When is a Graph Product of Groups Virtually-Free?

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Abstract

Those graph products of groups that are virtually-free are characterized.

1 Introduction

A finitely generated group G is called virtually-free if it has a free subgroup of finite index. Finitely generated virtually-free groups can be characterized in many different ways using notions from formal language theory [1, 13, 14], graph theory [3, 14], and mathematical logic [10, 14]. This makes virtually-free groups to an important research topic in combinatorial group theory. It is known that every virtually-free group is hyperbolic and that the class of virtually-free groups is closed under the operations of amalgamated free products and HNN extensions, when both operations are restricted to finite associated subgroups.

In this paper we study the operation of graph product. For a given finite undirected graph $X = (V, I)$ and a non-trivial group $G_v \neq 1$ for every node $v \in V$ we define the graph product $\mathbb{GP}(X, (G_v)_{v \in V})$ as the free product of the groups G_v subject to the relations $xyx^{-1}y^{-1} = 1$ for all $x \in G_u$, $y \in G_v$ such that $(u, v) \in I$. This operation generalizes both free and direct products. Graph products were first studied in [7]. Several nice closure properties are known for graph products. For instance, the following classes are all closed under graph products: residually finite groups [7], semihyperbolic groups [9], automatic groups [9]. In [12], a complete answer to the question under which conditions a graph product of hyperbolic groups is again hyperbolic was given: $\mathbb{GP}(X, (G_v)_{v \in V})$ is hyperbolic if and only if

(i) for every $v \in V$, G_v is hyperbolic,

(ii) if G_u and G_v are infinite groups and $u \neq v$, then $(u, v) \notin I$,

(iii) if G_u is an infinite group, G_v and G_w are finite groups such that $v \neq w$, and $(u, v), (u, w) \in I$, then also $(v, w) \in I$, and

(iv) the graph X does not contain an induced cycle of length 4.

The main result of this paper gives a similar characterization for virtually-free groups:

Theorem 1.1. *Let $G = \mathbb{GP}(X, (G_v)_{v \in V})$ be a graph product with $X = (V, I)$. Then G is virtually-free if and only if:*

(1) for every $v \in V$, G_v is virtually-free,

(2) if G_u and G_v are infinite groups and $u \neq v$, then $(u, v) \notin I$,

(3) if G_u is an infinite group, G_v and G_w are finite groups such that $v \neq w$, and $(u, v), (u, w) \in I$, then also $(v, w) \in I$, and

(4) the graph X is chordal.

Recall that a graph X is chordal if X does not contain an induced cycle of length at least 4, see e.g. [5] for background on graph theory. Note that condition (2) and (3) are identical to condition (ii) and (iii) in the corresponding characterization for hyperbolic groups and that condition (1) is just the obvious counterpart of condition (i). But condition (4) is strictly stronger than condition (iv) which only requires that X does not contain an induced cycle of length 4. In particular we see that a graph product $\mathbb{GP}(X, (G_v)_{v \in V})$, where every group G_v is finite and X is a cycle on 5 nodes, is hyperbolic but not virtually-free.

2 Preliminaries

For background in combinatorial group theory see [11]. In this paper we restrict to finitely generated groups. We will not mention this implict assumption in the following. Let $G = \mathbb{GP}(X, (G_v)_{v \in V})$ be a graph product as defined in Section 1, where $X = (V, I)$ is a finite undirected graph, i.e., $I \subseteq V \times V$ is a symmetric and irreflexive relation. Let $U \subseteq V$. The subgraph of X *induced* by U is $X{\restriction_U} = (U, I \cap (U \times U))$. We define the graph product $G{\restriction_U}$ as $\mathbb{GP}(X{\restriction_U}, (G_v)_{v \in U})$; it is a subgroup of G [7]. Moreover, there exists a canonical projection homomorphism $\pi : G \to G{\restriction_U}$ which is defined by $\pi(x) = 1$ for all $x \in G_v$ with $v \in V \setminus U$ and $\pi(x) = x$ for all $x \in G_u$ with $u \in U$. For groups G_1, G_2, and H, and injective homomorphisms $h_1 : H \to G_1$ and $h_2 : H \to G_2$, the *amalgamated free product* $G_1 *_H G_2$ is defined as the free product of G_1 and G_2 subject to the relations $h_1(x) = h_2(x)$ for every $x \in H$. W.l.o.g. we may assume that $G_1 \cap G_2 = H$ and that h_i is just the inclusion map. We will need the following two results on graph products:

Proposition 2.1 (Lemma 4.5 in [7]). *Let $G = \mathbb{GP}(X, (G_v)_{v \in V})$ be a graph product and let F be a finite subgroup of G. Then F is a subgroup of a conjugate of some subgroup $G{\restriction_U}$, where U is a clique of X.*

Proposition 2.2 (p 48 of [7]). *Let $G = \mathbb{GP}(X, (G_v)_{v \in V})$ be a graph product, $X = (V, I)$, let $u \in V$, and let $N(u) = \{v \in V \mid (u, v) \in I\}$ be the neighbors of u. Then*

$$G \simeq (G\restriction_{V \setminus \{u\}}) *_{G\restriction_{N(u)}} (G\restriction_{N(u)} \times G_u).$$

Let $G = G_1 *_H G_2$ be an amalgamated free product where w.l.o.g. $G_1 \cap G_2 = H$. A *reduced sequence* is a word $x_1 x_2 \cdots x_n$ such that either $n = 1$ and $x_1 \in G_1 \cup G_2$ or $n > 1$, $x_i \in (G_1 \setminus H) \cup (G_2 \setminus H)$ $(1 \leq i \leq n)$, and $x_i \in G_1 \setminus H \Leftrightarrow x_{i+1} \in G_2 \setminus H$ $(1 \leq i < n)$. Then for every $x \in G$ there exists a reduced sequence $x_1 x_2 \cdots x_n$ such that $x = x_1 x_2 \cdots x_n$ in the group G and moreover, if $y_1 y_2 \cdots y_m$ is another reduced sequence with $x = y_1 y_2 \cdots y_m$ in the group G then $n = m$. We call n the length of the reduced sequence $x_1 \cdots x_n$.

We will also need the standard concept of a graph of groups and its fundamental group, which should not be confused with the graph product. Graphs of groups and graph products are both based on a graph where each vertex has an associated group attached. But for the concept of a graph of groups we need a definition of graphs which differs from the one for undirected graphs (which was used for graph products). Hence, we use the term *multigraph* in the following. A multigraph is a tuple $Y = (V, E, s, t)$, where V is the set of vertices, E is the set of edges, $s : E \to V$ maps every edge e to its source vertex $s(e)$, and $t : E \to V$ maps every edge e to its target vertex $t(e)$. Note that we may have loops (i.e., $t(e) = s(e)$ for some $e \in E$) and multiple edges (i.e., $s(e_1) = s(e_2)$ and $t(e_1) = t(e_2)$ for some $e_1, e_2 \in E$ with $e_1 \neq e_2$) between vertices in a multigraph. We say that Y is *connected* if for all $u, v \in V$ there exists a sequence of vertices $v_1, v_2, \ldots, v_n$ $(n \geq 1)$ such that: (i) $v_1 = u$, (ii) $v_n = v$, and (iii) for all $1 \leq i < n$ there exists an edge $e_i \in E$ such that $\{s(e_i), t(e_i)\} = \{v_i, v_{i+1}\}$. Y is finite if both V and E are finite. We say that $Y = (V, E, s, t)$ is a *tree* if it is connected and $|E| = |V| - 1$. Equivalently, Y is a tree if and only if Y is connected, Y does not contain loops and multiple edges and the undirected graph resulting from Y in the obvious way is acyclic. A *covering tree* of a connected multigraph $Y = (V, E, s, t)$ is a tree of the form $T = (V, E_T, s\restriction_{E_T}, t\restriction_{E_T})$ with $E_T \subseteq E$; note that it contains all vertices of Y. Clearly, every covering tree of Y contains exactly $|V| - 1$ edges.

A *graph of groups* is a pair $(\mathcal{G}, Y)$, where $Y = (V, E, s, t)$ is a *finite connected* multigraph and $\mathcal{G}$ is a mapping that associates with every vertex $v \in V$ of Y a group $\mathcal{G}_v$ and with every edge $e \in E$ of Y a group $\mathcal{G}_e$. Moreover, we assume that every edge $e \in E$ is equipped with injective homomorphisms $\sigma_e : \mathcal{G}_e \to \mathcal{G}_{s(e)}$ and $\tau_e : \mathcal{G}_e \to \mathcal{G}_{t(e)}$. We say that $(\mathcal{G}, Y)$ is a *graph of finite groups* if every vertex group $\mathcal{G}_v$ is finite.

The *fundamental group* $\pi_1(\mathcal{G}, Y)$ of a graph of groups is defined as follows: Let $Y = (V, E)$ and let $T = (V, E_T)$ be any covering tree of Y. For every edge $e \in E \setminus E_T$ take a new generator t_e. Then $\pi_1(\mathcal{G}, Y)$ is the free product $*_{v \in V} \mathcal{G}_v * F$, where F is the free group generated by the new generators t_e $(e \in E \setminus E_T)$, subject to the following relations:

- $\sigma_e(x) = \tau_e(x)$ for all $x \in \mathcal{G}_e$ whenever $e \in E_T$
- $t_e^{-1} \sigma_e(x) t_e = \tau_e(x)$ for all $x \in \mathcal{G}_e$ whenever $e \in E \setminus E_T$

Amalgamated free products are a special case of this construction for the case that Y consists of a single edge between two different vertices. For every vertex group

$\mathcal{G}_v$ there is a canonical embedding of $\mathcal{G}_v$ into the fundamental group $\pi_1(\mathcal{G}, Y)$. In the following, we will identify $\mathcal{G}_v$ with its image under this embedding. In particular, we view a vertex group $\mathcal{G}_v$ as a subgroup of $\pi_1(\mathcal{G}, Y)$.

Later, it will turn out that we will only need the case that the graph $Y = (V, E, s, t)$ is itself a tree. In this case, the fundamental group $\pi_1(\mathcal{G}, Y)$ can be obtained from the vertex groups by a repeated application of amalgamated free products along edge groups, and we can assume that for every edge $e \in E$, we have $\mathcal{G}_e = \mathcal{G}_{s(e)} \cap \mathcal{G}_{t(e)}$ (as subgroups of $\pi_1(\mathcal{G}, Y)$).

From the definition of a fundamental group it follows immediately that for every graph of groups $(\mathcal{G}, Y)$ there exists a graph of groups $(\mathcal{H}, Z)$ defining the same fundamental group and such that for every edge e of Z with $s(e) \neq t(e)$ the edge group $\mathcal{H}_e$ is a proper subgroup of both $\mathcal{H}_{s(e)}$ and $\mathcal{H}_{t(e)}$. We will always assume this property in the following.

Lemma 2.3. *Let $G = A *_C B$ with $C \subsetneq A$ and $C \subsetneq B$. Then there is no $x \in G$ such that $x^{-1}Ax \subseteq B$.*

Proof. Assume that $x^{-1}Ax \subseteq B$ for some $x \in G$. Take a reduced sequence $x_1 x_2 \cdots x_n$ for x. First assume that $x_1 \in B \setminus C$. Let $a \in A \setminus C$. Then also the sequence $x_n^{-1} \cdots x_2^{-1} x_1^{-1} a x_1 x_2 \cdots x_n$ is reduced and it has length at least 3. Thus we cannot have $x^{-1}ax \in B$. Now assume that $x_1 \in A$. Since conjugation with x_1 is an isomorphism of the group A, we can find an element $a \in A$ such that $x_1^{-1} a x_1 \in A \setminus C$. But then either $n \geq 2$ and the sequence $x_n^{-1} \cdots x_2^{-1}(x_1^{-1} a x_1) x_2 \cdots x_n$ is reduced and of length at least 3 (thus $x^{-1}ax \notin B$) or $n = 1$. In the latter case we have $x^{-1}ax = x_1^{-1} a x_1 \in A \setminus C$, and hence again $x^{-1}ax \notin B$. $\square$

Proposition 2.4 (Chapt. I, Prop. 7.11 in [4]). *Let $G = \pi_1(\mathcal{G}, Y)$ for a graph $(\mathcal{G}, Y)$ of finite groups. Let F be a finite subgroup of G. Then F is a subgroup of a conjugate of one of the vertex groups $\mathcal{G}_v$.*

Theorem 2.5 (Chapt. IV, Corollary 1.9 in [4]). *A group G is virtually-free if and only if it is isomorphic to some fundamental group $\pi_1(\mathcal{G}, Y)$, where $(\mathcal{G}, Y)$ is a graph of finite groups.*

For a group G let $\langle\!\langle \mathrm{Tor}(G) \rangle\!\rangle$ be the normal subgroup of G that is generated by all elements in G of finite order.

Lemma 2.6. *Let $G = \pi_1(\mathcal{G}, Y)$ for a graph $(\mathcal{G}, Y)$ of finite groups (thus G is virtually-free). Then $G/_{\langle\!\langle \mathrm{Tor}(G) \rangle\!\rangle}$ is a free group of rank $|E| - c$, where c is the number of edges in an arbitrary covering tree of $Y = (V, E, s, t)$.*

Proof. Let $T = (V, E_T, s, t)$ by any covering tree of Y. Let $g : G \to G/_{\langle\!\langle \mathrm{Tor}(G) \rangle\!\rangle}$ be the canonical homomorphism. Since every vertex group $\mathcal{G}_v$ is finite, every $x \in \mathcal{G}_v$ is mapped by g to 1. Thus, only the generators t_e of G with $e \in E \setminus E_T$ survive under g. Moreover, each of the defining relations $\sigma_e(x) = \tau_e(x)$ (for $x \in \mathcal{G}_e$, $e \in E_T$) and $t_e^{-1} \sigma_e(x) t_e = \tau_e(x)$ (for $x \in \mathcal{G}_e$, $e \in E \setminus E_T$) of G becomes trivial under the homomorphism g. Thus, $G/_{\langle\!\langle \mathrm{Tor}(G) \rangle\!\rangle}$ is the free group generated by the $|E| - |E_T| = |E| - c$ many generators t_e ($e \in E \setminus E_T$). This proves the lemma. $\square$

We will use the following well-known facts about virtually-free groups:

- A direct product of two infinite groups is not virtually-free.
- Every finitely generated subgroup of a virtually-free group is virtually-free.
- If $G = G_1 *_A G_2$, where G_1 and G_2 are virtually free and A is a finite, then also G is virtually-free.
- If G is virtually-free and A is finite, then also $G \times A$ is virtually-free.

3 Sufficiency of condition (1)–(4)

Sufficiency of condition (1)–(4) is the easier part in the proof of Theorem 1.1. We first consider the case of a graph product of finite groups.

Lemma 3.1. *If $G = \mathbb{GP}(X, (G_v)_{v \in V})$ is a graph product such that X is chordal and every group G_v is finite, then $\mathbb{GP}(X, (G_v)_{v \in V})$ is virtually-free.*

Proof. Let $X = (V, I)$ be chordal. We prove by induction on the number of vertices of X that G is virtually-free. By [6] there exists a simplical vertex in X, i.e., a vertex $u \in V$ such that the neighborhood $N_u = \{v \in V \mid (u, v) \in I\}$ of u is a clique of X. Thus, $F = G\restriction_{N(u)} = \prod_{v \in N(u)} G_v$ and $F \times G_u$ are both finite (and hence virtually-free). By induction, $H = G\restriction_{V \setminus \{u\}}$ is virtually-free. By Proposition 2.2 we have $G \simeq H *_F (F \times G_u)$, so G is virtually-free. $\square$

Now assume that condition (1)–(4) in Theorem 1.1 are true for the graph product $G = \mathbb{GP}(X, (G_v)_{v \in V})$. Let n be the number of vertices $u \in V$ such that G_u is infinite. We show by induction on n that G is virtually-free. The case $n = 0$ is dealt in Lemma 3.1. Now assume that $n > 0$ and let $u \in V$ such that G_u is infinite. Since for the graph product $H = G\restriction_{V \setminus \{u\}}$ again the conditions (1)–(4) from the theorem are satisfied, it follows by induction that H is virtually free. Let $N_u = \{v \in V \mid (u, v) \in I\}$ be the neighborhood of u. By (2) from Theorem 1.1, G_v has to be finite for every $v \in N_u$, and by (3), N_u is a clique of X. Thus, $F = G\restriction_{N(u)} = \prod_{v \in N(u)} G_v$ is finite and hence, $F \times G_u$ is still virtually-free. We can conclude as in the proof of Lemma 3.1.

4 Necessity of condition (1)–(4)

The necessity of the conditions (1)–(3) from Theorem 1.1 is easy to see: For (1), just recall that every finitely generated subgroup of a virtually-free group is again virtually-free. Condition (2) has to be true, because the direct product of two infinite groups is not virtually-free. Similarly, if condition (3) would be false, then G would contain a subgroup of the form $G_u \times (G_v * G_w)$ with G_u infinite, and hence again a direct product of two infinite groups.

It remains to show that if $X = (V, I)$ is not chordal then $\mathbb{GP}(X, (G_v)_{v \in V})$ is not virtually-free. Thus, assume that (V, I) is not chordal, i.e., it contains an induced cycle of length at least 4. Since we may assume that condition (2) and (3) are satisfied, it follows, that already the subgraph of X that is induced by those $v \in V$ with G_v finite is non-chordal. Hence, since finitely generated subgroups of virtually-free groups are again virtually-free, it suffices to show the following lemma:

Lemma 4.1. *A graph product $G = \mathbb{GP}(X, (A_v)_{v \in V})$, where $X = (V, I)$ is an induced cycle C_n of length $n \geq 4$ and every A_v is finite, is not virtually-free.*

We will present two proofs of Lemma 4.1: a very short proof using two deep results concerning virtual cohomology dimension, and a second slightly more complicated but more elementary proof using graph of groups.

First proof of Lemma 4.1. It is known that a group is virtually-free if and only if it has virtual cohomology dimension, see e.g. [2]. On the other hand, Harlander and Meinert calculated the virtual cohomology dimension of graph products of finite groups [8]. In particular, if G is a graph product as described in Lemma 4.1, then its virtual cohomology dimension is 2 by [8, Thm. 6.6]. Hence, G cannot be virtually-free. □

Second proof of Lemma 4.1. Let $V = \{0, \ldots, n-1\}$ and $I = \{(i, i+1) \mid 0 \leq i \leq n-1\}$, where here and in the following addition is modulo n. In order to deduce a contradiction, assume that G is virtually-free. Thus $G = \pi_1(\mathcal{G}, Y)$ for a graph $(\mathcal{G}, Y)$ of finite groups. By Theorem 2.6, $G/_{\langle\langle \mathrm{Tor}(G) \rangle\rangle}$ is a free group of rank $m - c$, where m is the number of edges of Y and c is the number of edges in an arbitrary covering tree of Y. But since G is a graph product of finite groups, every generator of G has finite order. Thus, $\langle\langle \mathrm{Tor}(G) \rangle\rangle = G$, i.e., $G/_{\langle\langle \mathrm{Tor}(G) \rangle\rangle} = 1$. Hence $m = c$, i.e., Y is in fact a finite tree.

By Proposition 2.1 we know that the maximal finite subgroups of G are the conjugates of the groups $A_i \times A_{i+1}$ for $0 \leq i \leq n-1$. Thus, by Proposition 2.4, every vertex group $\mathcal{G}_v \leq G$ in the graph of groups $(\mathcal{G}, Y)$ is a conjugate of one of the groups $A_i \times A_{i+1}$ for $0 \leq i \leq n-1$. Now consider a leaf u of the tree Y (which exists, since, Y is a finite tree) and let $\mathcal{G}_u$ be $\gamma^{-1}(A_i \times A_{i+1})\gamma$ for some $\gamma \in G$. By conjugating all vertex and edge groups in the graph of groups $(\mathcal{G}, Y)$ by γ^{-1} we can assume that $\mathcal{G}_u = A_i \times A_{i+1}$. Let v be the unique neighbor of u in the tree Y, let $\mathcal{G}_v = \alpha^{-1}(A_j \times A_{j+1})\alpha$ for some $\alpha \in G$, and let e be the unique edge between u and v. Then we have $G = (A_i \times A_{i+1}) *_{\mathcal{G}_e} H$, where H is the fundamental group of the graph of groups that results by removing from $(\mathcal{G}, Y)$ the leaf u and the edge e.

Case 1. $\{i, i+1\} \cap \{j, j+1\} \neq \emptyset$: By Lemma 2.3 we cannot have $\mathcal{G}_u = A_i \times A_{i+1}$ and at the same time $\mathcal{G}_v = \alpha^{-1}(A_i \times A_{i+1})\alpha$. Thus, $|\{i, i+1\} \cap \{j, j+1\}| = 1$. W.l.o.g. assume that $j = i+1$, i.e., $\mathcal{G}_u = A_{j-1} \times A_j$ and $\mathcal{G}_v = \alpha^{-1}(A_j \times A_{j+1})\alpha$. We first claim that for the edge group $\mathcal{G}_e = (A_{j-1} \times A_j) \cap \alpha^{-1}(A_j \times A_{j+1})\alpha$ we have $\mathcal{G}_e \subseteq A_j$. To see this, assume that $x = a_{j-1}a_j = \alpha^{-1}a'_j a_{j+1}\alpha \in \mathcal{G}_e$ for $a_{j-1} \in A_{j-1}$, $a_j, a'_j \in A_j$, $a_{j+1} \in A_{j+1}$. Apply the projection homomorphism $\pi_{j-1} : G \to A_{j-1}$ to this identity:

$$a_{j-1} = \pi_{j-1}(a_{j-1}a_j) = \pi_{j-1}(\alpha^{-1}a'_j a_{j+1}\alpha) = \pi_{j-1}(\alpha^{-1})\pi_{j-1}(\alpha) = 1.$$

Hence, $a_{j-1} = 1$, i.e., $x = a_j \in A_j$.

Now consider an arbitrary $a \in A_{j-1} \setminus \{1\} \subseteq \mathcal{G}_u \setminus \mathcal{G}_e$, which exists since every group A_k is assumed to be non-trivial. Assume that $b^{-1}ab = a$ for some $b \in G$. Take a reduced sequence $x_1 x_2 \cdots x_n$ for b ($n \geq 1$) with respect to the amalgamated free product $G = \mathcal{G}_u *_{\mathcal{G}_e} H = (A_{j-1} \times A_j) *_{\mathcal{G}_e} H$. Thus,

$$x_n^{-1} \cdots x_2^{-1} x_1^{-1} a x_1 x_2 \cdots x_n = a. \tag{1}$$

If $x_1 \in H \setminus \mathcal{G}_e$ then the sequence $x_n^{-1} \cdots x_2^{-1} x_1^{-1} a x_1 x_2 \cdots x_n$ is again reduced and of length at least 3. But the length of the sequence a is 1, a contradiction to (1). Thus, $x_1 \in \mathcal{G}_u = A_{j-1} \times A_j$. Let $x_1 = a_{j-1} a_j$ with $a_{j-1} \in A_{j-1}$ and $a_j \in A_j$. Then for the middle factor $x_1^{-1} a x_1$ we obtain $x_1^{-1} a x_1 = a_{j-1}^{-1} a_j^{-1} a a_{j-1} a_j = a_{j-1}^{-1} a a_{j-1} \in A_{j-1} \setminus \{1\}$ (note that $a \neq 1$). Thus, $x_1^{-1} a x_1 \notin \mathcal{G}_e$ and the sequence $x_n^{-1} \cdots x_2^{-1} (x_1^{-1} a x_1) x_2 \cdots x_n$ is again reduced. But since by (1) this sequence equals a in G, we have $n = 1$, i.e., $b = x_1 \in A_{j-1} \times A_j$. We have shown that the centralizer $C(a)$ of a is included in $A_{j-1} \times A_j$ and hence finite. But since in the graph X there is no edge in I between the two neighbor groups A_{j-2} and A_j of A_{j-1}, we have $A_{j-2} * A_j \subseteq C(a)$, i.e., $C(a)$ is infinite. We have obtained a contradiction.

Case 2. $\{i, i+1\} \cap \{j, j+1\} = \emptyset$: In this case we have $\mathcal{G}_e = (A_i \times A_{i+1}) \cap \alpha^{-1}(A_j \times A_{j+1})\alpha = 1$. To see this, assume that $x = a_i a_{i+1} = \alpha^{-1} a_j a_{j+1} \alpha \in \mathcal{G}_e$ with $a_i \in A_i$, $a_{i+1} \in A_{i+1}$, $a_j \in A_j$, and $a_{j+1} \in A_{j+1}$. Then by applying the projection homomorphism $\pi_i : G \to A_i$ to this identity we obtain $a_i = 1$. Similarly, by applying the projection homomorphism $\pi_{i+1} : G \to A_{i+1}$ we get $a_{i+1} = 1$. Thus, indeed $x = a_i a_{i+1} = 1$. Analogously to case 1 we can then show that for all $a \in A_i \times A_{i+1}$ we have $C(a) \subseteq A_i \times A_{i+1}$, i.e., $C(a)$ is finite. This leads again to a contradiction. This concludes the proof of Lemma 4.1 and hence of Theorem 1.1. $\square$

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