thus the pair (Q_1, Q_2) is Turing-reducible to the single problem Q_2, i.e. the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$. □

The two following sections are stating some variants of the main Theorem 4. The variations consist in considering:

- other kinds of constraints: *positive* rational constraints, positive *subgroup* constraints, *constant* constraints,
- not only equations but also *disequations*
- the operation of free product with *amalgamation* instead of the operation of HNN-extension.

It turns out that all these variations lead to an analogue of Theorem 4.

10 Equations with positive rational constraints over $\mathbb{G}$

10.1 Positive rational constraints

We consider here constraints over the variables consisting of a rational subset of the monoid (while in Theorem 4 the constraints were consisting in *boolean combinations* of rational subsets). More formally, we call system of equations with *positive* rational constraints in the monoid $\mathbb{M}$, any system of equations $\mathcal{S}$ with constraints in $\mathcal{C} = \mathrm{Rat}(\mathbb{M})$.

Theorem 5. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with positive rational constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with positive rational constraints in $\mathbb{H}$.*

Sketch of proof: It suffices to adapt the reduction given in the proof of Theorem 4:

- The *strict normal* partitioned fta $\mathcal{A}$ with labeling set $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$ is replaced by, merely, a *normal* partitioned fta $\mathcal{A}$ with labeling set $\mathrm{Rat}(\mathbb{H})$. The existence of such a t-automaton recognizing the given positive constraints is ensured by point (1) Proposition 2.
- The AB-algebra $\mathbb{H}_t$ is replaced by the AB-algebra $\mathbb{H}_{t+}$.
- For every $W \in \mathcal{W}'_{\mathbb{H}}$: $\mathsf{C}_\gamma(W) = \{h \in \mathbb{H} \mid \gamma_w(W) \subseteq \gamma_+(h)\}$.
- For $W \in \mathcal{W}'_t \setminus \mathcal{W}'_{\mathbb{H}}$: $\mathsf{C}_\gamma(W) = \{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \gamma_w(W) \subseteq \gamma_+(s)\}$.
- For $W \in \mathcal{W}'_t \setminus \mathcal{W}'_{\mathbb{H}}$, each of the sets $\mathsf{C}_I(W), \mathsf{C}_\gamma(W), \mathsf{C}_\mu(W), \mathsf{C}_\delta(W)$ is recognized by a t-automaton with labels in $\mathrm{EQ}(\mathbb{H}, \mathrm{Rat}(\mathbb{H}))$.

□

10.2 Positive subgroup constraints

We consider here the case where $\mathbb{G}$ is the HNN-extension of a group $\mathbb{H}$ by an isomorphism $\varphi : A \to A$ from a finite subgroup A of $\mathbb{H}$ into itself. Let $C_1, \ldots, C_n$ be finitely generated subgroups of $\mathbb{H}$ containing A. Let us consider the sets

$$\mathcal{C}_{\mathbb{H}} = \{C_1, \ldots, C_n\} \text{ and } \mathcal{C}_{\mathbb{G}} = \{C_1, \ldots, C_n, \langle C_1, t\rangle, \ldots, \langle C_n, t\rangle\}.$$

The set of constraints $\mathcal{C}_{\mathbb{G}}$ turns out to be useful for the decidability of the positive first-order theory of $\mathbb{G}$ (see [LS05]).

Theorem 6. *Let $\mathbb{H}$ be a group and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups $A = B$. The satisfiability problem for systems of equations over $\mathbb{G}$ with constraints in $\mathcal{C}_{\mathbb{G}}$ is Turing-reducible to the SAT-problem for systems of equations over $\mathbb{H}$ with constraints in $\mathcal{C}_{\mathbb{H}}$.*

Sketch of proof: It suffices to adapt the reduction given in the proof of Theorem 4:

- The *strict normal* partitioned fta $\mathcal{A}$ with labeling set $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$ is replaced by a *normal* partitioned fta $\mathcal{A}$ with labeling set $\{\{a\} \mid a \in A\} \cup \mathcal{C}_{\mathbb{H}}$.
- The AB-algebra $\mathbb{H}_t$ is replaced by the AB-algebra $\mathbb{H}_{t+}$.

Let us describe more precisely the necessary adaptations. Given a finitely generated subgroup C_i, we consider the fta $\mathcal{A}_i$ that posesses exactly two states $(1, H)$ (its initial state) and $(1, 1)$ (its terminal state) and one transition $((1, H), C_i, (1, 1))$. This fta $\mathcal{A}_i$ is trivially partitioned, $\approx$-compatible $\sim$-saturated, unitary and it recognizes the subgroup C_i. Let us consider now the fta $\mathcal{B}_i$ obtained from the fta $\mathcal{G}_6$ by replacing each occurence of the label $\mathbb{H}$ by the label C_i. This fta $\mathcal{B}_i$ is partitioned, $\approx$-compatible (because $A \subseteq C_i$), $\sim$-saturated (because each of the four states $(A, T), (B, H), (B, T), (A, H)$ has a loop labeled by A) unitary (because, for every vertex-type θ, there exists only one state mapped to the type θ by τ) and recognizes the subgroup $\langle C_i, t\rangle$.

Let $\mathcal{A}$ be the direct product of all these partitioned fta $\mathcal{A}_i$ and $\mathcal{B}_i$. The fta $\mathcal{A}$ is a partitioned fta which is also $\approx$-compatible and $\sim$-saturated (by Lemma 5 of [LS08]) and unitary (because unitarity is also preserved by direct product). Given a finite set of constraints $\mu : \mathcal{U}^* \to \mathcal{C}_{\mathbb{G}}$, there exists $I_U \subseteq Q_{\mathcal{A}}, T_U \subseteq Q_{\mathcal{A}}$ such that

$$\mathsf{C}(U) = \{g \in \mathbb{G} \mid (I_U \times T_U) \cap \mu_{\mathcal{A},\mathbb{G}}(g) \neq \emptyset\}.$$

Any system of equations $\mathcal{S}_0$ over $\mathbb{G}$ with constraints in $\mathcal{C}_{\mathbb{G}}$ can thus be reduced to a disjunction $\bigvee_{j\in J} \mathcal{S}_{1,j}$ of systems of equations with rational constraints over $\mathbb{G}$ with set of variables $\mathcal{U}_1 \supseteq \mathcal{U}_0$, of the form (118), fulfiling points (a), (c), (d), and (e) of Proposition 5 and point (b), with the modification that the fta $\mathcal{A}$ is a normal partitioned fta (which might be non-strict) over the labelling set $\{\{a\} \mid a \in A\} \cup \mathcal{C}_{\mathbb{H}}$. Line 6 of the Algorithm Scheme amounts to the following:

- Find a solution in $\mathbb{H}$ to the system

  $$\sigma_{\mathbb{W}}(\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})) \cup \{(W_k, a_k^{-1}\bar{W}_k b_k^{-1}) \mid 1 \leq k \leq p\} \cup \{(\bar{W}_k, a_k W_k b_k) \mid 1 \leq k \leq p\}$$

  with the additional constraints $\mathsf{C}(W)$ for $W \in \mathcal{W}'_{\mathbb{H}}$; this is an instance of the SAT-problem for systems of equations with constraints in $\mathcal{C}_{\mathbb{H}}$.

– Find an element in the set $\mathsf{C}(W)$ for $W \in \mathcal{W}'_t - \mathcal{W}'_{\mathbb{H}}$; the set $\mathsf{C}(W)$ is recognized by a fta where each label is the set of solutions of a system of equations in $\mathbb{H}$ with constraints in $\mathcal{C}_{\mathbb{H}}$. This problem thus reduces to finitely many instances of the SAT-problem for systems of equations with constraints in $\mathcal{C}_{\mathbb{H}}$.

□

10.3 Constants

The sets of constraints corresponding to so-called equations *with constants* are

$$\mathcal{C}_{\mathbb{H}} = \{\{h\} \mid h \in \mathbb{H}\} \cup \{\mathbb{H}\} \text{ and } \mathcal{C}_{\mathbb{G}} = \{\{g\} \mid g \in \mathbb{G}\} \cup \{\mathbb{G}\}.$$

Theorem 7. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with constants in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations with constants in $\mathbb{H}$.*

Sketch of proof: It suffices to adapt the reduction given in the proof of Theorem 4:

– The *strict normal* partitioned fta $\mathcal{A}$ with labeling set $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$ is replaced by, merely, a *normal* partitioned fta $\mathcal{A}$ with labeling set $\mathcal{C}_{\mathbb{H}}$.
– The AB-algebra $\mathbb{H}_t$ is replaced by the AB-algebra $\mathbb{H}_{t+}$.

□

11 Equations and disequations with rational constraints over $\mathbb{G}$

We recall that the notion of systems of equations and disequations with rational constraints over a monoid has been defined in Section 2.6.

11.1 Rational constraints

Let us show how to reduce a system of equations and disequations (with rational constraints) over $\mathbb{G}$ to a systems of equations (with rational constraints) over $\mathbb{H}_t$ together with a system of equations/disequations (with rational constraints) over $\mathbb{H}$.
Let us start with a system of equations/disequations with rational constraints, over $\mathbb{G}$, which is in normal form (see Proposition 5):

$$((E_i)_{1 \le i \le n}, (\overline{E}_i)_{n+1 \le i \le 2n}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}}) \tag{361}$$

The equations E_i have the form

$$E_i : \quad (U_{i,1}, U_{i,2}U_{i,3}) \text{ for all } 1 \le i \le n$$

while the disequations $\overline{E}_i$ have the form

$$\overline{E}_i : \quad (U_{i,1}, U_{i,2}) \text{ for all } n+1 \le i \le 2n$$

where, for every $i \in [1,n]$, the symbols $U_{i,1}, U_{i,2}, U_{i,3}, U_{n+i,1}, U_{n+i,2}$ belong to the alphabet of unknowns $\mathcal{U}$. Let us consider the alphabet $\mathcal{V}_0 = [1,2n] \times [1,3] \times [1,7] \times [0,N_0]$ and the alphabet $\mathcal{W}$ constructed from this $\mathcal{V}_0$ in Section 3.6. We choose the integer N_0 in such a way that

$$\mathrm{Card}(\{W \in \mathcal{W} \mid \exists i,j,k, p_1(W) = (i,j,k,0)\}) < \frac{1}{2}\mathrm{Card}(\mathcal{V}_0) \tag{362}$$

(one can still take, as in the case of equations, $N_0 := 2\mathrm{Card}(\{-1,0,1\} \times \mathcal{T}_{HT} \times \mathsf{B}^2(\mathsf{Q}) \times \mathrm{PGI}\{A,B\}) + 1$ in order to achieve this inequality).

We consider all the vectors $(W_{i,j,k})$ where $1 \le i \le 2n, 1 \le j \le 3, 1 \le k \le 9$ of elements of $\mathcal{W} \cup \{1\}$ and all triple $(e_{i,1,2}, e_{i,2,3}, e_{i,3,1}) \in (A \cup B)^3$ such that: the vectors

$$(W_{i,j,k})_{1 \le i \le n, 1 \le j \le 3, 1 \le k \le 9}, \quad (e_{i,1,2}, e_{i,2,3}, e_{i,3,1})_{1 \le i \le n}$$

fulfill conditions (129-137) and their counterpart for disequations

$$(W_{i,j,k})_{n+1 \le i \le 2n, 1 \le j \le 2, 1 \le k \le 9}, \quad (e_{i,1,2})_{n+1 \le i \le 2n}$$

fulfill the analogous conditions:

$$p_1(W_{i,j,k}) = (i,j,k,0) \in \mathcal{V}_0 \text{ for } W_{i,j,k} \neq 1 \tag{363}$$

$$\gamma(\prod_{k=1}^{9} W_{i,j,k}) = (1,H,b,1,1) \text{ for some } b \in \{0,1\} \tag{364}$$

$$\mu(\prod_{k=1}^{9} W_{i,j,k}) = \mu_{\mathcal{U}}(U_{i,j}) \tag{365}$$

$$\gamma(\prod_{k=1}^{4} W_{i,1,k}) = \gamma(\prod_{k=1}^{4} W_{i,2,k}) \tag{366}$$

$$W_{i,j,5} \in \mathcal{W} \wedge \gamma_w(W_{i,j,5}) \text{ is a } H\text{-type} \tag{367}$$

$$e_{i,1,2} \in \mathrm{Gi}(W_{i,1,5}) = \mathrm{Gi}(W_{i,2,5}) \tag{368}$$

A vector $(\boldsymbol{W}, \boldsymbol{e})$ fulfilling (129-137) for the indices $i \in [1,n]$ and (363-368) for the indices $i \in [n+1,2n]$, is called an *admissible* vector. For every admissible vector $(\boldsymbol{W}, \boldsymbol{e})$ we define the following equations and disequations:

$$\prod_{k=1}^{9} W_{i,j,k} = \prod_{k=1}^{9} W_{i',j',k} \text{ if } U_{i,j} = U_{i',j'} \tag{369}$$

$$W_{i,1,1}W_{i,1,2}W_{i,1,3}W_{i,1,4}e_{i,1,2} = W_{i,2,1}W_{i,2,2}W_{i,2,3}W_{i,2,4} \tag{370}$$

for all $1 \le i \le 2n$

$$W_{i,2,6}W_{i,2,7}W_{i,2,8}W_{i,2,9} = e_{i,2,3}\overline{W}_{i,3,4}\overline{W}_{i,3,3}\overline{W}_{i,3,2}\overline{W}_{i,3,1} \tag{371}$$

$$W_{i,1,5}W_{i,1,6}W_{i,1,7}W_{i,1,8} = e_{i,1,3}W_{i,3,6}W_{i,3,7}W_{i,3,8}W_{i,3,9} \tag{372}$$

$$W_{i,1,5} = e_{i,1,2}W_{i,2,5}e_{i,2,3}W_{i,3,5}e_{i,3,1} \tag{373}$$

for all $1 \le i \le n$

$$\bigwedge_{d \in \mathrm{Ge}(W_{i,1,5})} W_{i,1,5} \cdot d \neq e_{i,1,2} \cdot W_{i,2,5} \tag{374}$$

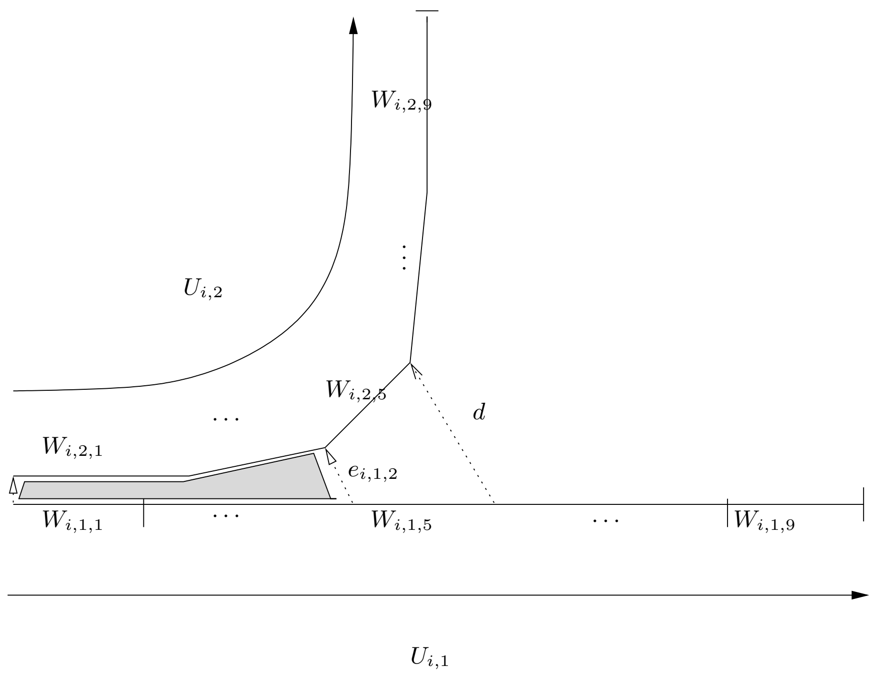


Fig. 20. Disequation cut into 3 parts

for all $n+1 \leq i \leq 2n$ such that $\tau\mathrm{e}(W_{i,1,5}) = \tau\mathrm{e}(W_{i,2,5})$.

We denote by $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ the sequence of equations (369-372), by $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ the sequence of equations and disequations (373-374). For every $(i,j) \in [1,n] \times [1,3] \cup [n+1, 2n] \times [1,2]$ we denote by $\overline{i,j}$ the smallest pair such that $U_{i,j} = U_{\overline{i,j}}$. By $\sigma_{\boldsymbol{W},\boldsymbol{e}} : \mathcal{U}^* \to \mathbb{W}$ we denote the unique monoid homomorphism such that,

$$\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,j}) = \prod_{k=1}^{9} W_{\overline{i,j},k}.$$

Lemma 53. *Let $\mathcal{S} = ((E_i)_{1 \leq i \leq n}, (\overline{E}_i)_{n+1 \leq i \leq 2n}, \mu_{\mathcal{A},\mathbb{G}}, \mu_{\mathcal{U}})$ be a system of equations and disequations over $\mathbb{G}$, with rational constraint. Let us suppose that $\mathcal{S}$ is in quadratic normal form. A monoid homomorphism*

$$\sigma : \mathcal{U}^* \to \mathbb{G}$$

is a solution of $\mathcal{S}$ if and only if, there exists an admissible choice $(\boldsymbol{W}, \boldsymbol{e})$ of variables of $\mathcal{W}_t$ and elements of $A \cup B$ and an AB-homomorphism

$$\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$$

solving simultaneously the system $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ of equations over $\mathbb{H}_t$ and the system $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ of equations and disequations over $\mathbb{H}$, and such that

$$\sigma = \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \sigma_t \circ \overline{\pi}_{\mathbb{G}}.$$

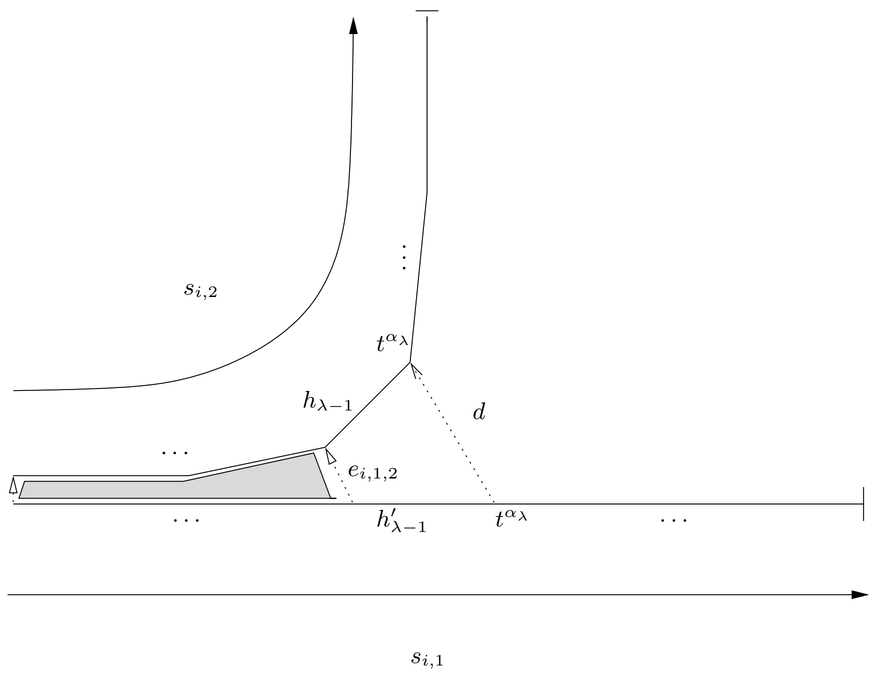


Fig. 21. Disequations, case 1

From $\mathbb{G}$-solutions to t-solutions Let $\sigma : \mathcal{U}^* \to \mathbb{G}$ be a monoid homomorphism solving the system $\mathcal{S}$. For every $1 \leq i \leq n$ we construct the vector $(W_{i,*,*}, e_{i,*,*})$ as in §5.2. Let us fix now some disequation from $\mathcal{S}$, i.e. some integer $n+1 \leq i \leq 2n$. Let us choose, for every $j \in [1,2]$, some $s_{i,j} \in \mathrm{Red}(\mathbb{H}, t)$ such that:

$$\sigma(U_{i,j}) = \pi_G(s_{i,j}).$$

Let us consider some decomposition of the form (5) for $s_{i,2}, s_{i,1}$:

$$s_{i,2} = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_\lambda} h_\lambda \cdots t^{\alpha_\ell} h_\ell, \tag{375}$$

$$s_{i,1} = h'_0 t^{\alpha'_1} h'_1 \cdots t^{\alpha'_\lambda} h'_\lambda \cdots t^{\alpha'_{\ell'}} h'_{\ell'}, \tag{376}$$

We know that $s_{i,1} \not\approx s_{i,2}$. Let us distinguish the possible forms for $s_{i,1}$, as represented on figures 21-23.

Case 1: there exists some integer $\lambda \in [2, \ell]$, some $e_{i,1,2} \in B(\alpha_{\lambda-1})$ such that

$$h_0 \cdots t^{\alpha_{\lambda-1}} h_{\lambda-1} = h'_0 \cdots t^{\alpha_{\lambda-1}} h'_{\lambda-1} e_{i,1,2}, \;\; \alpha_\lambda = \alpha'_\lambda$$

$$e_{i,1,2} h_{\lambda-1} \neq h'_{\lambda-1} d \quad \text{for all } d \in A(\alpha_\lambda).$$

We consider the following factors of $s_{i,2}, s_{i,1}$:

$$P_{i,2} = h_0 \cdots t^{\alpha_{\lambda-1}}, \;\; M_{i,2} = h_{\lambda-1}, \;\; S_{i,2} = t^{\alpha_\lambda} \cdots t^{\alpha_\ell} h_\ell.$$

$$P_{i,1} = h'_0 \cdots t^{\alpha_{\lambda-1}}, \;\; M_{i,1} = h'_{\lambda-1}, \;\; S_{i,1} = t^{\alpha_\lambda} \cdots t^{\alpha'_{\ell'}} h'_{\ell'}.$$

Following the lines of §5.2, these reduced sequences can be cut into nine factors $(P_{i,j,k})$, $1 \leq j \leq 2, 1 \leq k \leq 9$, and subsequently lifted to nine letters $(W_{i,j,k})$, $1 \leq j \leq 2, 1 \leq k \leq$

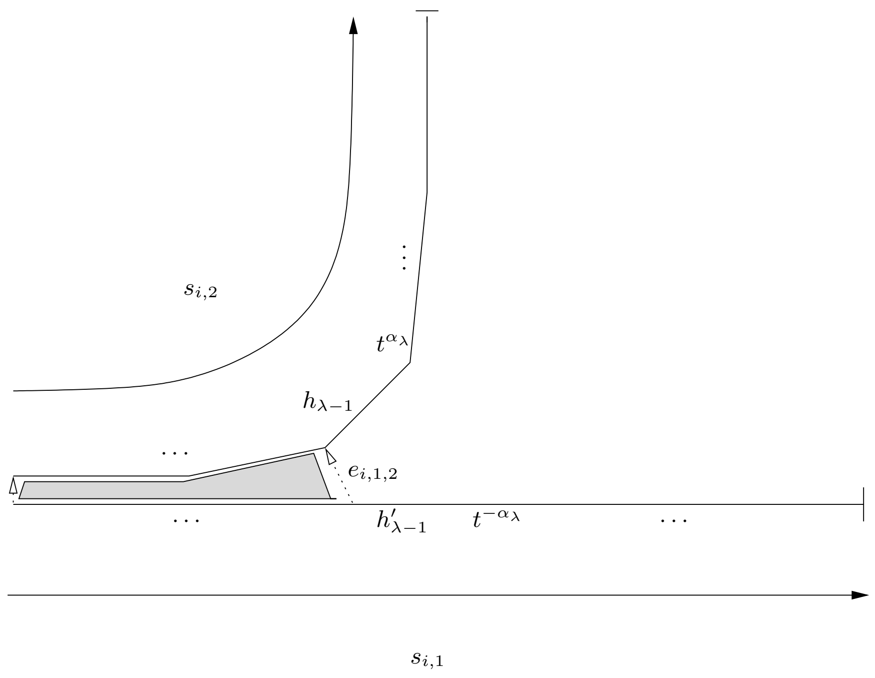


Fig. 22. Disequations, case 2

9, such that the vector $(W_{i,*,*}, e_{i,1,2})$ fulfills conditions (363-368) and the classes $([P_{i,*,*}]_\sim)$ fulfill equations and disequations (370),(374). We can define $\sigma_t(W_{i,j,k}) = [P_{i,j,k}]_\sim$; σ_t can be extended into an AB-homomorphism solving both systems $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$, $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ and such that

$$\sigma = \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \sigma_t \circ \overline{\pi}_{\mathbb{G}}.$$

Case 2: there exists $\lambda \in [2, \ell]$, some $e_{i,1,2} \in B(\alpha_{\lambda-1})$ such that

$$h_0 \cdots t^{\alpha_{\lambda-1}} h_{\lambda-1} = h'_0 \cdots t^{\alpha_{\lambda-1}} h'_{\lambda-1} e_{i,1,2}, \;\; \alpha_\lambda = -\alpha'_\lambda$$

We consider the factors $P_{i,2}, M_{i,2}, S_{i,2}, P_{i,1}, M_{i,1}$ defined by the same formulas as in case 1, and define

$$S_{i,1} = t^{-\alpha_\lambda} \cdots t^{\alpha'_{\ell'}} h'_{\ell'}.$$

This time we obtain a vector $(W_{i,*,*})$ such that $\tau\mathrm{e}(W_{i,1,5}) \neq \tau\mathrm{e}(W_{i,2,5})$. It follows there is no disequation (374) associated to this index i. The vector $(W_{i,*,*}, e_{i,1,2})$ fulfills conditions (363-368) and the classes $[P_{i,*,*}]_\sim)$ fulfill equation (370).
Case 3: there exists $\lambda \in [2, \ell]$, such that

$$s_{i,1} = h_0 \cdots t^{\alpha_{\lambda-1}} h_{\lambda-1}.$$

This case can be treated similarly as case 2. We just define $S_{i,1} = 1$ and, correspondingly $W_{i,1,k} = 1$, for $6 \leq k \leq 9$.
Case 4: there exists some integer $\lambda \in [2, \ell']$, such that

$$s_{i,2} = h_0 \cdots t^{\alpha_{\lambda-1}} h_{\lambda-1}.$$

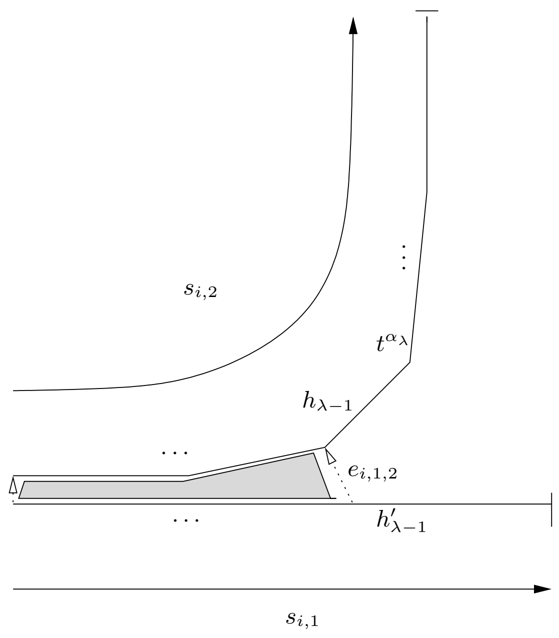


Fig. 23. Disequations, case 3

This case is obtained from case 3 by exchanging $s_{i,1}$ with $s_{i,2}$.
It remains to treat some degenerated cases.
Case 5: $\lambda = 1 \leq \ell$, fulfills one of the conditions defining cases 1-3 (except beeing smaller than 2) or $\lambda = 1 \leq \ell'$ fulfills the conditions defining case 4 (except beeing smaller than 2).
We take: $P_{i,1} = P_{i,2} = 1$, $e_{i,1,2} = 1$, $S_{i,1} = 1$ (under the condition of case 3), $S_{i,2} = 1$ (under the condition of case 4).
The construction is ended as in the degenerated cases of §5.2.
Case 6: $\ell = \ell' = 0$.
We take: $P_{i,1} = P_{i,2} = 1$, $e_{i,1,2} = 1$, $S_{i,1} = S_{i,2} = 1$.
The construction is ended as in the degenerated cases of §5.2.

From t-solutions to $\mathbb{G}$-solutions Let $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ be an AB-homomorphism solving both systems $\mathcal{S}_t(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ and $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$.
Owing to the proofs of §5.2, we just have to prove that for every $i \in [n+1, 2n]$,

$$\sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,1})) \not\approx \sigma_t(\sigma_{\boldsymbol{W},\boldsymbol{e}}(U_{i,2})).$$

Using equation (369) and the definition of $\sigma_{\boldsymbol{W},\boldsymbol{e}}$, the above inequalities are equivalent with

$$\sigma_t(\prod_{k=1}^{9} W_{i,1,k}) \not\approx \sigma_t(\prod_{k=1}^{9} W_{i,2,k}). \tag{377}$$

Equation (370) states that

$$\sigma_t(\prod_{k=1}^{4} W_{i,1,k}) e_{i,1,2} \approx \sigma_t(\prod_{k=1}^{4} W_{i,2,k}). \tag{378}$$

Let us distinguish several cases according to the values of $\tau\mathrm{e}(W_{i,j,5})$. Since, by (367), $\gamma_w(W_{i,j,5})$ are H-types, one of the following cases must occur.
Case 1: $\tau\mathrm{e}(W_{i,2,5}) = \tau\mathrm{e}(W_{i,1,5}) = (1,1)$
By (364), $\gamma(\prod_{k=6}^{9} W_{i,j,k}) = (1,1,0,1,1)$ for $j \in [1,2]$, which implies that, for every $j \in [1,2]$:

$$\sigma_t(\prod_{k=1}^{9} W_{i,j,k}) = \sigma_t(\prod_{k=1}^{5} W_{i,j,k}), \tag{379}$$

But disequation (374) reads:

$$\sigma_t(W_{i,1,5}) \neq_{\mathbb{H}_t} e_{i,1,2} \cdot \sigma_t(W_{i,2,5}). \tag{380}$$

The conjunction of (378)(379)(380) proves inequality (377).
Case 2: $\tau\mathrm{e}(W_{i,2,5}) = \tau\mathrm{e}(W_{i,1,5}) = (A,T)$
Since σ_t is a solution of equation (370) and disequation (374), two representatives of $\sigma_t(\prod_{k=1}^{5} W_{i,2,k})$, (resp. $\sigma_t(\prod_{k=1}^{5} W_{i,1,k})$) cannot be prefixes of two reduced sequences which are equivalent modulo $\sim$. Thus (377) is established.
Case 3: $\tau\mathrm{e}(W_{i,2,5}) = \tau\mathrm{e}(W_{i,1,5}) = (B,T)$
Same argument as for case 2.
Case 4: $\tau\mathrm{e}(W_{i,2,5}) \neq \tau\mathrm{e}(W_{i,1,5})$
This shows that the projections of $\sigma_t(\prod_{k=1}^{9} W_{i,2,k})$, $\sigma_t(\prod_{k=1}^{9} W_{i,1,k})$ on $\{t, t^{-1}\}^*$ are non-equal, so that, a fortiori, inequality (377) holds.

Theorem 8. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups A,B. The satisfiability problem for systems of equations and disequations with rational constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations and disequations with rational constraints in $\mathbb{H}$.*

In order to prove this theorem we must, as for proving Theorem 4, cope with the fact that we do not know an algorithm testing the satisfiability over $\mathbb{H}$ of an equation with the constraint $\mathbb{H} \setminus \mathsf{I}(\mathbb{H})$ (the set of non-units of the monoid $\mathbb{H}$). Therefore we use again the notion of *weak*-AB-homomorphism [5]. We first adapt Lemma 52 into the following

Lemma 54. *Let $\mathcal{DS} = \bigvee_{j\in J} \mathcal{S}_j$ be a finite disjunction of systems of equations and disequations over $\mathbb{G}$, with rational constraint where $\mathcal{S}_j = ((E_i)_{1\leq i\leq n}, (\overline{E}_i)_{n+1\leq i\leq 2n}, \mu_{\mathcal{A}}, \mu_{\mathcal{U},j})$. Let us suppose that $\mathcal{DS}$ is in closed quadratic normal form. A monoid homomorphism*

$$\sigma : \mathcal{U}^* \to \mathbb{G}$$

is a solution of $\mathcal{DS}$ if and only if, there exists and index $j \in J$, an admissible choice $(\boldsymbol{W}, \boldsymbol{e})$ of variables of $\mathcal{W}$ (resp. elements of $A \cup B$), an involution $\Phi \in \mathsf{HInv}$, an AB-homomorphism $\sigma_{\mathbb{W}} : \mathbb{W}'_t \to \mathbb{W}'_t/\Phi$ and a weak AB-homomorphism $\psi_t : \mathbb{W}'_t/\Phi \to \mathbb{H}_t$ such that:

(S1) *$\sigma_{\mathbb{W}}$ is a solution of the system of equations $\mathcal{S}_t(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$,*
(S2) *$\sigma_{\mathbb{W}} \circ \psi_t$ is a solution of the system of equations and disequations $\mathcal{S}_{\mathbb{H}}(\mathcal{S}_j, \boldsymbol{W}, \boldsymbol{e})$,*
(S3) *$\sigma = \sigma_{\boldsymbol{W},\boldsymbol{e}} \circ \sigma_{\mathbb{W}} \circ \psi_t \circ \overline{\pi}_{\mathbb{G}}$.*

[5] though we might also use a notion weaker than AB-hom and stronger than weak-AB-hom, since we are able to translate the conditions over γ, μ, δ by equations and disequations with rational constraints.

This lemma can be proved in the same way as Lemma 52.

Sketch of the proof of Theorem 8. It suffices to adapt the reductions given in the proof of Theorem 4 as follows:

- Lemma 52 is replaced by Lemma 54.
- Instead of a system of equations with rational constraints in $\mathbb{H}$ we use the system $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ of equations and disequations with rational contraints in $\mathbb{H}$ supplied by Lemma 53.

□

11.2 Positive rational constraints

Here we consider $\mathcal{C}_{\mathbb{G}} = \mathrm{Rat}(\mathbb{G})$ and $\mathcal{C}_{\mathbb{H}} = \mathrm{Rat}(\mathbb{H})$.

Theorem 9. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups A,B. The satisfiability problem for systems of equations and disequations with positive rational constraints in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations and disequations with positive rational constraints in $\mathbb{H}$*

In order to prove this theorem we must, as for proving Theorem 4, cope with the fact that we do not know an algorithm testing a set of the form $\{s \in \mathbb{H} * \{t, \bar{t}\}^* \mid \mu_t(s) = \mu(W)\}$ for emptiness: such a set is not known to be recognized by a t-automaton with labels which are defined by equations, disequations and *positive* rational constraints over $\mathbb{H}$. Therefore we use the notion of *weak*-AB-homomorphism.
Sketch of the proof of Theorem 9. It suffices to adapt the reductions given in the proof of Theorem 4 as follows:

- Lemma 52 is replaced by Lemma 54
- The initial positive constraints given by the system of equations and disequations, are defined by of some normal (possibly non-strict) fta's with labelling set $\mathrm{Rat}(\mathbb{H})$
- Instead of a system of equations with rational constraints in $\mathbb{H}$ we use the system $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \boldsymbol{W}, \boldsymbol{e})$ of equations and disequations with positive rational contraints in $\mathbb{H}$ supplied by Lemma 53.
- For every $W \in \mathcal{W}'_{\mathbb{H}}$, $\mathsf{C}(W)$ belongs to $\mathrm{DEQ}(\mathbb{H}, \mathrm{Rat}(\mathbb{H}))$.
- For every $W \in \mathcal{W}'_t \setminus \mathcal{W}'_{\mathbb{H}}$, $\mathsf{C}(W)$ is recognized by a fta with labels in $\mathrm{DEQ}(\mathbb{H}, \mathrm{Rat}(\mathbb{H}))$: this is clear for $\mathsf{C}_I(W), \mathsf{C}_\mu(W), \mathsf{C}_\delta(W)$; for $\mathsf{C}_\gamma(W)$, the trick consists just in seeing the subset $\mathbb{H} \setminus A$ (resp. $\mathbb{H} \setminus B$) as defined by the system of disequations $\bigwedge_{a \in A} v \neq a$ (resp. $\bigwedge_{b \in B} v \neq b$).

□

11.3 Constants

The set of constraints here are $\mathcal{C}_{\mathbb{G}} = \{\{g\} \mid g \in \mathbb{G}\} \cup \{\mathbb{G}\}$ and $\mathcal{C}_{\mathbb{H}} = \{\{h\} \mid h \in \mathbb{H}\} \cup \{\mathbb{H}\}$.

Theorem 10. *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations and disequations with constants in $\mathbb{G}$ is Turing-reducible to the SAT-problem for systems of equations and disequations with constants in $\mathbb{H}$.*

Sketch of proof: We adapt the reductions given in the proof of Theorem 4 as follows:

- Lemma 52 is replaced by Lemma 54
- for every $W \in \mathcal{W}'_t$, $\mathsf{C}(W)$ is recognized by an fta with labeling set $DEQ(\mathbb{H}, \mathcal{C}_{\mathbb{H}})$, i.e. the set of subsets of $\mathbb{H}$ which are definable by systems of equations and disequations with constants.

□

12 Equations and disequations over an amalgamated product

12.1 Equations and disequations

In this section we adapt the transfer result about the operation of HNN-extension (i.e. Theorem 8) as a transfer result about *free product with amalgamation* (see Theorem 11 below). We recalled in section 2 Theorem 1 for equations/disequations and the operation of free-product. Let us use here the same notation $\mathcal{L}(\mathcal{C}_{\mathbb{H}_1}, \mathcal{C}_{\mathbb{H}_2})$ and state a more general theorem.

Theorem 11. *Let us consider two cancellative monoids $\mathbb{H}_1, \mathbb{H}_2$, two finite subgroups $A_1 \leq \mathbb{H}_1, A_2 \leq \mathbb{H}_2$, and an isomorphism $\varphi : A_1 \to A_2$. The satisfiability problem for systems of equations and disequations with rational constraints in the amalgamated product $\mathbb{G} = \langle \mathbb{H}_1, \mathbb{H}_2; a = \varphi(a)\ (a \in A_1)\rangle$ is Turing-reducible to the pair of problems (S_1, S_2) where*

- *S_1 is the SAT-problem for systems of equations and disequations with rational constraints in $\mathbb{H}_1$ and*
- *S_2 is the SAT-problem for systems of equations and disequations with rational constraints in $\mathbb{H}_2$.*

Proof. Let us use the embedding $\eta : \mathbb{G} \to \widehat{\mathbb{G}}$ defined in subsubsection 2.2 by formula (11), where $\widehat{\mathbb{G}} := \langle \mathbb{H}_1 * \mathbb{H}_2, t; t^{-1}at = \varphi(a)(a \in A_1)\rangle$. By Lemma 9 the SAT problem for systems of equations and disequations with rational constraints in $\mathbb{G}$ is reduced to the same problem in $\widehat{\mathbb{G}}$. By Theorem 4 this problem reduces to the same problem in the product $\mathbb{H}_1 * \mathbb{H}_2$ and by Theorem 1 this last problem reduces to the same problem in every factor i.e. (S_1, S_2). □

Note that when $\mathcal{C}_{\mathbb{H}_i} = \{\{h\} \mid h \in \mathbb{H}_i\} \cup \{\mathbb{H}_i\}$ then $\mathcal{L}(\mathcal{C}_{\mathbb{H}_1}, \mathcal{C}_{\mathbb{H}_2}) \supseteq \{\{g\} \mid g \in \mathbb{G}\} \cup \{\mathbb{G}\}$. Hence, by similar arguments as above, one can prove the variant of Theorem 11 where equations and disequations *with constants* are considered.

12.2 Equations

We strongly believe that one can adapt the main transfer result about equations and the operation of HNN-extension (i.e. Theorem 4) as a transfer result about *free product with amalgamation* . Since we lack such a theorem even in the case of a free product, the most natural method would consist in adapting all the method developed in sections 3-6 to the case of a free product with amalgamation over finite subgroups. As well, analogues of Theorem 5, dealing with equations with positive rational constraints, and of Theorem 7, dealing with equations with constants, should hold for free products with amalgamation.

13 Equations and disequations over a graph of groups

Let us recall a graph of groups $\mathcal{G}$ is a directed graph $G = (V, E)$ endowed with a family of groups $(G_v)_{v\in V}$ and a family $(\varphi_e)_{e\in E}$ of maps, such that, for every $e \in E$, φ_e is a partial isomorphism from $G_{\iota(e)}$ into $G_{\tau(e)}$ (here $\iota(e)$ denotes the initial vertex of edge e and $\tau(e)$ denotes the terminal vertex of edge e). For every vertex v, we denote by $\pi_1(\mathcal{G}, v)$ the fundamental group of $\mathcal{G}$ with base-point v (see [Ser77] or [DD90] for background on graphs of groups).

Theorem 12. *Let $\mathcal{G} = (V, E, (G_v)_{v\in V}, (\varphi_e)_{e\in E})$ be a finite graph of groups where all the edge partial isomorphisms φ_e have finite domain and let $v_0 \in V$. The satisfiability problem for systems of equations and disequations with rational constraints in the fundamental group $\pi_1(\mathcal{G}, v_0)$ is Turing-reducible to the join of the problems $(S_v)_{v\in V}$ where S_v is the SAT-problem for systems of equations and disequations with rational constraints in G_v.*

Proof. We can assume that (V, E) is connected (otherwise adding some edges with trivial partial isomorphisms would preserve the fundamental group and make the graph connected). Let $T \subseteq E$ be a covering tree. The group $\pi_1(\mathcal{G}, v_0)$ can be obtained as a k-fold HNN-extension of a ℓ-fold free product with amalgamation of the vertex groups (G_v), where the associated (or amalgamated) subgroups are pairs of the form $(\mathrm{dom}(\varphi_e), \mathrm{im}(\varphi_e))$ for $e \in E$, $k = |T| - |E|$ and $\ell = |T|$. Hence, using k times Theorem 8 and ℓ times Theorem 11, we obtain the desired Turing reduction. □

Related works In [DG07], F. Dahmani and V. Guirardel are proving Theorem 2 by geometrical methods. Using this result, they get ([DG10]) an algorithm that solves equations in any word hyperbolic group (even with torsion). As mentioned in the introduction, A. Myasnikov and O. Karlampovich have shown that the full first-order theory of a free group of finite rank is decidable. Their solution includes ([KM05a,KM05b]) methods for solving equations in so-called fully residually finite groups (these groups generalize free groups).

Perspectives We think that part of the techniques exposed here will be extendable to the HNN-extensions where the subgroups A, B are infinite but assumed to be nicely embedded in the base group (or monoid) $\mathbb{H}$.

We extend in [LS05] our transfer theorems to the positive first-order theory of HNN-extensions (or free products with amalgamation) of groups. Whether an extension to the full first-order theory is true (or not) is a fascinating open question.

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