$\Gamma(A) \sim \Gamma(B)$?

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Abstract. The *bisimulation* problem for equational graphs of finite out-degree is shown to be decidable. We reduce this problem to the η-bisimulation problem for deterministic rational (vectors of) boolean series on the alphabet of a dpda $\mathcal{M}$. We then exhibit a *complete formal system* for deducing equivalent pairs of such vectors.

Keywords: bisimulation; equational graphs; deterministic pushdown automata; rational languages; finite dimensional vector spaces; matrix semi-groups; complete formal systems.

1 Introduction

1.1 Motivations

Processes In the context of concurrency theory , several notions of "behaviour of a process" and " behavioural equivalence between processes" have been proposed. Among them, the notion of *bisimulation* equivalence seems to play a prominent role (see [Mil89]). The question of whether this equivalence is *decidable* or not for various classes of infinite processes has been the subject of many works in the last ten years. The aim of this work is to show decidability of the bisimulation equivalence for the class of all processes defined by pushdown automata with decreasing ϵ-transitions (of course, we assume that ϵ-transitions are *not* visible, which implies that the graphs of the processes considered here, might have infinite in-degree). This problem was raised in [Cau95] (see Problem 6.2 of this reference) and in [Sti96] (as the bisimulation-problem for processes " of type -1").

Infinite graphs A wide class of graphs enjoying interesting decidability properties has been defined in [Cou89, Bau91, Bau92] (see [Cou90] for a survey). In particular it is known that the problem

$$\text{" are } \Gamma, \Gamma' \text{ isomorphic? "}$$

is decidable for pairs Γ, Γ' of equational graphs. It seems quite natural to investigate whether the problem

$$\text{" are } \Gamma, \Gamma' \text{ bisimilar? "}$$

is decidable for pairs Γ, Γ' of equational graphs. We show here that this problem is decidable for equational graphs of finite out-degree.

Formal languages Another classical equivalence relation between processes is the notion of *language* equivalence . The decidability of language equivalence for *deterministic* pushdown automata has been recently established in [Sén97a] (see also in [Sén97c, Sén97b] shorter expositions of this result). It was first noticed in [BBK87] that , in the case of deterministic processes, language equivalence and bisimulation equivalence are identical. Moreover deterministic pushdown automata can always be normalized (with preservation of the language) in such a way that ϵ-transitions are all decreasing. Hence the main result of this work is a generalisation of the decidability of the equivalence problem for dpda's.

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Mathematical generality More precisely, the present work *extends the notions* developped in [Sén97a] so as to obtain a more general result.
As a by-product of this extension, we obtain a deduction system which, in the deterministic case, seems *simpler* than the one presented in [Sén97a] (see system $\mathcal{B}_3$ in §10).
The present work can also be seen as a common generalization of 3 different results: the results of [Sti96, Jan97] establishing decidability of the bisimulation equivalence in two non-deterministic sub-classes of the class considered here, and the result of [Sén97a] dealing only with deterministic pda's (or processes).

1.2 Result

The main result of this work is the following theorem 10.7:
The bisimulation problem for rooted equational 1-graphs of finite out-degree is decidable.

1.3 Tools

We re-use here the notions developped in [Sén97a] (1-4) and introduce new ideas (5-7):

1. the *deduction systems* (which were in turn inspired by [Cou83a]).
2. the *deterministic boolean series* (which were in turn inspired by [HHY79]).
3. the *deterministic spaces* (which were elaborated around Meitus notion of linear independence ([Mei89, Mei92])).
4. the *analysis* of the proof-trees generated by a suitable strategy (which was somehow similar with the analysis of the parallel computations , interspersed with replacement-moves, done in [Val74, Rom85, Oya87]).
5. the notion of *η-bisimulation* over deterministic row-vectors of boolean series (which , in some sense, translates the usual notion of bisimulation to the d-space of row-vectors of series).
6. the notion of *oracle*, which is a choice of bisimulation for every pair of bisimilar vectors; the notion of triangulation of systems of linear equations is now "parametrized" by such an oracle $\mathcal{O}$ (see §5); as well, the strategies are now parametrized by an oracle too.
7. an *elimination* argument: roughly speaking, this argument shows that, in a proof-tree t, if we take into account not only the *branch* ending at a node x, but also the *whole* proof-tree, then the meta-rule $R5$

   $$\{(p, S, S')\} \mid\!\vdash\!- (p+2, S \odot x, S' \odot x')$$

   is not needed to show that $\mathrm{im}(t) \vdash\!- \{t(x)\}$. A nice (and unexpected) by-product of this elimination is that the *weights* can be removed from the equations (see systems $\mathcal{B}_2, \mathcal{B}_3$ in §10).

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2 Preliminaries

2.1 Graphs

Let X be a finite alphabet. We call *graph over* X any pair $\Gamma = (V_\Gamma, E_\Gamma)$ where V_Γ is a set and E_Γ is a subset of $V_\Gamma \times X \times V_\Gamma$. For every integer $n \in \mathbb{N}$, we call an n-graph every $n+2$-tuple $\Gamma = (V_\Gamma, E_\Gamma, v_1, \ldots, v_n)$ where (V_Γ, E_Γ) is a graph and $(v_1, \ldots, v_n)$ is a sequence of distinguished vertices: they are called the *sources* of Γ.
A 1-graph (V, E, v_1) is said to be *rooted* iff v_1 is a root of (V, E). A 2-graph (V, E, v_1, v_2) is said *bi-rooted* iff v_1 is a root, v_2 is a co-root of (V, E) and there is no edge going out of v_2 (this last technical condition will be useful for reducing the bisimilarity notion for graphs to an analogous notion on series, see §2.1, §2.3 and §3.2).

The *equational* graphs are the least solutions (in a suitable sense) of the systems of (hyperedge) graph-equations (see in [Cou90] precise definitions). Let us mention that the equational graphs of *finite degree* are exactly the *context-free* graphs defined in [MS85].

Bisimulations

Definition 2.1 *Let* $\Gamma = (V_\Gamma, E_\Gamma, v_1, \dots, v_n), \Gamma' = (V_{\Gamma'}, E_{\Gamma'}, v'_1, \dots, v'_n)$ *be two n-graphs over an alphabet* X. *Let* $\mathcal{R}$ *be some binary relation* $\mathcal{R} \subseteq V_\Gamma \times V_{\Gamma'}$.
$\mathcal{R}$ *is a* simulation *from* Γ *to* Γ' *iff*

1. $\mathrm{dom}(\mathcal{R}) = V_\Gamma$,
2. $\forall i \in [1,n], (v_i, v'_i) \in \mathcal{R}$,
3. $\forall v \in V_\Gamma, w \in V_\Gamma, v' \in V_{\Gamma'}, x \in X,$ *such that* $(v,x,w) \in E_\Gamma$ *and* $v\mathcal{R}v'$,
   there exists $w' \in V_{\Gamma'}$ *such that* $(v',x,w') \in E_{\Gamma'}$ *and* $w\mathcal{R}w'$.

$\mathcal{R}$ *is a* bisimulation *from* Γ *to* Γ' *iff* $\mathcal{R}$ *is a simulation from* Γ *to* Γ' *and* $\mathcal{R}^{-1}$ *is a simulation from* Γ' *to* Γ.

This definition corresponds to the standard one ([Par81, Mil89, Cau95]) in the case where $n = 0$. The n-graphs Γ, Γ' are said *bisimilar* , which we denote by $\Gamma \sim \Gamma'$, iff there exists a bisimulation $\mathcal{R}$ from Γ to Γ'.

Let us extend now this definition by means of a relational morphism between free monoids.

Definition 2.2 *Let* X, X' *be two alphabets. A binary relation* $\eta \subseteq X^* \times X'^*$ *is called a strong relational morphism from* X^* *to* X'^* *iff*

1. η *is a submonoid of* $X^* \times X'^*$
2. $\mathrm{dom}(\eta) = X^*, \mathrm{im}(\eta) = X'^*$
3. η *is generated (as a submonoid) by the subset* $\eta \cap (X \times X')$.

One can easily check that s.r. morphisms are preserved by inversion, composition and that any surjective map $\eta : X \to X'$ induces a s.r. morphism from X^* to X'^*. Let $\Gamma = (V_\Gamma, E_\Gamma, v_1, \dots, v_n)$ be an n-graph over the alphabet X, $\Gamma' = (V_{\Gamma'}, E_{\Gamma'}, v'_1, \dots, v'_n)$ be an n-graph over the alphabet X'. Let $\eta \subseteq X^* \times X'^*$ be some s.r. morphism, and let $\mathcal{R}$ be some binary relation $\mathcal{R} \subseteq V_\Gamma \times V_{\Gamma'}$.

Definition 2.3 $\mathcal{R}$ *is a* η-simulation *from* Γ *to* Γ' *iff*

1. $\mathrm{dom}(\mathcal{R}) = V_\Gamma$,
2. $\forall i \in [1,n], (v_i, v'_i) \in \mathcal{R}$,
3. $\forall v, w \in V_\Gamma, v' \in V_{\Gamma'}, x \in X,$ *such that* $(v,x,w) \in E_\Gamma$ *and* $v\mathcal{R}v'$,

$$\exists w' \in V_{\Gamma'}, x' \in \eta(x) \text{ such that } (v',x',w') \in E_{\Gamma'} \text{ and } w\mathcal{R}w'.$$

$\mathcal{R}$ *is a* η-bisimulation *iff* $\mathcal{R}$ *is a* η*-simulation and* $\mathcal{R}^{-1}$ *is a* η^{-1}*- simulation.*

For every $v \in V_\Gamma, v' \in V_{\Gamma'}$, we denote by $v \sim v'$ the fact that there exists some η-bisimulation $\mathcal{R}$ from Γ to Γ' such that $(v, v') \in \mathcal{R}$. In all this work, the composition of binary relations is denoted by $\circ$ and defined by: if $\mathcal{R}_1 \subseteq E \times F$ and $\mathcal{R}_2 \subseteq F \times G$ then,

$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{(x, z) \in E \times G | \exists y \in F, (x, y) \in \mathcal{R}_1, (y, z) \in \mathcal{R}_2\}. \tag{1}$$

Fact 2.4

1. *if* $\mathcal{R}$ *is a* η*-bisimulation, then* $\mathcal{R}^{-1}$ *is a* η^{-1}*-bisimulation*
2. *if* $\mathcal{R}_1$ *is a* η_1*-bisimulation and* $\mathcal{R}_2$ *is a* η_2*-bisimulation, then* $\mathcal{R}_1 \circ \mathcal{R}_2$ *is a* $\eta_1 \circ \eta_2$*-bisimulation*
3. *if for every* $i \in I, \mathcal{R}_i$ *is a* η*-bisimulation, then* $\bigcup_{i \in I} \mathcal{R}_i$ *is a* η*-bisimulation.*

2.2 Pushdown automata

A *pushdown automaton* on the alphabet X is a 7-tuple $\mathcal{M} = < X, Z, Q, \delta, q_0, z_0, F >$ where Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol, F is a finite subset of QZ^*, the set of *final* configurations, and δ, the transition function, is a mapping $\delta : QZ \times (X \cup \{\epsilon\}) \to \mathcal{P}_f(QZ^*)$.

Let $q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, f \in X^*$ and $a \in X \cup \{\epsilon\}$; we note $(qz\omega, af) \longmapsto_{\mathcal{M}} (q'\omega'\omega, f)$ if $q'\omega' \in \delta(qz, a)$. $\overset{*}{\longmapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{M}}$.

For every $q\omega, q'\omega' \in QZ^*$ and $f \in X^*$, we note $q\omega \overset{f}{\longrightarrow}_{\mathcal{M}} q'\omega'$ iff $(q\omega, f) \overset{*}{\longmapsto}_{\mathcal{M}} (q'\omega', \epsilon)$.

$\mathcal{M}$ is said *deterministic* iff, for every $z \in Z, q \in Q, x \in X$:

$$\mathrm{Card}(\delta(qz, \epsilon)) \in \{0, 1\} \tag{2}$$

$$\mathrm{Card}(\delta(qz, \epsilon)) = 1 \Rightarrow \mathrm{Card}(\delta(qz, x)) = 0, \tag{3}$$

$$\mathrm{Card}(\delta(qz, \epsilon)) = 0 \Rightarrow \mathrm{Card}(\delta(qz, x)) \leq 1. \tag{4}$$

$\mathcal{M}$ is said *real-time* iff, for every $q \in Q, z \in Z$, $\mathrm{Card}(\delta(qz, \epsilon)) = 0$.
A configuration $q\omega$ of $\mathcal{M}$ is said ϵ-bound iff there exists a configuration $q'\omega'$ such that $(q\omega, \epsilon) \longmapsto_{\mathcal{M}} (q'\omega', \epsilon)$; $q\omega$ is said ϵ-free iff it is not ϵ-bound.
A pda $\mathcal{M}$ is said *normalized* iff, it fufills conditions (2), (3) (see above) and (5),(6),(7):

$$q_0 z_0 \text{ is } \epsilon - \text{free} \tag{5}$$

and for every $q \in Q, z \in Z, x \in X$:

$$q'\omega' \in \delta(qz, x) \Rightarrow \mid \omega' \mid \leq 2, \tag{6}$$

$$q'\omega' \in \delta(qz, \epsilon) \Rightarrow \mid \omega' \mid = 0 \tag{7}$$

All the pda considered here are assumed to fulfill condition (5). A pda $\mathcal{M}$ will be said *bi-rooted* iff it fulfills (8) and (9):

$$\exists \bar{q} \in Q, F = \{\bar{q}\} \text{ and} \tag{8}$$

$$\forall q \in Q, \omega \in Z^*, f \in X^*, q_0 z_0 \xrightarrow{f}_{\mathcal{M}} q\omega \Rightarrow \exists g \in X^*, q\omega \xrightarrow{g}_{\mathcal{M}} \bar{q}. \tag{9}$$

The *language recognized* by $\mathcal{M}$ is

$$\mathrm{L}(\mathcal{M}) = \{w \in X^* \mid \exists c \in F, q_0 z_0 \xrightarrow{w}_{\mathcal{M}} c\}.$$

It is a "folklore" result that , given a deterministic pda $\mathcal{M}$, one can effectively compute another dpda $\mathcal{M}'$ which is normalized and fulfills:

$$\mathrm{L}(\mathcal{M}) = \mathrm{L}(\mathcal{M}').$$

2.3 Graphs and pushdown automata

Equational graphs and pushdown automata We call *transition-graph* of a pda $\mathcal{M}$, denoted $\mathcal{T}(\mathcal{M})$, the 0-graph:
$\mathcal{T}(\mathcal{M}) = (V_{\mathcal{T}(\mathcal{M})}, E_{\mathcal{T}(\mathcal{M})})$ where $V_{\mathcal{T}(\mathcal{M})} = \{q\omega \mid q \in Q, \omega \in Z^*, q\omega \text{ is } \epsilon - \text{free}\}$ and

$$E_{\mathcal{T}(\mathcal{A})} = \{(c, x, c') \in V_{\mathcal{T}(\mathcal{M})} \times V_{\mathcal{T}(\mathcal{M})} \mid c \xrightarrow{x}_{\mathcal{M}} c'\}. \tag{10}$$

We call *computation 1-graph* of the pda $\mathcal{M}$, denoted $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}})$, the subgraph of $\mathcal{T}(\mathcal{M})$ induced by the set of vertices which are accessible from the vertex $q_0 z_0$, together with the source $v_{\mathcal{M}} = q_0 z_0$. In the case where $\mathcal{M}$ is bi-rooted, we call *computation 2-graph* of the pda $\mathcal{M}$, denoted $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}}, \bar{v}_{\mathcal{M}})$, the graph $\mathcal{C}(\mathcal{M})$ defined just above, together with the sources $v_{\mathcal{M}} = q_0 z_0, \bar{v}_{\mathcal{M}} = \bar{q}$.

Theorem 2.5 *Let $\Gamma = (\Gamma_0, v_0)$ be a rooted 1-graph over X. The following conditions are equivalent:*

1. *Γ is equational and has finite out-degree.*
2. *Γ is isomorphic to the computation 1-graph $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}})$ of some normalized pushdown automaton $\mathcal{M}$.*

The formal proof of this theorem is quite technical and is omitted here.

Corollary 2.6 *Let $\Gamma = (\Gamma_0, v_0, \bar{v})$ be a bi-rooted 2-graph over X. The following conditions are equivalent:*

1. *Γ is equational and has finite out-degree.*
2. *Γ is isomorphic to the computation 2-graph $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}}, \bar{v}_{\mathcal{M}})$ of some bi-rooted normalized pushdown automaton $\mathcal{M}$.*

Bisimulation for non-deterministic (versus deterministic) graphs
In this paragraph, we reduce the classical notion of *bisimulation* for equational graphs to the notion of η-bisimulation for *deterministic* equational graphs, where η has been suitably choosen (see definition 2.3).

Lemma 2.7 *Let Γ_1 be some rooted equational 1-graph over a finite alphabet Y_1 and let $\#$ be a new letter $\# \notin Y_1$. Then one can construct an equational bi-rooted 2-graph Γ over the alphabet $Y = Y_1 \cup \{\#\}$ such that,*

1. $V_{\Gamma_1} \subseteq V_\Gamma$,
2. *for every $v, v' \in V_{\Gamma_1}$, (v, v' are bisimilar in Γ_1) iff (v, v' are bisimilar in Γ),*
3. *Γ_1 has finite out-degree iff Γ has finite out-degree.*

Sketch of proof: Let us define Γ_1 from Γ by:

$$V_\Gamma = V_{\Gamma_1} \cup \{\bar{v}\}, \quad E_\Gamma = E_{\Gamma_1} \cup \{(w, \#, \bar{v}) \mid w \in V_{\Gamma_1}\}, \quad \Gamma = (\Gamma_1, \bar{v}),$$

where $\bar{v}$ is a new vertex $\bar{v} \notin V_{\Gamma_1}$. One can easily check that Γ is equational iff Γ_1 is equational and that , provided Γ_1 is rooted, Γ is bi-rooted. Points (1) and (3) of the lemma are clear. One can check that the mapping $\mathcal{R} \mapsto \mathcal{R} \cup \{(\bar{v}, \bar{v})\}$ is a bijection from the set of all the bisimulations over Γ_1 (i.e. from Γ_1 to Γ_1) to the set of all the bisimulations over Γ. Hence point (2) is true. $\Box$

Let us consider finite aphabets X, Y , a length-preserving homomorphism $\psi : X^* \to Y^*$ and the s.r. morphism $\bar{\psi} = \psi \circ \psi^{-1} \subseteq X^* \times X^*$. A n-graph Γ over X will be said *$\bar{\psi}$-saturated* iff, for every $v \in V_\Gamma$, for every $(x, x') \in \bar{\psi}$,

$$(\exists v_1 \in V_\Gamma, (v, x, v_1) \in E_\Gamma) \Leftrightarrow (\exists v_1' \in V_\Gamma, (v, x', v_1') \in E_\Gamma).$$

Lemma 2.8 *Let Γ_1 be an equational bi-rooted 2-graph of finite out-degree over an alphabet Y. One can construct a finite alphabet X, a surjective length-preserving homomorphism $\psi : X^* \to Y^*$ and an equational, bi-rooted 2-graph Γ over the alphabet X, such that*

1. *Γ is deterministic,*
2. *Γ is $\bar{\psi}$-saturated,*
3. $V_{\Gamma_1} = V_\Gamma$,
4. *$\mathrm{Id} : V_\Gamma \to V_{\Gamma_1}$ is a ψ-bisimulation from Γ to Γ_1.*

Sketch of proof: By lemma 2.6, we can suppose that Γ_1 is the computation 2-graph $(\mathcal{C}(\mathcal{M}_1), v_{\mathcal{M}_1}, \bar{v}_{\mathcal{M}_1})$ of some bi-rooted normalized pushdown automaton $\mathcal{M}_1 =< Y, Z, Q, \delta_1, q_0, z_0, \{\bar{q}\} >$. Let us consider the following integers:

$$\forall q \in Q, \forall z \in Z, \forall y \in Y, t_1(qz, y) = \mathrm{Card}(\delta_1(qz, y)), \quad \bar{t_1} = \max\{t_1(qz, y) \mid q \in Q, z \in Z, y \in Y\}.$$

Let $X = Y \times [1, \bar{t_1}]$ and let $\psi : X \to Y$ be the first projection.
Let $\rho : QZ \times Y \times \mathbb{N} \to QZ^*$ such that $\mathrm{dom}(\rho) = \bigcup_{q \in Q, z \in Z, y \in Y} \{qz\} \times \{y\} \times [1, t_1(qz, y)]$ and

$$\rho(qz, y, \star) : \{qz\} \times \{y\} \times [1, t_1(qz, y)] \to \delta_1(qz, y)$$

is a bijection (for every q, z, y)). We then define $\mathcal{M} =< X, Z, Q, \delta, q_0, z_0, \{\bar{q}\} >$ by: for every $q \in Q, z \in Z, y \in Y, i \in [1, \bar{t_1}]$

$$\delta(qz, \epsilon) = \delta_1(qz, \epsilon) \text{ if } qz \text{ is } \epsilon-\text{bound}.$$

$\delta(qz, (y, i)) = \{q'\omega'\}$ if $\rho(qz, y, i) = q'\omega'$ or $(1 \leq t_1(qz, y) < i \leq \bar{t_1}$ and $\rho(qz, y, 1) = q'\omega')$.

The 2-graph $\Gamma = (\mathcal{C}(\mathcal{M}), v_{\mathcal{M}}, \bar{v}_{\mathcal{M}})$ fulfills the required properties. □

Let us remark that, by point (4) and by composition of η-bisimulations, for every $v, v' \in V_\Gamma$, v, v' are $\bar{\psi}$-bisimilar (w.r.t. Γ) iff v, v' are bisimilar (w.r.t. Γ_1).

2.4 Deterministic context-free grammars

Let $\mathcal{M}$ be some deterministic pushdown automaton (we suppose here that $\mathcal{M}$ is normalized). The *variable* alphabet $V_{\mathcal{M}}$ associated to $\mathcal{M}$ is defined as:

$$V_{\mathcal{M}} = \{[p, z, q] \mid p, q \in Q, z \in Z\}.$$

The *context-free* grammar $G_{\mathcal{M}}$ associated to $\mathcal{M}$ is then

$$G_{\mathcal{M}} =< X, V, P >$$

where
$V = V_{\mathcal{M}}$,
P is the set of all the pairs of one of the following forms:

$$([p, z, q], x[p', z_1, p''][p'', z_2, q]) \tag{11}$$

where $p, q, p', p'' \in Q, x \in X, p'z_1z_2 \in \delta(pz, x)$

$$([p, z, q], x[p', z', q]) \tag{12}$$

where $p, q, p' \in Q, x \in X, p'z' \in \delta(pz, x)$

$$([p, z, q], a) \tag{13}$$

where $p, q, \in Q, a \in X \cup \{\epsilon\}, q \in \delta(pz, a)$. $G_{\mathcal{M}}$ is a *strict-deterministic* grammar. (A general theory of this class of grammars is exposed in [Har78] and used in [HHY79]).

We call *mode* every element of $QZ \cup \{\epsilon\}$. For every $q \in Q, z \in Z$, qz is said *ϵ-bound* (respectively ϵ-free) iff condition (3) (resp. condition (4)) in the above definition of deterministic automata is realized. The mode ϵ is said ϵ-free. We define a mapping $\mu : V^* \to QZ \cup \{\epsilon\}$ by

$$\mu(\epsilon) = \epsilon \text{ and } \mu([p, z, q] \cdot \beta) = pz,$$

for every $p, q \in Q, z \in Z, \beta \in V^*$. For every $w \in V^*$ we call $\mu(w)$ the *mode* of the word w.

For technical reasons (which will be made clear in section 7), we suppose that Z contains a special symbol e such that, for every $q \in Q, \delta(qe, \epsilon) = \{q\}$ and $\mathrm{im}(\delta) \subseteq \mathcal{P}_f(Q(Z - \{e\})^*)$. Equivalently,

$$\forall q \in Q, ([q, e, q], \epsilon) \in P, \tag{14}$$

and

$$\forall (v, w) \in P, w \in (V - \{[q, e, q] | q \in Q\})^*. \tag{15}$$

2.5 Free monoids acting on semi-rings

Semi-ring $\mathsf{B} < W >$ Let $(\mathsf{B}, +, \cdot, 0, 1)$ where $\mathsf{B} = \{0, 1\}$ denote the semi-ring of "booleans". Let W be some alphabet. By $(\mathsf{B} < W >, +, \cdot, \emptyset, \epsilon)$ we denote the semi-ring of *boolean series* over W:
the set $\mathsf{B} < W >$ is defined as B^{W^*}; the sum and product are defined as usual; each word $w \in W^*$ can be identified with the element of B^{W^*} mapping the word w on 1 and every other word $w' \neq w$ on 0; every boolean series $S \in \mathsf{B} < W >$ can then be written in a unique way as:

$$S = \Sigma_{w \in W^*} S_w \cdot w,$$

where, for every $w \in W^*$, $S_w \in \mathsf{B}$.
The *support* of S is the language

$$\mathrm{supp}(S) = \{w \in W^* \mid S_w \neq 0\}.$$

In the particular case where the semi-ring of coefficients is B (which is the only case considered in this article) we sometimes identify the series S with its support. The usual ordering $\leq$ on B extends to $\mathsf{B} < W >$ by:

$$S \leq S' \text{ iff } \forall w \in W^*, S_w \leq S'_w.$$

We recall that for every $S \in \mathsf{B} < W >$, S^* is the series defined by:

$$S^* = \sum_{0 \leq n} S^n. \tag{16}$$

Given two alphabets W, W', a map $\psi : \mathsf{B} < W > \rightarrow \mathsf{B} < W' >$ is said *σ-additive* iff it fulfills: for every denumerable family $(S_i)_{i \in \mathbb{N}}$ of elements of $\mathsf{B} < W >$,

$$\psi(\sum_{i \in \mathbb{N}} S_i) = \sum_{i \in \mathbb{N}} \psi(S_i). \tag{17}$$

A map $\psi : \mathsf{B} < W > \rightarrow \mathsf{B} < W' >$ which is both a semi-ring homomorphism and a σ-additive map is usually called a *substitution.*

Actions of monoids Given a semi-ring $(\mathsf{S}, +, \cdot, 0, 1)$ and a monoid $(\mathsf{M}, \cdot, 1_M)$, a map $\circ : \mathsf{S} \times \mathsf{M} \rightarrow \mathsf{S}$ is called a *right-action* of the monoid M over the semi-ring S iff, for every $S, T \in \mathsf{S}, m, m' \in \mathsf{M}$:

$$0 \circ m = 0,\ \ S \circ 1_M = S,\ \ (S+T) \circ m = (S \circ m) + (T \circ m) \quad \text{and } S \circ (m \cdot m') = (S \circ m) \circ m'. \tag{18}$$

In the particular case where $\mathsf{S} = \mathsf{B} < W >$, $\circ$ is said to be a σ-right-action if it fulfills the additional property that, for every denumerable family $(S_i)_{i \in \mathbb{N}}$ of elements of S and $m \in \mathsf{M}$:

$$(\sum_{i \in \mathbb{N}} S_i) \circ m = \sum_{i \in \mathbb{N}} (S_i \circ m). \tag{19}$$

The action of W^* on $\mathsf{B} < W >$ We recall the following classical σ-right-action $\bullet$ of the monoid W^* over the semi-ring $\mathsf{B} < W >$: for all $S, S' \in \mathsf{B} < W >, u \in W^*$

$$S \bullet u = S' \Leftrightarrow \forall w \in W^*, (S'_w = S_{u \cdot w}),$$

(i.e. $S \bullet u$ is the *left-quotient* of S by u , or the *residual* of S by u).

For every $S \in \mathsf{B} < W >$ we denote by $\mathsf{Q}(S)$ the set of residuals of S:

$$\mathsf{Q}(S) = \{S \bullet u \mid u \in W^*\}.$$

We recall that S is said *rational* iff the set $\mathsf{Q}(S)$ is *finite*. We define the *norm* of a series $S \in \mathsf{B} < W >$, denoted $\|S\|$ by:

$$\|S\| = \mathrm{Card}(\mathsf{Q}(S)) \in \mathbb{N} \cup \{\infty\}.$$

The action of X^* on $\mathsf{B} < V >$ Let us fix now a deterministic (normalized) pda $\mathcal{M}$ and consider the associated grammar G. We define a σ-right-action $\otimes$ of the monoid $(X \cup \{e\})^*$ over the semi-ring $\mathsf{B} < V >$ by: for every $p, q \in Q, A \in Z, H \in \mathsf{B} < V >, \beta \in V^*, x \in X$

$$[p, A, q] \cdot \beta \otimes x = H \cdot \beta \text{ iff } H = \sum_{([p,A,q], x \cdot h) \in P_{\mathcal{M}}} h, \tag{20}$$

$$[p, A, q] \cdot \beta \otimes e = \beta \text{ iff } ([p, A, q], \epsilon) \in P_{\mathcal{M}}, \tag{21}$$

$$[p, A, q] \cdot \beta \otimes e = \emptyset \text{ iff } (\{[p, A, q]\} \times V^*) \cap P_{\mathcal{M}} = \emptyset, \tag{22}$$

$$\epsilon \otimes x = \emptyset, \quad \epsilon \otimes e = \emptyset. \tag{23}$$

A series $S \in \mathsf{B} < V >$ is said *ϵ-free* iff $\forall w \in V^*, S_w = 1 \Rightarrow \mu(w)$ is $\epsilon -$ free. We denote by $\mathsf{B}_\epsilon < V >$ the subset of ϵ-free series. We define the map $\rho_\epsilon : \mathsf{B} < V > \rightarrow \mathsf{B} < V >$ as the unique σ-additive map such that,

$$\rho_\epsilon(\emptyset) = \emptyset, \quad \rho_\epsilon(\epsilon) = \epsilon,$$

and for every $p \in Q, z \in Z, q \in Q, \beta \in V^*$,

$$\rho_\epsilon([p, z, q] \cdot \beta) = \rho_\epsilon(([p, z, q] \otimes e) \cdot \beta) \text{ if } pz \text{ is } \epsilon - \text{bound and,}$$

$$\rho_\epsilon([p, z, q] \cdot \beta) = [p, z, q] \cdot \beta \text{ if } pz \text{ is } \epsilon - \text{free.}$$

The above definition is sound because, by hypothesis (7), every $[p, z, q] \otimes e$ is either the unit series ϵ or the empty series $\emptyset$. One can notice that for every $w \in V^*$, $\rho_\epsilon(w) \in V^* \cup \{\emptyset\}$. We call ρ_ϵ the ϵ-reduction map. We then define $\odot$ as the unique σ-right-action of the monoid X^* over the semi-ring $\mathsf{B} < V >$ such that: for every $S \in \mathsf{B} < V >, x \in X$,

$$S \odot x = \rho_\epsilon(\rho_\epsilon(S) \otimes x).$$

One can notice that if $u \neq \epsilon$, then $S \odot u$ is ϵ-free. Let us consider the unique substitution $\varphi : \mathsf{B} < V > \rightarrow \mathsf{B} < X >$ fulfilling: for every $p, q \in Q, z \in Z$,

$$\varphi([p, z, q]) = \{u \in X^* \mid [p, z, q] \odot u = \epsilon\},$$

(in other words, φ maps every subset $L \subseteq V^*$ on the language generated by the grammar G from the set of axioms L).

Lemma 2.9 *For every* $S \in \mathsf{B} < V >, u \in X^*$,

1. $\varphi(S) = \varphi(\rho_\epsilon(S))$,
2. $\varphi(S \odot u) = \varphi(S) \bullet u$ *(i.e.* φ *is a morphism of right-actions).*

Proof: Let $p, q \in Q, z \in Z, \beta \in V^*, x \in X$. One can check on formulas (20 - 23) that:

– if $[p, z, q]$ is ϵ-bound, then

$$\varphi(([p, z, q] \cdot \beta) \otimes e) = \varphi([p, z, q] \cdot \beta)$$

– if $[p, z, q]$ is ϵ-free, then

$$\varphi(([p, z, q] \cdot \beta) \otimes x) = \varphi([p, z, q] \cdot \beta) \bullet x.$$

By induction, it follows that, $\forall w \in V^*$,

$$\varphi(\rho_\epsilon(w)) = \varphi(w), \quad \varphi(w \odot x) = \varphi(w) \bullet x.$$

By σ-additivity of φ, the lemma follows. □

We denote by $\equiv$ the kernel of φ i.e.: for every $S, T \in \mathsf{B} < V >$,

$$S \equiv T \Leftrightarrow \varphi(S) = \varphi(T).$$

3 Series and matrices

3.1 Deterministic series and matrices

We introduce here a notion of *deterministic* series which, in the case of the alphabet V associated to a dpda $\mathcal{M}$, generalizes the classical notion of *configuration* of $\mathcal{M}$. The main advantage of this notion is that, unlike for configurations, we shall be able to define *nice algebraic operations* on these series (this is done in section 3.3). Let us consider a pair $(W, \smile)$ where W is an alphabet and $\smile$ is an equivalence relation over W. We call $(W, \smile)$ a *structured* alphabet. The two examples we have in mind are:

– the case where $W = V$, the variable alphabet associated to $\mathcal{M}$ and $[p, A, q] \smile [p', A', q']$ iff $p = p'$ and $A = A'$ (see [Har78])
– the case where $W = X$, the terminal alphabet of $\mathcal{M}$ and $x \smile y$ holds for every $x, y \in X$ (see [Har78]).

Definitions

Definition 3.1 *Let* $S \in \mathsf{B} < W >$. *S is said* left-deterministic *iff either*

(1) $S = \emptyset$ *or*
(2) $S = \epsilon$ *or*
(3) $\exists i_0 \in [1, m], S_{i_0} \neq \emptyset$ *and* $\forall w, w' \in W^*$,

$$S_w = S_{w'} = 1 \Rightarrow [\exists A, A' \in W, w_1, w_1' \in W^*, A \smile A', w = A\cdot w_1 \text{ and } w' = A'\cdot w_1'].$$

A left-deterministic series S is said to have the type $\emptyset$ (resp. ϵ, $[A]\smile$) if case (1) (resp. (2), (3)) occurs.

Definition 3.2 *Let* $S \in \mathsf{B} < W >$. *S is said* deterministic *iff, for every* $u \in W^*$, $S \bullet u$ *is left-deterministic.*

This notion is the straighforward extension to the infinite case of the notion of (finite) *set of associates* defined in [HHY79].
We denote by $\mathsf{DB} < W >$ the subset of deterministic boolean series over W. Let us denote by $\mathsf{B}_{n,m} < W >$ the set of (n, m)-matrices with entries in the semi-ring $\mathsf{B} < W >$.

Definition 3.3 *Let* $m \in \mathbb{N}, S \in \mathsf{B}_{1,m} < W >$: $S = (S_1, \cdots, S_m)$. *S is said* left-deterministic *iff either*

(1) $\forall i \in [1, m], S_i = \emptyset$ *or*
(2) $\exists i_0 \in [1, m], S_{i_0} = \epsilon$ *and* $\forall i \neq i_0, S_i = \emptyset$ *or*
(3) $\forall w, w' \in W^*, \forall i, j \in [1, m], (S_i)_w = (S_j)_{w'} = 1 \Rightarrow [\exists A, A' \in W, w_1, w_1' \in V^*, A \smile A', w = A \cdot w_1$ *and* $w' = A' \cdot w_1']$.

A left-deterministic row-vector S is said to have the type $\emptyset$ (resp. (ϵ, i_0), $[A]\smile$) if case (1) (resp. (2), (3)) occurs.

The right-action $\bullet$ on $\mathsf{B} < W >$ is extended componentwise to $\mathsf{B}_{n,m} < W >$: for every $S = (s_{i,j})$, $u \in W^*$, the matrix $T = S \bullet u$ is defined by

$$t_{i,j} = s_{i,j} \bullet u.$$

The ordering $\leq$ on B is also extended componentwise to $\mathsf{B}_{n,m} < W >$.

Definition 3.4 *Let* $S \in \mathsf{B}_{1,m} < W >$. *S is said* deterministic *iff, for every* $u \in W^*$, $S \bullet u$ *is left-deterministic.*

We denote by $\mathsf{DB}_{1,m} < W >$ the subset of deterministic row-vectors of dimension m over $\mathsf{B} < W >$.

Definition 3.5 *Let* $S \in \mathsf{B}_{n,m} < W >$. *S is said* deterministic *iff, for every* $i \in [1, n]$, $S_{i,.}$ *is a deterministic row-vector.*

Let us notice first some easy facts about deterministic matrices.

Fact 3.6 *Let* $S \in \mathsf{DB} < W >$. *For every* $T \in \mathsf{B} < W >, u \in W^*$

(1) $T \leq S \Rightarrow T \in \mathsf{DB} < W >$
(2) $T = S \bullet u \Rightarrow T \in \mathsf{DB} < W >$

Residuals

Lemma 3.7 *Let* $S \in \mathsf{DB} < W >, T \in \mathsf{B} < W >, u \in W^*$. *If* $S \bullet u \neq \emptyset$ *then* $(S \cdot T) \bullet u = (S \bullet u) \cdot T$.

Proof: Let $S \in \mathsf{DB} < W >, T \in \mathsf{B} < W >, u \in W^*$, such that $S \bullet u \neq \emptyset$. Let $u', u'' \in W^*$ such that $u = u' \cdot u'', u'' \neq \epsilon$ and let $w \in \mathrm{supp}(S)$. If $w \bullet u' = \epsilon$ then $S \bullet u' = \epsilon$ (because $S \bullet u'$ is left-deterministic), hence $S \bullet u = \epsilon \bullet u'' = \emptyset$, which would contradict the hypothesis. It follows that

$$\forall u' \prec u, \forall w \in \mathrm{supp}(S), w \bullet u' \neq \epsilon.$$

Hence

$$\forall w_1 \in \mathrm{supp}(S), \forall w_2 \in \mathrm{supp}(T), (w_1 \cdot w_2) \bullet u = (w_1 \bullet u) \cdot w_2.$$

This proves that $(S \cdot T) \bullet u = (S \bullet u) \cdot T$. □

Lemma 3.8 *Let* $S \in \mathsf{DB} < W >, T \in \mathsf{B} < W >, u \in W^*$ *and* $U = S \cdot T$. *Exactly one of the following cases is true:*

(1) $S \bullet u \neq \emptyset$;
in this case $U \bullet u = (S \bullet u) \cdot T$.
(2) $S \bullet u = \emptyset, \exists u', u'', u = u' \cdot u'', S \bullet u' = \epsilon$;
in this case $U \bullet u = T \bullet u''$.
(3) $S \bullet u = \emptyset, \forall u' \preceq u, S \bullet u' \neq \epsilon$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Proof: Clearly, one of the hypotheses (1-3) must occur. Let us examine each one of these cases.
In case (1), by lemma 3.7, $U \bullet u = (S \bullet u) \cdot T$.
In case (2), $U \bullet u = (U \bullet u') \bullet u''$ and by case (1), $U \bullet u' = (S \bullet u') \cdot T$. It follows that $U \bullet u = T \bullet u''$.
In case (3), if $S = \emptyset$, the conclusion of the lemma is clearly true. Let us suppose now that $S \neq \emptyset$ and let $u' \prec u$ be the maximum prefix of u such that $S \bullet u' \neq \emptyset$. Then, there exist some $A \in W, u'' \in W^*$such that $u = u' \cdot A \cdot u''$ and there exist some $B_1, \cdots, B_q \in W, S_1, \cdots, S_q \in \mathsf{B} < W > -\{\emptyset\}$ such that $S \bullet u' = \sum_{1 \leq i \leq q} B_q \cdot S_q$ and $B_1 \smile \cdots \smile B_i \smile \cdots \smile B_q$(because $S \bullet u'$ is left-deterministic). By maximality of u', A does not belong to $\{B_1, \cdots, B_q\}$, hence

$$U \bullet u = ((\sum_{1 \leq i \leq q} B_q \cdot S_q) \bullet A) \bullet u'' = \emptyset \bullet u'' = \emptyset.$$

□

Lemma 3.9 *Let* $S \in \mathsf{DB}_{1,m} < W >, T \in \mathsf{B}_{m,1} < W >, u \in W^*$ *and* $U = S \cdot T$. *Exactly one of the following cases is true:*

(1) $\exists j, S_j \bullet u \notin \{\emptyset, \epsilon\}$;
in this case $U \bullet u = (S \bullet u) \cdot T$.
(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \bullet u' = \epsilon$;
in this case $U \bullet u = T_{j_0} \bullet u''$.
(3) $\forall j, S_j \bullet u = \emptyset, \forall u' \preceq u, S_j \bullet u' \neq \epsilon$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Proof: Let us note $S = (S_j)_{1 \leq j \leq m}, T = (T_j)_{1 \leq j \leq m}$. Clearly, one of the hypotheses (1-3) must occur. Let us examine each one of these cases.
In case (1), every 3-tuple (S_j, T_j, u) fulfills case (1) or (3) of lemma 3.8, hence $(S_j \cdot T_j) \bullet u = (S_j \bullet u) \cdot T_j$. Hence

$$U \bullet u = \sum_{1 \leq j \leq m} (S_j \cdot T_j) \bullet u = \sum_{1 \leq j \leq m} (S_j \bullet u) \cdot T_j = (S \bullet u) \cdot T.$$

In case (2), $S \bullet u'$ must be left-deterministic of type (ϵ, j_0) , hence $\forall j \neq j_0, S_j \bullet u' = \emptyset$. It follows that

$$U \bullet u = T_{j_0} \bullet u''.$$

In case (3), every 3-tuple (S_j, T_j, u) fulfills case (3) of lemma 3.8, hence $(S_j \cdot T_j) \bullet u = \emptyset = (S_j \bullet u) \cdot T_j$. It follows that

$$U \bullet u = \emptyset = (S \bullet u) \cdot T.$$

□

Lemma 3.10 *Let* $S \in \mathsf{DB}_{1,m} < W >, T \in \mathsf{B}_{m,s} < W >, u \in W^*$ *and* $U = S \cdot T$. *Exactly one of the following cases is true:*

(1) $\exists j, S_j \bullet u \notin \{\emptyset, \epsilon\}$
in this case $U \bullet u = (S \bullet u) \cdot T$.
(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \bullet u' = \epsilon$;
in this case $U \bullet u = T_{j_0} \bullet u''$.
(3) $\forall j, \forall u' \preceq u, S_j \bullet u = \emptyset, S_j \bullet u' \neq \epsilon$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Proof: Let us notice that for every $k \in [1, s]$:

$$U_k = S \cdot T_{*,k}, \tag{24}$$

and that the hypothesis of the 3 cases considered in lemma 3.9 depend on the vector S and the word u only (but not on the integer $k \in [1, s]$). In case (1), by lemma 3.9,$\forall k \in [1, s]$

$$U_k \bullet u = (S \bullet u) \cdot T_{*,k},$$

hence $U \bullet u = (S \bullet u) \cdot T$. Cases 2,3 can be treated in the same way. □

Lemma 3.11 *For every* $S \in \mathsf{B}_{n,m} < W >, T \in \mathsf{B}_{m,s} < W >$, *if* S *and* T *are both left-deterministic, then* $S \cdot T$ *is left-deterministic.*

Lemma 3.12 *For every* $S \in \mathsf{DB}_{n,m} < W >, T \in \mathsf{DB}_{m,s} < W >, S \cdot T \in \mathsf{DB}_{n,s} < W >$.

Proof: As the notion of deterministic matrix is defined row by row, it is sufficient to prove this lemma in the particular case where $n = 1$. Let us note $U = S \cdot T$. Let $u \in W^*$. Let us show that $U \bullet u$ is left-deterministic. Let us consider every one of the 3 cases considered in lemma 3.10 . In case (1) or (3),

$$U \bullet u = (S \bullet u) \cdot T,$$

and in case (2),

$$U \bullet u = T \bullet u''.$$

In both cases, by lemma 3.11, $U \bullet u$ is left-deterministic. □

W=V Let $(W, \smile)$ be the structured alphabet $(V, \smile)$ associated with $\mathcal{M}$ and let us consider a bijective numbering of the elements of Q: $(q_1, q_2, \ldots, q_{n_Q})$. Some particular "vectorial" notions turn out to be useful:

- we define a Q-*series* as a family $S = (S_q)_{q \in Q}$ such that the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ is deterministic
- we define a Q-*form* as a family $\Phi = (\Phi_q)_{q \in Q}$ of deterministic series; more generally a Q-λ-*form* (where $\lambda \in \mathbb{N} - \{0\}$) is a family of deterministic row-vectors: $\Phi = (\Phi_q)_{q \in Q}$ with $\Phi_q \in \mathsf{DB}_{1,\lambda} < V >$ for every $q \in Q$.

Given a Q-series S and a Q-λ-form Φ, their Q-product $S * \Phi$ is the deterministic row-vector defined by

$$S * \Phi = \sum_{q \in Q} S_q \cdot \Phi_q.$$

Given the above ordering of the elements of Q, one can identify the Q-series $(S_q)_{q \in Q}$ with the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ and the Q-λ-form $(\Phi_q)_{q \in Q}$ with the n_q-λ-matrix:

$$\begin{pmatrix} \Phi_{q_1} \\ \vdots \\ \Phi_{q_j} \\ \vdots \\ \Phi_{q_{n_Q}} \end{pmatrix}$$

The Q-product appears then to be just the ordinary product of matrices.

Let us define here handful notations for some particular row-vectors or Q-series. Let us use the *Kronecker symbol* $\delta_{i,j}$ meaning ϵ if $i = j$ and $\emptyset$ if $i \neq j$. For every $1 \leq n, 1 \leq i \leq n$, we define the row-vector ϵ_i^n as:

$$\epsilon_i^n = (\epsilon_{i,j}^n)_{1 \leq j \leq n} \text{ where } \forall j, \epsilon_{i,j}^n = \delta_{i,j}.$$

We call *unit row-vector* any vector of the form ϵ_i^n.
For every $1 \leq n, 1 \leq m$, we denote by $\emptyset^n \in \mathsf{DB}_{1,n} < V >$ the row-vector:

$$\emptyset^n = (\emptyset, \ldots, \emptyset),$$

and we denote by $\emptyset_m^n \in \mathsf{DB}_{m,n} < V >$ the matrix:

$$\begin{pmatrix} \emptyset^n \\ \vdots \\ \emptyset^n \\ \vdots \\ \emptyset^n \end{pmatrix}$$

For every $\omega \in Z^*, p, q \in Q$, $[p\omega q]$ is the deterministic series defined inductively by:

$$[p\epsilon q] = \emptyset \text{ if } p \neq q, [p\epsilon q] = \epsilon \text{ if } p = q,$$

$$[p\omega q] = \sum_{r \in Q} [pAr] \cdot [r\omega' q] \text{ if } \omega = A \cdot \omega' \text{ for some } A \in Z, \omega' \in Z^*.$$

By $[p\omega]$ we denote the Q-series:

$$[p\omega] = ([p\omega q])_{q \in Q}.$$

(In particular $[q_i] = \epsilon_i^{n_Q}$). These Q-series represent faithfully the configurations of $\mathcal{M}$ in the sense that, for every ϵ-free configurations $q\omega, q'\omega'$ and word $f \in X^*$,

$$q\omega \xrightarrow{f}_{\mathcal{M}} q'\omega' \text{ iff } [q\omega] \odot f = [q'\omega']. \tag{25}$$

By $[\omega]$ we denote the Q-Q-matrix:

$$[\omega] = ([p\omega q])_{p \in Q, q \in Q} \text{ such that for every } p, q, \Omega_{p,q} = [p\omega q].$$

Here also we identify $[\omega]$ with the matrix $([p_i \omega q_j])_{1 \leq i \leq n_Q, 1 \leq j \leq n_Q} \in \mathsf{DB}_{n_Q, n_Q} < V >$. The next lemmas relate the mapping ρ_ϵ and right-action $\odot$ with the right-action $\bullet$.

Lemma 3.13 *Let $S \in \mathsf{DB}_{1,\lambda} < V >$.*

1. *there exists $v \in V^*$ such that $\rho_\epsilon(S) = S \bullet v$.*
2. *$\rho_\epsilon(S) \equiv S$.*

Proof: We treate first the case where $\lambda = 1$ i.e. S is a series.
If for every $w \in \text{supp}(S), \rho_\epsilon(w) = \emptyset$, then $\rho_\epsilon(S) = \emptyset$, which is a residual of S, hence point (1) of the lemma is true. Moreover, in this case every $w \in \text{supp}(S)$ contains a letter $[p, z, q]$ which is ϵ-bound and such that $[p, z, q] \otimes e = \emptyset$, hence $S \equiv \emptyset = \rho_\epsilon(S)$, which establishes point (2) of the lemma.
Let us suppose now that there exists some $w_0 \in \text{supp}(S)$ such that $\rho_\epsilon(w_0) = w' \in V^*$. Then $w_0 = [p_1, z_1, q_1] \cdots [p_n, z_n, q_n] \cdot w'$, where $n \geq 0$ and for every

$i \in [1,n]$, $[p_i, z_i, q_i] \otimes e = \epsilon$. Let us set $v = [p_1, z_1, q_1] \cdots [p_n, z_n, q_n]$. We consider the set of words: $D(v) = \{v' \cdot [p_{j+1}, z_{j+1}, q'_{j+1}], 0 \leq j \leq n-1, q'_{j+1} \in Q, v' = v(j), q'_{j+1} \neq q_{j+1}\}$, and we set $S' = \sum_{w \in D(v)} w \cdot (S \bullet w)$. It is clear that, as S is deterministic:

$$S = v \cdot (S \bullet v) + S'. \tag{26}$$

Moreover one can check that, for every $w \in D(v)$, $\rho_\epsilon(w) = \emptyset$ (because the letters $[p_{j+1}, z_{j+1}, q'_{j+1}]$ fulfill $\rho_\epsilon([p_{j+1}, z_{j+1}, q'_{j+1}]) = \emptyset$). Hence $\rho_\epsilon(S') = \emptyset$. As S is deterministic, $S \bullet v$ must be left-deterministic of the same type as w', hence $S \bullet v$ is ϵ-free. Using now (26) we obtain:

$$\rho_\epsilon(S) = S \bullet v + \emptyset = S \bullet v. \tag{27}$$

Point (1) is then proved. Applying φ to the two members of equation (26) and using point (1) we obtain that

$$\varphi(S) = \varphi(v \cdot \rho_\epsilon(S)) = \varphi(v) \cdot \varphi(\rho_\epsilon(S)).$$

But, by the hypothesis on the letters $[p_i, z_i, q_i]$, $\varphi(v) = \epsilon$. It follows that $\varphi(S) = \varphi(\rho_\epsilon(S))$ i.e. point (2) is true.
Let us treate now the general case. Let $S = (S_1, \ldots, S_j, \ldots, S_\lambda)$. Let us consider $\bar{S} = \Sigma_{j=1}^{\lambda} S_j$. Let us apply the above arguments (and notations) on $\bar{S}$.
case 1: $\forall w \in \text{supp}(\bar{S}), \rho_\epsilon(w) = \emptyset$.
In that case $\rho_\epsilon(S) = \emptyset^\lambda = S \bullet v$ (for some $v \in V^*$) and $S \equiv \rho_\epsilon(S)$.
case 2: $\exists w_0 \in \text{supp}(\bar{S}), \rho_\epsilon(w_0) = w' \in V^*$.
For every $j \in [1, \lambda]$,

$$S_j = v \cdot (S_j \bullet v) + \Sigma_{w \in D(v)} w \cdot (S_j \bullet w). \tag{28}$$

where $\rho_\epsilon(v) = \epsilon, \rho_\epsilon(\Sigma_{w \in D(v)} w \cdot (S_j \bullet w)) = \emptyset$ and $S_j \bullet v$ is ϵ-free. It follows that for every $j \in [1, \lambda]$, $\rho_\epsilon(S_j) = S_j \bullet v$ hence

$$\rho_\epsilon(S) = S \bullet v. \tag{29}$$

We also know that $\varphi(v) = \epsilon, \varphi(\Sigma_{w \in D(v)} w \cdot (S_j \bullet w)) = \emptyset$, which together with (28) shows that:

$$\varphi(S) = \varphi(S \bullet v).$$

Hence points (1,2) of the lemma are proved. □

Remark 3.14 *Point (2) of the lemma is also a direct corollary of point (1) of lemma 2.9. The proof given here for point (2) will be re-used in the proof of lemma 4.7.*

Corollary 3.15

1. $\forall \lambda \in \mathbb{N} - \{0\}, \forall S \in \mathsf{DB}_{1,\lambda} < V >, \rho_\epsilon(S) \in \mathsf{DB}_{1,\lambda} < V >$.
2. $\forall \lambda \in \mathbb{N} - \{0\}, \forall S \in \mathsf{DRB}_{1,\lambda} < V >, \rho_\epsilon(S) \in \mathsf{DRB}_{1,\lambda} < V >$.

Lemma 3.16 *Let $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DB}_{1,\lambda} < V >, u \in (X \cup \{e\})^*$. One of the three following cases must occur:*

(1) $S \otimes u = \emptyset^\lambda$,
(2) $S \otimes u = \epsilon_j^\lambda$ for some $j \in [1, \lambda]$,
(3) $\exists u_1, u_2 \in (X \cup \{e\})^*, v_1 \in V^*, p, q \in Q, A \in Z, \omega \in Z^*, \Phi\ Q - \lambda - form$ such that

$$u = u_1 \cdot u_2, S \otimes u_1 = S \bullet v_1 = [qA] * \Phi, \quad S \otimes u = ([qA] \otimes u_2) * \Phi, \text{ and}$$

$$[qA] \otimes u_2 = [p\omega] \text{ with } |\omega| \geq 1.$$

Proof: Let $u \in (X \cup \{e\})^*$. Let us prove the lemma by induction on $|u|$.
$u = \epsilon$:
if $S \in \emptyset^\lambda \cup \{\epsilon_j^\lambda | 1 \leq j \leq \lambda\}$ then clearly the conclusion of case (1) or (2) is realized. Otherwise, S has a decomposition as $S = [qA] * \Phi$ and the conclusion of case (3) is realized with $u_1 = u_2 = \epsilon$,$v_1 = \epsilon$, $p = q$, $\omega = A$.
$u = u_0 \cdot a, a \in X \cup \{e\}$:
Let us consider the $u_1, u_2, v_1, p, q, A, \omega, \Phi$ given by the induction hypothesis on u_0.

$$(S \otimes u) \otimes a = (([qA] \otimes u_2) * \Phi) \otimes a \text{ and}$$

$$[qA] \otimes u_2 = [p\omega], |\omega| \geq 1.$$

Let $[p\omega] \otimes a = [p'\eta']$.
case 1: $|\eta'| \geq 1$.
Then $S \otimes ua = ([qA] \otimes u_2 a) * \Phi$. Hence conclusion (3) of the lemma is fulfilled by $u_1' = u_1, u_2' = u_2 a, v_1' = v_1, q' = q, A' = A, \omega' = \eta', \Phi' = \Phi$.
case 2: $|\eta'| = 0$.

$$S \otimes ua = \Phi_r.$$

subcase 1: $\Phi_r \in \{\emptyset^\lambda\} \cup \{\epsilon_j^\lambda | 1 \leq j \leq \lambda\}$.
Conclusion (1) or (2) of the lemma is then realized.
subcase 2: $\Phi_r = [r'B] * \Psi$ for some $r' \in Q, B \in Z, \Psi \in \mathsf{DRB}_{Q,\lambda} < V >$.
Then

$$S \otimes ua = [r'B] * \Psi; \quad S \bullet (v_1[qAr]) = \Phi_r = [r'B] * \Psi.$$

Conclusion (3) of the lemma is then realized by $u_1' = ua, u_2' = \epsilon, v_1' = v_1[qAr], q' = r', A' = B, \omega' = B, \Phi' = \Psi$. □

Lemma 3.17 *Let $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DB}_{1,\lambda} < V >, u \in X^+$. One of the three following cases must occur:*

(1) $S \odot u = \emptyset^\lambda$,
(2) $S \odot u = \epsilon_j^\lambda$ for some $j \in [1, \lambda]$,
(3) $\exists u_1, u_2 \in X^*, v_1 \in V^*, q \in Q, A \in Z, \Phi\ Q - \lambda - form$ such that

$$u = u_1 \cdot u_2, \rho_\epsilon(S) \odot u_1 = S \bullet v_1 = [qA] * \Phi \text{ and } S \odot u = ([qA] \odot u_2) * \Phi.$$

Proof: Suppose that $u = x_1 \cdots x_l$ with $l \geq 1$. Let $u' = e^{n_0} x_1 e^{n_1} \cdots x_l e^{n_l}$ such that $S \odot u = S \otimes u'$. If the hypothesis of case (1) or (2) is realized, it is clear that the corresponding conclusion is realized. Otherwise:

$$u' = u'_1 \cdot u'_2, \;\; S \otimes u'_1 = S \bullet v_1 = [qA] * \Phi, \;\; S \otimes u' = ([qA] \otimes u'_2) * \Phi \text{ and}$$

$$[qA] \otimes u'_2 = [p'\omega'], \;\; |\omega'| \geq 1.$$

Let $u_1 = \Pi_X(u'_1), u_2 = \Pi_X(u'_2)$, (where $\Pi_X : (X \cup \{e\})^* \to X^*$ is the projection on the subalphabet X). The condition $[qA] \otimes u'_2 \notin \{\emptyset^Q\} \cup \{\epsilon_p^Q | p \in Q\}$ implies that $[qA]$ is ϵ-free, hence that

$$S \otimes u'_1 = \rho_\epsilon(S) \odot u_1.$$

The condition that $S \otimes u' = S \odot u, |u| \geq 1$ implies that $S \otimes u$ is ϵ-free, hence that $[qA] \otimes u'_2$ is ϵ-free, so that:

$$[qA] \otimes u'_2 = [qA] \odot u_2.$$

Hence point (3) of the lemma is realized. □

Corollary 3.18

1. $\forall S \in \mathsf{DB}_{1,\lambda} < V >, u \in X^*, S \odot u \in \mathsf{DB}_{1,\lambda} < V >$.

2. $\forall S \in \mathsf{DRB}_{1,\lambda} < V >, u \in X^*, S \odot u \in \mathsf{DRB}_{1,\lambda} < V >$.

Proof: Let us consider case (3) of lemma 3.17. Due to the form of the rules generating the right-action $\otimes$ (see section 2.5), $[qA] \odot u_2$ is of the form $[p\omega]$ for some $p \in Q, \omega \in Z^*$. Hence $S \odot u$ is the Q-product of a Q-series by a Q-λ-form, which is a deterministic row-vector by lemma 3.12. □
We give now an adaptation of lemma 3.10 to the actions $\otimes, \odot$ in place of $\bullet$.

Lemma 3.19 *Let $S \in \mathsf{DB}_{1,m} < W >, T \in \mathsf{B}_{m,s} < W >, u \in (X \cup \{e\})^*$ and $U = S \cdot T$. Exactly one of the following cases is true:*

(1) $S \otimes u \notin \{\emptyset^m\} \cup \{\epsilon_j^m | 1 \leq j \leq m\}$
in this case $U \otimes u = (S \otimes u) \cdot T$.
(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S \otimes u' = \epsilon_{j_0}^s$;
in this case $U \otimes u = T_{j_0} \otimes u''$.
(3) $\forall j, \forall u' \preceq u, S \otimes u = \emptyset^m$ and $S \otimes u' \neq \epsilon_j^m$;
in this case $U \otimes u = \emptyset^s = (S \otimes u) \cdot T$.

Proof: The arguments used in the proofs of lemma 3.7, 3.8, 3.9, 3.10 can be adapted to $\otimes$ in place of $\bullet$. The only non-trivial adaptation is that of lines 6-7 of the proof of lemma 3.7: let us suppose that $u \in (X \cup \{e\})^*$ is such that

$$\forall u' \prec u, \forall w \in \mathrm{supp}(S), w \otimes u' \neq \epsilon, \tag{30}$$

and let us prove that

$$\forall w_1 \in \mathrm{supp}(S), \forall w_2 \in \mathrm{supp}(T), (w_1 \cdot w_2) \otimes u = (w_1 \otimes u) \cdot w_2. \qquad (31)$$

We prove by induction on $|u|$ that (30) implies (31) .
$|u| = 0$: by definition of a right-action, $\forall w \in W^*, w \otimes \epsilon = w$. Hence conclusion (31) is true.
$u = u_0 \cdot a$, where $u_0 \in (X \cup \{e\})^*, a \in X \cup \{e\}$:
Hypothesis (30) is fulfilled by u_0 too, hence, by induction hypothesis,

$$(w_1 \cdot w_2) \otimes u_0 = (w_1 \otimes u_0) \cdot w_2.$$

If $w_1 \otimes u_0 = \emptyset$, then , by the above equality $(w_1 \cdot w_2) \otimes u_0 = \emptyset$ too, hence

$$(w_1 \cdot w_2) \otimes u_0 a = \emptyset = (w_1 \otimes u_0 a) \cdot w_2,$$

hence (31) is true.
Otherwise, by hypothesis (30) $w_1 \otimes u_0 \notin \{\emptyset, \epsilon\}$, hence there exists $p, q \in Q, A \in Z$ such that

$$w_1 \otimes u_0 = [p, A, q] \cdot w_3.$$

By definitions (20-21-22)

$$([p, A, q] \cdot w_3 w_2) \otimes a = ([p, A, q] \otimes a) \cdot w_3 w_2,$$

hence

$$(w_1 \cdot w_2) \otimes u_0 a = (w_1 \otimes u_0 a) \cdot w_2.$$

□

Lemma 3.20 *Let* $S \in \mathsf{DB}_{1,m} < W >, T \in \mathsf{B}_{m,s} < W >, u \in X^+$ *and* $U = S \cdot T$. *Exactly one of the following cases is true:*

(1) $S \odot u \notin \{\emptyset^m\} \cup \{\epsilon_j^m | 1 \leq j \leq m\}$
in this case $U \odot u = (S \odot u) \cdot T$.
(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', \rho_\epsilon(S \odot u') = \epsilon^s_{j_0}$;
in this case $U \odot u = \rho_\epsilon(T_{j_0} \odot u'')$.
(3) $\forall j, \forall u' \preceq u, S \odot u = \emptyset^m$ *and* $\rho_\epsilon(S \odot u') \neq \epsilon_j^m$;
in this case $U \odot u = \emptyset^s = (S \odot u) \cdot T$.

Proof: Let $u = x_1 \cdots x_l$. Let us consider which case (as defined in the lemma) occurs.
case 1: $S \odot u = S \otimes \bar{u}$ with $\bar{u} = e^{n_0} \cdot x_1 e^{n_1} \cdots x_l e^{n_l}$.
By lemma 3.19, $U \otimes \bar{u} = (S \otimes \bar{u}) \cdot T$, and, as $(S \otimes \bar{u}) \cdot T$ is ϵ-free and $\notin \{\emptyset^m\} \cup \{\epsilon_j^m | 1 \leq j \leq m\}$,

$$U \otimes \bar{u} = U \odot u, S \otimes \bar{u} = S \odot u,$$

which shows that

$$U \odot u = (S \odot u) \cdot T.$$

case 2: let $\bar{u'} = e^{n'_0} \cdot x_1 e^{n'_1} \cdots x_i e^{n'_i}$, with $0 \leq i \leq l$, such that

$$\rho_\epsilon(S \odot u') = S \otimes \bar{u'}.$$

By lemma 3.19, case (2), where $u'' = \epsilon$

$$U \odot \bar{u'} = T_{j_0},$$

hence $U \odot u' = \rho_\epsilon(T_{j_0})$, hence

$$U \odot u = \rho_\epsilon(T_{j_0}) \odot u'' = \rho_\epsilon(T_{j_0} \odot u'').$$

case 3: let $\bar{u} = e^{n_0} \cdot x_1 e^{n_1} \cdots x_l e^{n_l}$ such that:

$$U \otimes \bar{u} = U \odot u.$$

The hypothesis of this case implies that, $\forall j, \forall u' \preceq u$,

$$S \otimes \bar{u} \neq \epsilon_j^m,$$

(because, as ϵ_j^m is ϵ-free, if $S \otimes \bar{u} = \epsilon_j^m$ then $\rho_\epsilon(S \odot u) = \epsilon_j^m$ too). Hence, by lemma 3.19

$$U \otimes \bar{u} = \emptyset^m.$$

Hence

$$U \odot u = \emptyset^m = (S \odot u) \cdot T.$$

□

The particular letters $[p, e, q]$ for $p, q \in Q$ play a special role in sections 7 and section 8: we use them as *marks* in the series (somehow like the ceilings of [Val74]). We define below a map ρ_e which removes the marks in the series. Let us define $\rho_e : \mathsf{DB} < V > \rightarrow \mathsf{B} < V >$ as the unique substitution such that:

$$\rho_e([p,e,q]) = \epsilon \quad \text{if } p = q, \quad \rho_e([p,e,q]) = \emptyset \quad \text{if } p \neq q, \quad \rho_e([p,A,q]) = [p,A,q] \quad \text{if } A \neq e.$$

We note $\bar{V}_e = \{[p,e,q] |\, p, q, \in Q\}, V_e = V - \bar{V}_e$. A deterministic series $S \in \mathsf{DB} < V >$ is said e-free iff its type is $(\emptyset)$ or (ϵ) or$([pA])$, with $A \neq e$.

Lemma 3.21 *For every* $S \in \mathsf{DB}_{1,\lambda} < V >$

(1) $\rho_e(S) \in \mathsf{DB}_{1,\lambda} < V >$
(2) $||\rho_e(S)|| \leq ||S||$.

Sketch of proof: We establish first that, for every $u \in V_e^*, \exists u' \in V^*$ such that

$$\rho_e(S) \bullet u = \rho_e(S \bullet u') \text{ and } S \bullet u' \text{ is } e\text{-free.} \tag{32}$$

Let us prove (32) by induction on $|u|$.
$|u|$**=0**: If $\rho_e(S) = \emptyset^\lambda$ then (32) is true: it suffices to choose some u' such that $S \bullet u' = \emptyset^\lambda$. Otherwise , $\rho_e(S) \neq \emptyset^\lambda$ and, using the determinism of S, one can show that there exists a maximal integer n such that:

$$\exists (p_i)_{1 \leq i \leq n} \in Q^n, S \bullet ([p_1, e, p_1] \cdots [p_n, e, p_n]) \neq \emptyset^\lambda.$$

Then $u' = [p_1, e, p_1] \cdots [p_n, e, p_n]$ (where $u' = \epsilon$ when $n = 0$) satisfies (32).
$|u|$**=m+1**: $u = u_1 \cdot v_1$ where $u_1 \in V_e^*, |u_1| = m, v_1 \in V_e$.
By induction hypothesis there exists $u_1' \in V^*$ such that

$$\rho_e(S) \bullet u_1 = \rho_e(S \bullet u_1') \text{ and } S \bullet u_1' \text{ is } e\text{-free.}$$

If $S \bullet u_1' \in \{\emptyset^\lambda, \epsilon_1^\lambda, \ldots, \epsilon_\lambda^\lambda\}$, then $u' = u_1' v_1$ satisfies (32). Otherwise let $([pA])$ be the type of $S \bullet u_1'$ $(p \in Q, A \neq e)$. We then have

$$\rho_e(S) \bullet u_1 = \rho_e(S \bullet u_1') \tag{33}$$

$$S \bullet u_1' = [pA] * \Phi \tag{34}$$

for some Q-λ-form Φ.
subcase 1: $v_1 = [p, A, q_1]$ (for some $q_1 \in Q$).
Let us consider the vector Φ_{q_1}: by induction hypothesis, there exists some $w_1' \in V^*$ such that

$$\rho_e(\Phi_{q_1}) \bullet \epsilon = \rho_e(\Phi_{q_1} \bullet w_1') \text{ and } \Phi_{q_1} \bullet w_1' \text{ is } e\text{-free.} \tag{35}$$

Combining equations (33,34,35) we see that $u' = u_1' \cdot [p, A, q_1] \cdot w_1'$ fulfills:

$$\begin{aligned}
\rho_e(S) \bullet (u_1 \cdot v_1) &= (\rho_e(S) \bullet u_1) \bullet v_1 \\
&= \rho_e(S \bullet u_1') \bullet [p, A, q_1] \\
&= \rho_e([pA] * \Phi) \bullet [p, A, q_1] \\
&= \rho_e(\Phi_{q_1}) \\
&= \rho_e(\Phi_{q_1} \bullet w_1')
\end{aligned}$$

where $S \bullet (u_1' \cdot [p, A, q_1] \cdot w_1') = \Phi_{q_1} \bullet w_1'$ and $\Phi_{q_1} \bullet w_1'$ is e-free. Hence (32) is fulfilled by our choice of u'.
subcase 2: $v_1 \notin \{[pAq_1] | q_1 \in Q\}$

$$\rho_e(S) \bullet u_1 v_1 = ([pA] * \rho_e(\Phi)) \bullet v_1 = \emptyset^\lambda.$$

$$\rho_e(S \bullet u_1' v_1) = \rho_e(([pA] * \Phi) \bullet v_1) = \rho_e(\emptyset^\lambda) = \emptyset^\lambda.$$

Hence $u' = u_1' \cdot v_1$ satisfies (32).
Let us prove the lemma now. By (32) every residual $\rho_e(S) \bullet u$ is left-deterministic of the same type as $S \bullet u'$. Hence $\rho_e(S)$ is deterministic. Moreover formula (32) shows that $\|\rho_e(S)\| \leq \|S\|$. □

3.2 Bisimulation of series

Up to the end of this section, we consider the structured alphabet V associated with a dpda $\mathcal{M}$ over X . We suppose a s.r. morphism $\eta \subseteq X^* \times X^*$ is given (see definition 2.2).

Series, words and graphs Let us give first a slight adaptation of definition 2.1 to the n-graph $(\mathsf{DRB}_{1,n} < V >, \odot, (\epsilon_i^n)_{1\leq i\leq n})$.

Definition 3.22 *Let $\mathcal{R}$ be some binary relation $\mathcal{R} \subseteq \mathsf{DRB}_{1,n} < V > \times \mathsf{DRB}_{1,n} < V >$. $\mathcal{R}$ is a $\sigma - \eta$-bisimulation iff*

1. $\forall (S, S') \in \mathcal{R}, \forall x \in X,$
$$\exists x' \in \eta(x), (S \odot x, S' \odot x') \in \mathcal{R} \text{ and } \exists x'' \in \eta^{-1}(x), (S \odot x'', S' \odot x) \in \mathcal{R},$$
2. $\forall S, S' \in \mathcal{R}, \forall i \in [1, n], (\rho_\epsilon(S) = \epsilon_i^n) \Leftrightarrow (\rho_\epsilon(S') = \epsilon_i^n)$.

We denote by $S \sim S'$ the fact that there exists some $\sigma - \eta$-bisimulation $\mathcal{R}$ such that $(S, S') \in \mathcal{R}$. One can notice that $\sim$ is the greatest $\sigma - \eta$-bisimulation (with respect to the inclusion ordering) over $\mathsf{DRB}_{1,n} < V >$. The σ-bisimulation relations can be conveniently expressed in terms of *word*-bisimulations.

Definition 3.23 *Let $S, S' \in \mathsf{DRB}_{1,n} < V >$ and $\mathcal{R} \subseteq X^* \times X^*$. $\mathcal{R}$ is a $w - \eta$-bisimulation with respect to (S, S') iff $\mathcal{R} \subseteq \eta$ and*

(1) totality: $\mathrm{dom}(\mathcal{R}) = X^*, \mathrm{im}(\mathcal{R}) = X^*,$
(2) extension: $\forall (u, u') \in \mathcal{R}, \forall x \in X,$
$$\exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R} \text{ and } \exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R}.$$
(3) coherence: $\forall (u, u') \in \mathcal{R}, \forall i \in [1, n], (\rho_\epsilon(S \odot u) = \epsilon_i^n) \Leftrightarrow (\rho_\epsilon(S' \odot u') = \epsilon_i^n),$
(4) prefix: $\forall (u, u') \in X^* \times X^*, \forall (x, x') \in X \times X, (u \cdot x, u' \cdot x') \in \mathcal{R} \Rightarrow (u, u') \in \mathcal{R}.$

(Condition (1) can be equivalently replaced by "$(\epsilon, \epsilon) \in \mathcal{R}$".) $\mathcal{R}$ is said to be a $w - \eta$-bisimulation of *order m* with respect to (S, S') iff it fulfills conditions (3-4) above and the modified conditions
(1'): $\mathrm{dom}(\mathcal{R}) = X^{\leq m}, \mathrm{im}(\mathcal{R}) = X^{\leq m},$
(2'): $\forall (u, u') \in \mathcal{R} \cap (X^{\leq m-1} \times X^{\leq m-1}), \forall x \in X,$
$$\exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R} \text{ and } \exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R}.$$

The $w - \eta$-bisimulations are also called $w - \eta$-bisimulations of *order* ∞. The two next lemmas are relating the notions of $w - \eta$-bisimulation (on words), $\sigma - \eta$-bisimulation (on series), and η-bisimulation (on the vertices of the computation 2-graph of $\mathcal{M}$).

Lemma 3.24 *Let $S, S' \in \mathsf{DRB}_{1,n} < V >$. The following properties are equivalent:*

(i) $S \sim S'$
(ii) *there exists $\mathcal{R} \subseteq X^* \times X^*$ which is a $w - \eta$-bisimulation w.r.t. (S, S')*
(iii) *$\forall m \in \mathbb{N}$, there exists $\mathcal{R}_m \subseteq X^{\leq m} \times X^{\leq m}$ which is a $w - \eta$-bisimulation of order m w.r.t. (S, S').*

Proof:
(i) $\Rightarrow$ (iii): Suppose that $\mathcal{S}$ is a σ-η-bisimulation w.r.t. (S,S'). Let us prove by induction on the integer m, the following property $\mathcal{P}(m)$:
$\exists \mathcal{R}_m, w-\eta-$ bisimulation of order m w.r.t. (S,S') such that

$$\forall (u,u') \in \mathcal{R}_m, (S \odot u, S' \odot u') \in \mathcal{S}. \tag{36}$$

m=0: Let $\mathcal{R}_0 = \{(\epsilon,\epsilon)\}$. $\mathcal{R}_0$ clearly fulfills points (1'),(2'),(4) of the above definition. Moreover, as $(S,S') \in \mathcal{S}$ where $\mathcal{S}$ fulfills condition (2) of definition 3.22, $\mathcal{R}_0$ fulfills point (3) of definition 3.23.
m=m'+1: Let $\mathcal{R}_{m'}$ be some $w-\eta$-bisimulation of order m' w.r.t. (S,S'). Let us define $\mathcal{R}_m = \mathcal{R}_{m'} \cup \{(u\cdot x, u'\cdot x') \mid (u,u') \in \mathcal{R}_{m'}, (S\odot ux, S'\odot u'x') \in \mathcal{S}$ and $(x,x') \in \eta\}$. Property (1) of $\mathcal{S}$ and property (1') of $\mathcal{R}_{m'}$ imply that

$$\mathrm{dom}(\mathcal{R}_m) = X^{\leq m}, \mathrm{im}(\mathcal{R}_m) = X^{\leq m}. \tag{37}$$

Property (1) of $\mathcal{S}$ and property (2') of $\mathcal{R}_{m'}$ imply that $\forall (u,u') \in \mathcal{R}_m \cap (X^{\leq m-1} \times X^{\leq m-1}), \forall x \in X$,

$$\exists x' \in \eta(x), (u\cdot x, u'\cdot x') \in \mathcal{R}_m \text{ and } \exists x'' \in \eta^{-1}(x), (u\cdot x'', u'\cdot x) \in \mathcal{R}_m. \tag{38}$$

Property (2) of $\mathcal{S}$ and property (3) of $\mathcal{R}_{m'}$ imply that

$$\forall (u,u') \in \mathcal{R}_m, \forall i \in [1,n], (\rho_\epsilon(S\odot u) = \epsilon_i^n) \Leftrightarrow (\rho_\epsilon(S'\odot u') = \epsilon_i^n). \tag{39}$$

Property (4) of $\mathcal{R}_{m'}$ and the definition of $\mathcal{R}_m$ imply that

$$\forall (u,u') \in X^* \times X^*, \forall (x,x') \in X \times X, (u\cdot x, u'\cdot x') \in \mathcal{R}_m \Rightarrow (u,u') \in \mathcal{R}_m. \tag{40}$$

Property (36) for $\mathcal{R}_{m'}$ and the definition of $\mathcal{R}_m$ imply that (36) is fulfilled by $\mathcal{R}_m$ too. Equations (37,38,39,40) prove that $\mathcal{R}_m$ is a w-η-bisimulation of order m w.r.t. (S,S'), hence $\mathcal{P}(m)$ is proved.
(iii) $\Rightarrow$ (ii): Let us notice that, as the alphabet X is finite, for every w-η-bisimulation $\mathcal{R}$ of order m w.r.t. (S,S'),

$$\mathrm{Card}\{\mathcal{R}' \subseteq X^*\times X^* \mid \mathcal{R} \subseteq \mathcal{R}' \text{ and } \mathcal{R}' \text{ is a } w-\eta-\text{bisimulation of order } m+1 \text{ w.r.t. } (S,S')\} < \infty.$$

Hence, by Koenig's lemma, if (iii) is true, then there exists an infinite sequence $(\mathcal{R}_m)_{m\in\mathbb{N}}$ such that for every $m \in \mathbb{N}$, $\mathcal{R}_m$ is a $w-\eta-$bisimulation of order m w.r.t. (S,S') and $\mathcal{R}_m \subseteq \mathcal{R}_{m+1}$. Let us define then

$$\mathcal{R} = \bigcup_{m\geq 0} \mathcal{R}_m.$$

$\mathcal{R}$ is a $w-\eta-$ bisimulation of order ∞ w.r.t. (S,S').
(ii) $\Rightarrow$ (i): Let $\mathcal{R}$ be a $w-\eta-$ bisimulation of order ∞ w.r.t. (S,S'). Let us define a relation $\mathcal{S}$ by:

$$\mathcal{S} = \{(S\odot u, S'\odot u') \mid (u,u') \in \mathcal{R}\}.$$

The totality property of $\mathcal{R}$ implies that $(S,S') \in \mathcal{S}$. The extension property of $\mathcal{R}$ implies that $\mathcal{S}$ fulfills condition (1) of definition 3.22 and the coherence property of $\mathcal{R}$ implies $\mathcal{S}$ fulfills condition (2). $\square$
Lemma 3.24 leads naturally to the following

Definition 3.25 *Let* $\lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >$. *We define the* divergence *between* S *and* S' *as:*

$$\mathrm{Div}(S, S') = \inf\{n \in \mathbb{N} \mid \mathcal{B}_n(S, S') = \emptyset\}.$$

(It is understood that $\inf(\emptyset) = \infty$*).*

Let us suppose that the dpda $\mathcal{M} =< X, Z, Q, \delta, q_0, z_0, \{\bar{q}\} >$ is normalized and bi-rooted. Let $\psi : X^* \to Y^*$ be a monoid homomorphism such that $\psi(X) \subseteq Y$ and let $\bar{\psi} = \psi \circ \psi^{-1}$ ($\bar{\psi}$, the kernel of ψ, is a s.r. morphism which is also an equivalence relation; this additional property will be used in the sequel). Let Γ be the computation 2-graph of $\mathcal{M}$ and let us suppose Γ is $\bar{\psi}$-saturated.

Lemma 3.26 *For every* $q, q' \in Q, \omega, \omega' \in Z^*$ *such that* $q\omega, q'\omega' \in V_\Gamma$*,*

$$q\omega \sim q'\omega' \Leftrightarrow [q\omega\bar{q}] \sim [q'\omega'\bar{q}].$$

Proof: Let $\theta : V_\Gamma \to \mathsf{DRB} < V >$ the mapping defined by: $\forall q \in Q, \forall \omega \in Z^*$, such that $q\omega \in V_\Gamma$,

$$\theta(q\omega) = [q\omega\bar{q}].$$

We first remark that θ is a graph-embedding in the sense that θ is injective and:

$$\forall v_1, v_2 \in V_\Gamma, \forall x \in X, (v_1, x, v_2) \in E_\Gamma \Leftrightarrow \theta(v_1) \odot x = \theta(v_2). \tag{41}$$

Let us denote by $\mathcal{B}(\Gamma)$ $(resp.\mathcal{B}(\mathsf{DRB} < V >))$ the set of all binary relations over V_Γ (resp. over $\mathsf{DRB} < V >$). We define mappings $\Theta : \mathcal{B}(\Gamma) \to \mathcal{B}(\mathsf{DRB} < V >)$ and $\bar{\Theta} : \mathcal{B}(\mathsf{DRB} < V >) \to \mathcal{B}(\Gamma)$ by: for every $\mathcal{R} \in \mathcal{B}(\Gamma), \mathcal{S} \in \mathcal{B}(\mathsf{DRB} < V >)$

$$\Theta(\mathcal{R}) = \theta^{-1} \circ \mathcal{R} \circ \theta \cup \{(\emptyset, \emptyset)\}, \quad \bar{\Theta}(\mathcal{S}) = \theta \circ \mathcal{S} \circ \theta^{-1} \cup \{(v, v) \mid v \in V_\Gamma\}.$$

We shall prove that

(I) if $\mathcal{R}$ is a $\bar{\psi}$-bisimulation then $\Theta(\mathcal{R})$ is a σ-$\bar{\psi}$-bisimulation,
(II) if $\mathcal{S}$ is a σ-$\bar{\psi}$-bisimulation then $\bar{\Theta}(\mathcal{S})$ is a $\bar{\psi}$-bisimulation.

(I):
Suppose $\mathcal{R}$ is a $\bar{\psi}$-bisimulation and $\mathcal{S} = \Theta(\mathcal{R})$. Let $(S, S') \in \mathcal{S}, x \in X$. Let us check property (1) of definition 3.22.
case I.1: $S = [q\omega\bar{q}], S' = [q'\omega'\bar{q}]$ and $[q\omega\bar{q}] \odot x \neq \emptyset$.
By (41) , there exist $q_1 \in Q, \omega_1 \in Z^*$ such that $(q\omega, x, q_1\omega_1) \in E_\Gamma$. As $\mathcal{R}$ is a $\bar{\psi}$-bisimulation, $\exists q_1' \in Q, \omega_1' \in Z^*, x' \in \bar{\psi}(x)$ such that

$$(q'\omega', x', q_1'\omega_1') \in E_\Gamma \text{ and } (q_1\omega_1, q_1'\omega_1') \in \mathcal{R}.$$

Hence, applying the embedding θ:

$$[q\omega\bar{q}] \odot x = [q_1\omega_1\bar{q}], [q'\omega'\bar{q}] \odot x' = [q_1'\omega_1'\bar{q}] \text{ and } ([q_1\omega_1\bar{q}], [q_1'\omega_1'\bar{q}]) \in \mathcal{S}.$$

Hence (1) is true.
case I.2: $S = [q\omega\bar{q}], S' = [q'\omega'\bar{q}]$ and $[q\omega\bar{q}] \odot x = \emptyset$.
By (41) there is no edge $(q\omega, x, v')$ in E_Γ (for any v'). As Γ is $\bar{\psi}$-saturated, for every $x' \in \bar{\psi}(x)$, there is no edge $(q\omega, x', v')$ in E_Γ (for any v'). But, as $\mathcal{R}^{-1}$ is a $\bar{\psi}^{-1}$- bisimulation, there is no edge $(q'\omega', x'', v')$ in E_Γ , for any $x'' \in \bar{\psi}(x)$ (we use here the fact that $\bar{\psi}^{-1} \circ \bar{\psi} = \bar{\psi}$). It follows that

$$\forall x' \in \bar{\psi}(x), [q'\omega'\bar{q}] \odot x' = \emptyset.$$

As $(\emptyset, \emptyset) \in \mathcal{S}$, (1) is true.
case I.3: $S = S' = \emptyset$.
(1) is trivially true.
Let us check property (2) of definition 3.22.
Let $(S, S') \in \theta^{-1} \circ \mathcal{R} \circ \theta$ and suppose that $S = \epsilon$. Then $\theta^{-1}(S) = \bar{q}$ and $(\theta^{-1}(S), \theta^{-1}(S')) \in \mathcal{R}$. As $\bar{q}$ is the only vertex of Γ having out-degree 0, $\theta^{-1}(S') = \bar{q}$ too. Hence (2) is true.
(II):
Suppose $\mathcal{S}$ is a σ-$\bar{\psi}$-bisimulation and $\mathcal{R} = \bar{\Theta}(\mathcal{S})$. Let us prove that $\mathcal{R}$ is a $\bar{\psi}$-bisimulation. Conditions (1) and (2) of definition 2.3 are true just because $\{(v, v) \mid v \in V_\Gamma\} \subseteq \mathcal{R}$. Let us consider $q, q_1, q' \in Q, \omega, \omega_1, \omega' \in Z^*, x \in X$ such that $q\omega, q_1\omega_1, q'\omega' \in V_\Gamma, (q\omega, x, q_1\omega_1) \in E_\Gamma$, $(q\omega, q'\omega') \in \mathcal{R}$ and let us check condition (3) of definition 2.3.
Using (41) we know that

$$[q\omega\bar{q}] \odot x = [q_1\omega_1\bar{q}]. \tag{42}$$

As $\mathcal{S}$ is a σ-$\bar{\psi}$-bisimulation, we get that

$$\exists x' \in \bar{\psi}(x), ([q_1\omega_1\bar{q}], [q'\omega'\bar{q}] \odot x') \in \mathcal{S}. \tag{43}$$

As $\bar{q}$ is a co-root of Γ , there exists $u \in X^*$ such that $q_1\omega_1 \xrightarrow{u}_{\mathcal{M}} \bar{q}$ which by (41) implies that

$$[q_1\omega_1\bar{q}] \odot u = \epsilon. \tag{44}$$

As $\mathcal{S}$ is a σ-$\bar{\psi}$-bisimulation, from (43) and (44) we deduce that, there exists $u' \in \bar{\psi}(u)$ such that

$$(\epsilon, [q'\omega'\bar{q}] \odot x'u') \in \mathcal{S}. \tag{45}$$

From condition (2) on σ-$\bar{\psi}$-bisimulations and (45) we deduce that $\epsilon = [q'\omega'\bar{q}] \odot x'u'$, which implies that

$$[q'\omega'\bar{q}] \odot x' \neq \emptyset. \tag{46}$$

It follows that $[q'\omega'\bar{q}] \odot x' = [q'_1\omega'_1\bar{q}]$ for some $q'_1 \in Q, \omega'_1 \in Z^*$. Assertion (43) and the definition of $\mathcal{R}$ then implies that

$$(q_1\omega_1, q'_1\omega'_1) \in \mathcal{R}. \tag{47}$$

Condition (3) is checked, showing that $\mathcal{R}$ is a $\bar{\psi}$-simulation. By the same arguments, $\mathcal{R}^{-1}$ is a $\bar{\psi}^{-1}$-simulation. □

Extension to matrices
Let $\delta, \lambda \in \mathbb{N} - \{0\}$. We extend the binary relation $\sim$ from vectors in $\mathsf{DRB}_{1,\lambda} < V >$ to matrices in $\mathsf{DRB}_{\delta,\lambda} < V >$ as follows: for every $T, T' \in \mathsf{DRB}_{\delta,\lambda} < V >$,

$$T \sim T' \Leftrightarrow \forall i \in [1, \delta], T_{i,*} \sim T'_{i,*}. \tag{48}$$

We call *w-η-bisimulation* of order $n \in \mathbb{N} \cup \{\infty\}$ with respect to (T, T') every

$$\mathcal{R} = (\mathcal{R}_i)_{i \in [1,\delta]} \text{ such that } \forall i \in [1, \delta], \mathcal{R}_i \in \mathcal{B}_n(T_{i,*}, T'_{i,*}).$$

We denote by $\mathcal{B}_n(T, T')$ the set of w-η-bisimulations of order n w.r.t. (T, T') . Some algebraic properties of this extended relation $\sim$ will be established in corollary 4.6.

Operations on w-bisimulations
The following operations on word-$\bar{\psi}$-bisimulations turn out to be useful.
right-product:
Let $\delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\delta} < V >, T \in \mathsf{DRB}_{\delta,\lambda} < V >$. For every $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R} \in \mathcal{B}_n(S, S')$ we define:

$$< S|\mathcal{R} > = [\{(u, u') \in \mathcal{R} \mid \forall v \preceq u, \forall i \in [1, \delta], \rho_\epsilon(S \odot v) \neq \epsilon_i^\delta\} \tag{49}$$
$$\cup \{(u \cdot w, u' \cdot w) \mid (u, u') \in \mathcal{R}, w \in X^*, \exists i \in [1, \delta], \rho_\epsilon(S \odot u) = \epsilon_i^\delta\}] \cap X^{\leq n} \times X^{\leq n}. \tag{50}$$

One can check that $< S|\mathcal{R} > \in \mathcal{B}_n(S \cdot T, S' \cdot T)$.
left-product:
Let $\delta, \lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\delta} < V >, T, T' \in \mathsf{DRB}_{\delta,\lambda} < V >$. For every $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R} \in \mathcal{B}_n(T, T')$ we define:

$$< S, \mathcal{R} > = [\{(u, u) \mid u \in X^*, \forall v \preceq u, \forall i \in [1, \delta], \rho_\epsilon(S \odot v) \neq \epsilon_i^\delta\} \tag{51}$$
$$\cup \{(u \cdot w, u \cdot w') \mid u \in X^*, \exists i \in [1, \delta], \rho_\epsilon(S \odot u) = \epsilon_i^\delta, (w, w') \in \mathcal{R}_i\}] \cap X^{\leq n} \times X^{\leq n} \tag{52}$$

One can check that $< S, \mathcal{R} > \in \mathcal{B}_n(S \cdot T, S \cdot T')$.
star:
Let $\lambda \in \mathbb{N} - \{0\}, S_1 \in \mathsf{DRB}_{1,1} < V >, S_1 \not\equiv \epsilon, (S_1, S) \in \mathsf{DRB}_{1,\lambda+1} < V >, T \in \mathsf{DRB}_{1,\lambda} < V >$. For every $n \in \mathbb{N} \cup \{\infty\}$ and $\mathcal{R} \in \mathcal{B}_n(S_1 \cdot T + S, T)$ we define:

$$\mathcal{R}_0 = \mathcal{R} \tag{53}$$

$$\mathcal{S}_0 = \begin{pmatrix} \mathcal{R}_0 \\ \vdots \\ \mathcal{R}_0 \end{pmatrix} \tag{54}$$

$$\forall k \geq 0, \mathcal{R}_{k+1} = < (S_1, S), \mathcal{S}_k > \circ \mathcal{R}_0 \tag{55}$$

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{R}_k \\ \vdots \\ \mathcal{R}_k \end{pmatrix} \tag{56}$$

and finally
$$\mathcal{R}^{<S_1,*>} = \bigcup_{k\geq 0} \mathcal{R}_k \cap X^{\leq k} \times X^{\leq k}. \tag{57}$$
One can check that, for every $k \geq 0$:
$$\mathcal{R}_k \in \mathcal{B}_n(S_1^{k+1} + \sum_{i=0}^{k} S_1^i \cdot S, T) \tag{58}$$
$$\mathcal{S}_k \in \mathcal{B}_n(\begin{pmatrix} S_1^{k+1} + \sum_{i=0}^{k} S_1^i \cdot S \\ \mathrm{I}_\lambda \end{pmatrix}, \begin{pmatrix} T \\ \mathrm{I}_\lambda \end{pmatrix}), \tag{59}$$
and finally $\mathcal{R}^{<S_1,*>} \in \mathcal{B}_n(S_1^* \cdot S, T)$.

Remark 3.27 *In fact operations could be more adequately defined on "pointed" w-bisimulations, i.e. on binary relations with sets of "terminal pairs of words" of type $i \in [1, \delta]$ corresponding to the pairs (u, u') such that $S \odot u = \epsilon_i^\delta, S' \odot u' = \epsilon_i^\delta$. The two different external operations $< S, \mathcal{R} >, < S|\mathcal{R} >$ could then be replaced by only one binary operation $< \mathcal{R}_1, \mathcal{R}_2 >$ over "pointed" w-bisimulations.*

Rational matrices, norm

Let us generalize the definition of *rationality* of series in $\mathsf{B} < W >$ to matrices. Given $M \in \mathsf{B}_{n,m} < W >$ we denote by $\mathsf{Q}(M)$ the set of *residuals* of M:
$$\mathsf{Q}(M) = \{M \bullet u \mid u \in W^*\}.$$
Similarly, we denote by $\mathsf{Q}_r(M)$ the set of *row-residuals* of M:
$$\mathsf{Q}_r(M) = \bigcup_{1\leq i\leq n} \mathsf{Q}(M_{i,*}).$$
M is said *rational* iff the set $\mathsf{Q}(M)$ is finite. One can check that it is equivalent to the property that every coefficient $M_{i,j}$ is rational, or to the property that $\mathsf{Q}_r(M)$ is finite. We denote by $\mathsf{RB}_{n,m} < W >$ (resp. $\mathsf{DRB}_{n,m} < W >$) the set of rational (resp. deterministic, rational) matrices over $\mathsf{B} < W >$. For every $M \in \mathsf{RB}_{n,m} < W >$, we define the norm of M as:
$$\|M\| = \mathrm{Card}(\mathsf{Q}_r(M)).$$

Lemma 3.28 *Let $A \in \mathsf{DB}_{n,m} < W >, B \in \mathsf{B}_{m,s} < W >$. Then $\|A \cdot B\| \leq \|A\| + \|B\|$.*

Proof: Let $A = (a_{i,k}), B = (b_{k,j}), C = A \cdot B, C = (c_{i,j})$. Let $1 \leq i \leq n, H \in \mathsf{Q}(C_{i,*})$. Let $u \in W^*$ such that
$$H = C_{i,*} \bullet u = (A_{i,*} \cdot B) \bullet u.$$
We apply lemma 3.10 to $S = A_{i,*}$ and $T = B$. If case (1) or (3) of lemma 3.10 is realized then
$$H = (A_{i,*} \bullet u) \cdot B.$$

If case (2) of lemma 3.10 is realized then

$$H = B_{k_0,*} \bullet u''.$$

The number of residuals H obtained by case (1) is less or equal than $\|A\|$ and the number obtained by case (2) is less or equal than $\|B\|$. This proves the inequality. $\Box$

Operations on row-vectors

Let us introduce two new operations on row-vectors and prove some technical lemmas about them.

Given $A, B \in \mathsf{B}_{1,m} < W >$ and $1 \leq j_0 \leq m$ we define the vector $C = A \Box_{j_0} B$ as follows:

if $A = (a_1, \ldots, a_j, \ldots, a_m), B = (b_1, \ldots, b_j, \ldots, b_m)$ then $C = (c_1, \ldots, c_j, \ldots, c_m)$ where

$$c_j = a_j + a_{j_0} \cdot b_j \text{ if } j \neq j_0\ ,\ \ c_j = \emptyset \text{ if } j = j_0.$$

Lemma 3.29 *Let $A, B \in \mathsf{B}_{1,m} < W >$ and $1 \leq j_0 \leq m$.*

1. *if A, B are left-deterministic, then $A \Box_{j_0} B$ is left-deterministic.*
2. *if A, B are deterministic, then $A \Box_{j_0} B$ is deterministic.*
3. *if A, B are deterministic, then $\|A \Box_{j_0} B\| \leq \|A\| + \|B\|$.*

Proof:

Let $C = A \Box_{j_0} B$.

1 Let us prove first that if A, B are both left-deterministic, then C is left-deterministic too.

If A is left-deterministic of type $[pz]$, then C is left-deterministic of the same type.

If A is left-deterministic of type (ϵ, j_1) with $j_1 \neq j_0$, then $C = A$, hence C is left-deterministic.

If A is left-deterministic of type (ϵ, j_0), then $C \leq B$, hence C is left-deterministic.

If A is left-deterministic of type $(\emptyset)$, then $C = \emptyset$, hence C is left-deterministic.

2 Let us suppose now that A is deterministic and let us examine a residual $C \bullet u$, for some $u \in W^*$. Lemma 3.8 applies on $S = a_{j_0}$ and $T = b_j$ for every $j \neq j_0$. But the case of the lemma fulfilled by (S, T_j, u) depends on (S, u) only.

Suppose $a_{j_0} \bullet u \neq \emptyset$ (case 1); in this case

$$C \bullet u = (A \bullet u) \Box_{j_0} B \tag{60}$$

Suppose $a_{j_0} \bullet u = \emptyset, \exists u', u'', u = u' \cdot u'', a_{j_0} \bullet u' = \epsilon$ (case 2); in this case

$$C \bullet u = < (B \bullet u'') | \emptyset^m_{j_0} > \tag{61}$$

where $\emptyset^m_{j_0}$ is the row vector $\epsilon^m_{j_0}$ in which $\emptyset$ and ϵ have been exchanged and $< *, * >$ is the "scalar product" defined by $< S, T >= \sum_{j=1}^m S_j \cdot T_j$.

Suppose $a_{j_0} \bullet u = \emptyset, \forall u' \preceq u, a_{j_0} \bullet u' \neq \epsilon$ (case 3); in this case, equation (60) is true again. When equation (60) is true, $C \bullet u$ is left-deterministic by part (1) of this proof, and when equation (61) is true, $C \bullet u$ is left-deterministic because B is assumed deterministic. We have proved that $C \in \mathsf{DB}_{1,m} < W >$.

3 The number of residuals of the form (60) is bounded above by $\|A\|$ and the number of residuals of the form (61) is bounded above by $\|B\|$. Hence $\|C\| \leq \|A\| + \|B\|$. $\Box$

Given $A \in \mathsf{DB}_{1,m} < W >$ and $1 \leq j_0 \leq m$ we define the vector $A' = \Box^*_{j_0}(A)$ as follows:

if $A = (a_1, \ldots, a_j, \ldots, a_m)$ then $A' = (a'_1, \ldots, a'_j, \ldots, a'_m)$ where

$$a'_j = a^*_{j_0} \cdot a_j \text{ if } j \neq j_0 \ , \ \ a'_j = \emptyset \text{ if } j = j_0.$$

Lemma 3.30 *Let $A \in \mathsf{DB}_{1,m} < W >$ and $1 \leq j_0 \leq m$.*
Then $\Box^*_{j_0}(A) \in \mathsf{DB}_{1,m} < W >$ and $\|\Box^*_{j_0}(A)\| \leq \|A\|$.

Proof: Let us examine a residual $A' \bullet u$, for some $u \in W^*$. Let $u' = \max\{v \preceq u \mid v \in a^*_{j_0}\}$. Let $u'' \in W^*$ such that $u = u' \cdot u''$. One can check that for every $S, T \in \mathsf{B} < W >$

$$(S \cdot T) \bullet u = (S \bullet u) \cdot T + \sum_{\substack{u = u_1 \cdot u_2, \\ \epsilon \in S \bullet u_1}} a_j \bullet u_2.$$

Applying this formula to $S = a^*_{j_0}$ and $T = a_j$, with $j \neq j_0$ we obtain

$$a'_j \bullet u = (a^*_{j_0} \bullet u) \cdot a_j + \sum_{\substack{u = u_1 \cdot u_2, \\ \epsilon \in a^*_{j_0} \bullet u_1}} a_j \bullet u_2. \tag{62}$$

As a_{j_0} is deterministic, one can check that

$$a^*_{j_0} \bullet u = (a_{j_0} \bullet u'') \cdot a^*_{j_0}.$$

As A is deterministic, if u_2 has some prefix u'_2 in a_{j_0}, then $a_j \bullet u'_2 = \emptyset$ so that $a_j \bullet u_2 = \emptyset$. Hence

$$\sum_{\substack{u = u_1 \cdot u_2, \\ \epsilon \in a^*_{j_0} \bullet u_1}} a_j \bullet u_2 = a_j \bullet u''.$$

Plugging the two last equations into (62) we obtain

$$a'_j \bullet u = (a_{j_0} \bullet u'') \cdot a^*_{j_0} \cdot a_j + a_j \bullet u'' \text{ (for } j \neq j_0), \quad \text{and } a'_j \bullet u = \emptyset \text{ (for } j = j_0)$$

which can be rewritten as

$$A' \bullet u = (A \bullet u'') \Box_{j_0} A' \tag{63}$$

Let us show that A' is left-deterministic. If A is left-deterministic of type $[pz]$, then A' is left-deterministic of the same type.

If A is left-deterministic of type (ϵ, j_1) with $j_1 \neq j_0$, then $A' = A$ (notice that $\emptyset^* = \epsilon$), hence A' is left-deterministic.

If A is left-deterministic of type (ϵ, j_0) or $(\emptyset)$, then $A' = \emptyset$, hence A' is left-deterministic.
By point (1) of lemma 3.29, the fact that $A \bullet u''$ and A' are both left-deterministic implies that $(A \bullet u'')\square_{j_0} A'$ is left-deterministic too. By formula (63), $A' \bullet u$ is left-deterministic. We have proved that $A' \in \mathsf{DB}_{1,m} < W >$.
Moreover, by formula (63), $\mathrm{Card}(\mathsf{Q}(A')) \leq \mathrm{Card}(\mathsf{Q}(A))$, i.e. $\|A'\| \leq \|A\|$. $\square$

3.3 Deterministic spaces

We elaborate here on the key-idea of [Mei89, Mei92].

Definitions Let $(W, \smile)$ be some structured alphabet. We call *deterministic universe* over $\mathsf{DB} < W >$ (d-universe over $\mathsf{DB} < W >$, in short), any 3-tuple $(E, \circ_l, \circ_r)$ such that, E is a set, $\circ_l$ is a map from $\bigcup \mathsf{DB}_{1,m} < W > \times E^m$ to E and $\circ_r$ is a map from $E \times W^*$ to E , such that, their componentwise extensions:

$$\circ_l : \bigcup \mathsf{DB}_{n,m} < W > \times E^m \to E^n \text{ and } \circ_r : E^n \times W^* \to E^n$$

are fulfilling

- (D1) $\forall M \in \mathsf{DB}_{n,m} < W >, M' \in \mathsf{DB}_{m,s} < W >, \mathbf{e} \in E^s, (M \cdot M') \circ_l \mathbf{e} = M \circ_l (M' \circ_l \mathbf{e})$
- (D2) $\forall j \in [1, m], \forall \mathbf{e} \in E^m, \epsilon_j^m \circ_l \mathbf{e} = e_j$
- (D3) $\forall u, u' \in W^*, \forall \mathbf{e} \in E^m, \mathbf{e} \circ_r (u \cdot u') = (\mathbf{e} \circ_r u) \circ_r u'$
- (D4) $\forall e \in E, e \circ_r \epsilon = e$
- (D5) $\forall S \in \mathsf{DB}_{1,n} < W >, \mathbf{e} \in E^n, u \in W^*$, one of the two following cases occurs:
  (C1) $\forall u' \preceq u$, $S \bullet u'$ is not a unit and then $(S \circ_l \mathbf{e}) \circ_r u = (S \bullet u) \circ_l \mathbf{e}$
  (C2) $\exists u', u'', u' \cdot u'' = u$, $S \bullet u' = \epsilon_j^n$ and then $(S \circ_l \mathbf{e}) \circ_r u = (S \bullet u') \circ_l (\mathbf{e} \circ_r u'')$.
- (D6) $\forall S \in \mathsf{DB}_{1,n} < W >, \mathbf{e} \in E^n, j \in [1, n]$, if $S \neq \epsilon_j^n$, then
  $S \circ_l e = e_j \Rightarrow \square_j^*(S) \circ_l e = e_j$.

An element $e' = S \circ_l \mathbf{e}$ where $S \in \mathsf{DB}_{1,n} < W >, \mathbf{e} = (e_1, \ldots, e_i, \ldots, e_n)$ is called a *linear combination* of the e_i's.

Remark 3.31 *The lefthand-side of the equation in (D1) makes sense because, by lemma 3.12, $(M \cdot M')$ is a deterministic matrix too.*

Remark 3.32 *A weaker version of axiom (D6) where j is supposed equal to* 1 *is in fact sufficient to imply, together with the other axioms, the full axiom (D6). It suffices to consider some permutation- matrices and to use axioms (D1,D2).*

A pair $(E', \circ_l')$ such that E' is a subset of E, E' is closed under left-action $\circ_l$ and $\circ_l'$ is the restriction of $\circ_l$ to E', is called a *deterministic space* (d-space in short) in the universe $(E, \circ_l, \circ_r)$. More generally, a pair $(E', \circ_l')$ is called a deterministic space over $\mathsf{DB} < W >$ iff there exists some d-universe over $\mathsf{DB} <$

$W>$, $(E,\circ_l,\circ_r)$, such that $(E',\circ_l')$ is a d-space in $(E,\circ_l,\circ_r)$. Given a family of elements of E, $(U_i)_{i\in I}$, we denote by $\mathsf{V}((U_i)_{i\in I})$ the smallest d-space in E containing $\{U_i \mid i \in I\}$. We call it the d-space generated by $(U_i)_{i\in I}$. One can check that $\mathsf{V}((U_i)_{i\in I})$ is simply the set of all the (finite) linear combinations of elements of $\{U_i \mid i \in I\}$.

The d-space $\boldsymbol{E}_n$
For every integer n, let us consider the triple $(E_n,\circ_l,\circ_r)$ where $E_n = \mathsf{DB}_{1,n} < W >$, $\circ_l$ is the matrix product and $\circ_r$ is the residual action $\bullet$. Axiom (D1) is fulfilled by E_n (it expresses just the associativity of the matrix product), axioms (D2-4) are easily verified, axiom (D5) is fulfilled as a corollary of lemma 3.9 and axiom (D6) is fulfilled as a consequence of the so-called "Arden's lemma". Hence $(E_n,\circ_l,\circ_r)$ is a particular d-universe (and also d-space) over $\mathsf{DB} < W >$.

Linear independence Following an analogy with classical linear algebra, we develop here a notion of *linear independence* in a d-universe E. The analogue of the field of coefficients here is the set of coefficients $\mathsf{DB} < W >$. This set is a monoid only, and there exists no inverse element for a non-null element S (except for ϵ). Nevertheless, for every non-null $S \in \mathsf{DB} < W >$, there exists $s \in W^*$ such that $S \bullet s = \epsilon$. By virtue of axiom (D5), the right-action of s (by $\circ_r$) will provide a kind of *inverse* to the left-action of S (by $\circ_l$). This idea will lead to an analogue, in the context of d-universes over $\mathsf{DB} < W >$, of the classical notions of linear independence and basis for vector spaces over a field F.
From a purely technical point of view, the two next lemmas (3.33 and 3.34) are *not used* in the sequel. We nevertheles believe they are useful for a general understanding of d-spaces. The proofs of these lemmas (especially the parts (1) $\Rightarrow$ (2) $\Rightarrow$ (3)) develop in a pure algebraic form some ideas which will be used in section 5 but in a more complex form. In this further section these ideas will be combined with considerations on *weights* of linear equations and on *deduction systems* over weighted linear equations.

Lemma 3.33 (fundamental lemma) *Let E be a d-universe over $\mathsf{DB} < W >$, $m \geq 1$ and let $\mathbf{U} = (U_1,\ldots,U_j,\ldots,U_m) \in E^m$. The following are equivalent*

1. $\exists \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathsf{DB}_{1,m} < W >, \boldsymbol{\alpha} \neq \boldsymbol{\beta}$, *such that*
$$\boldsymbol{\alpha} \circ_l U = \boldsymbol{\beta} \circ_l U$$
2. $\exists j_0 \in [1,m], \exists \boldsymbol{\gamma} \in \mathsf{DB}_{1,m} < W >, \boldsymbol{\gamma} \neq \epsilon^m_{j_0}$, *such that*
$$U_{j_0} = \boldsymbol{\gamma} \circ_l U$$
3. $\exists j_0 \in [1,m], \exists \boldsymbol{\gamma}' \in \mathsf{DB}_{1,m} < W >, \gamma'_{j_0} = \emptyset$, *such that*
$$U_{j_0} = \boldsymbol{\gamma}' \circ_l U$$

4. $\exists j_0 \in [1,m]$, *such that*

$$\mathsf{V}((U_j)_{1\le j\le m}) = \mathsf{V}((U_j)_{1\le j\le m, j\ne j_0}).$$

.

Proof: (1) $\Rightarrow$ (2)**:**
Let us suppose (1) and let us consider $u = \min(\bigcup_{1\le j\le m} \alpha_j \Delta \beta_j)$. Without loss of generality we can suppose that $u \in \alpha_{j_0} - \beta_{j_0}$.

$$(\boldsymbol{\alpha} \circ_l U) \circ_r u = (\boldsymbol{\beta} \circ_l U) \circ_r u. \tag{64}$$

As $\alpha_{j_0} \bullet u = \epsilon$ and $\boldsymbol{\alpha}$ is deterministic, $\boldsymbol{\alpha} \bullet u$ must be left-deterministic of type (ϵ, j_0),i.e. $\boldsymbol{\alpha} \bullet u = \epsilon^m_{j_0}$. Using successively axiom (D5, case (C2)), axiom (D4) and axiom (D2) we obtain

$$\begin{aligned}(\boldsymbol{\alpha} \circ_l U) \circ_r u &= \epsilon^m_{j_0} \circ_l (U \circ_r \epsilon)\\ &= \epsilon^m_{j_0} \circ_l U\\ &= U_{j_0}.\end{aligned} \tag{65}$$

Let us examine now the righthand-side of equality (64). Let $u' \prec u$. By minimality of u, $\boldsymbol{\beta} \bullet u'$ is a unit iff $\boldsymbol{\alpha} \bullet u'$ is a unit. But if $\boldsymbol{\alpha} \bullet u'$ is a unit, then $\boldsymbol{\alpha} \bullet u = \emptyset$, which is false. Hence $\boldsymbol{\beta} \bullet u'$ is not a unit. By hypothesis, $\boldsymbol{\beta} \bullet u$ is not a unit either. Hence, $\forall u' \preceq u$, $\boldsymbol{\beta} \bullet u'$ is not a unit . By axiom (D5, case C1)

$$(\boldsymbol{\beta} \circ_l U) \circ_r u = (\boldsymbol{\beta} \bullet u) \circ_l U. \tag{66}$$

Let us set $\boldsymbol{\gamma} = \boldsymbol{\beta} \bullet u$. $\boldsymbol{\gamma} \ne \epsilon^m_{j_0}$ and by equalities (64),(65), (66),

$$U_{j_0} = \boldsymbol{\gamma} \circ_l U.$$

(2) $\Rightarrow$ (3)**:**
Let us suppose (2). Using axiom (D6) we obtain that:

$$U_{j_0} = \square^*_{j_0}(\boldsymbol{\gamma}) \circ_l (U_1, \ldots, U_j, \ldots, U_m)$$

As the component j_0 of $\square^*_{j_0}(\boldsymbol{\gamma})$ is empty, point (3) is proved.
(3) $\Rightarrow$ (4)**:**
Let us suppose (3). Let us denote by $\hat{U}$ the vector $(U_1, \ldots, U_{j_0-1}, U_{j_0+1} \ldots, U_m) \in E^{m-1}$. Let us consider the matrix $\Gamma \in \mathsf{DB}_{m,m} < W >$ defined by:

$$\Gamma_{j,k} = \delta_{j,k} \text{ if } j \ne j_0; \;\; \Gamma_{j_0,k} = \gamma_k,$$

and let $\hat{\Gamma} \in \mathsf{DB}_{m,m-1} < W >$ be the matrix Γ where the j_0-th column has been removed. Every vector $\mathbf{e} = \boldsymbol{\alpha} \circ_l \mathbf{U}$ for some $\boldsymbol{\alpha} \in \mathsf{DB}_{1,m} < W >$ can be expressed as $\mathbf{e} = (\boldsymbol{\alpha} \cdot \hat{\Gamma}) \circ_l \hat{U}$. (One can also notice that $\mathbf{e} = (\boldsymbol{\alpha} \square_{j_0} \boldsymbol{\gamma}) \circ_l \hat{U}$).
Hence

$$\mathsf{V}((U_j)_{1\le j\le m}) \subseteq \mathsf{V}((U_j)_{1\le j\le m, j\ne j_0}).$$

As the reverse inclusion is trivially true, (4) is proved.
(4) $\Rightarrow$ (1)**:**

Let us suppose (4) is true for some integer j_0. The element U_{j_0} has clearly two different representations as linear combination of the U_j's. Hence (1) is true. $\Box$

A family $(U_i)_{i\in I}$ (in a d-universe E) is said to be *linearly dependent* iff one of cases (1-4) occurs for some finite subfamily $(U_{i_j})_{1\leq j\leq m}$ defined by some injection $j\mapsto i_j$. It is said *linearly independent* or *free* iff it is *not* linearly dependent. A family $(U_i)_{i\in I}$ (in a d-space V) is a *basis* of V iff it is both a generating family (i.e. $\mathsf{V}=\mathsf{V}((U_i)_{i\in I})$) and a free family.

Let us adapt now the above lemma to the congruence $\sim$ Let us extend the equivalence relation $\sim$ to d-spaces in the universe $\mathsf{DB}<V>$ by: if $\mathsf{V}_1,\mathsf{V}_2$ are d-spaces ,

$$\mathsf{V}_1\sim\mathsf{V}_2\Leftrightarrow\forall i,j\in\{1,2\},\forall S\in\mathsf{V}_i,\exists S'\in\mathsf{V}_j, S\sim S'.$$

Lemma 3.34 *Let $S_1,\ldots,S_j,\ldots,S_m\in\mathsf{DB}_{1,\lambda}<V>$. The following are equivalent*

1. $\exists\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathsf{DB}_{1,m}<V>,\boldsymbol{\alpha}\not\sim\boldsymbol{\beta}$, *such that*
$$\sum_{j=1}^{m}\alpha_j\cdot S_j\sim\sum_{j=1}^{m}\beta_j\cdot S_j$$
2. $\exists j_0\in[1,m],\exists\boldsymbol{\gamma}\in\mathsf{DB}_{1,m}<V>,\boldsymbol{\gamma}\not\sim\epsilon^m_{j_0}$, *such that*
$$S_{j_0}\sim\sum_{j=1}^{m}\gamma_j\cdot S_j$$
3. $\exists j_0\in[1,m],\exists\boldsymbol{\gamma}'\in\mathsf{DB}_{1,m}<V>,\gamma'_{j_0}\sim\emptyset$, *such that*
$$S_{j_0}\sim\sum_{j=1}^{m}\gamma'_j\cdot S_j$$
4. $\exists j_0\in[1,m]$, *such that*
$$\mathsf{V}((S_j)_{1\leq j\leq m})\sim\mathsf{V}((S_j)_{1\leq j\leq m,j\neq j_0}).$$

.

The equivalence between (1),(2) and (3) was first proved in [Mei89, Mei92], in the case where $E=\mathsf{DB}<X>$,the S_j's are configurations $q_j\omega$, with the same ω and $\bar{\psi}=\mathrm{Id}_{X^*}$ hence $\sim=\equiv$. This is the key-idea around which we have developped the notion of d-spaces.

Proof: Let us follow step by step the proof of lemma 3.33 and adapt it to $\sim$. We use now the set $\mathsf{DB}_{1,m}<W>$ endowed with the operation $\circ_l$ chosen in paragraph 3.3 and the right-action $\odot$. In a first step, we assume that all the vectors $\boldsymbol{\alpha},\boldsymbol{\beta},S$ are ϵ-free.

$(1) \Rightarrow (2)$:
Let us consider $\mathcal{R} \in \mathcal{B}_\infty(\boldsymbol{\alpha} \circ_l S, \boldsymbol{\beta} \circ_l S)$, $\nu = \mathrm{Div}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and

$$(u, v) = \min\{(w, w') \in \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu} \mid \exists j \in [1, m], (\boldsymbol{\alpha} \odot w = \epsilon_j^m) \Leftrightarrow (\boldsymbol{\beta} \odot w' \neq \epsilon_j^m)\}. \tag{67}$$

Let us suppose , for example, that $\boldsymbol{\alpha} \odot u = \epsilon_{j_0}^m$ while $\boldsymbol{\beta} \odot v \neq \epsilon_{j_0}^m$ and let $\boldsymbol{\gamma} = \boldsymbol{\beta} \odot u$. As $(u, v) \in \mathcal{R} \in \mathcal{B}_\infty(\boldsymbol{\alpha} \circ_l S, \boldsymbol{\beta} \circ_l S)$

$$(\boldsymbol{\alpha} \circ_l S) \odot u \sim (\boldsymbol{\beta} \circ_l S) \odot v. \tag{68}$$

Using lemma 3.20 we obtain:

$$(\boldsymbol{\alpha} \circ_l S) \odot u = S_{j_0} \tag{69}$$

Let us examine now the righthand-side of equality (68). Let $(u', v') \prec (u, v)$ with $|u'| = |v'|$. By condition (4) in definition 3.23 $(u', v') \in \mathcal{R}$ [3] and by minimality of v, $\boldsymbol{\beta} \odot v'$ is a unit iff $\boldsymbol{\alpha} \odot u'$ is a unit. But if $\boldsymbol{\alpha} \odot u'$ is a unit, then $\boldsymbol{\alpha} \odot u = \emptyset$, which is false. Hence $\boldsymbol{\beta} \odot v'$ is not a unit. By hypothesis, $\boldsymbol{\beta} \odot v$ is not a unit either. Hence, $\forall v' \preceq v$, $\boldsymbol{\beta} \odot v'$ is not a unit . By lemma 3.20

$$(\boldsymbol{\beta} \circ_l S) \odot v = (\boldsymbol{\beta} \odot v) \circ_l S. \tag{70}$$

Let us plug equalities (69) and (70) in equivalence (68) and let us define $\boldsymbol{\gamma} = \boldsymbol{\beta} \odot v$. We obtain:

$$S_{j_0} \sim \boldsymbol{\gamma} \circ_l S, \boldsymbol{\gamma} \not\sim \epsilon_{j_0}^m.$$

$(2) \Rightarrow (3)$:

$$S_{j_0} \sim \gamma_{j_0} \cdot S_{j_0} + (\sum_{j \neq j_0} \gamma_j \cdot S_j), \quad \gamma_{j_0} \not\sim \epsilon.$$

By corollary 4.6, point C1, we can deduce that

$$S_{j_0} \sim \sum_{j \neq j_0} \gamma_{j_0}^* \gamma_j \cdot S_j = \Box_{j_0}^*(\gamma) \circ_l S.$$

Taking $\gamma' = \Box_{j_0}^*(\gamma)$ we obtain

$$S_{j_0} \sim \gamma' \circ_l S \text{ where } \gamma'_{j_0} = \emptyset.$$

$(3) \Rightarrow (4)$:
Let us denote by $\hat{S}$ the vector $(S_1, \ldots, S_{j_0-1}, S_{j_0+1} \ldots, S_m) \in \mathsf{DB}_{m-1,1} < V >$. If $T = \boldsymbol{\alpha} \circ_l S$ then $T \sim (\boldsymbol{\alpha} \Box_{j_0} \boldsymbol{\gamma}') \circ_l \hat{S}$.
$(4) \Rightarrow (1)$:
Let us suppose (4) is true for some integer j_0. The element S_{j_0} is clearly equivalent (mod $\sim$) to two linear combinations of the S_j's with non-equivalent vectors of coefficients (mod $\sim$). Hence (1) is true.

[3] here is the main place where this condition (4) is used

Let us consider now the general case where some vector $\boldsymbol{\alpha}, \boldsymbol{\beta}, S$ might be ϵ-bound. By lemma 3.13, point (2),

$$\boldsymbol{\alpha} \equiv \rho_\epsilon(\boldsymbol{\alpha}), \boldsymbol{\beta} \equiv \rho_\epsilon(\boldsymbol{\beta}), S \equiv \rho_\epsilon(S).$$

Moreover, as the s.r.morphism $\bar{\psi}$ is an equivalence relation, $\equiv \subseteq \sim$. Hence the lemma in the general case follows from the lemma in the ϵ-free case. □
Let us call *d-submonoid* of $\mathsf{DB} < W >$ any $K \subseteq \mathsf{DB} < W >$ such that,

(1)$\epsilon \in K$,
(2) for every $A \in \mathsf{DB}_{1,m} < W > \cap \mathsf{M}_{1,m}(K), B \in \mathsf{DB}_{m,1} < W > \cap \mathsf{M}_{m,1}(K)$, $A \cdot B \in K$,
(3) for every $A \in \mathsf{DB}_{1,m} < W > \cap \mathsf{M}_{1,m}(K), j_0 \in [1, m], \Box^*_{j_0}(A) \in \mathsf{M}_{1,m}(K)$,
(4) for every $A \in \mathsf{DB}_{1,m} < W > \cap \mathsf{M}_{1,m}(K), v \in W^*, A \bullet v \in \mathsf{M}_{1,m}(K)$.

The notions of universe and space over $\mathsf{DB} < V >$ can be adapted to K in a straightforward way and lemma 3.33 remains true in this context. In the case where $W = V$, we call *stable d-submonoid* of $\mathsf{DB} < W >$ every d-submonoid fulfilling

(5) for every $A \in \mathsf{DB}_{1,m} < W > \cap \mathsf{M}_{1,m}(K), \rho_\epsilon(A) \in \mathsf{M}_{1,m}(K)$,
(6) for every $A \in \mathsf{DB}_{1,m} < W > \cap \mathsf{M}_{1,m}(K), u \in X^*, A \odot u \in \mathsf{M}_{1,m}(K)$.

Lemma 3.34 remains true for any d-space over a stable d-submonoid K. One can check that $K = \mathsf{DRB} < V >$ fulfills conditions (1-6) above, i.e. is a stable d-submonoid. This set of coefficients K will be used in the sequel.

Dimension The results of this paragraph are not used in the sequel and can be skipped by the reader interested in the main decidability result only. We present them for the algebraic pleasure (and also because they might be useful for future works).

Lemma 3.35 *Let $(U_i)_{i \in I}$ be a finite family generating a d-space V. Then there is a subfamily $(U_{i_j})_{1 \leq j \leq m}$ which is a basis of V.*

Proof: Immediate from lemma 3.33. □

Corollary 3.36 (existence of a basis for spaces of finite type) *Let V be some d-space admitting a finite generating family. Then V has a basis.*

Example 3.37 $\mathsf{DB}_{1,2} < W >$ *admits the basis $(\epsilon^2_1, \epsilon^2_2)$.*

Remark 3.38 *The above statement does hold in general, i.e. for spaces which are not of finite type. For example, let us consider the universe (and d-space) $\mathsf{DB}_{2,1} < W >$ over $\mathsf{DB} < W >$. Every generating family of $\mathsf{DB}_{2,1} < W >$ is linearly dependent (we leave the proof as an exercise for the reader). Hence this d-space has no basis. But $\mathsf{DRB}_{2,1} < W >$ has a basis over $\mathsf{DRB} < W >$.*

Lemma 3.39 (dimension invariance) *Let* $(U_i)_{1\le i\le n}$,$(U'_j)_{1\le j\le m}$ *be two bases of some d-space* V*. Then* $n = m$*.*

Proof: Let us suppose that $(U_i)_{1\le i\le n}$,$(U'_j)_{1\le j\le m}$ are two basis of some d-space V. Then there are two matrices $A \in \mathsf{DB}_{n,m} < W >, B \in \mathsf{DB}_{m,n} < W >$ such that:

$$\mathbf{U} = A \circ_l \mathbf{U}' \text{ and } \mathbf{U}' = B \circ_l \mathbf{U},$$

where $\mathbf{U}$ (resp. $\mathbf{U}'$) is the column-vector corresponding to $(U_i)_{1\le i\le n}$ (resp. to $(U'_j)_{1\le j\le m}$). It follows that $\mathbf{U} = (A \cdot B) \circ_l \mathbf{U}$, which implies, by the freeness of $\mathbf{U}$ that $A \cdot B = \mathrm{I}_n$ (the unit matrix of order n). Let us denote by ξ the semi-ring homomorphism $\mathsf{B} < W > \longrightarrow \mathsf{B}$ mapping every non-empty set on 1 and the empty set on 0. We denote by the same symbol $\xi : \mathsf{DB}_{n,m} < W > \longrightarrow \mathsf{M}_{n,m}(\mathsf{B})$ the componentwise extensions of the above map. We then have:

$$\xi(A) \cdot \xi(B) = \mathrm{I}_n. \tag{71}$$

Let us consider the binary relations R_A, R_B defined by $\xi(A)$ (resp. $\xi(B)$):

$$(i, j) \in R_A \Leftrightarrow \xi(A)_{i,j} = 1 \text{ and } (j, i) \in R_B \Leftrightarrow \xi(B)_{j,i} = 1.$$

Identity (71) means that $R_A \circ R_B = \mathrm{Id}_{[1,n]}$. This is possible only if $m \ge n$. Similarly we must have $n \ge m$, so that $n = m$. □

This proves, in particular, that , independently of the chosen d-universe E over $\mathsf{DB} < W >$, every finitely generated d-space (in E) must be isomorphic to some $\mathsf{DB}_{1,n} < W >$ (by lemma 3.36) where n is unique (by lemma 3.39).

3.4 Height, defect and linearity

Let us consider an integer $\lambda \ge 1$ which we fix up to the end of this section. We define here notions of height and defect (for deterministic rational vectors) and a subsequent notion of (d, d')-linearity which will play a crucial role in section 8. We then relate these "size notions" with the notion of norm.

Definitions Let $S \in \mathsf{DRB}_{1,\lambda} < V >$. We call *linear decomposition of* S any pair $([p\omega], \Phi)$ where $p \in Q, \omega \in Z^*, \Phi \in \mathsf{DB}_{Q,\lambda} < V >$ such that

$$S = [p\omega] * \Phi.$$

We denote by $\mathcal{D}(S)$ the set of all linear decompositions of S. We define the *right-defect* of S (rd(S) for short) by

$$\mathrm{rd}(S) = \min\{\|\Phi\| \mid \exists p \in Q, \omega \in Z^*, ([p\omega], \Phi) \in \mathcal{D}(S)\}.$$

One can easily see that the right-defect of S is finite and not greater than $\|S\|$. We call *minimal* decomposition of S the decomposition $([p_0\omega_0], \Phi_0)$ which makes the triple $(p, |\omega|, \|\Phi\|)$ minimal for the right-to-left lexicographic ordering in $Q \times \mathbb{N} \times \mathbb{N}$. We then define the *linear-height* of S (noted $|S|$) as the integer

$|S| = |\omega_0|$ (and it is clear that $\mathrm{rd}(S) = \|\Phi_0\|$).

The height and right-defect of a $Q - \lambda$-form are defined similarly.
Let $\Phi \in \mathsf{DRB}_{Q,\lambda} < V >$. We call *linear decomposition of* Φ any pair $([\omega], \Psi)$ where $\omega \in Z^*, \Psi \in \mathsf{DB}_{Q,\lambda} < V >$ such that

$$\Phi = [\omega] * \Psi.$$

$\mathcal{D}(\Phi)$ denotes the set of all linear decompositions of Φ. The *minimal* decomposition of Φ is the $([\omega_0], \Psi_0)$ which makes the pair $(|\omega|, \|\Psi\|)$ minimal for the right-to-left lexicographic ordering in $\mathbb{N} \times \mathbb{N}$. The integers $|\Phi|, \mathrm{rd}(\Phi)$ are then defined by:

$$|\Phi| = |\omega_0|, \mathrm{rd}(\Phi) = \|\Psi_0\|.$$

We say that S is *marked* iff S contains some occurence of some letter $[qeq] \in V$ (we assumed the existence of such a "dummy " letter $e \in Z$ in section 2.4).

Definition 3.40 *Let $S \in \mathsf{DRB}_{1,\lambda} < V >$ and $d, d' \in \mathbb{N}, d \geq 1$.*

1. *S is said $(0, d')$-linear iff $\mathrm{rd}(S) \leq d'$ and S is not marked*
2. *S is said (d, d')-linear iff, either it is $(0, d')$-linear or S has a decomposition $S = \sum_{q \in Q}[phq] \cdot [qeq] \cdot \Phi_q$ where every Φ_q is a $(0, d')$-linear series and $|h| \leq d$.*

In case 2 we call the vectors Φ_q the d'-linear components of S. In case 1 we consider that S itself is the unique linear component of S. It should be clear that the set of d'-linear components are independent of the value of d (for d large enough) and that it is uniquely defined: it is empty if S is not (d, d')-linear for any d, otherwise it is equal to

$$\{S \bullet u \mid u \in V^*, S \bullet u \text{ is unmarked and, } \forall u' \prec u, S \bullet u \text{ is marked }\}$$

We denote by $\mathsf{DBlin}^{d'}_{1,\lambda} < V >$ (resp. $\mathsf{DRBlin}^{d'}_{1,\lambda} < V >$) the set of row-vectors in $\mathsf{DB}_{1,\lambda} < V >$ (resp. $\mathsf{DRB}_{1,\lambda} < V >$) which are $(0, d')$-linear.

Examples:

No vector can be $(0 - 0)$-linear. S is $(0, 1)$-linear iff $S = \emptyset^\lambda$. S is $(0, 2)$-linear iff there exists $j_0 \in [1, \lambda], \omega \in (Z - \{e\})^*, Q' \subseteq Q$, such that

$$S_{j_0} = \sum_{q \in Q'} [p\omega q] \quad \text{and} \quad \forall j \in [1, \lambda] - \{j_0\}, S_j = \emptyset.$$

Hence one can view the $(0, d)$-linear vectors as vectors which have a structure " not too far" from the *linear* structure of the configurations of the initial dpda $\mathcal{M}$.

(We illustrate on figure 1 the above definitions).

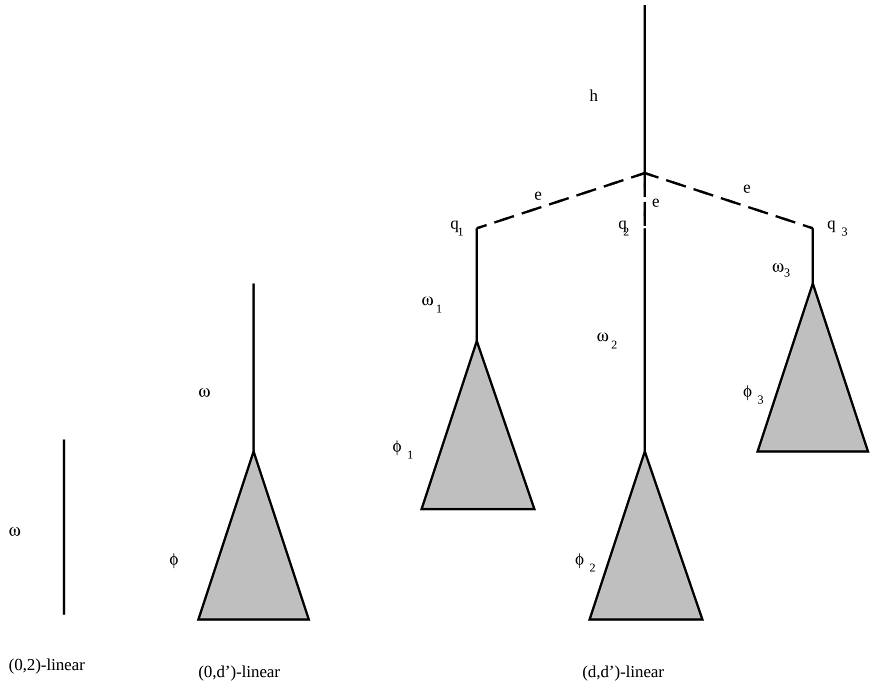

Fig. 1. (d, d')-linearity.

3.5 Height and norm

Lemma 3.41 *Let $S \in \mathsf{DRB}_{1,\lambda} < V >, x \in X, d, d' \in \mathbb{N}$.*

1. $\mathrm{rd}(S \odot x) \leq \mathrm{rd}(S)$
2. *S is (d, d')-linear $\Rightarrow S \odot x$ is $(d+1, d')$-linear*
3. $\|S \odot x\| \leq \|S\| + |Q|$.

Here $|Q|$ denotes the cardinality of the set Q.
Sketch of proof: Point (1) follows from lemma 3.20 and 3.17. Points (2),(3) follow from the hypothesis that the dpda $\mathcal{M}$ is normalized. □

Lemma 3.42 *Let $B, A \in Z$, $\Phi \in \mathsf{DB}_{Q,\lambda} < V >$. If $\|[A] * \Phi\| > \|\Phi\|$ then, $\forall q \in Q, [qBA] * \Phi \notin \mathsf{Q}_r([A] * \Phi)$.*

Proof: Suppose that $\exists q \in Q$, such that $[qBA] * \Phi \in \mathsf{Q}_r([A] * \Phi)$. Let us fix some $q \in Q, u_q \in V^*, q' \in Q$, fulfilling

$$[qBA] * \Phi = ([q'A] * \Phi) \bullet u_q.$$

case 1: $u_q = \epsilon$.
Hence $q = q', B = A, [qB] * [A] * \Phi = [qB] * \Phi$. It follows that $[A] * \Phi = \Phi$ and

finally $||[A] * \Phi|| = ||\Phi||$.
case 2: $u_q \neq \epsilon, \Phi \neq \emptyset_Q^\lambda$ [4] .
Then $u_q = [q'Ar]v_q$ for some $r \in Q, v_q \in V^*$.

$$[qBA] * \Phi = (([q'A] * \Phi) \bullet [q'Ar]) \bullet v_q = \Phi_r \bullet v_q.$$

It follows that , $\forall p \in Q$,

$$[pA] * \Phi = \Phi_r \bullet v_q \bullet [qBp] \in \mathsf{Q}_r(\Phi),$$

hence $||[A] * \Phi|| = ||\Phi||$.
case 3: $u_q \neq \epsilon, \Phi = \emptyset_Q^\lambda$.
Then $||[A] * \Phi|| = ||\Phi|| = 1$.
In all cases the lemma is proved by contraposition. □

Lemma 3.43 *Let* $\omega \in Z^+, A', A \in Z, p \in Q$, $\Phi \in \mathsf{DB}_{Q,\lambda} < V >$. *If* $||[A] * \Phi|| > ||\Phi||$, *then*

(1) $||[\omega A] * \Phi|| = |Q| \cdot |\omega| + ||[A] * \Phi||$
(2) $||[pA'\omega A] * \Phi|| = 1 + |Q| \cdot |\omega| + ||[A] * \Phi||$.

Proof: Let us prove point (1) by induction on $n = |\omega|$.
$n = 0$:
formula (1) is obvious in this case.
$n = 1$: $\omega = B$.
$||[\omega A] * \Phi|| = ||[B] * ([A] * \Phi)||$. By lemma 3.42,

$$\mathsf{Q}_r([BA] * \Phi) = \{[qBA] * \Phi \mid q \in Q\} \,\dot{\cup}\, \mathsf{Q}_r([A] * \Phi),$$

(where $\dot{\cup}$ denotes a *disjoint* union) hence $||[BA] * \Phi|| = |Q| + ||[A] * \Phi||$.
$n = m + 1, m \geq 1$: $\omega = CB\omega'$ for some $C, B \in Z, \omega' \in Z^*$.
$||[\omega A] * \Phi|| = ||[CB] * ([\omega' A] * \Phi)||$. As a consequence of the induction hypothesis, $||[B] * [\omega' A] * \Phi|| > ||[\omega' A] * \Phi)||$, hence, by lemma 3.42

$$||[\omega A] * \Phi|| = |Q| + ||[B\omega' A] * \Phi||.$$

By induction hypothesis, $||[B\omega' A] * \Phi)|| = m \cdot |Q| + ||[A] * \Phi||$, hence

$$||[\omega A] * \Phi|| = (m + 1) \cdot |Q| + ||[A] * \Phi||.$$

Let us prove now point (2).
By the above induction, $||[A'\omega A] * \Phi|| = |Q| + ||[\omega A] * \Phi||$, i.e. $\forall q \in Q, [qA'\omega A] * \Phi \notin \mathsf{Q}_r([\omega A] * \Phi)$. Hence

$$\mathsf{Q}_r([pA'\omega A] * \Phi) = \{[pA'\omega A] * \Phi\} \,\dot{\cup}\, \mathsf{Q}_r([\omega A] * \Phi).$$

It follows that

$$||[pA'\omega A] * \Phi|| = 1 + ||[\omega A] * \Phi|| = 1 + |\omega| \cdot |Q| + ||[A] * \Phi||.$$

□

[4] by $\emptyset_Q^\lambda$ we denote the $Q - \lambda -$form which has all its entries equal to $\emptyset$.

Lemma 3.44 *Let* $\omega \in Z^*, B \in Z, p \in Q$, $\Phi \in \mathsf{DB}_{Q,\lambda} < V >$. *Then* $||[pB\omega] * \Phi|| \leq 1 + |Q| \cdot |\omega| + ||\Phi||$.

Proof: One can notice that $||[pB\omega]|| = 1 + |Q| \cdot |\omega|$. Hence , by lemma 3.28,

$$||[pB\omega] * \Phi|| \leq ||[pB\omega]|| + ||\Phi|| = 1 + |Q| \cdot |\omega| + ||\Phi||.$$

□

Lemma 3.45 *Let* $S \in \mathsf{DRB} < V >$. *Then* $1 + |Q| \cdot (|S| - 2) + \mathrm{rd}(S) \leq ||S|| \leq 1 + |Q| \cdot |S| + \mathrm{rd}(S)$.

Proof: The upper-bound on $||S||$ follows from lemma 3.44 . The lower bound follows from lemma 3.43 point (2) in the case where $|S| \geq 2$ and is clear for $|S| \leq 1$.
□
The following lemma serves as a crucial technical argument in section 8.

Lemma 3.46 *Let* $U = [ph] * \Phi, H = U \odot u$ *where* $p \in Q, h \in Z^*, |h| \geq 1, \Phi$ *is a* $Q - \lambda -$ *form* , $|\Phi| \geq 1, u \in X^*, |u| \leq k$. *Let us suppose that* $||H|| \geq 1 + k|Q| + ||\Phi||$. *Then*
$H = ([ph] \odot u) * \Phi$ *where* $[ph] \odot u = [q\omega]$ *for some* $q \in Q, |\omega| \geq k$.

Intuitive meaning: If U is a deterministic rational vector admitting a decomposition over the $Q-\lambda$-form Φ and, $U \odot u$ has a sufficiently large norm (compared to $||\Phi||$ and $|u|$), then the action of u on U cannot have “touched” the form Φ.
Proof: If $|u| = 0$ the conclusion of the lemma is clearly true.Let us suppose now that $|u| \geq 1$. By lemma 3.20, one of the following 3 cases occurs.
case 1: $\forall r \in Q, [phr] \odot u \notin \{\emptyset, \epsilon\}$.
Hence, $\exists \omega \in Z^*, \exists q \in Q, [ph] \odot u = [q\omega], H = [q\omega] * \Phi$ and, by lemma 3.44,

$$||H|| \leq 1 + |Q|(|\omega| - 1) + ||\Phi||.$$

The hypothesis about $||H||$ implies then: $|\omega| \geq k + 1$. Hence $|\omega| \geq k$.
case 2: $\exists q \in Q, \exists u', u'' \in X^*, u = u'u'', \rho_\epsilon([phq] \odot u') = \epsilon$.
Hence $H = \rho_\epsilon(\Phi_q \odot u'')$ where $|u''| \leq k$.
subcase 2.1: $u'' = \epsilon$.
Then

$$||H|| = ||\rho_\epsilon(\Phi_q)|| \leq ||\Phi_q|| \leq ||\Phi|| < 1 + k|Q| + ||\Phi||.$$

subcase 2.2: $u'' \neq \epsilon$.
By lemma 3.17, $\exists u''_1, u''_2 \in X^*, u'' = u''_1 \cdot u''_2, \exists r \in Q, A \in Z, \Phi'\ Q - \lambda -$ form such that

$$\rho_\epsilon(\Phi_q \odot u''_1) = [rA] * \Phi' \in \mathsf{Q}_r(\Phi) \text{ and } \Phi_q \odot u'' = ([rA] \odot u''_2) * \Phi'.$$

As $|u''_2| \leq |u''| \leq k$, we have:

$$\begin{aligned} ||H|| = ||\Phi_q \odot u''|| &\leq ||[rA] * \Phi'|| + k|Q| \\ &\leq ||\Phi|| + k|Q| \\ &< 1 + k|Q| + ||\Phi||, \end{aligned}$$

contradicting the hypothesis about $||H||$. This case is impossible.
case 3: $H = \emptyset$.
This contradicts also the hypothesis. □

Lemma 3.47 *Let* $D \geq 0$. *Let* $\Phi = (\Phi_q)_{q \in Q}$ *be a* $Q - \lambda$ *-form and let* $S \in \mathsf{V}((\Phi_q)_{q \in Q})$ *such that:*

(1) $||\Phi|| \geq D + |Q|, |\Phi| \geq 2$
(2) $\mathrm{rd}(S) \leq D$.

Then, $\exists \omega \in Z^*, \exists p \in Q, S = [p\omega] * \Phi$.

Proof: Let S fulfilling hypotheses (1),(2). S can be written

$$S = \sum_{q \in Q} \alpha_q \Phi_q = [p_0 \omega_0] * \Psi,$$

where $([p_0\omega_0], \Psi)$ is the minimal decomposition of S. Let us suppose that

$$\exists q \in Q, \exists u \in \alpha_q, \exists u', u'' \in V^*, u = u' \cdot u'' \text{ and } S \bullet u' = \Psi_q. \tag{72}$$

Then $\Psi_q \bullet u'' = \Phi_q$, hence $\mathsf{Q}_r(\Psi) \supseteq \mathsf{Q}_r(\Phi_q)$. As $|\Phi| \geq 2$, we must have $\Phi = [A] * \Phi'$ for some $A \in Z, |\Phi'| \geq 1$, hence by lemma 3.42

$$\mathrm{Card}(\mathsf{Q}(\Phi_q)) = \mathrm{Card}(\mathsf{Q}_r(\Phi)) - |Q| + 1. \tag{73}$$

It follows that

$$||\Psi|| \geq ||\Phi_q|| = ||\Phi|| - |Q| + 1 \geq D + 1.$$

But this contradicts the fact that $\mathrm{rd}(S) \leq D$. We have established that (72) is impossible. In other words, case (2) of lemma 3.10 cannot occur in the action of a word $u \in \alpha_q$ on the linear combination $\sum_{q \in Q} \alpha_q \Phi_q$. Hence $\forall q \in Q, \forall u \in \alpha_q, \exists q_u \in Q, \omega_u \in Z^+$,

$$S \bullet u = ([p_0\omega_0] \bullet u) * \Psi = [q_u \omega_u] * \Psi. \tag{74}$$

Let us notice that ω_0, ω_u have the same rightmost letter A_0. For such q, u, by equation (73) we have the following equality:

$$||S \bullet u|| = ||\Phi_q|| = ||\Phi|| - |Q| + 1 \tag{75}$$

The minimality of decomposition $([p_0\omega_0], \Psi)$ implies that $||[A_0] * \Psi|| > ||\Psi||$, the hypothesis $|\Phi| \geq 2$ implies $|\Phi_q| \geq 2$ hence $|\omega_u| \geq 2$ and lemma 3.43 point (2) gives the equality:

$$||S \bullet u|| = 1 + |Q|(|\omega_u| - 2) + ||[A_0] * \Psi|| \tag{76}$$

These two equations (75),(76) show that there exists some unique integer $1 \leq n \leq |\omega|$, such that

$$\forall q \in Q, \forall u \in \alpha_q, |\omega_u| = n.$$

But all the words ω_u are suffixes of ω_0, hence there exists some unique words $\omega_0', \omega_0'', \omega_0 = \omega_0' \cdot \omega_0''$ such that

$$\forall q \in Q, \forall u \in \alpha_q, S \bullet u = [q\omega_0''] * \Psi = \Phi_q.$$

It follows that $[\omega_0''] * \Psi = \Phi$ and $S = [p_0\omega_0'] * \Phi$. □

4 Deduction systems

4.1 General formal systems

We follow here the general philosophy of [HHY79, Cou83a]. Let us call *formal system* any triple $\mathcal{D} =< \mathcal{A}, H, \models\!\!- >$ where $\mathcal{A}$ is a denumerable set called the *set of assertions*, H, the *cost function* is a mapping $\mathcal{A} \rightarrow \mathbb{N} \cup \{\infty\}$ and $\models\!\!-$, the *deduction relation* is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$; $\mathcal{A}$ is given with a fixed bijection with $\mathbb{N}$ (an "encoding" or "Gödel numbering") so that the notions of recursive subset, recursively enumerable subset, recursive function, ... over $\mathcal{A}, \mathcal{P}_f(\mathcal{A})$, ... are defined, up to this fixed bijection ; we assume that $\mathcal{D}$ satisfies the following axiom:

(A 1) $\forall (P, A) \in \models\!\!-$, (min $\{H(p), p \in P\} < H(A)$) or $(H(A) = \infty)$.

(We let $\min(\emptyset) = \infty$). We call $\mathcal{D}$ a *deduction system* iff $\mathcal{D}$ is a formal system satisfying the additional axiom:

(A 2) $\models\!\!-$ is recursively enumerable.

In the sequel we use the notation $P \models\!\!- A$ for $(P, A) \in \models\!\!-$. We call *proof* in the system $\mathcal{D}$, *relative to the set of hypotheses* $\mathcal{H} \subseteq \mathcal{A}$, any subset $P \subseteq \mathcal{A}$ fulfilling :

$$\forall p \in P, (\exists Q \subseteq P, Q \models\!\!- p) \text{ or } (p \in \mathcal{H}).$$

We call P a *proof* iff

$$\forall p \in P, (\exists Q \subseteq P, Q \models\!\!- p)$$

(i.e. iff P is a proof relative to $\emptyset$).

Let us define the total map $\chi : \mathcal{A} \rightarrow \{0, 1\}$ and the partial map $\overline{\chi} : \mathcal{A} \rightarrow \{0, 1\}$ by :

$$\chi(A) = 1 \text{ if } H(A) = \infty, \chi(A) = 0 \text{ if } H(A) < \infty,$$

$$\overline{\chi}(A) = 1 \text{ if } H(A) = \infty, \overline{\chi} \text{ is undefined if } H(A) < \infty.$$

(χ is the "truth-value function", $\overline{\chi}$ is the "1-value function").

Lemma 4.1 *Let P be a proof relative to $\mathcal{H} \subseteq H^{-1}(\infty)$ and $A \in P$. Then $\chi(A) = 1$.*

In other words : if an assertion is provable from true hypotheses, then it is true.
Proof: Let P be a proof. We prove by induction on n that,

$$\mathcal{P}(n) : \forall p \in P, H(p) \geq n.$$

It is clear that, $\forall p \in P, H(p) \geq 0$. Suppose that $\mathcal{P}(n)$ is true. Let $p \in P - \mathcal{H} : \exists Q \subseteq P, Q \models\!\!- p$. By induction hypothesis, $\forall q \in Q, H(q) \geq n$ and by (A1),$H(p) \geq n + 1$. It follows that : $\forall p \in P - \mathcal{H}, H(p) = \infty$. But by hypothesis, $\forall p \in \mathcal{H}, H(p) = \infty$. □

A formal system $\mathcal{D}$ will be said *complete* iff, conversely, $\forall A \in \mathcal{A}, \chi(A) = 1 \Longrightarrow$ there exists some *finite* proof P such that $A \in P$. (In other words, $\mathcal{D}$ is complete iff every true assertion is "finitely" provable).

Lemma 4.2 *: If $\mathcal{D}$ is a complete deduction system, $\overline{\chi}$ is a recursive partial map.*

Proof: Let $i \mapsto P_i$ be some recursive function whose domain is $\mathbb{N}$ and whose image is $\mathcal{P}_f(\mathcal{A})$. Let $h : (\mathcal{P}_f(\mathcal{A}) \times \mathcal{A} \times \mathbb{N}) \to \{0,1\}$ be a total recursive function such that :

$$P \vdash A \text{ iff } \exists n \in \mathbb{N}, h(P, A, n) = 1$$

(such an h exists, because the r.e. sets are the projections of the recursive sets, see [Rog67]).

The following (informal) semi-algorithm computes $\overline{\chi}$ on the assertion A :

1. i := 0 ; n := 0 ; s := $i+n$;
2. P := P_i;
3. b := $\min_{p \in P}\{\max_{Q \subseteq P}\{h(Q,p,n)\}\}$;
4. c:= $(A \in P)$;
5. **if** $(b \wedge c)$ **then** $(\overline{\chi}(A) = 1$; **stop**);
6. **if** $i = 0$ **then** $(i := s+1$; $n := 0$; $s := i+n)$
   else $(i := i-1$; $n := n+1)$;
7. **goto** 2 ;

□

In order to define deduction relations from more elementary ones, we set the following definitions.

Let $\vdash \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$. For every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set :

- $P \overset{[0]}{\vdash} Q$ iff $P \supseteq Q$
- $P \overset{[1]}{\vdash} Q$ iff $\forall q \in Q, \exists R \subseteq P, R \vdash q$
- $P \overset{<0>}{\vdash} Q$ iff $P \overset{[0]}{\vdash} Q$
- $P \overset{<1>}{\vdash} Q$ iff $\forall q \in Q, (\exists R \subseteq P, R \vdash q)$ or $(q \in P)$
- $P \overset{<n+1>}{\vdash} Q$ iff $\exists R \in \mathcal{P}_f(A), P \overset{<1>}{\vdash} R$ and $R \overset{<n>}{\vdash} Q$ (for every $n \geq 0$).
- $\overset{<*>}{\vdash} = \bigcup_{n \geq 0} \overset{<n>}{\vdash}$.

Given $\vdash_1, \vdash_2 \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(\mathcal{A})$, for every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set :

$$P(\vdash_1 \circ \vdash_2)Q \text{ iff } \exists R \subseteq \mathcal{A}, (P \vdash_1 R) \wedge (R \vdash_2 Q).$$

4.2 System $\mathcal{B}_0$

Let us define here a particular formal system $\mathcal{B}_0$ "Taylored for the σ- $\bar{\psi}$-bisimulation problem for deterministic series".
Let us fix two finite alphabets X, Y, a surjection $\psi : X \rightarrow Y$ (which induces a surjection $X^* \rightarrow Y^*$ denoted by the same symbol ψ) and its kernel $\bar{\psi} = \mathrm{Ker}\psi \subseteq X^* \times X^*$ (see section 3.2). We also fix a dpda $\mathcal{M}$ over the terminal alphabet X and consider the variable alphabet V associated to $\mathcal{M}$ (see section 3.1) and the sets $\mathsf{DRB}_{\delta,\lambda} < V >$ (the sets of Deterministic Rational Boolean matrices over V^*, with δ rows and λ columns). The set of assertions is defined by :

$$\mathcal{A} = \bigcup_{\lambda \geq 1} \mathbb{N} \times \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V >$$

i.e. an assertion is here a *weighted equation* over $\mathsf{DRB}_{1,\lambda} < V >$ for some integer λ.
For every $n \geq 0$ we define

$$\bar{\mathcal{B}}_n = \{\mathcal{R} \subseteq \bar{\psi} \mid \mathcal{R} \text{ fulfills conditions } (1'), (2') \text{ and } (4) \text{ of definition } 3.23\}. \quad (77)$$

We call the elements of $\bar{\mathcal{B}}_n$ the *admissible* relations of order n over $X^* \times X^*$. For every pair $(S, S') \in \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V >$, and $n \in \mathbb{N} \cup \{\infty\}$ we define:

$$\mathcal{B}_n(S, S') = \{\mathcal{R} \subseteq \bar{\psi} \mid \mathcal{R} \text{ is a } w - \bar{\psi} - \text{bisimulation of order n w.r.t. } (S, S')\}. \quad (78)$$

The "cost-function" $H : \mathcal{A} \rightarrow \mathbb{N} \cup \{\infty\}$ is defined by :

$$H(n, S, S') = n + 2 \cdot \mathrm{Div}(S, S'),$$

where $\mathrm{Div}(S, S')$ is the *divergence* between S and S' (definition 3.25). We recall it is defined by :

$$\mathrm{Div}(S, S') = \inf\{n \in \mathbb{N} \mid \mathcal{B}_n(S, S') = \emptyset\}.$$

(We recall $\inf(\emptyset) = \infty$).

Let us notice that, by lemma 3.24 :

$$\chi(n, S, S') = 1 \Longleftrightarrow S \sim S'.$$

We define a binary relation $||\!\!-\!\!- \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

(R0)

$$\{(p, S, T)\} \,||\!\!-\!\!-\, (p + 1, S, T)$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >$,

(R1)

$$\{(p, S, T)\} \,||\!\!-\!\!-\, (p, T, S)$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >$,

(R2)

$$\{(p, S, S'), (p, S', S'')\} \mid\!\vdash\!\!- (p, S, S'')$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >$,

(R3)

$$\emptyset \mid\!\vdash\!\!- (0, S, S)$$

for $S \in \mathsf{DRB}_{1,\lambda} < V >$,

(R'3)

$$\emptyset \mid\!\vdash\!\!- (0, S, T)$$

for $S \in \mathsf{DRB}_{1,1} < V >, T \in \{\emptyset, \epsilon\}, S \sim T$,

(R4)

$$\{(p+1, S \odot x, T \odot x') \mid (x, x') \in \mathcal{R}_1\} \mid\!\vdash\!\!- (p, S, T)$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >, (S \not\equiv \epsilon \wedge T \not\equiv \epsilon)$ and $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$,

(R5)

$$\{(p, S, S')\} \mid\!\vdash\!\!- (p+2, S \odot x, S' \odot x')$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >, (x, x') \in \bar{\psi}, S \sim S' \wedge S \odot x \sim S' \odot x'$,

(R6)

$$\{(p, S_1 \cdot T + S, T)\} \mid\!\vdash\!\!- (p, S_1^* \cdot S, T)$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S_1 \in \mathsf{DRB}_{1,1} < V >, S_1 \not\equiv \epsilon, (S_1, S) \in \mathsf{DRB}_{1,\lambda+1} < V >, T \in \mathsf{DRB}_{1,\lambda} < V >$,

(R7)

$$\{(p, S, S'), (p, T, T')\} \mid\!\vdash\!\!- (p, (S, T), (S', T'))$$

for $p \in \mathbb{N}, \lambda, \mu \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >, T, T' \in \mathsf{DRB}_{1,\mu} < V >, (S, T), (S', T') \in \mathsf{DRB}_{1,\lambda+\mu} < V >$, where ($\lambda = 1$ and $S \in \{\emptyset, \epsilon\}$) or ($\mu = 1$ and $T \in \{\emptyset, \epsilon\}$),

(R8)

$$\{(p, S, S')\} \mid\!\vdash\!\!- (p, S \cdot T, S' \cdot T)$$

for $p \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\delta} < V >, T \in \mathsf{DRB}_{\delta,\lambda} < V >$,

(R9)

$$\{(p, T_{i,*}, T'_{i,*}) \mid 1 \leq i \leq \delta\} \mid\!\vdash\!\!- (p, S \cdot T, S \cdot T')$$

for $p \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\delta} < V >, T, T' \in \mathsf{DRB}_{\delta,\lambda} < V >$,

(R10)

$$\{(p, U_1, \epsilon)\} \mid\!\vdash\!\!- (p, U_1^* \cdot T, T)$$

for $p \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, U_1 \in \mathsf{DRB}_{1,1} < V >, (U_1, T) \in \mathsf{DRB}_{1,\lambda+1} < V >$.

Remark 4.3 *We do not claim that this formal sytem is* recursively enumerable*: due to rule (R5), establishing this property is as difficult as to solve the general bisimulation problem for equational graphs of finite out-degree. This difficulty will be overcome in section 10 by an* elimination *lemma .*

Lemma 4.4 *: Let $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$ such that $P \mid\!\vdash\!\!- A$. Then $min\{H(p) \mid p \in P\} \leq H(A)$.*

Let us introduce a notation: for every $n \in \mathbb{N} \cup \{\infty\}, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >$,

$$S \sim_n S' \Leftrightarrow \mathcal{B}_n(S, S') \neq \emptyset.$$

Proof: Let us check this property for every type of rule.
R0: $p + 2 \cdot \mathrm{Div}(S, T) \leq p + 1 + 2 \cdot \mathrm{Div}(S, T)$.
R1: $p + 2 \cdot \mathrm{Div}(S, T) = p + 2 \cdot \mathrm{Div}(T, S)$.
R2: as the weight p is the same in all the considered equations, we are reduced to prove that :
$\forall n \in \mathbb{N}, S \sim_n S' \wedge S' \sim_n S" \Longrightarrow S \sim_n S"$. This is true because, if $\mathcal{R} \in \mathcal{B}_n(S, S')$ and $\mathcal{R}' \in \mathcal{B}_n(S', S")$, then $\mathcal{R} \circ \mathcal{R}' \in \mathcal{B}_n(S, S")$.
R3: Let us notice that $\mathrm{Id}_{X^*} \subseteq \bar{\psi}$. It follows that $\infty = \mathrm{Div}(S, S)$.
R'3: By hypothesis, $H(0, S, T) = \infty$.
R4: Let $n \in \mathbb{N}$ such that:

$$\forall (x, x') \in \mathcal{R}_1, n \leq \mathrm{Div}(S \odot x, S' \odot x').$$

Let us choose, for every $(x, x') \in \mathcal{R}_1$, some $\mathcal{R}_{x,x'} \in \mathcal{B}_n(S \odot x, S' \odot x')$. Let us define then

$$\mathcal{R} = \bigcup_{(x,x') \in \mathcal{R}_1} (x, x') \cdot \mathcal{R}_{x,x'}.$$

$\mathcal{R}$ belongs to $\mathcal{B}_{n+1}(S, S')$. It follows that

$$\min\{\mathrm{Div}(S \odot x, S' \odot x') \mid (x, x') \in \mathcal{R}_1\} + 1 \leq \mathrm{Div}(S, S')$$

hence that

$$\min\{H(p + 1, S \odot x, S' \odot x') \mid (x, x') \in \mathcal{R}_1\} \leq H(p, S, S') - 1.$$

R5: By hypothesis, $H(p + 2, S \odot x, S' \odot x') = \infty$.
R6: Let $n \in \mathbb{N}$ such that:

$$n \leq \mathrm{Div}(S_1 \cdot T + S, T).$$

Let $\mathcal{R} \in \mathcal{B}_n(S_1 \cdot T + S, T)$. Let $\mathcal{R}' = \mathcal{R}^{<S_1,*>}$ (see definition (57) in §3.2). As we have

$$\mathcal{R}' \in \mathcal{B}_n(S_1^* \cdot S, T),$$

we get the inequality : $\mathrm{Div}(S_1 \cdot T + S, T) \leq \mathrm{Div}(S_1^* \cdot S, T)$.
R7: Suppose $S \in \{\emptyset, \epsilon\}$ and let us consider $n \leq \min\{\mathrm{Div}(S, S'), \mathrm{Div}(T, T')\}$. One can check that, due to the particular value of S,

$$\bar{\mathcal{B}}_n \subseteq \mathcal{B}_n(S, S').$$

It follows that

$$\mathcal{B}_n(T, T') \subseteq \mathcal{B}_n((S, T), (S', T')).$$

This proves the required inequality. The case where $T \in \{\emptyset, \epsilon\}$ can be treated similarly.
R8: Let $n \leq \mathrm{Div}((S, S')$ and $\mathcal{R} \in \mathcal{B}_n(S, S')$. Let us consider: $\mathcal{R}' =< S|\mathcal{R} >$ (see definition (50) in §3.2). As we have $\mathcal{R}' \in \mathcal{B}_n(S \cdot T, S' \cdot T)$, the required inequality is proved.
R9: Let $n \leq \min\{\mathrm{Div}(T_{i,*}, T'_{i,*}) \mid 1 \leq i \leq \delta\}$ and, for every $i \in [1, \delta]$, let $\mathcal{R}_i \in \mathcal{B}_n(T_{i,*}, T'_{i,*})$. Let us consider $\mathcal{R}' =< S, \mathcal{R} >$ (see definition (52) in §3.2). As we know that

$$\mathcal{R}' \in \mathcal{B}_n(S \cdot T, S \cdot T'),$$

the required inequality is proved.
R10: Either $\mathrm{Div}(U_1, \epsilon) = 0$ and the inequality $\mathrm{Div}(U_1, \epsilon) \leq \mathrm{Div}(U_1^* \cdot T, T)$ is clearly true or, $\mathrm{Div}(U_1, \epsilon) \geq 1$. In this last case $\{(\epsilon, \epsilon)\} \in \mathcal{B}_0(U_1, \epsilon)$, hence $\rho_\epsilon(U_1) = \epsilon$, i.e. by lemma 3.13 $U_1 \equiv \epsilon$. This implies that $U_1^* \cdot T \equiv T$, hence that $\mathrm{Id}_{X^*} \in \mathcal{B}_\infty(U_1^* \cdot T, T)$, so that $\mathrm{Div}(U_1^* \cdot T, T) = \infty$ and the required inequality is true. □
Let us define $\vdash\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$,

$$P \vdash\!\!- A \Longleftrightarrow P \overset{<*>}{|\!\!\vdash\!\!-} \circ \overset{[1]}{|\!\!\vdash\!\!-}_{0,3,4} \circ \overset{<*>}{|\!\!\vdash\!\!-} \{A\}.$$

where $|\!\!\vdash\!\!-_{0,3,4}$ is the relation defined by R_0, R_3, R'_3, R_4 only. We let

$$\mathcal{B}_0 =< \mathcal{A}, H, \vdash\!\!- > .$$

Lemma 4.5 *: $\mathcal{B}_0$ is a formal system.*

Proof: Using lemma 4.4, one can show by induction on n that :

$$P \overset{<n>}{|\!\!\vdash\!\!-} Q \Longrightarrow \forall q \in Q, \min\{H(A) \mid A \in P\} \leq H(q).$$

The proof of lemma 4.4 also reveals that :

$$P \,|\!\!\vdash\!\!-_{\{0,3,4\}} q \Longrightarrow (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

It follows that, for every $m, n \geq 0$:

$$P \overset{<n>}{|\!\!\vdash\!\!-} Q \overset{[1]}{|\!\!\vdash\!\!-}_{0,3,4} R \overset{<m>}{|\!\!\vdash\!\!-} q \Longrightarrow (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

Hence axiom $(A1)$ is fulfilled. □
Let us remark the following algebraic corollaries of lemma 4.4.

Corollary 4.6

(C1) $\forall \lambda \in \mathbb{N} - \{0\}, S_1 \in \mathsf{DRB}_{1,1} < V >, S_1 \not\equiv \epsilon, (S_1, S) \in \mathsf{DRB}_{1,\lambda+1} < V >$ $, T \in \mathsf{DRB}_{1,\lambda} < V >,$

$$S_1 \cdot T + S \sim T \Longrightarrow S_1^* \cdot S \sim T$$

(C2) $\forall \lambda, \mu \in \mathbb{N}-\{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >, T, T' \in \mathsf{DRB}_{1,\mu} < V >, (S,T), (S',T') \in \mathsf{DRB}_{1,\lambda+\mu} < V >,$ *where*$(\lambda = 1$ *and* $S \in \{\emptyset, \epsilon\})$ *or* $(\mu = 1$ *and* $T \in \{\emptyset, \epsilon\})$,

$$S \sim S' \textit{ and } T \sim T' \Longrightarrow (S,T) \sim (S',T')$$

(C3) $\forall \delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\delta} < V >, T \in \mathsf{DRB}_{\delta,\lambda} < V >,$

$$S \sim S' \Longrightarrow S \cdot T \sim S' \cdot T$$

(C4) $\forall \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,1} < V >, T \in \mathsf{DRB}_{1,\lambda} < V >,$

$$[S \cdot T \sim S' \cdot T \textit{ and } T \not\equiv \emptyset^{\lambda}] \Longrightarrow S \sim S'$$

(C5) $\forall \delta, \lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\delta} < V >, T, T' \in \mathsf{DRB}_{\delta,\lambda} < V >,$

$$T \sim T' \Longrightarrow S \cdot T \sim S \cdot T'.$$

Proof: Statement (Ci) (for $1 \leq i \leq 5$) is a direct corollary of the fact that the value of H at the left-hand side of some rule (Rj) is smaller or equal to the value of H at the right-hand side of rule (Rj): (C1) is justified by (R6), (C2) by (R7), (C3) by (R8),(C5) by (R9).
Let us prove (C4): suppose that $\lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,1} < V >, T \in \mathsf{DRB}_{1,\lambda} < V >$ and

$$S \cdot T \sim S' \cdot T \text{ and } S \not\sim S'. \tag{79}$$

Let $\mathcal{R} \in \mathcal{B}_{\infty}(S \cdot T, S' \cdot T)$ and let

$$(u, u') = \min\{(v, v') \in \mathcal{R} \mid (\rho_{\epsilon}(S \odot v) = \epsilon) \Leftrightarrow (\rho_{\epsilon}(S' \odot v') \neq \epsilon)\}.$$

From the hypothesis that $\mathcal{R} \in \mathcal{B}_{\infty}(S \cdot T, S' \cdot T)$, we get that

$$\forall (v, v') \in \mathcal{R}, \;\; (S \cdot T) \odot v \sim (S' \cdot T) \odot v',$$

and by the choice of (u, u') we obtain that:

$$T \sim (S' \odot u') \cdot T \text{ or } (S \odot u) \cdot T \sim T,$$

which, by $C1$, implies:

$$T \sim (S' \odot u')^{*} \cdot \emptyset^{\lambda} \text{ or } (S \odot u)^{*} \cdot \emptyset^{\lambda} \sim T,$$

i.e.

$$T \sim \emptyset^{\lambda}. \tag{80}$$

We have proved that (79) implies (80), hence (C4). □

4.3 Congruence closure

Let us consider the subset $\mathcal{C}$ of the rules of $\mathcal{B}_0$, consisting of all the instances of the metarules R0,R1,R2,R3,R'3,R6,R7,R8,R9,R10. We also denote by $|\!\!\vdash\!\!-_{\mathcal{C}} \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$ the set of all instances of these meta-rules. We are interested here (and later in section 10.1) in special subsets of $\mathcal{A}$ which express an ordinary weighted equation (p, S, S') together with an admissible binary relation $\mathcal{R}$ of finite order (which is a *candidate* to be a w-$\bar{\psi}$-bisimulation w.r.t. (S, S')).
For every $p, n \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >, \mathcal{R} \in \bar{\mathcal{B}_n}$, we use the notation:

$$[p, S, S', \mathcal{R}] = \{(p + |u|, S \odot u, S' \odot u') \mid (u, u') \in \mathcal{R}\}. \qquad (81)$$

One can check the following properties.
composition:
for every $p, n \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >, \mathcal{R}_1, \mathcal{R}_2 \in \bar{\mathcal{B}_n}$,

$$[p, S, S', \mathcal{R}_1] \cup [p, S', S'', \mathcal{R}_2] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} [p, S, S'', \mathcal{R}_1 \circ \mathcal{R}_2]$$

star:
for every $p, n \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S_1 \in \mathsf{DRB}_{1,1} < V >, S_1 \not\equiv \epsilon, (S_1, S) \in \mathsf{DRB}_{1,\lambda+1} < V >, T \in \mathsf{DRB}_{1,\lambda} < V >, \mathcal{R} \in \bar{\mathcal{B}_n}$,

$$[p, S_1 \cdot T + S, T, \mathcal{R}] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} [p, S_1^* \cdot S, T, \mathcal{R}^{<S_1,*>}]$$

right-product:
for every $p, n \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\delta} < V >, T \in \mathsf{DRB}_{\delta,\lambda} < V >, \mathcal{R} \in \bar{\mathcal{B}_n}$,

$$[p, S, S', \mathcal{R}] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} [p, S \cdot T, S' \cdot T, < S|\mathcal{R} >]$$

left-product:
for every $p, n \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\delta} < V >, T, T' \in \mathsf{DRB}_{\delta,\lambda} < V >, \mathcal{R}_1, \dots, \mathcal{R}_\delta \in \bar{\mathcal{B}_n}$,

$$\bigcup_{1 \leq i \leq \delta} [p, T_{i,*}, T'_{i,*}, \mathcal{R}_i] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} [p, S \cdot T, S \cdot T', < S, \mathcal{R} >].$$

Given a subset $P \in \mathcal{P}_f(\mathcal{A})$, we call *congruence closure* of P, denoted by $\mathrm{Cong}(P)$, the set:

$$\mathrm{Cong}(P) = \{A \in \mathcal{A} \mid P \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{A\}\} \qquad (82)$$

As well, for every integer $q \geq 0$ we define:

$$\mathrm{Cong}_q(P) = \{A \in \mathcal{A} \mid P \overset{<q>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{A\}\} \qquad (83)$$

Let us prove now a lemma which will be useful to reduce properties of general equations to the case of equations with ϵ-free coefficients (see §5.2).

Lemma 4.7 *Let $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\lambda} < V >$. Then* $\emptyset \overset{<*>}{\mid\!\vdash\!-} (0, S, \rho_\epsilon(S))$.

Proof: Let us use the notations of the proof of lemma 3.13. If $\rho_\epsilon(S) = \emptyset^\lambda$, then $S \sim \emptyset^\lambda$, hence by rules (R'3, R7), $\emptyset \overset{<*>}{\mid\!\vdash\!-} (0, S, \rho_\epsilon(S))$. Otherwise, by equations (28, 29), we know that:

$$S = v \cdot \rho_\epsilon(S) + \sum_{w \in D(v)} w \cdot (S \bullet w). \tag{84}$$

The form of v and of the elements of $D(v)$ shows that, using rule (R'3) and then finitely many times rules (R9) and (R2):

$$\emptyset \overset{<*>}{\mid\!\vdash\!-} (0, v, \epsilon) \tag{85}$$

$$\forall w \in D(v), \emptyset \overset{<*>}{\mid\!\vdash\!-} (0, w, \emptyset) \tag{86}$$

using then (R7) and (R8) we get

$$\emptyset \overset{<*>}{\mid\!\vdash\!-} (0, S, \rho_\epsilon(S)).$$

□

4.4 Extension to matrices

Let us consider an even more restricted set of rules $\mathcal{C}_0 \subseteq \mathcal{C}$: $\mathcal{C}_0$ is the set of all rules instantiating one of the metarules $R0, R1, R2, R3, R6, R7, R8, R9, R10$. Notice that this set of rules is *independant of the automaton* $\mathcal{M}$.
In the following, given matrices $S, T \in \mathsf{DRB}_{\delta,\lambda} < V >$, we use the notation $[\pi, S, T]$ as an abbreviation for the set of assertions $\{(\pi, S_{i,*}, T_{i,*}) \mid 1 \leq i \leq \delta\}$.

Lemma 4.8 *Let $\delta, \lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{\delta,\delta} < V >, T, U \in \mathsf{DRB}_{\delta,\lambda} < V >$, such that $(S, T) \in \mathsf{DRB}_{\delta,\delta+\lambda} < V >$ and $\forall i, j \in [1, \delta], S_{i,j} \neq \epsilon$. Then*

$$[\pi, S \cdot U + T, U] \overset{<*>}{\mid\!\vdash\!-}_{\mathcal{C}_0} [\pi, S^* \cdot T, U].$$

Proof: Let us prove this lemma by induction over δ. If $\delta = 1$, the property is clearly true by rule $R6$. Let us suppose now that $\delta = 2$. By definition , $[\pi, S \cdot U + T, U]$ consists of the assertions

$$(\pi, S_{i,i} \cdot U_{i,*} + S_{i,\bar{\imath}} \cdot U_{\bar{\imath},*} + T_{i,*}, U_{i,*}), \tag{87}$$

where $\{i, \bar{\imath}\} = \{1, 2\}$. Applying rule $(R6)$ for each $i \in \{1, 2\}$ we obtain a $\mathcal{C}_0$-deduction from (87) to the following system

$$(\pi, S^*_{i,i}(S_{i,\bar{\imath}} \cdot U_{\bar{\imath},*} + T_{i,*}), U_{i,*}). \tag{88}$$

Applying rules $(R8, R9)$, we obtain a $\mathcal{C}_0$-deduction from (88) to the following system

$$(\pi, S^*_{i,i}(S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}}(S_{\bar{\imath},i} \cdot U_{i,*} + T_{\bar{\imath},*}) + T_{i,*}), U_{i,*}). \tag{89}$$

Applying again rule $(R6)$ for each $i \in \{1, 2\}$, we obtain a $\mathcal{C}_0$-deduction from (89) to the following system

$$(\pi, (S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot S_{\bar{\imath},i})^* \cdot (S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot T_{\bar{\imath},*} + S^*_{i,i} \cdot T_{i,*}), U_{i,*}). \tag{90}$$

But, in the particular case where $U = S^* \cdot T$, the equalities corresponding to assertions (87-88-89-90) are also true (because rules $R6, R8, R9$ are valid for equality as well as for bisimulation), hence the second projection of the last assertion (90) is always equal to

$$(S^* \cdot T)_{i,*}.$$

Hence the deduction from (87) to (90) shows that

$$[\pi, S \cdot U + T, U] \ \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, S^* \cdot T, U] \tag{91}$$

Let us suppose now that $\delta \geq 2$. Let us consider block decompositions,

$$S = (S_{i,j})_{i,j \in \{1,2\}}, T = (T_{i,k})_{\substack{i \in \{1,2\} \\ k \in [1,\lambda]}}, U = (U_{i,k})_{\substack{i \in \{1,2\} \\ k \in [1,\lambda]}}$$

where $\delta_1, \delta_2 \in [1, \delta - 1], \delta_1 + \delta_2 = \delta$, $\forall i, j \in \{1, 2\}, k \in [1, \lambda], S_{i,j} \in \mathsf{DRB}_{\delta_i, \delta_j} < V >, T_{i,k}, U_{i,k} \in \mathsf{DRB}_{\delta_i, 1} < V >$. Let us consider the same formulas as above and let us replace every invocation of rule $(R6)$ by an invocation of the induction hypothesis. We then have proved that (91) is true. $\Box$

Lemma 4.9 *Let $\delta, \lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{\delta,\delta} < V >, T \in \mathsf{DRB}_{\delta,\lambda} < V >$, such that $(S, T) \in \mathsf{DRB}_{\delta,\delta+\lambda} < V >$. Then*

$$[\pi, S, \mathrm{I}_\delta] \ \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, S^* \cdot T, T].$$

Proof: Let us prove this lemma by induction over δ. If $\delta = 1$, the property is clearly true by rule $(R10)$. Let us suppose now that $\delta = 2$. We know, by formula (90) obtained in the proof of lemma 4.8 that, for every $i \in \{1, 2\}$:

$$(S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot S_{\bar{\imath},i})^* \cdot H_{i,*} = (S^* \cdot T)_{i,*} \tag{92}$$

where

$$H_{i,*} = S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot T_{\bar{\imath},*} + S^*_{i,i} \cdot T_{i,*}.$$

By rules $(R8, R9)$, owing to the fact that $(\mathrm{I}_\delta)_{i,\bar{\imath}} = \emptyset$,

$$[\pi, S, \mathrm{I}_\delta] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, \emptyset, S_{i,\bar{\imath}}) \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot S_{\bar{\imath},i}, \emptyset). \tag{93}$$

Let us notice that the vector $(S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot S_{\bar{\imath},i}, H_{i,*})$ is deterministic. Hence using (93) and lemma 4.8 , we obtain that:

$$(\pi, S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot S_{\bar{\imath},i}, \emptyset) \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, (S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot S_{\bar{\imath},i})^* \cdot H_{i,*}, H_{i,*}) \tag{94}$$

By rules $(R8, R9)$, owing to the fact that $(\mathrm{I}_\delta)_{i,\bar{\imath}} = \emptyset$,

$$[\pi, S, \mathrm{I}_\delta] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, \emptyset, S_{i,\bar{\imath}}) \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, H_{i,*}, S^*_{i,i} \cdot T_{i,*}). \tag{95}$$

As $(\mathrm{I}_\delta)_{i,i} = \epsilon$, using rule $(R10)$ we obtain:

$$[\pi, S, \mathrm{I}_\delta] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, S_{i,i}, \epsilon) \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, S^*_{i,i} \cdot T_{i,*}, T_{i,*}). \tag{96}$$

By deductions (93,94,95,96) we have, for every $i, \bar{\imath} \in \{1,2\}$ such that $\{i, \bar{\imath}\} = \{1,2\}$

$$[\pi, S, \mathrm{I}_\delta] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, (S^*_{i,i} \cdot S_{i,\bar{\imath}} \cdot S^*_{\bar{\imath},\bar{\imath}} \cdot S_{\bar{\imath},i})^* \cdot H_{i,*}, T_{i,*}), \tag{97}$$

i.e.

$$[\pi, S, \mathrm{I}_\delta] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, S^* \cdot T, T].$$

Let us suppose now that $\delta \geq 2$. We can conclude, by the same argument as in the last step of the proof of lemma 4.8, (i.e. a block decomposition of (S, T)), that the property to be proved is true. □

Lemma 4.10 *Let* $\pi \in \mathbb{N}, \delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{\delta,\delta} < V >, T, T' \in \mathsf{DRB}_{\delta,\lambda} < V >$, *such that* $(S, T), (S', T') \in \mathsf{DRB}_{\delta,\delta+\lambda} < V >$. *Then*

$$[\pi, (S, T), (S', T')] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, S^* \cdot T, S'^* \cdot T'].$$

Proof: We treat three special cases and then the general case.
Let us say that a matrix $M \in \mathsf{DRB}_{\delta,\lambda} < V >$ is *unit-free* iff , $\forall (i,j) \in [1,\delta] \times [1,\lambda], S_{i,j} \neq \epsilon$.
case 1: S is unit-free.
By rule $(R3)$:

$$\emptyset \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (0, S'^* \cdot T', S'^* \cdot T') = (0, S' \cdot (S'^* \cdot T') + T', S'^* \cdot T'). \tag{98}$$

By $(R8)$:

$$(\pi, (S, T), (S', T')) \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (\pi, S \cdot (S'^* \cdot T') + T, S' \cdot (S'^* \cdot T') + T'). \tag{99}$$

Using (99,98) and rule $(R2)$ we get:

$$(\pi,(S,T),(S',T')) \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} (\pi, S\cdot(S'^*\cdot T')+T, S'^*\cdot T'). \tag{100}$$

According to lemma 4.8 where we take $U = S'^*\cdot T'$ we have:

$$(\pi, S\cdot(S'^*\cdot T')+T, S'^*\cdot T') \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} (\pi, S^*\cdot T, S'^*\cdot T'). \tag{101}$$

Putting together deductions (100,102) we obtain finally:

$$(\pi,(S,T),(S',T')) \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} (\pi, S^*\cdot T, S'^*\cdot T'), \tag{102}$$

as required.
case 2: $S = \mathrm{I}_\delta$.
Let us notice that $S^* = \mathrm{I}_\delta$. By lemma 4.9,

$$(\pi,(S,T),(S',T')) \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} (\pi, \mathrm{I}_\delta, S') \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} (\pi, S^*\cdot T', S'^*\cdot T'). \tag{103}$$

Using rule $(R8)$ we also have

$$(\pi,(S,T),(S',T')) \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} (\pi, T, T') = (\pi, S^*\cdot T, S^*\cdot T'). \tag{104}$$

By composition of deductions (104,103) and use of rule $(R2)$ we obtain:

$$(\pi,(S,T),(S',T')) \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} (\pi, S^*\cdot T, S'^*\cdot T').$$

case 3:

$$S = \begin{pmatrix} S_{1,1} & S_{1,2} \\ \emptyset_{\delta_2,\delta_1} & \mathrm{I}_{\delta_2} \end{pmatrix}, \quad T = \begin{pmatrix} T_{1,*} \\ \emptyset_{\delta_2,\lambda} \end{pmatrix}$$

where $\delta_1, \delta_2 \in [0,\delta], \delta_1+\delta_2 = \delta, \forall j \in \{1,2\}, k \in [1,\lambda], S_{1,j} \in \mathsf{DRB}_{\delta_1,\delta_j} < V >, S_{1,1}$ is unit-free $, T_{1,k} \in \mathsf{DRB}_{\delta_1,1} < V >$. Let us then consider the block-decompositions:

$$S' = \begin{pmatrix} S'_{1,1} & S'_{1,2} \\ S'_{2,1} & S'_{2,2} \end{pmatrix}, \quad T' = \begin{pmatrix} T'_{1,*} \\ T'_{2,*} \end{pmatrix}$$

corresponding to the same dimensions. Let us notice that

$$[\pi,(S,T),(S',T')] \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} [\pi, \emptyset_{\delta_2,\delta_1}, S'_{2,1}]. \tag{105}$$

Let us note $U = S^*\cdot T, U' = S'^*\cdot T'$. Let us consider the formula (88) relating the coefficients of U (resp. U') with the coefficients of (S,T) (resp. (S',T')). We know that

$$U'_2 = S'^*_{2,2}\cdot(S'_{2,1}\cdot U'_{1,*}+T'_{2,*}). \tag{106}$$

By lemma 4.9, we have:

$$[\pi,(S,T),(S',T')] \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} [\pi, S'_{2,2}, \mathrm{I}_{\delta_2}] \overset{<*>}{|\!\!\vdash}_{\mathcal{C}_0} [\pi, S'^*_{2,2}\cdot(S'_{2,1}\cdot U'_{1,*}+T'_{2,*}), S'_{2,1}\cdot U'_{1,*}+T'_{2,*}] \tag{107}$$

Using deduction (105) and rule ($R8$) we get:

$$\begin{aligned}[\pi,(S,T),(S',T')] &\overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, S'_{2,1}\cdot U'_{1,*}+T'_{2,*},T'_{2,*}]\\ &\overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, S'_{2,1}\cdot U'_{1,*}+T'_{2,*},T_{2,*}]\\ &= \quad [\pi, S'_{2,1}\cdot U'_{1,*}+T'_{2,*},U_{2,*}]. \end{aligned} \tag{108}$$

Combining together deductions (107,108) we obtain:

$$[\pi,(S,T),(S',T')] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, U'_{2,*},U_{2,*}]. \tag{109}$$

We also have the following deductions:

$$\emptyset \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, U', S'\cdot U'+T'] \text{ (by rule } R3) \tag{110}$$

$$[\pi,(S,T),(S',T')] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, U', S{\cdot}U'{+}T] \text{ (by deduction (110) and rules } R1, R2, R8)$$

$$[\pi,(S,T),(S',T')] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, U'_{1,*}, S_{1,1}\cdot U'_{1,*}+S_{1,2}\cdot U'_{2,*}+T_{1,*}]$$

(just by restricting the previous right-hand side)

$$[\pi,(S,T),(S',T')] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, U'_{1,*}, S^*_{1,1}\cdot(S_{1,2}\cdot U'_{2,*}+T_{1,*})] \quad \text{(by lemma 4.8)}$$

$$\begin{aligned}[\pi,(S,T),(S',T')] &\overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, U'_{1,*}, S^*_{1,1}\cdot(S_{1,2}\cdot U_{2,*}+T_{1,*})] \quad \text{(by deduction 109)}\\ &= \quad [\pi, U'_{1,*}, U_{1,*}]. \end{aligned} \tag{111}$$

By deductions (111,109), the conclusion of the lemma is proved.
case 4: general case.
Let $n = \delta\cdot\delta!$. One can show that, up to a conjugation by a permutation-matrix, S^n has some block-decomposition:

$$S^n = \begin{pmatrix} C_{1,1} & C_{1,2} \\ \emptyset_{\delta_2,\delta_1} & \mathrm{I}_{\delta_2}\end{pmatrix},$$

where $\delta_1,\delta_2 \in [0,\delta], \delta_1+\delta_2=\delta, C_{1,j} \in \mathsf{DRB}_{\delta_1,\delta_j} < V >, C_{1,1}$ is unit-free . Let us also consider the matrices:

$$H = \begin{pmatrix} S & T \\ \emptyset_{\delta,\delta} & \mathrm{I}_\lambda\end{pmatrix}, \quad H' = \begin{pmatrix} S' & T' \\ \emptyset_{\delta,\delta} & \mathrm{I}_\lambda\end{pmatrix}.$$

We then have

$$[\pi,(S,T),(S',T')] \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, H, H']$$

$$
\begin{aligned}
&\overset{<*>}{|\!\!\longmapsto}_{\mathcal{C}_0} [\pi, H^n, H'^n] \text{ by } (R2, R8, R9)\\
&\supseteq \quad [\pi, (S^n, \sum_{k=0}^{n-1} S^k \cdot T), (S'^n, \sum_{k=0}^{n-1} S'^k \cdot T')]\\
&\overset{<*>}{|\!\!\longmapsto}_{\mathcal{C}_0} [\pi, (S^n)^* \cdot \sum_{k=0}^{n-1} S^k \cdot T, (S'^n)^* \cdot \sum_{k=0}^{n-1} S'^k \cdot T'] (\text{ by case } 3)\\
&= \quad [\pi, S^* \cdot T, S'^* \cdot T'].
\end{aligned}
$$

□

Let us call a *deterministic rational* substitution $\mathsf{B} < V > \longrightarrow \mathsf{B} < V >$ any substitution τ fulfilling:

$$\forall q \in Q, \forall z \in Z, \tau([qz]) \in \mathsf{DRB}_{1,Q} < V > . \tag{112}$$

Lemma 4.11 *Let* $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\lambda} < V >$, *let* $\tau : \mathsf{B} < V > \longrightarrow \mathsf{B} < V >$ *be a deterministic rational substitution and let* $\pi \in \mathbb{N}$.

$$\{(\pi, [qz], \tau([qz]) \mid q \in Q\} \overset{<*>}{|\!\!\longmapsto}_{\mathcal{C}_0} (\pi, S, \tau(S)).$$

Sketch of proof: Let $S \in \mathsf{DRB}_{1,\lambda} < V >$. There exist $A \in \mathsf{DRB}_{1,\delta} < V >, B \in \mathsf{DRB}_{\delta,\delta} < V >, C \in \mathsf{DRB}_{\delta,\lambda} < V >$ such that $(B, C) \in \mathsf{DRB}_{\delta,\delta+\lambda} < V >$ and

- $A = \epsilon_1^\delta$
- $\forall i \in [1, \delta], (B_{i,*} = \emptyset^\delta)$ or $(\exists i_0 \in [1, \delta], B_{i,*} = \epsilon_{i_0}^\delta)$ or $(\exists q \in Q, \exists z \in Z, \forall j \in [1, \delta], B_{i,j} \leq \sum_{r \in Q} [qzr])$
- $\forall i \in [1, \delta], C_{i,*} = \emptyset^\lambda$ or $\exists k \in [1, \lambda], C_{i,*} = \epsilon_k^\lambda$

and

$$S = A \cdot B^* \cdot C.$$

B is the matrix-expression of a trim deterministic finite automaton recognizing $S_1 + S_2 + \cdots + S_\lambda$, A expresses the fact that the only initial state of this automaton is 1 and $C_i = \epsilon_k^\lambda$ expresses that the state i is a final state recognizing a subset of S_k.(This way of seeing automata is classical, see [ABB96],and goes back to [Con71]; a systematic treatment of automata theory within this formalism is done in [Kui96]).

$$\tau(S) = \tau(A) \cdot (\tau(B))^* \cdot \tau(C).$$

The special form of the lines of matrix B is such that:

$$\forall i \in [1, \delta], \{(\pi, [qz], \tau([qz]) | q \in Q\} \overset{<*>}{|\!\!\longmapsto}_{\mathcal{C}_0} (\pi, B_{i,*}, \tau(B_{i,*})).$$

By lemma 4.10:

$$[\pi, (B, C), (\tau(B), \tau(C))] \overset{<*>}{|\!\!\longmapsto}_{\mathcal{C}_0} [\pi, B^* \cdot C, (\tau(B))^* \cdot \tau(C)],$$

and as $A = \tau(A)$, we clearly have

$$[\pi, B^* \cdot C, (\tau(B))^* \cdot \tau(C)] \overset{<*>}{|\!\vdash\!\!-}_{\mathcal{C}_0} [\pi, A \cdot B^* \cdot C, \tau(A) \cdot (\tau(B))^* \cdot \tau(C)] = [\pi, S, \tau(S)].$$

□

Corollary 4.12 *Let* $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\lambda} < V >$*. Then*

$$\emptyset \overset{<*>}{|\!\vdash\!\!-}_{\mathcal{C}} (0, S, \rho_e(S)).$$

4.5 Strategies

One key-step of this work is the statement that $\mathcal{B}_0$ is complete (theorem 10.6). We prove this completeness result by exhibiting a "strategy" $\mathcal{S}$ which, for every true assertion (p, S, S'), constructs a finite $\mathcal{B}_0$-proof of this assertion. Let $\mathcal{D} =< \mathcal{A}, H, \vdash\!\!- >$ be a formal system. We call a *strategy* for $\mathcal{D}$ any map $\mathcal{S} : \mathcal{A}^+ \to \mathcal{P}(\mathcal{A}^*)$ such that:
(S1) if $B_1 \cdots B_m \in \mathcal{S}(A_1 A_2 \cdots A_n)$ then $\exists Q \subseteq \{A_i \mid 1 \leq i \leq n-1\}$ such that

$$\{B_j \mid 1 \leq j \leq m\} \cup Q \vdash\!\!- A_n,$$

(S2) if $B_1 \cdots B_m \in \mathcal{S}(A_1 A_2 \cdots A_n)$ then

$$\min\{H(A_i) \mid 1 \leq i \leq n\} = \infty \Longrightarrow \min\{H(B_j) \mid 1 \leq j \leq m\} = \infty.$$

Remark 4.13 *It may happen that* $\epsilon \in \mathcal{S}(A_1 A_2 \cdots A_n)$ *(and correspondingly, that* $m = 0$ *in the above conditions): it just means that* $\{A_1, \ldots, A_{n-1}\} \vdash\!\!- A_n$*. It may also happen that* $\mathcal{S}(A_1 A_2 \cdots A_n) = \emptyset$*: it means , intuitively, that* $\mathcal{S}$ *"does not know" how to extend a proof (with hypothesis), with the only information that the given proof contains the assertions* $A_1, A_2, \cdots A_n$*.*

Remark 4.14 *Axiom (A1) on systems is similar to the "monotonicity" condition of [HHY79] or axiom (2.4.2') of [Cou83a]. Axiom (S2) on strategies is similar to the "validity" condition of [HHY79] or property (2.4.1') of [Cou83a].*

Given a strategy $\mathcal{S}$, we define $\mathcal{T}(\mathcal{S}, A)$, the set of proof-trees associated to the strategy $\mathcal{S}$ and the assertion A as the set of all the trees t fulfilling the following properties:

$$\varepsilon \in dom(t), \quad t(\varepsilon) = A, \tag{113}$$

and, for every path $x_0 x_1, \cdots x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \leq i \leq n-1$) if x_{n-1} has m sons $x_{n-1} \cdot 1, \cdots . x_{n-1} \cdot m \in \mathrm{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \leq j \leq m$) then

$$(B_1 \cdots B_m) \in \mathcal{S}(A_1 \cdots A_n) \text{ or } m = 0. \tag{114}$$

The proof-tree t is said *closed* iff it fulfills the additional condition: for every path $x_0 x_1, \cdots x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \leq i \leq n-1$) if x_{n-1} has m sons $x_{n-1} \cdot 1, \cdots, x_{n-1} \cdot m \in \mathrm{dom}(t)$ with labels $t(x_{n-1} \cdot j) = B_j$ (for $1 \leq j \leq m$) then

$$m = 0 \Rightarrow ((\exists i \in [1, n-1], A_i = A_n) \text{ or } (\varepsilon \in \mathcal{S}(A_1 \cdots A_n))) \tag{115}$$

A node $x \in \mathrm{dom}(t)$ is said *closed* iff it is an internal node or it is a leaf fulfilling property (115) above.
The proof-tree t is said *repetition-free* iff, for every $x, x' \in \mathrm{dom}(t)$,

$$[x \preceq x' \text{and } t(x) = t(x')] \quad \Rightarrow \quad x = x' \text{ or } x' \text{ is a leaf }.$$

For every tree t let us define:

$$\mathcal{L}(t) = \{t(x) | \forall y \in \mathrm{dom}(t), x \preceq y \Rightarrow x = y\}, \;\; \mathcal{I}(t) = \{t(x) | \exists y \in \mathrm{dom}(t), x \prec y\}.$$

(Here $\mathcal{L}$ stands for "leaves" and $\mathcal{I}$ stands for "internal nodes").

Lemma 4.15 *If $\mathcal{S}$ is a strategy for the deduction-system $\mathcal{D}$ then, for every true assertion A and every $t \in \mathcal{T}(\mathcal{S}, A)$*
(1) the set of labels of t is a $\mathcal{D}$-proof, relative to the set $\mathcal{L}(t) - \mathcal{I}(t)$.
(2) every label of a leaf is true.

Proof: Let us suppose that $H(A) = \infty$. Let $t \in \mathcal{T}(\mathcal{S}, A), P = \mathrm{im}(t)$ (the set of labels of t), $\mathcal{H} = \mathcal{L}(t) - \mathcal{I}(t)$.
Using (S2), one can prove by induction on the depth of $x \in \mathrm{dom}(t)$ that, $H(t(x)) = \infty$. Point (2) is then proved. Let x be an internal node of t, with sons $x \cdot 1, x \cdot 2, \cdots, x \cdot m$ $(m \geq 1)$, and with ancestors $y_1, y_2, \cdots, y_{n-1}, y_n = x$ $(n \geq 1)$, such that

$$t(y_1) \cdots t(y_n) = A_1 \cdots A_n, \quad t(x_1) \cdots t(x_m) = B_1 \cdots B_m.$$

By definition of $\mathcal{T}(\mathcal{S}, A)$,

$$B_1 \cdots B_m \in \mathcal{S}(A_1 \cdots A_n)$$

and by condition (S1):

$$\exists Q \subseteq \{A_i \mid i \leq n-1\}, \text{ such that } \{B_j \mid 1 \leq j \leq m\} \bigcup Q \vdash\!\!- A_n.$$

It follows that for every $p \notin \mathcal{H}, \exists R \subseteq P, R \vdash\!\!- p$, hence

$$\forall p \in P, (\exists R \subseteq P, R \vdash\!\!- p) \text{ or } p \in \mathcal{H}.$$

Point (1) is proved. □
For every $\mathcal{D}$-strategy $\mathcal{S}$, we use the notation:

$$\mathcal{T}(\mathcal{S}) = \bigcup_{A \in H^{-1}(\infty)} \mathcal{T}(\mathcal{S}, A).$$

We call a *global strategy* w.r.t. $\mathcal{S}$ any total map $\hat{\mathcal{S}} : \mathcal{T}(\mathcal{S}) \rightarrow \mathcal{T}(\mathcal{S})$ such that:

$$\forall t \in \mathcal{T}(\mathcal{S}), t \preceq \hat{\mathcal{S}}(t). \tag{116}$$

$\hat{\mathcal{S}}$ is a *terminating* global strategy iff:

$$\forall A_0 \in H^{-1}(\infty), \exists n_0 \in \mathbb{N}, \hat{\mathcal{S}}^{n_0}(A_0) = \hat{\mathcal{S}}^{n_0+1}(A_0), \tag{117}$$

$\hat{\mathcal{S}}$ is a *closed* global strategy iff:

$$\forall A_0 \in H^{-1}(\infty), \forall n \in \mathbb{N}, \hat{\mathcal{S}}^{n}(A_0) \text{ is closed} \Longleftrightarrow \hat{\mathcal{S}}^{n}(A_0) = \hat{\mathcal{S}}^{n+1}(A_0), \tag{118}$$

(where the assertion A_0 is identified with the tree reduced to one node whose label is A_0).

Lemma 4.16 : *Let $\mathcal{D}$ be a formal system, $\mathcal{S}$ a strategy for $\mathcal{D}$ and $\hat{\mathcal{S}}$ a global strategy w.r.t. $\mathcal{S}$. If $\hat{\mathcal{S}}$ is terminating and $\hat{\mathcal{S}}$ is closed, then $\mathcal{D}$ is complete.*

Proof: Let $A_0 \in \mathcal{A}$. Under the hypothesis of the lemma, $\exists n_0 \in \mathbb{N}$ such that (117) and (118) are both true. Hence $t_\infty = \hat{\mathcal{S}}^{n_0}(A_0)$ is a closed proof-tree for $\mathcal{S}$. By lemma 4.15 $\mathrm{im}(t_\infty)$ is a $\mathcal{D}$-proof relative to the set $\mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$. Let x be a leaf such that $t_\infty(x) \in \mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$. Let $A_0, A_1, \ldots, A_n = t_\infty(x)$ be the word labelling the path from the root to x. As x is closed and $t_\infty(x) \in \mathcal{L}(t_\infty) - \mathcal{I}(t_\infty)$ by (115), $\varepsilon \in \mathcal{S}(A_1 \cdots A_n)$ hence $\{A_1, \ldots, A_{n-1}\} \vdash\!\!\!- t_\infty(x)$. It follows that $\mathrm{im}(t_\infty)$ is a $\mathcal{D}$-proof . $\Box$

5 Triangulations

Let $S_1, S_2, \cdots, S_d$ be a family of deterministic row-vectors over the structured alphabet V (i.e. $S_i \in \mathsf{DRB}_{1,\lambda} < V >$ where $\lambda \in \mathbb{N} - \{0\}$). We recall V is the alphabet associated with some dpda $\mathcal{M}$ as defined in section 2.4.
Let us consider a sequence $\mathcal{S}$ of n "weighted" linear equations :

$$(\mathcal{E}_i) : p_i, \sum_{j=1}^{d} \alpha_{i,j} S_j \ , \ \sum_{j=1}^{d} \beta_{i,j} S_j \tag{119}$$

where $p_i \in \mathbb{N} - \{0\}$, and $A = (\alpha_{i,j}), B = (\beta_{i,j})$ are deterministic rational matrices of dimension (n, d), with indices $m \leq i \leq m + n - 1, 1 \leq j \leq d$.

For any weighted equation, $\mathcal{E} = (p, S, S')$, we recall the "cost" of this equation is : $H(\mathcal{E}) = p + 2 \cdot \mathrm{Div}(S, S')$.

Let us define an *oracle* on deterministic vectors as a mapping $\mathcal{O} : \bigcup_{\lambda \geq 1} \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V > \rightarrow \mathcal{P}(X^* \times X^*)$ such that:

$$\forall (S, S') \in \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V >, S \sim S' \Rightarrow \mathcal{O}(S, S') \in \mathcal{B}_\infty(S, S').$$

In other words, an oracle is a *choice* of w-$\bar{\psi}$-bisimulation for every pair of equivalent vectors (modulo $\sim$). Let us denote by Ω the set of all oracles. Let us fix an oracle $\mathcal{O}$ throughout this section.

We associate to every system (119) another equation, $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$, which "translates the equations of $\mathcal{S}$ into equations over the coefficients $(\alpha_{i,j}, \beta_{i,j})$ only"[5]. The general idea of the construction of $\mathrm{INV}^{(\mathcal{O})}$ consists in iterating the transformation used in the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in lemma 3.34, i.e. the classical idea of *triangulating* a system of linear equations. Of course we must deal with the weights and relate the construction with the deduction system $\mathcal{B}_0$.

5.1 Restricted systems

We assume here that

$$\forall j \in [1, d], S_j \not\equiv \emptyset^{\lambda}, \tag{120}$$

and

$$\forall i \in [m, m+n-1], \forall j \in [1, d], \alpha_{i,j}, \beta_{i,j} \text{ are } \epsilon - \text{free}. \tag{121}$$

A system $\mathcal{S}$ fulfilling both hypotheses (120), (121) will be called a *restricted* system of weighted linear equations.

Let us define $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$, $\mathrm{W}^{(\mathcal{O})}(\mathcal{S}) \in \mathbb{N} \cup \{\perp\}$, $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \in \mathbb{N}$ by induction on n. $\mathrm{W}^{(\mathcal{O})}(\mathcal{S})$ is the *weight* of $\mathcal{S}$. $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ is the *weak codimension* of $\mathcal{S}$.
Case 1 : $\alpha_{m,*} \sim \beta_{m,*}$

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = (\mathrm{W}^{(\mathcal{O})}(\mathcal{S}), \alpha_{m,*}, \beta_{m,*}), \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = p_m - 1,\ \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0.$$

Case 2 : $\alpha_{m,*} \not\sim \beta_{m,*}, n \geq 2, p_{m+1} - p_m \geq 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*}) + 1$
Let us consider $\mathcal{R} = \mathcal{O}(\sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j)$, $\nu = \mathrm{Div}(\alpha_{m,*}, \beta_{m,*})$ and

$$(u, u') = \min\{(v, v') \in \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu} \mid \exists j \in [1, d], (\alpha_{m,*} \odot v = \epsilon_j^{\lambda}) \Leftrightarrow (\beta_{m,*} \odot v' \neq \epsilon_j^{\lambda})\}. \tag{122}$$

As $\alpha_{m,*}, \beta_{m,*}$ are ϵ-free, there exists $j_0 \in [1, d]$ such that $(\alpha_{m,*} \odot u = \epsilon_{j_0}^{\lambda}) \Leftrightarrow (\beta_{m,*} \odot u' \neq \epsilon_{j_0}^{\lambda})$.
Subcase 1 : $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u' = \emptyset$.
Let us consider the equation

$$(\mathcal{E}'_m) : p_m + 2 \cdot |u|, S_{j_0}, \sum_{j=1}^{d} (\beta_{m,j} \odot u') S_j$$

and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by :

$$(\mathcal{E}'_i) : p_i, \sum_{j \neq j_0} (\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j} \odot u)) S_j\ ,\ \sum_{j \neq j_0} (\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j} \odot u)) S_j$$

[5] The function INV defined in [Sén97a] was an "elaborated version" of the *inverse* systems defined in [Mei89, Mei92] in the case of a single equation. We consider here a *relativization* of this notion to some oracle O.

where the above equation is seen as as an equation between two linear combinations of the S_i's where the j_0-th coefficient is $\emptyset$ on both sides. We then define :

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}'), \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{W}^{(\mathcal{O})}(\mathcal{S}'), \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}') + 1.$$

Subcase 2 : $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u' \neq \emptyset$.

Let us consider the equation

$$(\mathcal{E}'_m) : p_m + 2 \cdot |u|, S_{j_0} \ , \ \sum_{\substack{j=1 \\ j \neq j_0}}^{d} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u') S_j$$

and define a new system of weighted equations $\mathcal{S}' = (\mathcal{E}'_i)_{m+1 \leq i \leq m+n-1}$ by :

$$(\mathcal{E}'_i) : p_i, \sum_{j \neq j_0} [(\alpha_{i,j} + \alpha_{i,j_0} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u')] S_j \ , \ \sum_{j \neq j_0} [(\beta_{i,j} + \beta_{i,j_0} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u')] S_j .$$

(Here also the above equation is seen as an equation between two linear combinations of the S_i's, $1 \leq i \leq d$).We then define :

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}'), \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{W}^{(\mathcal{O})}(\mathcal{S}'), \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}') + 1.$$

Subcase 3 : $\alpha_{m,j_0} \odot u = \emptyset, \beta_{m,j_0} \odot u' = \varepsilon$
(analogous to subcase 1).
Subcase 4 : $\alpha_{m,j_0} \odot u \neq \emptyset, \beta_{m,j_0} \odot u' = \varepsilon$
(analogous to subcase 2).
Case 3 : $\alpha_{m,*} \not\sim \beta_{m,*}, n = 1$.
We then define:

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \perp, \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = \perp, \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0,$$

where $\perp$ is a special symbol which can be understood as meaning "undefined".
Case 4 : $\alpha_{m,*} \not\sim \beta_{m,*}, n \geq 2, p_{m+1} - p_m \leq 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*})$.
We then define:

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \perp, \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) = \perp, \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0.$$

Lemma 5.1 *: Let $\mathcal{S}$ be a restricted system of linear equations. If $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$ then, $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$ is a weighted linear equation with deterministic coefficients.*

Proof: Follows from lemmas 3.29,3.30 and the formula defining $\mathcal{S}'$ from $\mathcal{S}$. □

Lemma 5.2 *: Let $\mathcal{S}$ be a restricted system of linear equations. If $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$ then:*

1. $\{\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1\} \models\!\!- \mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$

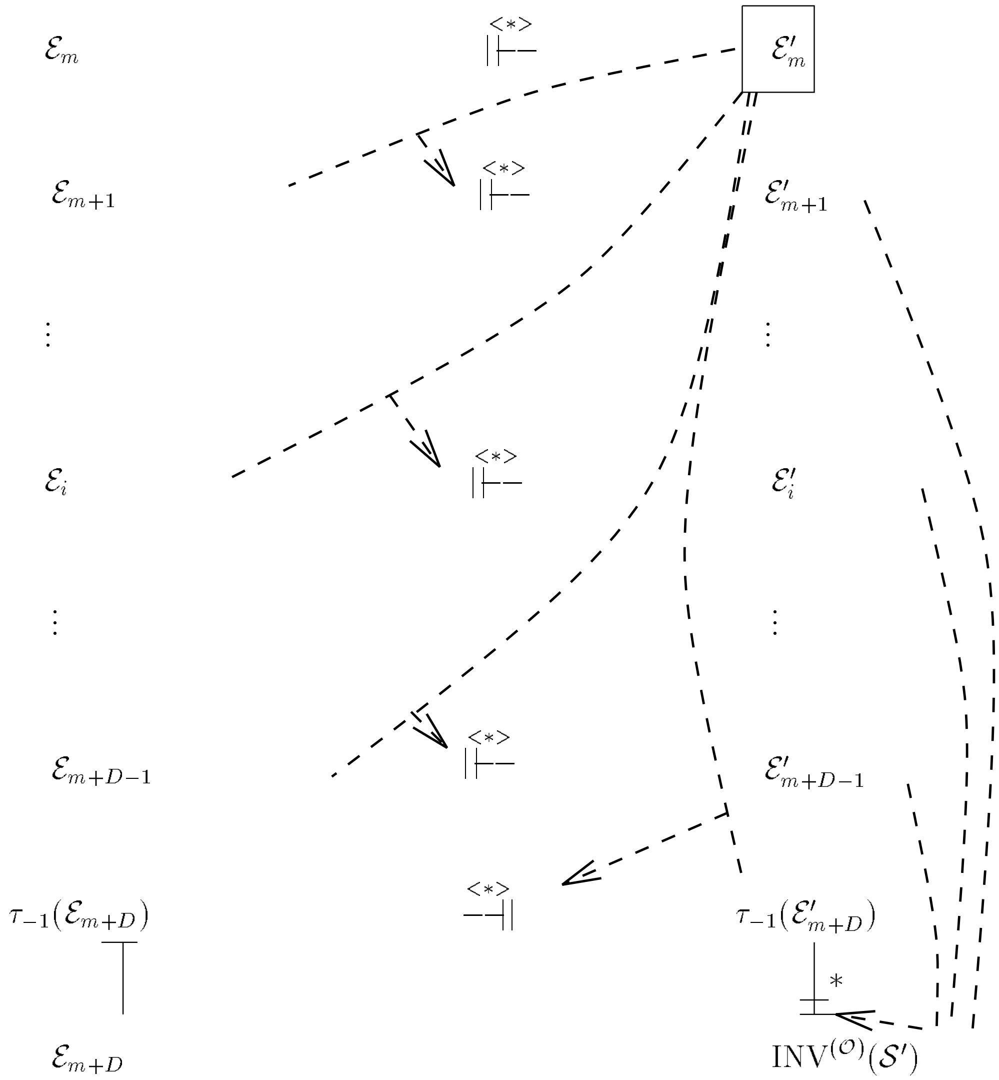

Fig. 2. Proof of lemma 5.2

2. $\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} = \infty \Longrightarrow H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) = \infty$.

Proof: See on figure 2 the "graph of the deductions" we use for proving point (1). Let us prove by induction on $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ the following strengthened version of point (1):

$$\{\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1\} \stackrel{<*>}{\Vdash} \tau_{-1}(\mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}) \quad (123)$$

where, for every integer $k \in \mathbb{Z}$, $\tau_k : \{(p, S, S') \in \mathcal{A} \mid p \geq -k\} \rightarrow \mathcal{A}$ is the *translation* map on the weights: $\tau_k(p, S, S') = (p + k, S, S')$.
if $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0$: as $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp, \mathcal{S}$ must fulfill the hypothesis of case 1.

$$\mathcal{E}_m = (p_m, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j) = \mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$$

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = (p_m - 1, \alpha_{m,*}, \beta_{m,*}).$$

Using rules (R8) we obtain :

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \overset{<*>}{|\!\!\vdash\!\!-} (p_m - 1, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d}, \beta_{m,j} S_j) = \tau_{-1}(\mathcal{E}_m).$$

if $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = m + 1, n \geq 0$: $\mathcal{S}$ must fulfill case 2.
• Suppose **case 2, subcase 1** occurs.
As the relation $\mathcal{R}$ used in the construction of $\mathcal{E'}_m$ from $\mathcal{E}_m$ is a w-$\bar{\psi}$-bisimulation w.r.t. the pair of sides of equation $\mathcal{E}_m$, using (R5) we obtain a deduction :

$$\mathcal{E}_m \overset{<2\cdot|u|>}{|\!\!\vdash\!\!-} \mathcal{E'}_m. \tag{124}$$

Using (R2,R9) we get that, for every $i \in [m + 1, m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})]$

$$\{\mathcal{E}_i, \mathcal{E'}_m\} \overset{<*>}{|\!\!\vdash\!\!-} (\max\{p_i, p_m{+}2 \mid u \mid\}, \sum_j (\alpha_{i,j}{+}\alpha_{i,j_0}(\beta_{m,j} \odot u')) S_j, \sum_j (\beta_{i,j}{+}\beta_{i,j_0}(\beta_{m,j} \odot u')) S_j)$$

but the hypothesis of case 2 implies that $\max\{p_{m+1}, p_m{+}2 \mid u \mid\} = p_{m+1}$ and the fact that $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S'})$ is defined implies that $\forall i \in [m{+}1, m{+}\mathrm{D}^{(\mathcal{O})}(\mathcal{S})], p_i \geq p_{m+1}$, hence, $\max\{p_i, p_m + 2 \mid u \mid\} = p_i$ and the right-hand side of the above deduction is exactly $\mathcal{E'}_i$. Hence,

$$\forall i \in [m + 1, m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})], \;\; \{\mathcal{E}_i, \mathcal{E'}_m\} \overset{<*>}{|\!\!\vdash\!\!-} \mathcal{E'}_i. \tag{125}$$

Using deductions (124) and (125), we obtain that:

$$\{\mathcal{E}_i \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1\} \overset{<*>}{|\!\!\vdash\!\!-} \{\mathcal{E'}_i \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1\}. \tag{126}$$

By induction hypothesis :

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S'}) \cup \{\mathcal{E}'_i \mid m + 1 \leq i \leq m + 1 + \mathrm{D}^{(\mathcal{O})}(\mathcal{S'}) - 1\} \overset{<*>}{|\!\!\vdash\!\!-} \tau_{-1}(\mathcal{E'}_{m+1+\mathrm{D}^{(\mathcal{O})}(\mathcal{S'})})$$

which is equivalent to

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \cup \{\mathcal{E}'_i \mid m + 1 \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1\} \overset{<*>}{|\!\!\vdash\!\!-} \tau_{-1}(\mathcal{E'}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}). \tag{127}$$

As $p_m + 2 \cdot |u| \leq p_{m+1} - 1 \leq p_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} - 1$, we have also the following inverse deduction (which is similar to deduction (125)):

$$\{\mathcal{E}'_m, \tau_{-1}(\mathcal{E}'_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})})\} \overset{<*>}{|\!\!\vdash\!\!-} \tau_{-1}(\mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}). \tag{128}$$

Combining together deductions (126) (127) and (128), we have proved (123). Using rule (R0), this last deduction leads to point (1) of the lemma.

• Suppose now that **case 2, subcase 2** occurs.

$$\mathcal{E}'_m = (p_m + 2 \mid u \mid, S_{j_0}, \sum_{\substack{j=1 \\ j \neq j_0}}^{d} (\beta_{m,j_0} \odot u')^* \beta_{m,j} \cdot S_j).$$

Using (R5) and then (R6) , (this is possible because $\beta_{m,j_0} \odot u' \not\equiv \epsilon$), we obtain, analogously with (124):

$$\mathcal{E}_m \overset{<2\cdot|u|+1>}{|\!\!\vdash\!\!-} \mathcal{E}'_m. \tag{129}$$

All the other arguments used in previous subcase remain valid here, hence we may conclude that (123) is true again, hence that statement (1) of the lemma is true. The two other subcases can obviously be treated in the same way.

Let us prove statement (2) of the lemma.
We prove by induction on $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ the statement:

$$\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} = \infty \Longrightarrow H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) = \infty. \tag{130}$$

if $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0$: as $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$, case 1 must occur. $\alpha_{m,*} \sim \beta_{m,*}$ implies that $H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) = \infty$, hence the statement is true.
if $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = p+1, p \geq 0$: as $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \geq 1$ and $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$, case 2 must occur.
Using deductions (124) and (125) established above we obtain that :

$$\{\mathcal{E}_i \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} \overset{<*>}{|\!\!\vdash\!\!-} \{\mathcal{E}'_i \mid m+1 \leq i \leq m+1+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}')\},$$

which proves that

$$\min\{H(\mathcal{E}_i) \mid m \leq i \leq m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} \leq \min\{H(\mathcal{E}'_i) \mid m+1 \leq i \leq m+1+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}')\}. \tag{131}$$

As $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}') = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1$, we can use the induction hypothesis:

$$\min\{H(\mathcal{E}'_i) \mid m+1 \leq i \leq m+1+D(\mathcal{S}')\} = \infty \Longrightarrow H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}')) = \infty. \tag{132}$$

As $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}')$, (131,132) imply statement (130). □

Lemma 5.3 *: Let $\mathcal{S}$ be a restricted system of linear equations satisfying the hypothesis of case 2. Then, $\forall i \in [m+1, m+n-1]$,*
$\| \alpha'_{i,*} \| \leq \| \alpha_{i,*} \| + \| \beta_{m,*} \| + | Q || u |, \| \beta'_{i,*} \| \leq \| \beta_{i,*} \| + \| \beta_{m,*} \| + | Q || u |$.

Proof: Follows from lemmas 3.29,3.30,3.41 and the formula defining $\mathcal{S}'$ from $\mathcal{S}$. $\square$

Let us consider the function F defined by :

$$F(d,n) = \max\{\mathrm{Div}(A,B) \mid A, B \in \mathsf{DRB}_{1,d} < V >, \| A \| \leq n, \| B \| \leq n, A \not\sim B\}.$$

For every integer parameters $K_1, K_2, K_3, K_4 \in \mathbb{N} - \{0\}$, we define integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ by :

$$\delta_m = 0, \ell_m = 0, L_m = K_2, s_m = K_3 \cdot K_2 + K_4, S_m = 0, \Sigma_m = 0, \tag{133}$$

$$\begin{cases} \delta_{i+1} &= 2 \cdot F(d, s_i + \Sigma_i) + 1 \\ \ell_{i+1} &= 2 \cdot \delta_{i+1} + 3 \\ L_{i+1} &= K_1 \cdot (L_i + \ell_{i+1}) + K_2 \\ s_{i+1} &= K_3 \cdot L_{i+1} + K_4 \\ S_{i+1} &= s_i + \Sigma_i + \mid Q \mid F(d, s_i + \Sigma_i) \\ \Sigma_{i+1} &= \Sigma_i + S_{i+1} \end{cases} \tag{134}$$

for $m \leq i \leq m+n-2$.

These sequences are intended to have the following meanings when K_1, K_2, K_3, K_4 are chosen to be the constants defined in section 6 and the equations $(\mathcal{E}_i)$ are labelling nodes of a N-stacking sequence (see section 8.4):

$\delta_{i+1} \leq$ increase of weight between $\mathcal{E}_i, \mathcal{E}_{i+1}$
$\ell_{i+1} \geq$ increase of depth between $\mathcal{E}_i, \mathcal{E}_{i+1}$
$\mathrm{L}_{i+1} \geq$ increase of depth between $\mathcal{E}_m, \mathcal{E}_{i+1}$
$s_{i+1} \geq$ size of the coefficients of $\mathcal{E}_{i+1}$
$S_{i+1} \geq$ size of the coefficients of $\mathcal{E}_{i+1}^{(i+1-m)}$ (these systems are introduced below in the proof of lemma 5.4)
$\Sigma_{i+1} \geq$ increase of the coefficients between $\mathcal{E}_k^{(i-m)}, \mathcal{E}_k^{(i+1-m)}$(for $k \geq i+1$).

For every weighted, deterministic rational linear equation $\mathcal{E} = (p, \sum_{j=1}^d \alpha_j S_j \ , \ \sum_{j=1}^d \beta_j S_j)$, we define

$$\||\mathcal{E}\|| = max\{\| \alpha \|, \| \beta \|\}.$$

Lemma 5.4 *Let $\mathcal{S} = (\mathcal{E}_i)_{m \leq i \leq m+d-1}$ be a restricted system of d weighted equations such that $H(\mathcal{E}_i) = \infty$ (for every i) and :*

(1) $\forall i \in [m, m+d-1], \||\mathcal{E}_i\|| \leq s_i$
(2) $\forall i \in [m, m+d-2], \mathrm{W}(\mathcal{E}_{i+1}) - \mathrm{W}(\mathcal{E}_i) \geq \delta_{i+1}$.

Then $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp, \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \leq d-1, \||\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\|| \leq \Sigma_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} + s_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$.

Proof: (Figure 3 might help the reader to follow the definitions below). Let us define a sequence of systems $\mathcal{S}^{(i-m)} = (\mathcal{E}_k^{(i-m)})_{m \leq i \leq k \leq m+d-1}$, where $i \in [m, m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})]$, by induction :

$$
\begin{array}{llllll}
\mathcal{E}_m = \mathcal{E}_m^{(0)} & \boxed{(\mathcal{E}_m^{(0)})'} & & & & \\
\mathcal{E}_{m+1} = \mathcal{E}_{m+1}^{(0)} & \mathcal{E}_{m+1}^{(1)} & \ddots & & & \\
\vdots & \vdots & & \boxed{(\mathcal{E}_{i-1}^{(i-1-m)})'} & & \\
\mathcal{E}_i = \mathcal{E}_i^{(0)} & \mathcal{E}_i^{(1)} & \cdots & \mathcal{E}_i^{(i-m)} & \ddots & \\
\vdots & \vdots & & \vdots & & \boxed{(\mathcal{E}_{m-1+D}^{(D-1)})'} \\
\mathcal{E}_{m+D} = \mathcal{E}_{m+D}^{(0)} & \mathcal{E}_{m+D}^{(1)} & \cdots & \mathcal{E}_{m+D}^{(i-m)} & \vdots & \mathcal{E}_{m+D}^{(D)}
\end{array}
$$

Fig. 3. Proof of lemma 5.4

- $\mathcal{E}_k^{(0)} = \mathcal{E}_k$ for $m \leq k \leq m+d-1$
- if case 1 or case 3 or case 4 is realized, $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = 0$, hence $\mathcal{S}^{(i-m)}$ is well-defined for $m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})$
- if case 2 is realized then we set : $\forall i \geq m+1, \mathcal{E}_k^{(i-m)} = ({\mathcal{E}'}_k)^{(i-m-1)}$, for $m+1 \leq k \leq m+d-1$.

Let us prove by induction on $i \in [m, m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})]$ that, $\forall k \in [i, m+d-1]$:

$$||| \, \mathcal{E}_k^{(i-m)} \, ||| \leq s_k + \Sigma_i. \tag{135}$$

$i = m$: in this case

$$||| \, \mathcal{E}_k^{(i-m)} \, ||| = ||| \, \mathcal{E}_k \, ||| \leq s_k = s_k + \Sigma_m.$$

$i + 1 \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})$: in this case, by lemma 5.3,

$$||| \, \mathcal{E}_k^{(i+1-m)} \, ||| \leq ||| \, \mathcal{E}_k^{(i-m)} \, ||| + ||| \, \mathcal{E}_i^{(i-m)} \, ||| + | \, Q \, || \, u \, |$$

where $\mathcal{R} = \mathcal{O}(\sum_{j=1}^{d} \alpha_{i,j}^{(i-m)} S_j, \sum_{j=1}^{d} \beta_{i,j}^{(i-m)} S_j)$, $\nu = \mathrm{Div}(\alpha_{i,*}^{(i-m)}, \beta_{i,*}^{(i-m)})$, and $(u, u') = \min\{(v, v') \in \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu} \mid \exists j \in [1, d], (\alpha_{i,*}^{(i-m)} \odot v = \epsilon_j^{\lambda}) \Leftrightarrow (\beta_{i,*}^{(i-m)} \odot v' \neq \epsilon_j^{\lambda})\}$.
By definition of F and the induction hypothesis :

$$| u | \leq F(d, ||| \mathcal{E}_i^{(i-m)} |||) \leq F(d, s_i + \Sigma_i).$$

Hence

$$\begin{aligned} ||| \mathcal{E}_k^{(i+1-m)} ||| \leq (s_k + \Sigma_i) + (s_i + \Sigma_i) + | Q | F(d, s_i + \Sigma_i) &= (s_k + \Sigma_i) + S_{i+1} \\ &= s_k + \Sigma_{i+1}. \end{aligned}$$

Let us notice that $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ is always an integer and that this proof is valid for $m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}), i \leq k \leq m + d - 1$.
Let us prove now that $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$. Let us consider the system $(\mathcal{E}_k^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))})_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \leq k \leq m+d-1}$.
If $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = d - 1$, $\mathcal{E}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}$ fulfills either case 1 or case 3 of the definition of $\mathrm{INV}^{(\mathcal{O})}$.
Using the successive deductions (124)(125) established in the proof of lemma 5.2 we get that:

$$\{\mathcal{E}_i \mid m \leq i \leq m + d - 1\} \overset{<*>}{|\!\vdash\!\!-} \{\mathcal{E}_{m+d-1}^{(d-1)}\}.$$

Using now the hypothesis that $H(\mathcal{E}_i) = \infty$ (for $m \leq i \leq m + d - 1$), we obtain:

$$H(\mathcal{E}_{m+d-1}^{(d-1)}) = \infty. \tag{136}$$

For any system of equations $\mathcal{S}$, let us define the *support* of the system as

$$\mathrm{supp}(\mathcal{S}) = \{j \in [1, d] \mid \sum_{i=m}^{m+n-1} \alpha_{i,j} + \beta_{i,j} \neq \emptyset\}.$$

Let us consider $\delta = \mathrm{Card}(\mathrm{supp}(\mathcal{S}^{(d-1)})$. One can prove by induction on i that:

$$\mathrm{Card}(\mathrm{supp}(\mathcal{S}^{(i-m)}) \leq d - i + m,$$

hence

$$\delta = \mathrm{Card}(\mathrm{supp}(\mathcal{S}^{(d-1)}) \leq d - (d - 1) = 1.$$

- If $\delta = 1, \mathrm{supp}(\mathcal{S}^{(d-1)}) = \{j_0\}$, for some $j_0 \in [1, d]$.
  By corollary 4.6 point C4 and hypothesis (120), the implication $[(\alpha_{m+d-1,j_0}^{(d-1)} S_{j_0} \sim \beta_{m+d-1,j_0}^{(d-1)} S_{j_0}) \Longrightarrow \alpha_{m+d-1,j_0}^{(d-1)} \sim \beta_{m+d-1,j_0}^{(d-1)}]$ holds. Hence , by (136), $\alpha_{m+d-1,j_0}^{(d-1)} \sim \beta_{m+d-1,j_0}^{(d-1)}$, i.e. $\mathcal{S}^{(d-1)})$ fulfills case 1, so that

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}^{(d-1)}) \neq \perp.$$

- If $\delta = 0, \mathrm{supp}(\mathcal{S}) = \emptyset$.
  Then $\alpha_{m+d-1,*}^{(d-1)} = \beta_{m+d-1,*}^{(d-1)} = \emptyset^d$. Here also $\mathcal{S}^{(d-1)}$ fulfills case 1.

If $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) < d-1$, by hypothesis :

$\mathrm{W}(\mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})+1}) - \mathrm{W}(\mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}) \geq \delta_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})+1} = 2F(d, s_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} + \Sigma_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}) + 1.$

If $\alpha^{\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*} \sim \beta^{\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*}$, then $\mathcal{E}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$ fulfills case 1 of the definition of $\mathrm{INV}^{(\mathcal{O})}$, hence $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$.
Otherwise, let us consider:

$$\mathcal{R} = \mathcal{O}(\sum_{j=1}^{d} \alpha^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),j} S_j, \sum_{j=1}^{d} \beta^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),j} S_j),$$

$$\nu = \mathrm{Div}(\alpha^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*}, \beta^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*}), \text{ and}$$

$$(u, u') = \min\{(v, v') \in \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu} \mid \exists j \in [1, d], (\alpha^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*} \odot v = \epsilon^{\lambda}_{j}) \Leftrightarrow (\beta^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S}),*} \odot v' \neq \epsilon^{\lambda}_{j})\}.$$

By definition of F and inequality (135),

$$\mid u \mid \leq F(d, ||| \mathcal{E}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} |||) \leq F(d, s_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} + \Sigma_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}).$$

Hence $p_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})+1} - p_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} \geq 2 \mid u \mid +1$ i.e. the hypothesis of case 2 is realized. This proves that $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}) \geq 1$ while in fact, $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}) = 0$. This contradiction shows that this last case ($\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) < d-1$ and $\mathcal{E}^{(\mathrm{D}^{(\mathcal{O})}(\mathcal{S}))}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$ not fulfilling case 1 of definition of $\mathrm{INV}^{(\mathcal{O})}$) is impossible. We have proved point (2) of the lemma. □

5.2 General systems

We consider now the general case where assumptions (120) and (121) are removed. Let us suppose that

$$\exists d_1 \in [1, d], S_{d_1} \not\equiv \emptyset. \tag{137}$$

Up to some permutation of the column indices (such a permutation leaves function H invariant), we can suppose that there exists $\hat{d} \in [1, d]$ such that:

$$\forall j \in [1, \hat{d}], S_j \not\equiv \emptyset, \;\; \forall j \in [\hat{d}+1, d], S_j \equiv \emptyset \tag{138}$$

We then associate to the original system $\mathcal{S}$ a new system $\hat{\mathcal{S}}$ of n weighted linear equations:

$$(\hat{\mathcal{E}}_i) : p_i, \sum_{j=1}^{\hat{d}} \rho_\epsilon(\alpha_{i,j}) \cdot S_j \; , \; \sum_{j=1}^{\hat{d}} \rho_\epsilon(\beta_{i,j}) \cdot S_j$$

where $m \leq i \leq m+n-1$.
We then define:

$$\text{If } \hat{d}=0 \text{ then } \mathrm{INV}^{(\mathcal{O})}(\mathcal{S})=\perp,\ \mathrm{W}^{(\mathcal{O})}(\mathcal{S})=\perp,\ \mathrm{D}^{(\mathcal{O})}(\mathcal{S})=0.$$

$$\text{If } \hat{d}\geq 1 \text{ then } \mathrm{INV}^{(\mathcal{O})}(\mathcal{S})=\mathrm{INV}^{(\mathcal{O})}(\hat{\mathcal{S}}),\ \mathrm{W}^{(\mathcal{O})}(\mathcal{S})=\mathrm{W}^{(\mathcal{O})}(\hat{\mathcal{S}}),\ \mathrm{D}^{(\mathcal{O})}(\mathcal{S})=\mathrm{D}^{(\mathcal{O})}(\hat{\mathcal{S}}).$$

Let us show that lemmas 5.1,5.2,5.4 remain true in the general case.

Lemma 5.5 (preliminary lemma) *For every $i \in [m, m+n-1]$*

(1) $\hat{\mathcal{E}}_i \ \overset{<*>}{|\!\!\vdash\!\!-}\ \mathcal{E}_i$,
(2) $\mathcal{E}_i \ \overset{<*>}{|\!\!\vdash\!\!-}\ \hat{\mathcal{E}}_i$.

Proof: By lemma 4.7, $\forall i \in [m, m+n-1]$,

$$\emptyset \ \overset{<*>}{|\!\!\vdash\!\!-}\ (0, \alpha_{i,*}, \rho_\epsilon(\alpha_{i,*})),$$

whose combination with (R8) gives, $\forall i \in [m, m+n-1]$:

$$\emptyset \ \overset{<*>}{|\!\!\vdash\!\!-}\ (0, \sum_{j=1}^{d} \alpha_{i,j} \cdot S_j, \sum_{j=1}^{d} \rho_\epsilon(\alpha_{i,j}) \cdot S_j), \tag{139}$$

Using rule (R'3) (for all the triples $(0, S_j, \emptyset), j \in [\hat{d}+1, d]$), combined with rule (R8) we get, $\forall i \in [m, m+n-1]$,

$$\emptyset \ \overset{<*>}{|\!\!\vdash\!\!-}\ (0, \sum_{j=1}^{d} \rho_\epsilon(\alpha_{i,j}) \cdot S_j, \sum_{j=1}^{\hat{d}} \rho_\epsilon(\alpha_{i,j}) \cdot S_j). \tag{140}$$

Using then rules (R1, R2), deductions (139, 140) and their analogues for the right-hand sides, we obtain points (1) and (2) of the lemma. □

Lemma 5.6 *: Let $\mathcal{S}$ be a system of linear equations. If $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$ then, $\forall \mathcal{E} \in \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}), \mathcal{E}$ is a weighted linear equation with deterministic coefficients.*

Proof: As $\hat{\mathcal{S}}$ is a restricted system, this follows from lemma 5.1. □

Lemma 5.7 *: Let $\mathcal{S}$ be a system of linear equations. If $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp$ then :*

1. $\{\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\} \cup \{\mathcal{E}_i \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) - 1\} \vdash\!\!- \mathcal{E}_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$,
2. $\min\{H(\mathcal{E}_i) \mid m \leq i \leq m + \mathrm{D}^{(\mathcal{O})}(\mathcal{S})\} = \infty \Longrightarrow H(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) = \infty$.

Proof: Point (1) follows from lemma 5.2 point (1) and from lemma 5.5. Point (2) follows from lemma 5.2 point (2) and from lemma 5.5, point (2). □

Lemma 5.8 *Let* $\mathcal{S} = (\mathcal{E}_i)_{m \le i \le m+d-1}$ *be a system of* d *weighted equations such that* $H(\mathcal{E}_i) = \infty$ *(for every* i*) and:*

(0) $\exists j \in [1, d], S_j \not\equiv \emptyset$,
(1) $\forall i \in [m, m+d-1], ||| \mathcal{E}_i ||| \le s_i$,
(2) $\forall i \in [m, m+d-2], W(\mathcal{E}_{i+1}) - W(\mathcal{E}_i) \ge \delta_{i+1}$.

Then $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \ne \perp, \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \le d-1, \forall \mathcal{E} \in \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}), ||| \mathcal{E} ||| \le \Sigma_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} + s_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}$.

Proof: By hypothesis (0), $\hat{\mathcal{S}}$ is defined and is a restricted system of linear equations. Moreover, using lemma 3.13 for every $i \in [m, m+d-1], ||| \hat{\mathcal{E}_i} ||| \le ||| \mathcal{E}_i |||$. Hence $\hat{\mathcal{S}}$ fulfills the hypothesis of lemma 5.4 and the conclusion of this previous lemma applied on the system $\hat{\mathcal{S}}$ gives :

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \ne \perp, \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) \le \hat{d}-1, \forall \mathcal{E} \in \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}), ||| \mathcal{E} ||| \le \Sigma_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})} + s_{m+\mathrm{D}^{(\mathcal{O})}(\mathcal{S})}.$$

□

6 Constants

Let us *fix* a birooted dpda $\mathcal{M}$, a s.r. morphism $\bar{\psi}$ and an initial equation $A_0 = (\Pi_0, S_0^-, S_0^+) \in \mathbb{N} \times \mathsf{DRB}_{1,\lambda_0} \times \mathsf{DRB}_{1,\lambda_0}$ in the corresponding set of assertions. This short section is devoted to the definition of some integer *constants*: these integers are constant in the sense that they are depending only on this dpda $\mathcal{M}$ and initial equation $A_0 = (\Pi_0, U_0^-, U_0^+)$. The *motivation* of each of these definitions will appear later on, in different places for the different constants. The equations below provide merely an overview of the dependencies betweens these constants and allow to check that the definitions are sound (i.e. there is no hidden loop in the dependencies).

$$k_0 = \max\{\nu([pAq]) \mid p, q \in Q, A \in Z, [pAq] \not\equiv \emptyset\}, \quad k_1 = \max\{2k_0 + 1, 3\}, \tag{141}$$

$$D_1 = 4 \cdot k_0 + 3, \quad k_2 = (D_1 + 4) \cdot k_1 + k_0 + 1. \tag{142}$$

k_1 is used in the definition of strategy T_B (section7), D_1 appears as an upper-bound on the left-defect of series in lemma 8.6 and k_2 is used in the definition of a "security band" before lemma 8.12.

$$K_1 = k_1 + 1, \quad K_2 = (D_1 + 1) \cdot k_1^2 + (D_2 + 1) \cdot k_1 + 1. \tag{143}$$

These constants K_1, K_2 appear in lemma 8.15.

$$K_3 = k_0 |Q|, \quad K_4 = k_0 \cdot |Q|^2 + (k_2 + k_1 + 7) \cdot |Q| + 3. \tag{144}$$

These constants K_3, K_4 appear in lemma 8.16.

$$d_0 = 2 \cdot |Q| \cdot (\mathrm{Card}(X^{\le k_1}) + 1). \tag{145}$$

d_0 appears as an upper-bound on the dimension of the d-space V_1 defined by equation (187) and used in lemma 8.15. We consider now the integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leq i \leq m+n-1}$ defined by the relations (134) of section 5 where the parameters $K_1, \ldots, K_4$ are chosen to be the above constants and $m = 1, n = d = d_0$. Equivalently, they are defined by:

$$\delta_1 = 0, \ell_1 = 0, L_1 = K_2, s_1 = K_3 \cdot K_2 + K_4, S_1 = 0, \Sigma_1 = 0, \tag{146}$$

$$\begin{cases} \delta_{i+1} & = 2 \cdot F(d_0, s_i + \Sigma_i) + 1 \\ \ell_{i+1} & = 2 \cdot \delta_{i+1} + 3 \\ L_{i+1} & = K_1 \cdot (L_i + \ell_{i+1}) + K_2 \\ s_{i+1} & = K_3 \cdot L_{i+1} + K_4 \\ S_{i+1} & = s_i + \Sigma_i + \mid Q \mid F(d_0, s_i + \Sigma_i) \\ \Sigma_{i+1} & = \Sigma_i + S_{i+1} \end{cases} \tag{147}$$

for $1 \leq i \leq d_0 - 1$.

$$D_2 = \max\{\Sigma_{d_0} + s_{d_0}, \|S_0^-\|, \|S_0^+\|\}, \tag{148}$$

$\Sigma_{d_0} + s_{d_0}$ appears in the conclusion of lemma 5.8 when we take $d = d_0$ in the hypothesis and suppose that $\mathrm{D}^{(\mathcal{O})}(\mathcal{S})$ has its maximal possible value i.e. $\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = d_0 - 1$. It is used as an upper-bound on the right-defect in the definition of the trees τ analysed in part 8 (inequation (168)).

$$N_0 = 1 + (k_2 + 2)|Q| + D_2. \tag{149}$$

N_0 appears as a lower bound for the norm in the definition of a N-stacking sequence (section 8.4, condition (173)).

$$\lambda_2 = \max\{\lambda_0, d_0\}, \tag{150}$$

$$C_2 = \mathrm{Card}\{U \in \mathsf{DRB}_{1,\lambda_2} < V >, \|U\| \leq D_2\}, \tag{151}$$

$$K_6 = 5 \cdot [(C_2 \cdot |Q||Z|^{k_2+D_2+3})^{|Q|} \cdot |Q||Z|^{D_1}]^2, \;\; K_5 = (K_6 + 1) \cdot k_0 \cdot |Q|. \tag{152}$$

K_5, K_6 appear in lemma 8.18 and C_2 is used in the proof of lemma 8.18.

7 Strategies for $\mathcal{B}_0$

Let us define strategies for the particular system $\mathcal{B}_0$.

7.1 Strategies

We shall define first auxiliary strategies $T_{cut}, T_{\emptyset}, T_{\varepsilon}$, and then for every oracle $\mathcal{O} \in \Omega$ auxiliary strategies $T_A^{(\mathcal{O})}, T_B^{(\mathcal{O})}, T_C^{(\mathcal{O})}$, we define the strategies T_A, T_B, T_C and finally the "compound" strategies $\mathcal{S}_{AB}^{(\mathcal{O})}, \mathcal{S}_{ABC}^{(\mathcal{O})}, \mathcal{S}_{AB}, \mathcal{S}_{ABC}$. Let us fix here some total ordering on $X : x_1 < x_2 < \cdots < x_\alpha$.

T_{cut}:

$B_1 \cdots B_m \in T_{cut}(A_1 \cdots A_n)$ iff $\exists i \in [1, n-1], \exists S, T,$

$$A_i = (p_i, S, T), A_n = (p_n, S, T), p_i < p_n \text{ and } m = 0^6.$$

$T_{\emptyset}$:

$B_1 \cdots B_m \in T_{\emptyset}(A_1 A_2 \cdots A_n)$ iff $\exists S, T,$

$$A_n = (p, S, T), p \geq 0, S \equiv T \equiv \emptyset^\lambda \text{ and } m = 0.$$

T_{ε}:

$B_1 \cdots B_m \in T_{\varepsilon}(A_1 \cdots A_n)$ iff

$$A_n = (p, S, T), p \geq 0, S \equiv T \equiv \varepsilon_i^\lambda (\text{ for some } i \in [1, \lambda]) \text{ and } m = 0.$$

Let us consider an oracle $\mathcal{O} \in \Omega$.

$T_A^{(\mathcal{O})}$:

$B_1 \cdots B_m \in T_A^{(\mathcal{O})}(A_1 \cdots A_n)$ iff

$A_n = (p, S, T), \mid X \mid \leq m \leq \mid X \mid^2, B_1 = (p+1, S \odot x_1, T \odot x'_1), \cdots, B_m = (p+1, S \odot x_m, T \odot x'_m),$

where $S \not\equiv \varepsilon, T \not\equiv \varepsilon, \mathcal{O}(S, T) \cap X \times X = \{(x_1, x'_1), \ldots, (x_i, x'_i), \ldots, (x_m, x'_m)\}$.

$T_B^{(\mathcal{O}),+}$:

$B_1 \cdots B_m \in T_B^+(A_1 \cdots A_n)$ iff $n \geq k_1 + 1, A_{n-k_1} = (\pi, \overline{U}, U')$, (where $\overline{U}$ is unmarked)

$$U' = \sum_{q \in Q} [\overline{p} A q] \cdot V_q \quad (\text{ for some } \overline{p} \in Q)$$

$A_i = (\pi + k_1 + i - n, U_i, U'_i)$ for $n - k_1 \leq i \leq n$, $(U_i)_{n-k_1 \leq i \leq n}$ is a derivation, $(U'_i)_{n-k_1 \leq i \leq n}$ is a "stacking derivation" (see definitions in §8.2),

$$U'_n = \sum_{q \in Q} [p \tau q] \cdot V_q, \quad \text{for some } p \in Q, \tau \in Z^+,$$

$m = 1, B_1 = (\pi + k_1 - 1, V, V'), V = U_n,$

$$V' = \sum_{q \in Q'} [p \tau q] \cdot [q e q] \cdot (\overline{U} \odot u_q)$$

where $Q' = \{q \in Q \mid [\overline{p} A q] \not\equiv \emptyset\}, \forall q \in Q', u'_q = \min(\varphi([\overline{p} A q]))$, and if $\mathcal{R} = \mathcal{O}(S, T), \forall q \in Q', u_q = \min\{\mathcal{R}^{-1}(u'_q)\}$.

---

6 i.e. $B_1 \cdots B_m = \epsilon$

$T_B^{(\mathcal{O}),-}$:

$T_B^{(\mathcal{O}),-}$ is defined in the same way as $T_B^{(\mathcal{O}),+}$ by exchanging the left series (S^-) and right series (S^+) in every assertion (p, S^-, S^+).

$T_C^{(\mathcal{O})}$:

$B_1 \cdots B_m \in T_C^{(\mathcal{O})}(A_1 \cdots A_n)$ iff there exists $d \in [1, d_0], D \in [0, d-1], \lambda \in \mathbb{N} - \{0\}, S_1, S_2, \cdots, S_d \in \mathsf{DRB}_{1,\lambda} < V >, 1 \leq \kappa_1 < \kappa_2 < \cdots < \kappa_{D+1} = n$, such that,

(C1) every equation $\mathcal{E}_i = A_{\kappa_i} = (p_{\kappa_i} S^-_{p_{\kappa_i}}, S^+_{p_{\kappa_i}})$ is a weighted equation over $S_1, S_2, \cdots, S_d$, with $p_{\kappa_i} \geq 1$,

(C2)$\mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = D$ (where $\mathcal{S} = (\mathcal{E}_i)_{1 \leq i \leq D+1}$),

(C3)$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}) \neq \perp, ||| \,\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\, ||| \leq \Sigma_{d_0} + s_{d_0}$,

(C4) $m = 1$ and $B_1 = \rho_e(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}))$ (where ρ_e is the obvious extension of ρ_e to weighted pairs of deterministic row-vectors; in other words the result of $T_C^{(\mathcal{O})}$ is $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$ where the marks have been removed).

We then set, for every $W \in \mathcal{A}^+$:

$$T_A(W) = \bigcup_{\mathcal{O} \in \Omega} T_A^{(\mathcal{O})}(W),$$

$$T_B^+(W) = \bigcup_{\mathcal{O} \in \Omega} T_B^{(\mathcal{O}),+}(W), \;\; T_B^-(W) = \bigcup_{\mathcal{O} \in \Omega} T_B^{(\mathcal{O}),-}(W),$$

$$T_C(W) = \bigcup_{\mathcal{O} \in \Omega} T_C^{(\mathcal{O})}(W).$$

Lemma 7.1 *: $T_{cut}, T_\emptyset, T_\varepsilon, T_A$ are $\mathcal{B}_0$-strategies.*

Proof:
T_{cut} : (S1) is true by rule R0. (S2) is trivially true.
$T_\emptyset$: (S1) is true by rule R'3. (S2) is trivially true.
T_ε: (S1) is true by rule R'3. (S2) is trivially true.
T_A : by rule (R4), $\{B_j \mid 1 \leq j \leq m\} \,|\!\!\vdash\!\!-_4\, A_n$, which proves (S1). Suppose $H(A_n) = \infty$ i.e. $S \sim T$. Then, $\forall j \in [1, m], S \odot x_j \sim T \odot x'_j$, so that $\min\{H(B_j) \mid 1 \leq j \leq m\} = \infty$. (S2) is proved.
□

Lemma 7.2 *: T_B^+, T_B^- are $\mathcal{B}_0$-strategies.*

Proof: Let us show that T_B^+ is a $\mathcal{B}_0$-strategy.
Let us use the notation of the definition of $T_B^{(\mathcal{O}),+}$. Let $\mathcal{H} = \{(\pi, \overline{U}, U'), (\pi + k_1 - 1, V, V')\}$. Let us show that

$$\mathcal{H} \;\overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{B}_0} (\pi + k_1 - 1, U_n, U'_n). \tag{153}$$

Using rule (R5) we obtain: $\forall q \in Q'$,

$$\{(\pi, \overline{U}, U')\} = \{(\pi, \overline{U}, \sum_{r \in Q} [\overline{p}Ar] \cdot V_r)\} \overset{<*>}{|\!\vdash\!\!-}_{R5} (\pi + 2 \cdot \mid u_q \mid, \overline{U} \odot u_q, U' \odot u'_q)$$

$$\overset{<*>}{|\!\vdash\!\!-}_{R0} (\pi + 2 \cdot k_0, \overline{U} \odot u_q, U' \odot u'_q)$$

$$= \quad (\pi + 2 \cdot k_0, \overline{U} \odot u_q, V_q). \quad (154)$$

By rule $R'3$, for every q such that $[\overline{p}Aq] \equiv \emptyset$,

$$\emptyset \vdash\!\!- (0, [\overline{p}Aq], \emptyset). \quad (155)$$

Let us show that, for every q such that $[\overline{p}Aq] \equiv \emptyset$,

$$\emptyset \overset{<*>}{|\!\vdash\!\!-}_{\mathcal{C}} (0, [p\tau q], \emptyset). \quad (156)$$

From the equations $[\overline{p}Aq] \equiv \emptyset$ and $[\overline{p}Aq] \odot u = [p\tau q]$ (for some $u \in X^{k_1}$) we get that $[p\tau q] \equiv \emptyset$. Hence, by rule (R'3), (156) is true. From this deduction, we obtain

$$\emptyset \overset{<*>}{|\!\vdash\!\!-}_{\mathcal{C}} (\pi + 2k_0, U'_n, \sum_{[\overline{p}Aq] \not\equiv \emptyset} [p\tau q] \cdot V_q). \quad (157)$$

Using rule (R'3), for every $q \in Q$,

$$\emptyset \,|\!\vdash\!\!-_{R'3} (0, [qeq], \varepsilon). \quad (158)$$

Using (158) and (154) we obtain :

$$\{(\pi, \overline{U}, U')\} \overset{<*>}{|\!\vdash\!\!-}_{\mathcal{C}} (\pi + 2k_0, \sum_{[\overline{p}Aq] \not\equiv \emptyset} [p\tau q] \cdot V_q, \sum_{[\overline{p}Aq] \not\equiv \emptyset} [p\tau q][qeq] \cdot (\overline{U} \odot u_q)). \quad (159)$$

By (157) (159) and (R0,R2,R3) we get:

$$\{(\pi, \overline{U}, U')\} \overset{<*>}{|\!\vdash\!\!-} (\pi + 2k_0, U'_n, V'). \quad (160)$$

Let us recall that $U_n = V$. Hence, by $(R1, R2)$

$$\{(\pi + k_1 - 1, V, V'), (\pi + 2k_0, U'_n, V')\} \overset{<*>}{|\!\vdash\!\!-}_{\mathcal{C}} (\pi + k_1 - 1, U_n, U'_n). \quad (161)$$

By (160,161),(153) is proved. Using now (153) and rule $(R0)$, we obtain:

$$\mathcal{H} \overset{<*>}{|\!\vdash\!\!-}_{\mathcal{C}} (\pi + k_1 - 1, U_n, U'_n) \vdash\!\!-_{R0} (\pi + k_1, U_n, U'_n). \quad (162)$$

i.e. T_B^+ fulfills (S1).
Let us suppose now that $\forall i \in [n - k_1, n], U_i \sim U'_i$. Then, by (160), $U'_n \sim V'$ and by hypothesis $V = U_n \sim U'_n$. Hence $V \sim V'$. This shows that T_B^+ fulfills (S2).
An analogous proof can obviously be written for T_B^-. □

Lemma 7.3 *Let (p, S, S') be a weighted equation , i.e. $p \in \mathbb{N}, \lambda \in \mathbb{N}-\{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >$. Then $\{(p, S, S')\} \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{(p, \rho_e(S), \rho_e(S'))\}$ and $\{(p, \rho_e(S), \rho_e(S'))\} \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{(p, S, S')\}$.*

Proof: This is an easy consequence of corollary 4.12. □

Lemma 7.4 *For every $\mathcal{O} \in \Omega$, $T_C^{\mathcal{O}}$ is a $\mathcal{B}_0$-strategy.*

Proof: By lemma 5.7, point (1), combined with lemma 7.3, (S1) is proved. By lemma 5.7, point (2), combined with lemma 7.3, (S2) is proved. □

Let us define the strategy $\mathcal{S}_{ABC}$ by : for every $W = A_1 A_2 \cdots A_n$,

(0) if $T_{cut}(W) \neq \emptyset$, then $\mathcal{S}_{ABC}(W) = T_{cut}(W)$
(1) elsif $T_{\emptyset}(W) \neq \emptyset$, then $\mathcal{S}_{ABC}(W) = T_{\emptyset}(W)$
(2) elsif $T_{\varepsilon}(W) \neq \emptyset$, then $\mathcal{S}_{ABC}(W) = T_{\varepsilon}(W)$
(3) elsif $T_B^+(W) \cup T_B^-(W) \neq \emptyset$, then $\mathcal{S}_{ABC}(W) = T_B^+(W) \cup T_B^-(W) \cup T_C(W)$
(4) else $\mathcal{S}_{ABC}(W) = T_A(W) \cup T_C(W)$

The strategy $\mathcal{S}_{AB}$ is obtained from $\mathcal{S}_{ABC}$ by removing the occurence of T_C in cases (3)(4).

7.2 Global strategy

Let us define a global strategy $\hat{\mathcal{S}}_{ABC}$ w.r.t. the strategy $\mathcal{S}_{ABC}$. Let us fix (until the end of this article) a total well-ordering $\sqsubseteq$ over the set of oracles Ω. We need now three technical definitions.

Definition 7.5 *Let $P \in \mathcal{P}_f(\mathcal{A}), \mathcal{O} \in \Omega$ and $\bar{\pi} \in \mathbb{N} \cup \{\infty\}$. $\mathcal{O}$ is said* $\bar{\pi}$-consistent with *P iff, for every $(\pi, S, S') \in \mathrm{Cong}(P)$, and every $n \in \mathbb{N}$, if*

$$\pi + n - 1 < \bar{\pi},$$

then, the binary relation $\mathcal{R}_n = \mathcal{O}(S, S') \cap X^{\leq n} \times X^{\leq n}$ fulfills

$$[\pi, S, S', \mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

We use the notation:

$$\Omega(\bar{\pi}, P) = \{\mathcal{O} \in \Omega \mid \mathcal{O} \text{ is } \bar{\pi} - \text{consistent with } P\}.$$

Definition 7.6 *Let P be a finite subset of $\mathcal{A}$, and let $\bar{\pi} \in \mathbb{N} \cup \{\infty\}$. P is said* $\bar{\pi}$-consistent *iff, there exists some oracle $\mathcal{O} \in \Omega$, which is $\bar{\pi}$-consistent with P.*

For every proof tree $t \in \mathcal{T}(\mathcal{S}_{ABC})$, we denote by $\bar{\Pi}(t)$ the integer:

$$\bar{\Pi}(t) = \min\{\pi \in \mathbb{N} \mid \exists x \in \mathrm{dom}(t), x \text{ is not closed for } \mathcal{S}_{ABC}, \exists S, S', t(x) = (\pi, S, S')\}. \tag{163}$$

(we admit here that $\min(\emptyset) = \infty$.)

Definition 7.7 *Let t be a finite proof-tree for the strategy $\mathcal{S}_{ABC}$, $t \in \mathcal{T}(\mathcal{S}_{ABC})$. t is said* consistent *iff,* $\mathrm{im}(t)$ *is $\bar{\Pi}(t)$-consistent.*

Let us consider some tree $t \in \mathcal{T}(\mathcal{S}_{ABC})$ which is consistent and not closed. Let $\bar{\pi} = \bar{\Pi}(t)$, let x be the smallest unclosed node of weight $\bar{\pi}$. Let

$$W = A_1 \cdots A_n \tag{164}$$

be the word labelling the path from the root to x in t. (One can notice that, as x is not closed, $T_{cut}(W) \cup T_{\emptyset}(W) \cup T_{\varepsilon}(W) = \emptyset$). We define a tree of height one, $\hat{\Delta}(t)$ as follows:

(0) if $\exists \mathcal{O} \in \Omega(\bar{\pi}, \mathrm{im}(t)), T_C^{(\mathcal{O})}(W) \neq \emptyset$ then

$$\mathcal{O}_0 = \min\{\mathcal{O} \in \Omega(\bar{\pi}, \mathrm{im}(t)), T_C^{(\mathcal{O})}(W) \neq \emptyset\}, \;\; \hat{\Delta}(t) = A_n(T_C^{(\mathcal{O}_0)}(W)),$$

(1) elsif $T_B^{+}(W) \neq \emptyset$ then

$$\mathcal{O}_0 = \min(\Omega(\bar{\pi}, \mathrm{im}(t))), \;\; \hat{\Delta}(t) = A_n(T_B^{(\mathcal{O}_0),+}(W)),$$

(2) elsif $T_B^{-}(W) \neq \emptyset$ then

$$\mathcal{O}_0 = \min(\Omega(\bar{\pi}, \mathrm{im}(t))), \;\; \hat{\Delta}(t) = A_n(T_B^{(\mathcal{O}_0),-}(W)),$$

(3) else

$$\mathcal{O}_0 = \min(\Omega(\bar{\pi}, \mathrm{im}(t))), \;\; \hat{\Delta}(t) = A_n(T_A^{(\mathcal{O}_0)}(W)).$$

(In the above definition by $A(W')$, where $A \in \mathcal{A}, W' \in \mathcal{A}^{+}$ we mean the tree of height one with root labelled by A and whose sequence of leaves is the word W').

$$\hat{\mathcal{S}}_{ABC}(t) = t[\hat{\Delta}(t)/x], \tag{165}$$

i.e. $\hat{\mathcal{S}}_{ABC}(t)$ is obtained from t by substituting $\hat{\Delta}(t)$ at the leaf x.

Lemma 7.8 *For every $t \in \mathcal{T}(\mathcal{S}_{ABC})$, if t is consistent, then $\hat{\Delta}(t)$ is defined.*

Proof: By the definition of consistency the oracle $\mathcal{O}_0$ is always defined (i.e. $\Omega(\bar{\pi}, \mathrm{im}(t)) \neq \emptyset$), and for the word W defined above $T_{\varepsilon}(W) = \emptyset \Rightarrow \forall \mathcal{O} \in \Omega, T_A^{(\mathcal{O})}(W) \neq \emptyset$, hence one of cases (0-3) must occur. □
If t is not consistent or is closed then we define:

$$\hat{\mathcal{S}}_{ABC}(t) = t. \tag{166}$$

Lemma 7.9 *$\hat{\mathcal{S}}_{ABC}$ is a global strategy for $\mathcal{S}_{ABC}$.*

Sketch of proof: By lemma 7.8 $\hat{\mathcal{S}}_{ABC}$ is defined on every $t \in \mathcal{T}(\mathcal{S}_{ABC})$. It suffices to check that, in every case, the word constitued by the leaves of $\hat{\Delta}(t)$ belongs to $\mathcal{S}_{ABC}(W)$ (where W is the word considered in (164)). □

8 Tree analysis

This section is devoted to the analysis of the proof-trees τ produced by the strategy $\mathcal{S}_{AB}$ defined in section 7. The main results are lemma 8.17 and 8.18 whose combination asserts that if some path (from a node x to a node y) of τ is such that its origin has a "small norm" and its length is "large enough", then there exists some ancestor of y at which T_C has a non-empty value. This key technical result will ensure termination of the global strategy $\hat{\mathcal{S}}_{ABC}$ (see section 9).

8.1 Depth and weight

In this paragraph we show that the *weight* and the *depth* of a given node are closely related. Let us say that the strategy T "occurs at" node x iff,

$$\tau(x) \in T(\tau(x[0]) \cdot \tau(x[1]) \cdots \tau(x[|x|-1])),$$

i.e. the label of x belongs to the image of the path from ϵ (included) to x (excluded) by the strategy T. Let us notice that T_B^-, T_B^+ are defined in such a way that:
if (U_{i+1}^-, U_{i+1}^+) is the result of a T_B^α-transformation with arguments $j < i$, then $i - j = k_1 \geq 3$. Hence in every interval $[i+1, i+4]$, $\forall \alpha \in \{-,+\}$, at most one integer j is such that T_B^α occurs at j.

Lemma 8.1 *Let $\alpha \in \{-,+\}, A_1, \ldots, A_n \in H^{-1}(\infty)$. Suppose that $T_B^\alpha(A_1 \cdots A_n) \neq \emptyset$ and $T_\emptyset(A_1 \cdots A_n) = \emptyset$. Then, $\forall i \in [n-k_1, n], A_i \notin T_B^{-\alpha}(A_1 \cdots A_{i-1})$.*

Proof: Let us use the notation of §7.1. As $(U_i^{-\alpha})_{n-k_1 \leq i \leq n}$ is a derivation, and $U_{n-k_1}^{-\alpha}$ is unmarked, $\forall i \in [n-k_1, n], U_i^{-\alpha}$ is unmarked. Let us suppose that

$$(\pi_i, U_i^{-\alpha}, U_i^{+\alpha}) \in T_B^{-\alpha}(A_1 \cdots A_{i-1}). \tag{167}$$

Then

$$U_i^{-\alpha} = \sum_{q \in Q_i'} [p_i \tau_i q][qeq] \cdot (U_{i-k_1-1}^{+\alpha} \odot u_q'),$$

for some $Q_i' \subseteq Q, p_i \in Q, \tau_i \in Z^+, u_q' \in X^{\leq k_0}$.
As $U_i^{-\alpha}$ is unmarked, we can conclude that,

$$\forall q \in Q_i', [p_i \tau_i q][qeq] \cdot (U_{i-k_1-1}^{+\alpha} \odot u_q') = \emptyset^\lambda,$$

hence $U_i^{-\alpha} = \emptyset$. It follows that

$$\forall j \in [i, n], U_j^{-\alpha} = \emptyset, \;\; U_j^{+\alpha} \equiv \emptyset,$$

hence $T_\emptyset(A_1 \cdots A_n) = \epsilon$, contradicting the hypothesis. Hence (167) is impossible, which was to be proved. □
By lemma 8.1 and the considerations just above we conclude that, in every interval $[i+1, i+4]$, at most one integer j is such that $(T_B^- \cup T_B^+)$ occurs at j.

Lemma 8.2 *: Let τ be a proof-tree associated to the strategy $\mathcal{S}_{AB}$. Let $x, x' \in dom(\tau), x \preceq x'$. Then* $| W(x') - W(x) | \leq |x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3$.

(We recall the *depth* of a node x is just its length $|x|$). We denote by $W(x)$ the weight of x which we define as the first component of $\tau(x)$ i.e. the weight of the equation labelling x.
Proof: Let x, x' be such that $|x'| = |x| + 1$. Then $W(x') - W(x) \in \{-1, +1\}$, hence the inequality $| W(x') - W(x) | \leq |x'| - |x|$ is fulfilled by such nodes. The general case follows by induction on $(|x'| - |x|)$.
Let us prove now the other inequality. We distinguish two cases.
Case 1 : $|x'| - |x| \leq 3$.
Then $|x'| - |x| \leq 2 \cdot (W(x') - W(x)) + 3$ (because there is at most one T_B step in a sequence of length ≤ 3).
Case 2: $|x'| - |x| \geq 4$.
Let $x = x_0, x_1, \cdots, x_q, x'$ be the sequence of nodes such that $|x'| - |x| = 4 \cdot q + r, 0 \leq r < 4$ and $\forall i \in [0, q-1], |x_{i+1}| - |x_i| = 4$.

By lemma 8.1, in every set $\{y \in \mathrm{dom}(\tau) \mid x_i \prec y \preceq x_{i+1}\}$ at most one node z is such that T_B occurs at z. Hence $W(x_{i+1}) - W(x_i) \geq 2$.
It follows that :

$$\begin{aligned}
|x'| - |x| &= \sum_{i=0}^{q-1} [|x_{i+1}| - |x_i|] + |x'| - |x_q| \\
&\leq \sum_{i=0}^{q-1} 2(W(x_{i+1} - W(x_i)) + |x'| - |x_q| \\
&\leq 2(W(x_q) - W(x)) + 2(W(x') - W(x_q)) + 3 \text{ (by the first case)} \\
&\leq 2(W(x') - W(x)) + 3.
\end{aligned}$$

□

8.2 Stacking derivations

For every $u \in X^*$ we define the binary relation $\uparrow (u)$ over $\mathsf{DB}_{1,\lambda} < V >$ by: for every $S, S' \in \mathsf{DB}_{1,\lambda} < V >, S \uparrow (u) S' \Leftrightarrow \exists A \in Z, \omega \in Z^+, p, q \in Q, \Psi \in \mathsf{DB}_{Q,\lambda} < V >$ such that

$$S = [pA] * \Psi, [pA] \odot u = [q\omega], S' = [q\omega] * \Psi.$$

It is clear that if $S \uparrow (u) S'$ then $S \odot u = S'$ and that the converse is not true in general. A sequence of deterministic row-vectors $S_0, S_1, \ldots, S_n$ is a *derivation* iff there exist $x_1, \ldots, x_n \in X$ such that $S_0 \odot x_1 = S_1, \ldots, S_{n-1} \odot x_n = S_n$. The *length* of this derivation is n. If $u = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ we call $S_0, S_1, \ldots, S_n$ the derivation *associated* with (S, u). We denote this derivation by $S_0 \xrightarrow{u} S_n$.
A derivation $S_0, S_1, \ldots, S_n$ is said to be *stacking* iff it is the derivation associated

to a pair (S,u) such that $S = S_0$ and $S_0 \uparrow (u) S_n$. A derivation $S_0, S_1, \ldots, S_n$ is said to be a *sub-derivation* of a derivation $S'_0, S'_1, \ldots, S'_m$ iff there exists some $i \in [0,m]$ such that, $\forall j \in [1,n], S_j = S'_{i+j}$. (If we identify derivations with special words over the alphabet $\mathsf{DB} < V >$, a sub-derivation of $S'_0, S'_1, \ldots, S'_m$ is a factor of $S'_0, S'_1, \ldots, S'_m$).

Lemma 8.3 *Let $S, S' \in \mathsf{DRB}_{1,\lambda} < V >, w \in X^*$, such that $S \odot w = S'$, $\rho_\epsilon(S) = S$ and $\forall v \preceq w, \|S \odot v\| \geq \|S\|$. Then the derivation $S \xrightarrow{w} S'$ is stacking.*

Sketch of proof: Easy. □

Lemma 8.4 *Let $S, S' \in \mathsf{DRB}_{1,\lambda} < V >, w \in X^*, k \in \mathbb{N}$, such that $S \odot w = S'$ and $\|S'\| \geq \|S\| + (k-1) \cdot |Q| + 1$. Then the derivation $S \xrightarrow{w} S'$ contains some stacking sub-derivation of length k.*

Sketch of proof: Let $S = S_0, \ldots, S_i, \ldots, S_n = S'$ be the derivation associated to (S,w). Let $i_0 = \max\{i \in [0,n] \mid \|S_i\| = \min\{\|S_j\| \mid 0 \leq j \leq n\}\}$. Let $w = w'w''$ where $|w'| = i_0$. As $\|S_n\| - \|S_{i_0}\| \geq \|S_n\| - \|S_0\| \geq (k-1) \cdot |Q| + 1$, using lemma 3.41 point (3) we must have $|w''| \geq k$. Let $w'' = w''_1 w''_2$ with $|w''_1| = k$. By definition of i_0, $\forall i \in [i_0+1, i_0+k], \|S_i\| \geq \|S_{i_0}\| + 1$. By lemma 8.3, the sub-derivation $S_{i_0}, \ldots, S_{i_0+k}$ (associated to (S_{i_0}, w''_1)) is stacking. □

Lemma 8.5 *Let $S, S' \in \mathsf{DRB}_{1,\lambda} < V >, w \in X^*, k, d, d' \in \mathbb{N}$, such that S is ϵ-free, (d,d')-linear and:*

(1) the derivation $S \xrightarrow{w} S'$ contains no stacking sub-derivation of length k.
(2) $|w| \geq d \cdot k$.

Then S' is $(0,d')$-linear.

Proof: If S is $(0,d')$-linear, then the result follows from lemma 3.41, point (1). Otherwise,

$$S = \sum_{q \in Q} [p\omega q][qeq] T_q$$

for some $\omega \in Z^*, 1 \leq |\omega| \leq d, (T_q)_{q \in Q} \in \mathsf{DRB}_{Q,\lambda} < V >$ such that $\forall q \in Q, \mathrm{rd}(T_q) \leq d'$. Let $S \xrightarrow{w} S' = (S_0, \ldots, S_n)$. By induction on l, using hypothesis (1), one can show that: for every $l \in [0, |\omega| - 1]$, either

$$\exists m \leq k \cdot l, \exists \omega_m \in Z^+, |\omega_m| \leq |\omega| - l \text{ and } S_m = \sum_{q \in Q} [p_m \omega_m q][qeq] T_q,$$

or

$$\exists m \leq k \cdot l, \exists q \in Q, S_m = \rho_\epsilon(T_q).$$

Similarly

$$\exists m_0 \leq k \cdot |\omega|, \exists q \in Q, S_{m_0} = \rho_\epsilon(T_q).$$

Hence S_{m_0} is $(0, d')$-linear. Using lemma 3.41 we obtain that $S_n = S'$ is $(0, d')$-linear too.
□

Let us recall the values of some constants (defined in section 6):

$$k_0 = \max\{\nu([pAq]) \mid p, q \in Q, A \in Z, [pAq] \not\equiv \emptyset\}, \quad k_1 = \max\{2k_0 + 1, 3\},$$
$$D_1 = 4k_0 + 3, \quad k_2 = (D_1 + 4) \cdot k_1 + k_0 + 1,$$
$$d_0 = 2 \cdot |Q| \cdot \mathrm{Card}(X^{\leq k_1}), \quad N_0 = 1 + (k_2 + 2)|Q| + D_2.$$

We fix throughout the remaining of this section a tree $\tau \in \mathcal{T}(\mathcal{S}_{AB}, (\pi_0, U_0^-, U_0^+))$ (i.e. τ is a proof tree associated to the assertion $(\pi_0, U_0^-, U_0^+))$ by the strategy $\mathcal{S}_{AB}$). We suppose that

$$\mathrm{rd}(U_0^-) \leq D_2, \quad \mathrm{rd}(U_0^+) \leq D_2, \quad U_0^-, U_0^+ \text{ are both unmarked }, \tag{168}$$

$$U_0^-, U_0^+ \in \mathsf{DRB}_{1,\lambda} < V > \text{ with } \lambda \leq \lambda_2. \tag{169}$$

We recall that, formally, τ is a map $\mathrm{dom}(\tau) \to \mathbb{N} \times \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V >$ such that $\mathrm{dom}(\tau) \subseteq \{1, \ldots, |X|^2\}^*$ is closed under prefix and under "left-brother" (i.e. $w \cdot (i+1) \in \mathrm{dom}(\tau) \Rightarrow w \cdot i \in \mathrm{dom}(\tau)$). We denote by $pr_{2,3}$: $\mathbb{N} \times \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V > \to \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V >$ the projection $(\pi, U, U') \mapsto (U, U')$. By τ_s we denote the tree obtained from τ by forgetting the weights: $\tau_s = \tau \circ pr_{2,3}$.

8.3 Linearity

Lemma 8.6 *For every label (π, U^-, U^+) of τ,*

(1) $\forall \alpha \in \{-, +\}, U^\alpha$ is (D_1, D_2)-linear.
(2) if U^α is unmarked, then U^α is $(0, D_2)$-linear
(3) $\exists \alpha \in \{-, +\}, U^\alpha$ is unmarked.

Proof: Let $x \in \mathrm{dom}(\tau), \tau(x) = (\pi, U^-, U^+)$. We denote by $x(i)$ the prefix of length i of the node x. For every $0 \leq i \leq |x|$ we note $\tau(x(i)) = (\pi_i, U_i^-, U_i^+) = A_i$. (Hence $(A_i)_{0 \leq i \leq |x|}$ is the sequence of labels on the path from ϵ to x). We prove points (1-3) by induction on $|x|$ (the depth of node x).
$|x| = 0$: by hypothesis (168) points (1-3) are true.
$|x| = n + 1$:
case 1: (π, U^-, U^+) belongs to the result of the application of T_B^α on $(\pi_i, U_i^-, U_i^+), (\pi_n, U_n^-, U_n^+)$ where $i = n - k_1$.
Then $U_i^{-\alpha}$ is unmarked (by definition of T_B^α) and $(0, D_2)$-linear (by induction hypothesis). It follows that U^α is $(2k_0 + 2, D_2)$-linear. Moreover, by definition of $\mathcal{S}_{AB}$, $U_n^\alpha \not\equiv \emptyset$, hence U^α is marked. On the other hand, there exists some $w \in X^{k_1}$ such that $U^{-\alpha} = U_i^{-\alpha} \odot w$. By induction hypothesis $\mathrm{rd}(U_i^{-\alpha}) \leq D_2$ and by lemma 3.41 point (1), $\mathrm{rd}(U^{-\alpha}) \leq D_2$. Moreover , as no letter $[qeq]$ can be introduced by the action $\odot$ (see hypothesis (15) in section 2.4) , $U^{-\alpha}$ is

unmarked.
case 2: $(\pi, U^-, U^+) = (\pi_n + 1, U_n^- \odot x, U_n^+ \odot x')$ for some $x, x' \in X$.
If for every $1 \leq i \leq n$, (π_i, U_i^-, U_i^+) is not the result of an application of T_B^α (for any α), then U_{n+1}^-, U_{n+1}^+ are both $(0, D_2)$-linear. Otherwise, there is a maximal integer, m_0, such that $(\pi_{m_0}, U_{m_0}^-, U_{m_0}^+)$ is the result of an application of T_B. Let $\alpha \in \{-,+\}$ such that T_B^α occurs at m_0. We have:

$$U_{m_0}^{-\alpha} \text{ is unmarked and } (0, D_2) - \text{linear}, \tag{170}$$

$$U_{m_0}^{+\alpha} = \sum_{q \in Q} [p\omega q][qeq] T_q, \tag{171}$$

for some $\omega \in Z^+, 1 \leq |\omega| \leq 2k_0 + 2, (T_q)_{q \in Q} \in \mathsf{DRB}_{Q,\lambda} < V >$, where every T_q is unmarked and $(0, D_2)$-linear. As $U^{-\alpha} = U_{m_0}^{-\alpha} \odot w$ for some $w \in X^{n+1-m_0}$, property (170) implies that $U^{-\alpha}$ is unmarked and $(0, D_2) - \text{linear}$.
If there exists some $i \in [m_0, n+1]$ such that $U_i^{+\alpha} = \rho_\epsilon(T_q)$ then, by hypothesis $U_i^{+\alpha}$ is unmarked and $(0, D_2)$-linear. It follows that in this case $U^{+\alpha}$ is unmarked and $(0, D_2)$-linear.
If there exists no $i \in [m_0, n+1]$ such that $U_i^{+\alpha} = \rho_\epsilon(T_q)$ then,

$$\forall i \in [m_0, n+1], \exists p_i \in Q, \omega_i \in Z^+, \;\; U_i^{+\alpha} = \sum_{q \in Q} [p_i \omega_i q][qeq] T_q.$$

Let $m_0 \leq i_0 \leq n+1$ such that $|\omega_{i_0}|$ is minimal. If $|\omega_{n+1}| - |\omega_{i_0}| > 2k_0 + 1$, then $T_B^\alpha(A_0 \ldots A_i)$ would be non-empty for some $i_0 < i < n+1$, contradicting the maximality of m_0. Hence $|\omega_{n+1}| - |\omega_{i_0}| \leq 2k_0 + 1$, and since $|\omega_{i_0}| \leq |\omega_{m_0}| \leq 2k_0 + 2$ we have $|\omega_{n+1}| \leq 4k_0 + 3$. Hence $U^{+\alpha}$ is (D_1, D_2)-linear. □

8.4 N-stacking sequences

We show here that the tree τ is somewhat "smooth" in the sense that its labels cannot be varying in a too chaotic way along a given branch. We shall establish that every "sufficiently long" branch must contain a "reasonnably short factor" (a "N-stacking sequence") where at least d_0 labels (U, U') are belonging to the same d-space V of dimension $\leq d_0$ with coordinates not greater than s_{d_0} (over some fixed generating family of cardinality $\leq d_0$). Let us define now this notion of stacking sequence which is , roughly speaking, an extension to sequences of *pairs* (U_i, U_i') appearing in τ_s, of the above notion of stacking derivation.
For every $U, U' \in \mathsf{DRB}_{1,\lambda} < V >$ we set:

$$|||U||| = \max\{\|\tilde{U}\|, \;\; \tilde{U} \in \mathsf{Q}(U) \text{ and } \tilde{U} \text{ is unmarked } \}$$

$$|||(U, U')||| = \max\{|||U|||, |||U'|||\}.$$

Let $x \in \mathrm{dom}(\tau)$. We define now a kind of norm on nodes which in some sense "erases" the short-term variations of $|||(*, *)|||$:

$$\mathrm{N}(x) = \max\{|||\tau_s(x(j))|||, j \in \mathbb{N}, |x| - k_1 \leq j \leq |x|\}.$$

Lemma 8.7 *Let $(x_i)_{i\in I}$ (where I is a beginning section of $\mathbb{N}$) be some branch of τ.*

(1) The function $N : I \to \mathbb{N}$ is $(k_0 \cdot |Q|)$-up-Lipschitz.
(2) The restriction $N : J \to \mathbb{N}$ to any interval $J \subseteq I$ such that neither T_B^-, nor T_B^+ occur in J, is $|Q|$-up-Lipschitz.

(In the above lemma and in the proof below we use the simplified notation $\mathrm{N}(i)$ for the integer $\mathrm{N}(x_i)$).
Proof: Let us prove that for every i such that $i+1 \in I$

$$\mathrm{N}(i+1) \leq \mathrm{N}(i) + k_0|Q|. \tag{172}$$

Let $\alpha \in \{-,+\}$. Let us consider four cases.
case 1: $U^\alpha_{i+1} = U^\alpha_i \odot x$ (for some $x \in X, U^\alpha_i$ unmarked).
Then $\|U^\alpha_{i+1}\| \leq \|U^\alpha_i\| + |Q| \leq \mathrm{N}(i) + |Q|$.
case 2: U^α_{i+1} is obtained by T^α_B transformation.

$$U^\alpha_{i+1} = \sum_{q\in Q'} [p_i\omega_i q][qeq](U^{-\alpha}_{i-k_1} \odot u_q)$$

where $Q' \subseteq Q, 1 \leq |u_q| \leq k_0$. Then we have

$$\begin{aligned} |||\ U^\alpha_{i+1}\ ||| = \max_{q\in Q'}\{\|U^{-\alpha}_{i-k_1} \odot u_q\|\} &\leq \|U^{-\alpha}_{i-k_1}\| + k_0|Q| \\ &\leq \mathrm{N}(i) + k_0|Q| \end{aligned}$$

case 3: $U^\alpha_{i+1} = U^\alpha_i \odot x$ (for some $x \in X, U^\alpha_i, U^\alpha_{i+1}$ are marked).

$$U^\alpha_i = \sum_{q\in Q} [p_i\omega_i q][qeq]V_q$$

where the V_q are unmarked

$$U^\alpha_{i+1} = \sum_{q\in Q} ([p_i\omega_i q] \odot x)[qeq]V_q$$

hence $|||\ U^\alpha_{i+1}\ ||| = \max_{q\in Q}\{\|V_q\|\} = |||\ U^\alpha_i\ ||| \leq \mathrm{N}(i)$.
case 4: $U^\alpha_{i+1} = U^\alpha_i \odot x$ (for some $x \in X, U^\alpha_i$ marked, U^α_{i+1} unmarked).

$$U^\alpha_i = \sum_{q\in Q} [p_i\omega_i q][qeq]V_q$$

where the V_q are unmarked and ϵ-free $U^\alpha_{i+1} = V_{q_0}$, for some $q_0 \in Q$, hence

$$|||\ U^\alpha_{i+1}\ ||| = \|U^\alpha_{i+1}\| = \|V_{q_0}\| \leq \max_{q\in Q}\{\|V_q\|\} = |||\ U^\alpha_i\ ||| \leq \mathrm{N}(i).$$

As in every case , $|||\ U^\alpha_{i+1}\ ||| \leq \mathrm{N}(i) + k_0|Q|$, inequality (172) and point (1) of the lemma are proved.
The discussion above also shows that, if no T_B- transformation occurs in J, then for every J such that $j \in J, j+1 \in J$, $\mathrm{N}(j+1) \leq \mathrm{N}(j) + |Q|$, which proves point

(2) of the lemma. $\square$
Let $\sigma = (x_i)_{i\in I}$ be a path in τ, where $I \subseteq \mathbb{N}$ is a non-empty interval and $i_0 = \min(I)$. We call σ a *N-stacking sequence* iff:

$$\forall i \in I, \mathrm{N}(x_i) \geq \mathrm{N}(x_{i_0}) \text{ and } \mathrm{N}(x_{i_0}) \geq N_0. \tag{173}$$

From now on and until lemma 8.18 , we fix a N-stacking sequence $\sigma = (x_i)_{i\in I}$. We call $\mathrm{Card}(I) - 1$ the *length* of σ (denoted $|\sigma|$). Let us use the simplified notation $\mathrm{N}(i)$ for $\mathrm{N}(x_i)$ and let us note $\tau_s(x_i) = (U_i^-, U_i^+)$.

Lemma 8.8 $\forall i \in \mathbb{N}$, *if* $i + k_1 \in I$, *then* $\exists j \in [i, i+k_1], |||\ (U_j^-, U_j^+)\ ||| \geq \mathrm{N}(i_0)$.

Proof: This follows from the fact that $\mathrm{N}(i + k_1) \geq \mathrm{N}(i_0)$. $\square$

Lemma 8.9 *Let* $i \in \mathbb{N}$, *such that* $i + k_1 + 1 \in I$. *If* T_B *occurs at node* x_{i+1} *then* $|||\ (U_{i+1}^-, U_{i+1}^+)\ ||| \geq \mathrm{N}(i_0) - k_1|Q|$.

Proof: Suppose that $|||\ (U_{i+1}^-, U_{i+1}^+)\ ||| \leq \mathrm{N}(i_0) - k_1|Q| - 1$. Then, T_B cannot occur inside $[i+1, i+1+k_1]$ (see lemma 8.1). Hence , by lemma 8.7 $\forall j \in [i+1, i+1+k_1], |||\ (U_j^-, U_j^+)\ ||| \leq \mathrm{N}(i_0) - 1$. Finally $\mathrm{N}(i+1+k_1) \leq \mathrm{N}(i_0) - 1$, contradicting the fact that σ is N-stacking. $\square$

Lemma 8.10 *Suppose that* T_B^α *occurs at node* x_{i+1} *where* $i + k_1 + 1 \in I$.. *Then* $U_{i+1}^\alpha = \sum_{q\in Q'} [p_i \mu_i q][qeq](\overline{U_{i+1}^\alpha} \odot u_q)$, *for some* $p_i \in Q, Q' \subseteq Q, \mu_i \in (Z - \{e\})^*, 1 \leq |\mu_i|, |u_q| \leq k_0, \|\overline{U_{i+1}^\alpha}\| \geq \mathrm{N}(i_0) - 2k_1|Q|$.

Proof: We distinguish two cases.
case 1: $|||\ U_{i+1}^\alpha\ ||| = \max\{|||\ U_{i+1}^-\ |||, |||\ U_{i+1}^+\ |||\}$.
By lemma 8.9, $|||\ U_{i+1}^\alpha\ ||| \geq \mathrm{N}(i_0) - k_1|Q|$. So, $\exists q \in Q', \|\overline{U_{i+1}^\alpha} \odot u_q\| \geq \mathrm{N}(i_0) - k_1|Q|$, hence $\|\overline{U_{i+1}^\alpha}\| \geq \mathrm{N}(i_0) - (k_1 + k_0)|Q| \geq \mathrm{N}(i_0) - 2k_1|Q|$.
case 2: $|||\ U_{i+1}^{-\alpha}\ ||| = \max\{|||\ U_{i+1}^-\ |||, |||\ U_{i+1}^+\ |||\}$.
Hence $\|U_{i-k_1}^{-\alpha}\| \geq \|U_{i+1}^{-\alpha}\| - k_1|Q| = |||\ U_{i+1}^{-\alpha}\ ||| - k_1|Q| \geq \mathrm{N}(i_0) - 2k_1|Q|$. As $\overline{U_{i+1}^\alpha} = U_{i-k_1}^{-\alpha}$, the lemma is proved. $\square$
We define an integer i_1 by:

$$i_1 = \min\{i \in [i_0 - k_1, i_0] \cap \mathbb{N},\ |||\ (U_i^-, U_i^+)\ ||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|\}, \tag{174}$$

if T_B does not occur in $[i_0 - k_1, i_0]$;

$$i_1 \text{ is the unique element of } [i_0 - k_1, i_0] \cap \mathbb{N} \text{ where } T_B \text{ occurs} \tag{175}$$

if T_B occurs in $[i_0 - k_1, i_0]$;
Let us notice that, by lemma 8.8, i_1 is always defined (i.e. the set used in the r.h.s. of definition (174) cannot be empty).

For every $\alpha \in \{-, +\}$, we define a Q-series $[p^\alpha \omega^\alpha]$ and a Q-λ-form Φ^α as follows.

case 1: $U_{i_1}^{\alpha}$ is unmarked and $\|U_{i_1}^{\alpha}\| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.
As $\mathrm{rd}(U_{i_1}^{\alpha}) \leq D_2$, $U_{i_1}^{\alpha}$ has a minimal decomposition:

$$U_{i_1}^{\alpha} = [p^{\alpha}\omega_1^{\alpha}] * \Phi_1^{\alpha} \text{ with } \|\Phi_1^{\alpha}\| \leq D_2.$$

Using the inequality

$$\|U_{i_1}^{\alpha}\| \geq \mathrm{N}(i_0) - (k_2+1)|Q| \geq D_2,$$

we conclude that ω_1^{α} admits a decomposition $\omega_1^{\alpha} = \omega^{\alpha}\omega_2^{\alpha}$ such that

$$U_{i_1}^{\alpha} = [p^{\alpha}\omega^{\alpha}] * \Phi^{\alpha}, \;\; \Phi^{\alpha} = [\omega_2^{\alpha}] * \Phi_1^{\alpha}, \tag{176}$$

with

$$\mathrm{N}(i_0) - (k_2+1)|Q| \leq \|\Phi^{\alpha}\| < \mathrm{N}(i_0) - k_2|Q|, \;\; \|\Phi^{\alpha}\| \geq D_2 + |Q|, \;\; |\Phi^{\alpha}| \geq 1. \tag{177}$$

case 2: $U_{i_1}^{\alpha}$ is marked and $|||\, U_{i_1}^{\alpha} \,||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.
Hence $U_{i_1}^{\alpha}$ can be written as

$$U_{i_1}^{\alpha} = \sum_{q \in Q'} [p_1^{\alpha}\mu_1^{\alpha} q][qeq](\overline{U_{i_1}^{\alpha}} \odot u_q)$$

for some $p_1^{\alpha} \in Q, Q' \subseteq Q, 1 \leq |\mu_1^{\alpha}| \leq D_1, |u_q| \leq k_0, \overline{U_{i_1}^{\alpha}} = U_j^{-\alpha}, j < i_1$.
Let $q \in Q'$ such that $|||\, U_{i_1}^{\alpha} \,||| = \|\overline{U_{i_1}^{\alpha}} \odot u_q\|$. We have:

$$\|\overline{U_{i_1}^{\alpha}}\| \geq \|\overline{U_{i_1}^{\alpha}} \odot u_q\| - |u_q||Q| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q| - k_0|Q| = \mathrm{N}(i_0) - k_2|Q|.$$

From this inequality we conclude, as in case 1, that $\overline{U_{i_1}^{\alpha}}$ has a decomposition:

$$\overline{U_{i_1}^{\alpha}} = [p^{\alpha}\omega^{\alpha}] * \Phi^{\alpha}, \tag{178}$$

with

$$\mathrm{N}(i_0) - (k_2+1)|Q| \leq \|\Phi^{\alpha}\| < \mathrm{N}(i_0) - k_2|Q|, \;\; \|\Phi^{\alpha}\| \geq D_2 + |Q|, \;\; |\Phi^{\alpha}| \geq 1. \tag{179}$$

case 3: $|||\, U_{i_1}^{\alpha} \,||| < \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.
In this case , $[p^{\alpha}\omega^{\alpha}], \Phi^{\alpha}$ are both undefined.

Remark 8.11 *If definition (174) applies then, the inequality* $|||\, (U_{i_1}^{-}, U_{i_1}^{+}) \,||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$ *implies that there exists at least one* $\alpha \in \{-, +\}$ *such that* $[p^{\alpha}\omega^{\alpha}], \Phi^{\alpha}$ *are both defined. If definition (175) applies then, lemma 8.10 implies the same result.*

Let us define now the following families of vectors and d-spaces of vectors (i.e. in the universe $\mathsf{DRB}_{1,\lambda} < V >$ and with coefficients in $\mathsf{DRB}_{1,n} < V >$, $n \in \mathbb{N}$):

$$\mathcal{G}_0^{\alpha} = \{\Phi_q^{\alpha} \odot u \mid q \in Q, u \in X^{\leq k_0}\} \cup \{\rho_{\varepsilon}(\Phi_q^{\alpha}) \mid q \in Q\}, \tag{180}$$

$$W^{\alpha} = \mathsf{V}((\Phi_q^{\alpha})_{q \in Q}), \;\; V^{\alpha} = \mathsf{V}(\mathcal{G}_0^{\alpha}), \tag{181}$$

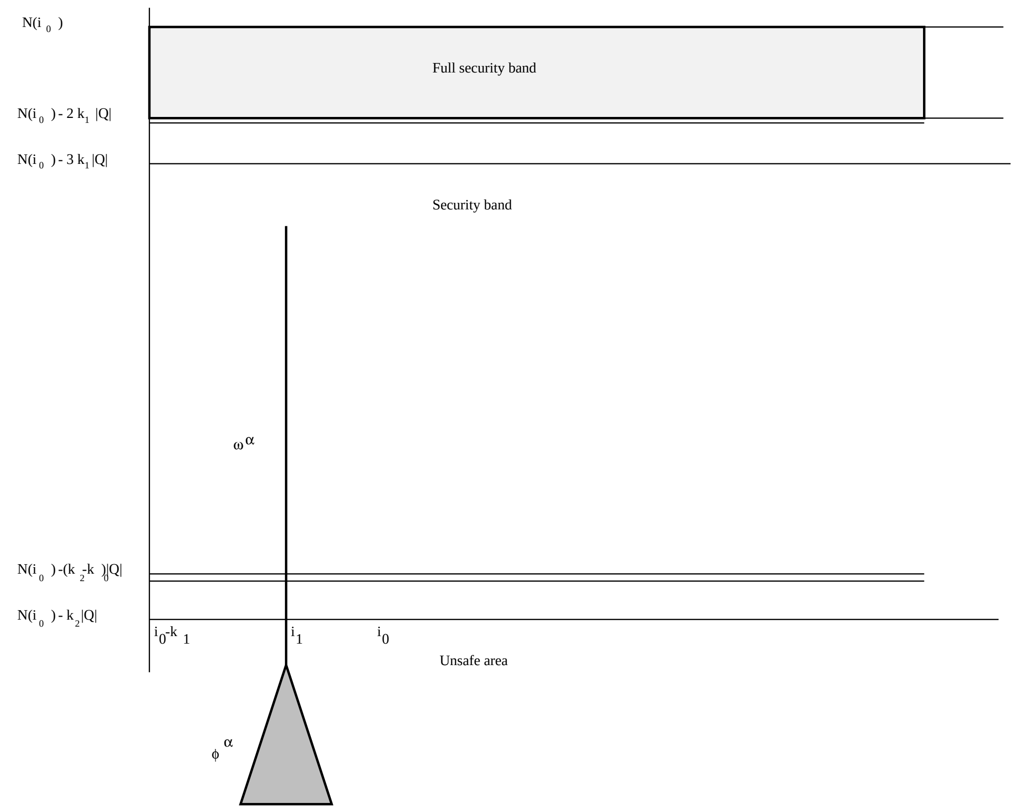

Fig. 4. the generating family.

for every $\alpha \in \{-,+\}$;

$$\tilde{W} = W^- \cup W^+, \ \ \tilde{V} = V^- \cup V^+, \ \ \bar{V} = V^- + V^+, \tag{182}$$

where , for every d-spaces V_1, V_2 in the same universe, $V_1 + V_2$ denotes the smallest d-space containing $V_1 \cup V_2$. We illustrate on figure 4 the above definitions. The band $\{U \in \mathsf{DRB}_{1,\lambda} < V > |\mathrm{N}(i_0) - 2k_1|Q| \leq \mathrm{N}(U)\}$ is a *full security band* in the sense that every U_i^α in this band *belongs to* $\bar{V}$ (lemma 8.13). The band $\{U \in \mathsf{DRB}_{1,\lambda} < V > | \ \mathrm{N}(i_0) - (k_2 - k_0)|Q| \leq ||| \ U \ ||| < \mathrm{N}(i_0)\}$ is a *security band* in the sense that if [U_j^α is in this band , $j < i$, U_i^α belongs to the full security band, and $U_j^\alpha \longrightarrow U_i^\alpha$], then U_j^α *belongs to* $\bar{V}$ (property $\mathcal{P}(\alpha, i, j)$ established in the proof of lemma 8.13).

Lemma 8.12 *Let* $\alpha \in \{-,+\}$, $i \in I, j \in \mathbb{N}, i \geq j \geq i_1$ *such that* $||| \ U_i^\alpha \ ||| \geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q|$ *and* T_B^α *does not occur in* $[j+1, i]$. *Then* $||| \ U_j^\alpha \ ||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$.

Proof: Let us suppose α, i, j fulfill the hypothesis of the lemma and let us suppose that the following inequality is realized:

$$||| U_j^\alpha |||< \mathrm{N}(i_0) - (k_2 - k_0)|Q|. \tag{183}$$

As $||| U_i^\alpha |||\geq \mathrm{N}(i_0) - 2k_1|Q|$, the following integers are well-defined:

$$i_2 = -1 + \min\{j' \in [j + k_1, i] \mid \mathrm{N}(i_0) - 3k_1|Q| \leq ||| U_{j'}^\alpha |||< \mathrm{N}(i_0) - (3k_1 - 1)|Q|\},$$

$$i_3 = -1 + \min\{j' \in [i_2, i] \mid \mathrm{N}(i_0) - 2k_1|Q| \leq ||| U_{j'}^\alpha |||\}.$$

As for every $j' \in [j, i_2 - k_1]$, $||| U_{j'}^\alpha |||< \mathrm{N}(i_0) - 3k_1|Q|$, by lemma 8.10, $T_B^{-\alpha}$ does not occur in $[j + k_1 + 1, i_2 + 1]$. But $||| U_{i_2}^\alpha ||| - ||| U_j^\alpha |||\geq (k_2 - k_0 - 3k_1 - 1)|Q| = (D_1 + 1) \cdot k_1 \cdot |Q|$, hence $i_2 - j \geq (D_1 + 1) \cdot k_1$ so that the interval $[j + k_1, i_2]$ has a length greater or equal to $D_1 \cdot k_1$. Applying lemma 8.5 we conclude that $U_{i_2}^{-\alpha}$ is unmarked. (Let us notice that $U_{i_2}^\alpha$ is unmarked too, just because $||| U_{i_2}^\alpha |||>||| U_j^\alpha |||$ while $U_{i_2}^\alpha = U_j^\alpha \odot w$ for some $w \in X^{i_2 - j}$).
As $||U_{i_3}^\alpha|| \geq ||U_{i_2}^\alpha|| + (k_1 - 1) \cdot |Q| + 1$, by lemma 8.4 the derivation $U_{i_2}^\alpha \longrightarrow U_{i_3}^\alpha$ must contain a stacking subderivation

$$U_{i_2+k}^\alpha \uparrow (u) U_{i_2+k+k_1}^\alpha,$$

for some $u \in X^{k_1}$. As $U_{i_2}^{-\alpha}$ is unmarked, there exists some $j' \in [i_2 + k_1 + 1, i_3 + 1]$ such that either T_B^α occurs at j', which contradicts the hypothesis of the lemma, or $T_B^{-\alpha}$ occurs at j' which contradicts lemma 8.10. (We illustrate our argument on figure 5). □

Lemma 8.13 *Let* $i \geq i_1, i + k_1 \in I, \alpha \in \{-,+\}$ *such that* $||| U_i^\alpha |||\geq \mathrm{N}(i_0) - 2k_1|Q|$. *Then* $U_i^\alpha \in \tilde{V}$.

Proof: Let us consider the following property $\mathcal{P}(\alpha, i, j)$:

$$\{j \leq i \text{ and } ||| U_i^\alpha |||\geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q| \text{ and no } T_B^\alpha \text{ occurs in } [j+1, i]\} \tag{184}$$

$$\Longrightarrow$$

$$\{(\text{ if } U_j^\alpha \text{ is unmarked then } U_j^\alpha \in \tilde{W}) \text{ and } (\text{ if } U_j^\alpha \text{ is marked then, there exists some } \alpha' \in \{-,+\}$$

$$\text{such that , every linear component of } U_j^\alpha \text{ is in } (W^{\alpha'} \cap \mathsf{DRBlin}_{1,\lambda}^{D_2} < V >) \odot X^{<1,k_0>}\}. \tag{185}$$

(Here we denote by $X^{<m,m'>}$ the set $\{u \in X^*, m \leq |u| \leq m'\}$).
We prove by lexicographic induction on the pair (i, j) that:

$$\forall i \geq i_1, \forall j \geq i_1, [\forall \alpha \in \{-,+\}, \mathcal{P}(\alpha, i, j)]. \tag{186}$$

Let us consider a pair $(i, j) \in \mathbb{N} \times \mathbb{N}, i \geq i_1, j \geq i_1$ and some $\alpha \in \{-,+\}$, fulfilling the lefthand-side of the implication $\mathcal{P}(\alpha, i, j)$ (i.e. fulfilling condition (184)).
case 1: $j = i_1$.

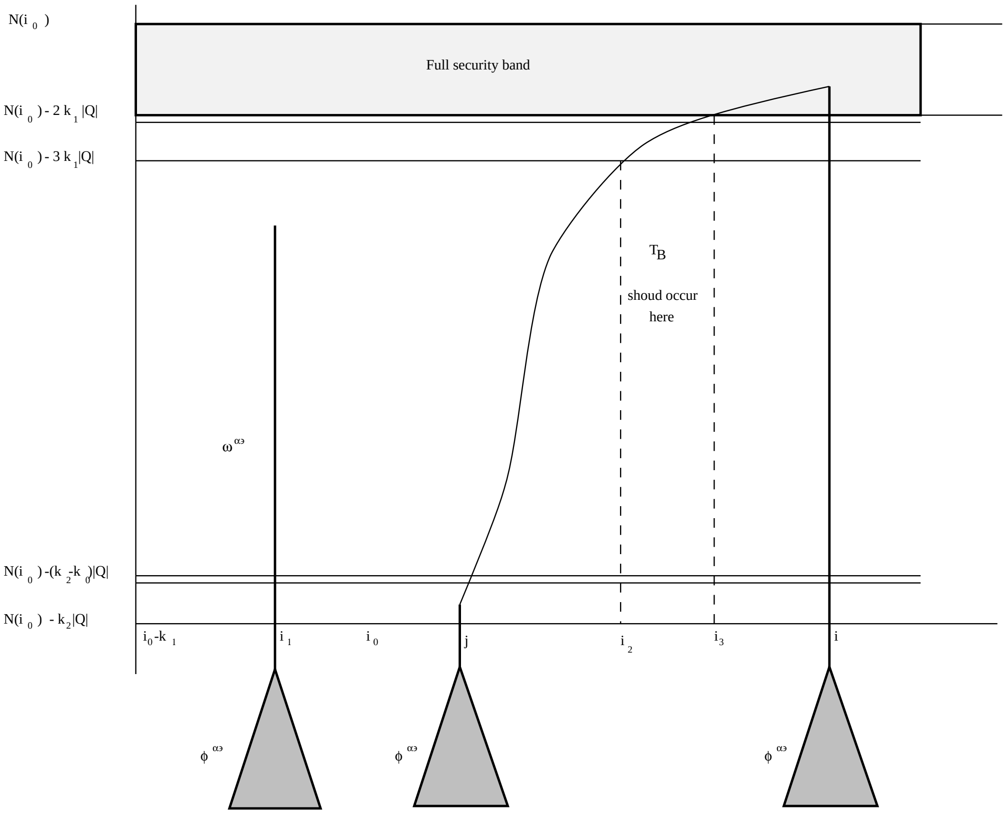

Fig. 5. U_j^α out of security band is impossible.

If U_j^α is unmarked, by lemma 8.12 $|||\ U_j^\alpha\ ||| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q|$. Hence $U_j^\alpha \in W^\alpha$ (by case 1 of the definition of Φ^α).
If U_j^α is marked, then, by case 2 of the definition of Φ^α, every linear component of U_j^α is in $(W^\alpha \cap \mathsf{DRBlin}_{1,\lambda}^{D_2} < V >) \odot X^{<1,k_0>}$.
case 2: $j > i_1$ and T_B^α occurs at j .

$$U_j^\alpha = \sum_{q \in Q'} [p_j \omega_j q][qeq](\overline{U_j^\alpha} \odot u_q),$$

for some $Q' \subseteq Q, \omega_j \in (Z - \{e\})^+, p_j \in Q, |u_q| \leq k_0$, and $\overline{U_j^\alpha} = U_{j-k_1-1}^{-\alpha}$.
By lemma 8.10

$$|||\ \overline{U_j^\alpha}\ ||| \geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q|.$$

It follows that $j - k_1 - 1 \geq i_1$ (by minimality of i_1 in the case where T_B does not occur in $[i_0 - k_1, i_0] \cap \mathbb{N}$ and because two successive occurrences of T_B must be at distance $\geq k_1 + 1$ in the case where T_B occurs at i_1). As $(j - k_1 - 1, j - k_1 - 1) < (i, j)$, by induction hypothesis, $\mathcal{P}(\alpha, j - k_1 - 1, j - k_1 - 1)$ is true: $U_{j-k_1-1}^{-\alpha}$ is unmarked (by definition of T_B^α) and $|||\ U_{j-k_1-1}^{-\alpha}\ ||| \geq \mathrm{N}(i_0) - 2 \cdot k_1 \cdot |Q|$, hence $U_{j-k_1-1}^{-\alpha} \in W^{\alpha'}$ (for some α'). It follows that

U_j^α is marked and for every $q \in Q', \overline{U_j^\alpha} \odot u_q \in (W^{\alpha'} \cap \mathsf{DRBlin}_{1,\lambda}^{D_2} < V >) \odot X^{<1,k_0>}$.

case 3: $j > i_1$ and T_B^α does not occur at j .
subcase 1: $U_{j-1}^\alpha, U_j^\alpha$ are both marked.
$U_{j-1}^\alpha \odot x = U_j^\alpha$ for some $x \in X$. By lemma 8.6,

$$U_{j-1}^\alpha = \sum_{q \in Q'} [p_{j-1}\omega_{j-1}q][qeq]T_q$$

for some $\omega_{j-1} \in (Z - \{e\})^+, 1 \leq |\omega_{j-1}| \leq D_1, (T_q)_{q \in Q} \in \mathsf{DRB}_{Q,\lambda} < V >$, where every T_q is unmarked and $(0, D_2)$-linear. It follows that

$$U_j^\alpha = \sum_{q \in Q'} ([p_{j-1}\omega_{j-1}q] \odot x)[qeq]T_q$$

By the induction hypothesis $\mathcal{P}(\alpha, i, j-1)$ we get: there exists $\alpha' \in \{-, +\}$ such that $\forall q \in Q', T_q \in (W^{\alpha'} \cap \mathsf{DRBlin}_{1,\lambda}^{D_2} < V >) \odot X^{<1,k_0>}$. Hence (185) is true for (α, i, j).
subcase 2: $U_{j-1}^\alpha, U_j^\alpha$ are both unmarked.
By induction hypothesis $U_{j-1}^\alpha \in \tilde{W}$, i.e. $U_{j-1}^\alpha \in \mathsf{V}(\{\Phi_q^{\alpha'} \mid q \in Q\})$ for some $\alpha' \in \{-, +\}$. By definition of $\Phi^{\alpha'}$ we have:

(1) $\|\Phi^{\alpha'}\| \geq D_2 + |Q|,\ \ |\Phi^{\alpha'}| \geq 1$,
(2) $\mathrm{rd}(U_{j-1}^\alpha) \leq D_2$.

Hence, by lemma 3.47, $\exists \omega_{j-1} \in Z^*, \exists p_{j-1} \in Q, U_{j-1}^\alpha = [p_{j-1}\omega_{j-1}] * \Phi^{\alpha'}$. As $\|U_{j-1}^\alpha\| > \|\Phi^{\alpha'}\|$, we must have $|\omega_{j-1}| \geq 1$. Let us apply lemma 3.46 on $U = U_{j-1}^\alpha, \Phi = \Phi^{\alpha'}, H = U_j^\alpha, u = x, k = 1$. One can check that, by lemma 8.12 the hypothesis

$$\|H\| \geq 1 + k \cdot |Q| + \|\Phi\|$$

is fulfilled. Hence $U_j^\alpha = ([p_{j-1}\omega_{j-1}] \odot x) * \Phi^{\alpha'}$, which proves that $U_j^\alpha \in \tilde{W}$.
subcase 3: U_{j-1}^α is marked while U_j^α is unmarked.
By lemma 8.6

$$U_{j-1}^\alpha = \sum_{q \in Q} [p_{j-1}\omega_{j-1}q][qeq]T_q$$

for some $\omega_{j-1} \in (Z - \{e\})^+, 1 \leq |\omega_{j-1}| \leq D_1, (T_q)_{q \in Q} \in \mathsf{DRB}_{Q,\lambda} < V >$, where every T_q is unmarked and $(0, D_2)$-linear. It follows that

$$U_j^\alpha = \rho_\epsilon(T_{q_0})$$

for some $q_0 \in Q$. By induction-hypothesis there exists $\alpha' \in \{-, +\}$ such that $T_{q_0} \in (W^{\alpha'} \cap \mathsf{DRBlin}_{1,\lambda}^{D_2} < V >) \odot X^{<1,k_0>}$, i.e. there exist $\overline{T_{q_0}} \in W^{\alpha'} \cap \mathsf{DRBlin}_{1,\lambda}^{D_2} < V >, u_{q_0} \in X^{<1,k_0>}$ such that:

$$T_{q_0} = \overline{T_{q_0}} \odot u_{q_0}.$$

(Let us notice that this equality implies that $T_{q_0} = \rho_\epsilon(T_{q_0})$).
$\overline{T_{q_0}} \in \mathsf{V}((\Phi_q^{\alpha'})_{q\in Q})$, and by definition of $\Phi^{\alpha'}$ we have:

(3) $||\Phi^{\alpha'}|| \geq D_2 + |Q|, |\Phi^{\alpha'}| \geq 1$
(4) $\mathrm{rd}(\overline{T_{q_0}}) \leq D_2$.

Applying lemma 3.47, we obtain that

$$\exists \omega_0 \in Z^*, \exists p_0 \in Q, \overline{T_{q_0}} = [p_0\omega_0] * \Phi^{\alpha'}.$$

Using the inequality given by lemma 8.12 (for U_j^α) and inequality (177 or 179) we obtain that

$$||U_j^\alpha|| \geq \mathrm{N}(i_0) - (k_2 - k_0)|Q| \geq [\mathrm{N}(i_0) - (k_2 - k_0)|Q| - 1 - k_0 \cdot |Q|] + 1 + k_0 \cdot |Q|$$
$$\geq ||\Phi^{\alpha'}|| + 1 + k_0 \cdot |Q|,$$

which is equivalent to

$$||\overline{T_{q_0}} \odot u_{q_0}|| \geq ||\Phi^{\alpha'}|| + 1 + k_0 \cdot |Q|.$$

Hence the hypothesis of lemma 3.46 are met by $U = \overline{T_{q_0}}, H = T_{q_0}, p = p_0, h = \omega_0, \Phi = \Phi^{\alpha'}, u = u_{q_0}, k = k_0$. Lemma 3.46 then concludes that

$$T_{q_0} = ([p_0\omega_0] \odot u_{q_0}) * \Phi^{\alpha'} \text{ where } [p_0\omega_0] \odot u_{q_0} = [q'\omega'], \text{ for some } q' \in Q, |\omega'| \geq k_0.$$

As $U_j^\alpha = T_{q_0}$, this establishes that $U_j^\alpha \in W^{\alpha'}$, hence that $\mathcal{P}(\alpha, i, j)$ is true.
Let us observe finally that $\mathcal{P}(\alpha, i, i)$ shows that, if U_i^α is unmarked then $U_i^\alpha \in \tilde{W} \subseteq \bar{V}$ else $U_i^\alpha \in V^{\alpha'} \subseteq \bar{V}$. Hence the lemma is proved. □

Lemma 8.14 *Suppose $i \geq i_1, i + k_1 \in I, ||U_i^\alpha|| \geq \mathrm{N}(i_0) - 2k_1|Q|, U_i^\alpha$ unmarked. Then $\exists \alpha' \in \{-, +\}, p \in Q, \omega \in Z^*, U_i^\alpha = [p\omega] * \Phi^{\alpha'}$ with $1 \leq |\omega| \leq k_0(i - i_0) + (k_2 + k_1 + 2)$.*

Proof: By property $\mathcal{P}(\alpha, i, i)$ established in the proof of lemma 8.13, $\exists \alpha' \in \{-, +\}, p \in Q, \omega \in Z^*$,

$$U_i^\alpha = [p\omega] * \Phi^{\alpha'}.$$

By lemma 3.43, as $|\Phi^{\alpha'}| \geq 1$,

$$||U_i^\alpha|| = 1 + (|\omega| - 1)|Q| + ||\Phi^{\alpha'}||.$$

If $i \geq i_0$, by lemma 8.7

$$||U_i^\alpha|| \leq \mathrm{N}(i) \leq (i - i_0)k_0|Q| + \mathrm{N}(i_0) \leq (i + k_1 - i_0)k_0|Q| + \mathrm{N}(i_0).$$

If $i \in [i_1, i_0]$,

$$||U_i^\alpha|| \leq \mathrm{N}(i_0) \leq (i + k_1 - i_0)k_0|Q| + \mathrm{N}(i_0).$$

By inequality (177 or 179),

$$||\Phi^{\alpha'}|| \geq \mathrm{N}(i_0) - (k_2 + 1)|Q|.$$

Hence we get

$$\begin{aligned}(|\omega|-1)|Q| \leq 1+(|\omega|-1)|Q| &\leq \|U_i^\alpha\| - \|\Phi^{\alpha'}\| \\ &\leq (i+k_1-i_0)k_0|Q| + \mathrm{N}(i_0) - (\mathrm{N}(i_0)-(k_2+1)|Q|) \\ &= (i+k_1-i_0)k_0|Q| + (k_2+1)|Q|\end{aligned}$$

Hence $|\omega| \leq (i-i_0)k_0 + (k_2+k_1+2)$. □
Let us define some additional spaces of vectors:

$$\mathcal{G}_1^\alpha = \{\Phi_q^\alpha \odot u \mid q \in Q, u \in X^{\leq k_1}\} \cup \{\rho_\epsilon(\Phi_q^\alpha) \mid q \in Q\}, \;\; V_1^\alpha = \mathsf{V}(\mathcal{G}_1^\alpha), \;\; \bar{V}_1 = V_1^- + V_1^+ \tag{187}$$

We recall that:

$$K_1 = k_1 + 1, \;\; K_2 = (D_1+1)\cdot k_1^2 + (D_1+1)\cdot k_1 + 1.$$

Lemma 8.15 *Let $L \geq 0$ such that $(i_0 + K_1 \cdot L + K_2) + k_1 \in I$. Then, there exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ such that, $U_i^- \in \bar{V}_1, U_i^+ \in \bar{V}_1$.*

Proof: Let us establish that

$$\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2], \exists \alpha \in \{-,+\}, T_B^\alpha \text{ occurs at } i. \tag{188}$$

If $\exists \alpha \in \{-,+\}, T_B^\alpha$ occurs in $[i_0 + L, i_0 + L + D_1 k_1]$ then (188) is true.
Otherwise, by lemma 8.6 and lemma 8.5, both $U^\alpha_{i_0+L+D_1\cdot k_1}$ are unmarked. If some α-stacking sequence of length k_1 occurs in the interval $I(i_0, L) = [i_0 + L + D_1 k_1, i_0 + K_1 L + K_2 - 1]$, then some $T_B^{\alpha'}$ occurs at some $i \in [i_0 + L + D_1 k_1 + 1, i_0 + K_1 L + K_2]$ and (188) is true.
Let us suppose now that, $\forall \alpha \in \{-,+\}$, no α-stacking sequence of length $\geq k_1$ occurs in the interval $I(i_0, L)$. By lemma 8.7 we must then have:

$$\|| \, (U^-_{i_0+L+D_1k_1}, U^+_{i_0+L+D_1k_1}) \, \|| \leq \mathrm{N}(i_0) + |Q|(L + D_1 k_1),$$

and , by induction on ℓ, using the fact that no α-stacking sequence of length k_1 occurs in $I(i_0, L)$, we obtain that: $\forall \alpha \in \{-,+\}, \forall \ell \leq L + D_1 k_1 + k_1, \exists k^\alpha \in [L + D_1 k_1, L + D_1 k_1 + \ell k_1]$,

$$\|U^\alpha_{(i_0+k^\alpha)}\| \leq \|U^\alpha_{i_0+L+D_1k_1}\| - |Q|\ell.$$

Taking $\bar{\ell} = L + D_1 k_1 + k_1$, we obtain some $k^-, k^+ \in [L + D_1 k_1, L + D_1 k_1 + \bar{\ell} k_1]$ such that $\forall \alpha \in \{-,+\}$,

$$\|U^\alpha_{i_0+k^\alpha}\| \leq \|U^\alpha_{i_0+L+D_1k_1}\| - |Q|\bar{\ell} \leq \mathrm{N}(i_0) - |Q|k_1. \tag{189}$$

Let $\hat{k} = \max\{k^-, k^+\}, \;\; \hat{i_0} = i_0 + \hat{k}$.

$$\begin{aligned}\hat{i_0} \leq i_0 + (L + D_1 k_1) + (L + D_1 k_1 + k_1)k_1 &= i_0 + (k_1+1)L + D_1 k_1 + (D_1+1)k_1^2 \\ &= i_0 + K_1 L + K_2 - (k_1 + 1)\end{aligned}$$

As there is no α-stacking sequence of length k_1 in $I(i_0, L)$, with the help of lemma 8.4, we obtain

$$\forall \alpha \in \{-,+\}, \forall k \in [0,k_1], ||U^{\alpha}_{\hat{i_0}+k}|| \leq \mathrm{N}(i_0) - |Q|,$$

hence $\mathrm{N}(\hat{i_0}+k_1) \leq \mathrm{N}(i_0) - |Q|$. As this inequality would contradict the hypothesis that σ is N-stacking, assertion (188) is proved.
Let us consider an integer i satisfying (188). Suppose that T^{α}_B occurs at i . By lemma 8.10, $||\overline{U^{\alpha}_i}|| \geq \mathrm{N}(i_0) - 2k_1|Q|$. One can check that, by the same argument as in the proof of lemma 8.13, case 2, necessarily $i - k_1 - 1 \geq i_1$. By the property $\mathcal{P}(\alpha, i', j')$ (where we choose $i' = j' = i - k_1 - 1$), established in the proof of lemma 8.13, $\overline{U^{\alpha}_i} \in \tilde{W}$. Hence $U^{\alpha}_i \in \tilde{V} \subseteq \overline{V_1}$ and $U^{-\alpha}_i \in \overline{V_1}$. □
Let us give now a stronger version of lemma 8.15 where we analyze the *size of the coefficients* of the linear combinations whose existence is proved in lemma 8.15. We recall that:

$$K_3 = k_0|Q|, \;\; K_4 = k_0 \cdot |Q|^2 + (k_2 + k_1 + 7) \cdot |Q| + 3.$$

Let us fix a total ordering on $\mathcal{G}_1$:

$$\mathcal{G}_1 = \{\theta_1, \theta_2, \ldots, \theta_d\}, \text{ where } d = \mathrm{Card}(\mathcal{G}_1).$$

Let us remark that $d \leq 2 \cdot |Q| \cdot (\mathrm{Card}(X^{\leq k_1}) + 1) = d_0$.

Lemma 8.16 *Let $L \geq 0$ such that $(i_0 + K_1 L + K_2) + k_1 \in I$. There exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ and, for every $\alpha \in \{-,+\}$, there exists a deterministic rational family $(\beta^{\alpha}_{i,j})_{1 \leq j \leq d}$ fulfilling*

(1) $U^{\alpha}_i = \sum_{j=1}^{d} \beta^{\alpha}_{i,j} \cdot \theta_j$
(2) $||\beta^{\alpha}_{i,*}|| \leq K_3 \cdot (i - i_0) + K_4$.

Proof: We have already established property (188) i.e. $\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2], \exists \alpha \in \{-,+\}, T^{\alpha}_B$ occurs at i. Then

$$U^{-\alpha}_i = \bar{U} \odot u, \text{ for some } u \in X^{k_1}$$

$$U^{+\alpha}_i = \sum_{q \in Q'} [phq][qeq](\bar{U} \odot u_q), \text{ for some } Q' \subseteq Q, h \in Z^*, u_q \in X^{<1,k_0>}$$

$$\bar{U} = U^{-\alpha}_{i-k_1-1} = [r\omega] * \Phi^{\alpha'}, \text{ for some } r \in Q, \omega \in Z^*, \alpha' \in \{-,+\}$$

and by lemma 8.10 and lemma 8.14

$$|\omega| \leq k_0(i - i_0 - k_1 - 1) + (k_2 + k_1 + 2). \tag{190}$$

Let us analyze first the coefficients of $U^{-\alpha}_i$ expressed as a linear combination of the $\{\Phi^{\alpha'}_q \odot w \mid 0 \leq |w| \leq k_1\} \cup \{\rho_\epsilon(\Phi^{\alpha'}_q) \mid q \in Q\}$.
Either $U^{-\alpha}_i = ([r\omega] \odot u) * \Phi^{\alpha'}$ and then

$$\begin{aligned} ||[r\omega] \odot u|| \leq ||[r\omega]|| + |Q||u| &\leq 1 + (|\omega| - 1)|Q| + |u||Q| \\ &\leq 1 + [k_0(i - i_0 - k_1 - 1) + (k_2 + k_1 + 1) + k_1] \cdot |Q| \\ &\leq k_0|Q|(i - i_0) + (k_2 + 2)|Q|. \end{aligned}$$

or $U_i^{-\alpha} = \rho_\epsilon(\Phi_q^{\alpha'} \odot u'')$ for some $q \in Q$, u'' suffix of u and then

$$\|(\emptyset, \ldots, \emptyset, \epsilon, \emptyset, \ldots, \emptyset)\| = 2 \leq k_0|Q|(i - i_0) + (k_2 + 2)|Q|.$$

Let us analyze now the coefficients of $U_i^{+\alpha}$. In order to deal with matrices we fix some total orderings of the sets Q and $\mathcal{G}_1^{\alpha'}$:

$$Q = \{q_1, q_2, \ldots, q_n\}, \text{ where } n = \mathrm{Card}(Q), \;\; \mathcal{G}^{\alpha'} = \{\xi_1, \xi_2, \ldots, \xi_m\}, \text{ where } m = \mathrm{Card}(\mathcal{G}^{\alpha'}).$$

Let us consider the following matrices $A \in \mathsf{DRB}_{1,n} < V >, B \in \mathsf{DRB}_{n,m} < V >$ where

$$\begin{array}{ll}
a_{i,j} = [phq_j][q_jeq_j] & \\
b_{j,l} = [r\omega q_k] \odot u_{q_j}, & \text{if } \; q_j \in Q', [r\omega] \odot u_{q_j} \notin \{\emptyset^n\} \cup \{\epsilon^n_{k'} \mid 1 \leq k' \leq n\}, \xi_l = \Phi^{\alpha'}_{q_k}, \\
b_{j,l} = \epsilon & \text{if } \; q_j \in Q', u_{q_j} = u'_j \cdot u''_j, [r\omega] \odot u'_j = \epsilon^n_k \text{ and } \xi_l = \rho_\epsilon(\Phi^{\alpha'}_{q_k} \odot u''_j). \\
b_{j,l} = \emptyset & \text{if } \; (j,l) \text{ does not fulfill any of the two above conditions.}
\end{array}$$

Then

$$U_i^{-\alpha} = A \cdot B \cdot \bar{\xi}$$

where $\bar{\xi} \in \mathsf{DRB}_{m,1} < V >$ is defined by

$$\bar{\xi}_{\ell,1} = \xi_\ell \;\; (\text{ for } 1 \leq \ell \leq m).$$

Let us analyze the sizes of A, B.
It is clear that $\|A\| = \|[phe]\| = (1 + |h|)|Q| + 2 \leq (2 + k_1)|Q| + 2$.
Let us consider $B_{j,*} = (b_{j,\ell})_{1 \leq \ell \leq m}$.
If $[r\omega] \odot u_{q_j} = [s\omega'_j]$ then

$$\omega'_j = \mu_j \omega''_j \text{ for some } |\mu_j| \leq k_0, \omega''_j \text{ suffix of } \omega.$$

If $\exists k, [r\omega q_k] \odot u'_j = \epsilon, u_{q_j} = u'_j u''_j$ then

$$\mathsf{Q}(B_{j,*}) = \{\epsilon^m_k, \emptyset^m\} \text{ for some } k \in [1, m]$$

If none of these two cases is realized then

$$\mathsf{Q}(B_{j,*}) = \{\emptyset^m\}.$$

It follows that

$$\begin{aligned}
\mathrm{Card}(\cup_i \mathsf{Q}(B_{i,*})) &\leq |Q|(k_0|Q| + 1) + |Q|(|\omega| + 1) + |Q| + 1 \\
&\leq k_0|Q|^2 + (|\omega| + 3)|Q| + 1.
\end{aligned}$$

Using the upper-bound (190) on $|\omega|$ we obtain:

$$\begin{aligned}
\|B\| &\leq k_0|Q|^2 + |Q||\omega| + 3|Q| + 1 \\
&\leq (k_0|Q|^2 + 3|Q| + 1) + |Q|[k_0(i - i_0 - k_1 - 1) + k_2 + k_1 + 2].
\end{aligned}$$

By lemma 3.28 we obtain

$$\begin{aligned}||A\cdot B|| \leq ||A||+||B|| &\leq [(2+k_1)|Q|+2]+[(k_0|Q|^2+3|Q|+1)+|Q|(k_0(i-i_0-k_1-1)+k_2+k_1+2)]\\ &\leq k_0|Q|^2+(k_2+k_1+7)|Q|+3+k_0|Q|(i-i_0)\\ &= K_3(i-i_0)+K_4.\end{aligned}$$

□

Lemma 8.17 *There exists* $i_0 \leq \kappa_1 < \kappa_2 < \ldots < \kappa_d$ *and deterministic rational vectors* $(\beta^{\alpha}_{i,j})_{1\leq j\leq d}$ *(for every* $i \in [1,d]$*) such that*

(0) $W(\kappa_1) \geq 1$
(1) $\forall i, \forall \alpha, U^{\alpha}_{\kappa_i} = \sum_{j=1}^{d} \beta^{\alpha}_{i,j}\theta_j \in \overline{V_1}$
(2) $\forall i, \forall \alpha, ||\beta^{\alpha}_{i,*}|| \leq s_i$
(3) $\forall i, W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}$

where the sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \sigma_i)$ are those defined by relations (146, 147) in section 6.
Proof: Let us consider the additional property
(4) $\kappa_i - i_0 \leq L_i$.
We prove by induction on i the conjunction $(1)\wedge(2)\wedge(3)\wedge(4)$.
i=1:
By lemma 8.16, there exists $\kappa_1 \in [i_0, i_0+K_2]$ such that $\forall \alpha \in \{-,+\}, \exists$ a deterministic vector $(\beta^{\alpha}_{1,j})_{1\leq j\leq d}$, such that

$$U^{\alpha}_{\kappa_1} = \sum_{j=1}^{d} \beta^{\alpha}_{1,j}\theta_j$$

and in addition $||\beta^{\alpha}_{1,*}|| \leq K_3K_2 + K_4 = s_1$.
i → i+1:
Suppose that $\kappa_1 < \kappa_2 < \ldots < \kappa_i$ are fulfilling $(1)\wedge(2)\wedge(3)\wedge(4)$. By lemma 8.16, there exists $\kappa_{i+1} \in [i_0 + L_i + \ell_{i+1}, i_0 + K_1(L_i+\ell_{i+1}) + K_2]$ such that $\forall \alpha \in \{-,+\}, \exists$ a deterministic vector $(\beta^{\alpha}_{i+1,j})_{1\leq j\leq d}$, such that

$$U^{\alpha}_{\kappa_{i+1}} = \sum_{j=1}^{d} \beta^{\alpha}_{i+1,j}\theta_j \tag{191}$$

and in addition

$$\begin{aligned}||\beta^{\alpha}_{i+1,*}|| \leq K_3(K_1(L_i+\ell_{i+1})+K_2)+K_4 &= K_3L_{i+1}+K_4\\ &= s_{i+1}\end{aligned} \tag{192}$$

By lemma 8.2

$$2(W(\kappa_{i+1})-W(\kappa_i))+3 \geq \kappa_{i+1}-\kappa_i \geq \ell_{i+1} = 2\delta_{i+1}+3$$

hence

$$W(\kappa_{i+1}) - W(\kappa_i) \geq \delta_{i+1}. \tag{193}$$

At last

$$\kappa_{i+1} - i_0 \leq K_1(L_i + l_{i+1}) + K_2 = L_{i+1}. \tag{194}$$

The above properties (191-192-193-194) prove the required conjunction.
It remains to prove point (0): the integer κ_1 introduced by lemma 8.16 is such that T_B occurs at κ_1, hence

$$\begin{aligned} W(\kappa_1) &= W(\kappa_1 - k_1 - 1) + k_1 - 1 \\ &\geq W(\kappa_1 - k_1 - 1) + 2 \geq 1. \end{aligned}$$

□
We recall that:

$$C_2 = \mathrm{Card}\{U \in \mathsf{DRB}_{1,\lambda_2} < V >, \|U\| \leq D_2\},$$

$$K_6 = 5 \cdot [(C_2 \cdot |Q||Z|^{k_2+D_2+3})^{|Q|} \cdot |Q||Z|^{D_1}]^2, \quad K_5 = (K_6 + 1) \cdot k_0 \cdot |Q|,$$

and by hypothesis (169)

$$\lambda \leq \lambda_2.$$

Lemma 8.18 *Let $(x_i)_{i \in I}$ be a path in τ (we suppose $I \subseteq \mathbb{N}$ is a non-empty interval). Let $L > 0$. One of the following cases is true:*

(0) $N(i_0) \geq N_0$, where $i_0 = \min(I)$
(1) $|I| \leq K_5 \cdot L + K_6$
(2) $(x_i)_{i \in I}$ contains a N-stacking sequence of length $\geq L$.

Proof: Suppose that neither (0) nor (2) is realized. The set $\{\tau_s(x_i) | i \in I\}$ can contain at most $[(C_2 \cdot |Q||Z|^{k_2+D_2+3})^{|Q|} \cdot |Q||Z|^{D_1}]^2$ pairs (U^-, U^+) such that $\||(U^-, U^+)\|| < N_0$ (because they are (D_1, D_2)-linear). But no pair can appear more than 5 times on a given path , due to the inequality of lemma 8.2 and to the cut rule. Hence

$$\mathrm{Card}\{i \in I | \mathrm{N}(i) < N_0\} \leq K_6.$$

Let us consider now an interval $J = [i, i+\ell] \subseteq I$ which is a maximal sub-interval of I on which N takes values in $[N_0, +\infty[$. As N is $k_0|Q|$-up-Lipschitz, and either $i = i_0$ or $\mathrm{N}(i-1) < N_0$ we have:

$$\mathrm{N}(i) < N_0 + k_0|Q|.$$

As J does not contain any N-stacking sequence of length L,

$$\ell \leq L \cdot k_0 \cdot |Q| - 1.$$

Finally, I contains at most $(K_6 + 1)$ maximal sub-intervals J on which N takes values in $[N_0, +\infty[$. It follows that

$$|I| \leq \mathrm{Card}(I) \leq K_6 + (K_6 + 1) \cdot L \cdot k_0 \cdot |Q| = K_5 \cdot L + K_6,$$

i.e. property (1) is realized. □

9 Termination

Lemma 9.1 *: $\hat{\mathcal{S}}_{ABC}$ is terminating on every unmarked assertion A_0: if $A_0 \in \mathcal{A}$ is unmarked, then, $\exists n_0 \in \mathbb{N}, \hat{\mathcal{S}}_{ABC}^{n_0+1}(A_0) = \hat{\mathcal{S}}_{ABC}^{n_0}(A_0)$.*

Proof: Suppose $A_0 \in \mathcal{A}$, A_0 is true, A_0 is unmarked and

$$\forall n \in \mathbb{N}, \hat{\mathcal{S}}_{ABC}^{n}(A_0) \prec \hat{\mathcal{S}}_{ABC}^{n+1}(A_0). \tag{195}$$

Let us consider all the constants associated to this precise A_0 and to the dpda $\mathcal{M}$ in section 6. Let us note : $t_n = \hat{\mathcal{S}}_{ABC}^{n}(A_0)$ (for every $n \in \mathbb{N}$) and let

$$t_\infty = l.u.b.\{t_n \mid n \in \mathbb{N}\}.$$

Let us notice that, by definition (166), the strict inequality (195) implies that

$$\forall n \in \mathbb{N}, t_n \text{ is consistent.} \tag{196}$$

Let us denote by x_n the node of t_n such that $t_{n+1} = t_n[\hat{\Delta}(t_n)/x_n]$. Let us notice that , as every x_n is unclosed in t_n, one can prove by induction that every t_n is repetition-free. Hence

$$t_\infty \text{ is repetition-free.} \tag{197}$$

By Koenig's lemma, t_∞ contains an infinite branch $y_0 y_1 \cdots y_s \cdots$ whose (infinite) labelling word is $A_0 A_1 \cdots A_s \cdots$ (where $A_s = t_\infty(y_s)$).
Condition (C3) in the definition of $T_C^{(\mathcal{O})}$, combined with lemma 3.21, shows that

$$\text{every equation produced by } T_C \text{ has size} \leq D_2, \tag{198}$$

hence that the number of possible unweighted equations produced by T_C is *finite*. Hence T_C occurs only a finite number of times on this branch (because t_∞ is repetition-free (197) and T_{cut} cannot occur on an infinite branch). Let n_0 be the last point where T_C occurs (or $n_0 = 0$ if T_C never occurs on this branch). $(y_{n_0+i})_{i \geq 0}$ is a branch of a tree $t' \in \mathcal{T}(\mathcal{S}_{AB}, A_{n_0})$. Let us notice also that

$$\text{every equation produced by } T_C \text{ is unmarked,} \tag{199}$$

(by condition (C4) in the definition of $T_C^{(\mathcal{O})}$, see section 7), and

$$\text{every equation produced by } T_C \text{ has a length } \lambda \leq \lambda_2, \tag{200}$$

because it has a length $\leq d_0$ and $d_0 \leq \lambda_2$ by definition (150) in section 6. Moreover, the root A_0 of t_∞ is supposed to have a size $\leq D_2$ (by definition (148), in section 6), to be unmarked (by the hypothesis of the lemma), and to have a length $\lambda_0 \leq \lambda_2$ (by definition (150) in section 6). Hence, in either case, t' fulfills the hypotheses (168)(169) stated in section 8.2 and assumed in section 8.4. We may apply now the results obtained in section 8.4.
By lemma 8.18, the branch $(y_{n_0+i})_{i \geq 0}$ must contain an infinite N-stacking sequence. Let us remark that, as $T_\emptyset$ does not occur (otherwise the branch would be finite) every equation (π, U^-, U^+) labelling this branch is such that $U^- \not\equiv$

$\emptyset, U^+ \not\equiv \emptyset$. By lemma 8.17 such an N-stacking sequence contains a subsequence $(A_{\kappa_1}, A_{\kappa_2}, \cdots, A_{\kappa_d})$ with $d \leq d_0$, fulfilling hypotheses (1,2) of lemma 5.8, and by the above remark it fulfills hypothesis (0) too. Let $n_i \in \mathbb{N}$ such that $x_{n_i} = y_{\kappa_i}$, for $1 \leq i \leq d$. By (196), $\Omega(\bar{\Pi}(t_{n_d}), \mathrm{im}(t_{n_d})) \neq \emptyset$. Let us consider some

$$\mathcal{O} \in \Omega(\bar{\Pi}(t_{n_d}), \mathrm{im}(t_{n_d})).$$

Let $\mathcal{S}_d = (A_{\kappa_i})_{1 \leq i \leq d}$ and $D = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}_d)$. By lemma 5.8,

$$\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_d) \neq \perp, D \in [0, d-1] \text{ and } \;|||\, \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_d) \,||| \leq \Sigma_{d_0} + s_{d_0}. \tag{201}$$

Let $\mathcal{S}_{D+1} = (A_{\kappa_i})_{1 \leq i \leq D+1}$. By the inequality (2) of lemma 5.8,

$$\bar{\Pi}(t_{n_{D+1}}) \leq \bar{\Pi}(t_{n_d}),$$

and it is straightforward that

$$\mathrm{im}(t_{n_{D+1}}) \subseteq \mathrm{im}(t_{n_d}),$$

hence,

$$\mathcal{O} \in \Omega(\bar{\Pi}(t_{n_{D+1}}), \mathrm{im}(t_{n_{D+1}})). \tag{202}$$

Let $W_{D+1} = A_0 \cdot A_1 \cdots A_{\kappa_1} \cdots A_{\kappa_{D+1}}$ (the word from the root to $y_{\kappa_{D+1}}$). Let us notice that

$$\mathrm{D}^{(\mathcal{O})}(\mathcal{S}_{D+1}) = \mathrm{D}^{(\mathcal{O})}(\mathcal{S}_d) = D, \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_{D+1}) = \mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_d). \tag{203}$$

By (201),(203),

$$\rho_e(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S}_{D+1})) \in T_C^{(\mathcal{O})}(W_{D+1}). \tag{204}$$

By (202)(204), the set $\{\mathcal{O} \in \Omega(\bar{\Pi}(t_{n_{D+1}}), \mathrm{im}(t_{n_{D+1}})), T_C^{(\mathcal{O})}(W_{D+1}) \neq \emptyset\}$ is not empty, so that case (0) of the definition of $\hat{\Delta}(t)$ (see section 7) is fulfilled and

$$\hat{\Delta}(t_{n_{D+1}}) = A_{n_{D+1}}(T_C^{(\mathcal{O}_0)}(W_{D+1})),$$

i.e. T_C occurs at $y_{\kappa_{D+1}+1}$. This is a contradiction with the minimality of n_0. We have proved that hypothesis (195) is impossible. Hence the lemma is proved. □

10 Elimination

10.1 System $\mathcal{B}_1$

We prove here that the new formal system $\mathcal{B}_1$ obtained by *elimination* of metarule (R5) in $\mathcal{B}_0$ is recursively enumerable and complete. The decidability of the bisimulation problem follows.

Let $\mathcal{B}_1 =< \mathcal{A}, H, \models\!\!-_{\mathcal{B}_1} >$ where $\mathcal{A}, H$, are the same as in $\mathcal{B}_0$, but the *elementary deduction relation* $||\!\!-_{\mathcal{B}_1}$ is the relation generated by the subset of metarules $R0, R1, R2, R3, R'3, R4, R6, R7, R8, R9, R10$, i.e. all the metarules of $\mathcal{B}_0$ except $R5$. The deduction relation $\models\!\!-_{\mathcal{B}_1}$ is now defined by:

$$\models\!\!-_{\mathcal{B}_1} = \overset{<*>}{||\!\!-}_{\mathcal{B}_1} \circ \overset{[1]}{||\!\!-}_{R0,R3,R'3,R4} \circ \overset{<*>}{||\!\!-}_{\mathcal{B}_1}.$$

Lemma 10.1 *: $\mathcal{B}_1$ is a deduction system.*

Sketch of proof: As $\vdash\!\!-_{\mathcal{B}_1} \subseteq \vdash\!\!-_{\mathcal{B}_0}$, property (A1) is fulfilled by $\vdash\!\!-_{\mathcal{B}_1}$.
By the well-known decidability properties for finite-automata, rules $R0, R1, R2, R3, R4, R6, R7, R8, R9, R10$ are recursively enumerable. By a well-known decidability result on context-free grammars, one can compute the subsets: $V_\emptyset = \{v \in V \mid \varphi(v) = \emptyset\}$, $V_\varepsilon = \{v \in V \mid \varphi(v) = \epsilon\}$. But the property $S \equiv \emptyset$ (resp $S \equiv \epsilon$) reduces to $S \leq V^* \cdot V_\emptyset \cdot V^*$ (resp. $S \leq {V_\varepsilon}^* + V^* \cdot V_\emptyset \cdot V^*$), hence $R'3$ is recursive. Hence property (A2) is fulfilled by $\mathcal{B}_1$. □

Completeness

Definition 10.2 *Let P be a finite subset of $\mathcal{A}$ and let $\bar{\pi} \in \mathbb{N}$. P is said* locally $\bar{\pi}$-consistent *iff, for every $(\pi, S, S') \in P$, if*

$$\pi < \bar{\pi},$$

then, there exists $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$ such that

$$[\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(P).$$

Lemma 10.3 *Let P be a finite subset of $\mathcal{A}$ and let $\bar{\pi} \in \mathbb{N}$. If P is locally $\bar{\pi}$-consistent, then P is $\bar{\pi}$-consistent.*

Proof: Let us consider, for every integers $n \geq 0, p \geq 0$, the following property $\mathcal{Q}(n,p)$: $\forall \pi \in \mathbb{N}, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >$,

$$(\pi, S, S') \in \mathrm{Cong}_p(P) \text{ and } \pi + n - 1 < \bar{\pi} \Rightarrow \exists \mathcal{R}_n \in \mathcal{B}_n(S, S'), [\pi, S, S', \mathcal{R}_n] \subseteq \mathrm{Cong}(P). \tag{205}$$

Let us prove by induction on (n,p) that

$$\forall (n,p) \in \mathbb{N} \times \mathbb{N}, \mathcal{Q}(n,p). \tag{206}$$

<u>$n = 0, p = 0$:</u>
The only possible value of $\mathcal{R}_0 \in \mathcal{B}_0(S, S')$ is $\mathcal{R}_0 = \{(\epsilon, \epsilon)\}$, and $[\pi, S, S', \mathcal{R}_0] = \{(\pi, S, S')\} \subseteq \mathrm{Cong}_0(P)$.
<u>$p > 0$:</u>
There exists a subset $Q \subseteq \mathcal{P}_f(\mathcal{A})$, such that

$$P \overset{<p-1>}{|\!\vdash\!\!-}_{\mathcal{C}} Q \text{ and } Q \overset{<1>}{|\!\vdash\!\!-}_{\mathcal{C}} \{(\pi, S, S')\}.$$

As every rule of $\mathcal{B}_0$ increases the weight, we can suppose that every assertion of Q has a weight $\leq \pi$. Hence, by induction hypothesis,

$$\forall (\pi', T, T') \in Q, \exists \mathcal{R}_n \in \mathcal{B}_n(T, T'), [\pi', T, T', \mathcal{R}_n] \subseteq \mathrm{Cong}(P). \tag{207}$$

Let us consider the type of rule used in the last step, $Q \overset{<1>}{|\!\vdash\!\!-}_{\mathcal{C}} \{(\pi, S, S')\}$, of the above deduction.

R0: $(\pi-1,S,S')\in Q$.

By (207), $\exists \mathcal{R}_n \in \mathcal{B}_n(S,S')$,

$$[\pi-1,S,S',\mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

As $[\pi-1,S,S',\mathcal{R}_n] \overset{<1>}{|\!\!\vdash\!\!-}_{\mathcal{C}} [\pi,S,S',\mathcal{R}_n]$,

$$[\pi,S,S',\mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

R1:$(\pi,S',S)\in Q$.

(analogous to the above case)

R2:$(\pi,S,T),(\pi,T,S')\in Q$.

By (207), $\exists \mathcal{R}_n \in \mathcal{B}_n(S,T), \mathcal{R}'_n \in \mathcal{B}_n(T,S')$,

$$[\pi,S,T,\mathcal{R}_n] \subseteq \mathrm{Cong}(P), [\pi,T,S',\mathcal{R}'_n] \subseteq \mathrm{Cong}(P).$$

Using the properties mentionned in section 4.3, we get that:

$$[\pi,S,S',\mathcal{R}_n \circ \mathcal{R}'_n] \subseteq \mathrm{Cong}(P).$$

R3,R'3,R10:

In this case, $\mathcal{R}_n = \mathrm{Id} \cap X^{\leq n} \times X^{\leq n} \in \mathcal{B}_n(S,S')$, and

$$[\pi,S,S',\mathcal{R}_n] \subseteq \{(\pi,S,S')\} \cup \{(\pi+k,T,T), 1 \leq k \leq n, T \in \mathsf{DRB}_{1,\lambda} < V >\} \subseteq \mathrm{Cong}(P).$$

R6:$(\pi, S_1 \cdot S' + U, S')\in Q, S = S_1^* \cdot U$.

By (207), $\exists \mathcal{R}_n \in \mathcal{B}_n(S_1 \cdot S' + U, S')$,

$$[\pi, S_1 \cdot S' + U, S', \mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

Using the properties mentionned in section 4.3, we get that:

$$\begin{aligned}[\pi,S,S',\mathcal{R}_n^{<S_1,*>}] &= [\pi, S_1^* \cdot U, S', \mathcal{R}_n^{<S_1,*>}] \\ &\subseteq \mathrm{Cong}[\pi, S_1 \cdot S' + U, S', \mathcal{R}_n] \\ &\subseteq \mathrm{Cong}(Q) \subseteq \mathrm{Cong}(P).\end{aligned}$$

R7:$(\pi,\emptyset,S'_1),(\pi,S_2,S'_2)\in Q, S=(\emptyset,S_2), S'=(S'_1,S'_2)$.

By (207), $\exists \mathcal{R}'_n \in \mathcal{B}_n(\emptyset,S'_1), \mathcal{R}_n \in \mathcal{B}_n(S_2,S'_2)$,

$$[\pi,\emptyset,S'_1,\mathcal{R}'_n] \subseteq \mathrm{Cong}(P),\ \ [\pi,S_2,S'_2,\mathcal{R}_n] \subseteq \mathrm{Cong}(P). \tag{208}$$

Let us notice that the subset $[\pi,\emptyset,S'_1,\mathcal{R}'_n]$ depends on $\mathrm{im}(\mathcal{R}'_n)$ only, hence $[\pi,\emptyset,S'_1,\mathcal{R}'_n] = [\pi,\emptyset,S'_1,\mathcal{R}_n]$, and

$$[\pi,\emptyset,S'_1,\mathcal{R}_n] \subseteq \mathrm{Cong}(P). \tag{209}$$

Using inclusions (209, 208), and rule R7, we conclude that

$$[\pi,(\emptyset,S_2),(S'_1,S'_2),\mathcal{R}_n] \subseteq \mathrm{Cong}(P).$$

The three other cases where $S=(\epsilon,S'_1)$ or $S=(S'_1,\emptyset)$ or $S=(S'_1,\epsilon)$ can be treated similarly.

R8:$(\pi, S_1, S_1') \in Q, S = S_1 \cdot T, S' = S_1' \cdot T$.
By (207), $\exists \mathcal{R}_n \in \mathcal{B}_n(S_1, S_1')$,

$$[\pi, S_1, S_1', \mathcal{R}_n] \subseteq \text{Cong}(P).$$

Using the properties mentionned in section 4.3, we get that:

$$\begin{aligned}[\pi, S, S', < S_1|\mathcal{R}_n >] &= [\pi, S_1 \cdot T, S_1' \cdot T, < S_1|\mathcal{R}_n >] \\ &\subseteq \text{Cong}([\pi, S_1, S_1', \mathcal{R}_n]) \\ &\subseteq \text{Cong}(Q) \subseteq \text{Cong}(P).\end{aligned}$$

R9:$\forall i \in [1, \delta], (\pi, T_{i,*}, T_{i,*}') \in Q, S = S_1 \cdot T, S' = S_1 \cdot T'$.
By (207), $\exists \mathcal{R}_{1,n}, \ldots, \mathcal{R}_{\delta,n} \in \mathcal{B}_n(T_{i,*}, T_{i,*}')$, such that

$$[\pi, T_{i,*}, T_{i,*}', \mathcal{R}_{i,n}] \subseteq \text{Cong}(P).$$

Using the properties mentionned in section 4.3, we get that:

$$\begin{aligned}[\pi, S, S', < S, \mathcal{R}_{*,n} >] &= [\pi, S_1 \cdot T, S_1 \cdot T', < S, \mathcal{R}_{*,n} >] \\ &\subseteq \text{Cong}(\bigcup_{1 \leq i \leq \delta} [\pi, T_{i,*}, T_{i,*}', \mathcal{R}_{i,n}]) \\ &\subseteq \text{Cong}(Q) \subseteq \text{Cong}(P).\end{aligned}$$

In all cases $\mathcal{Q}(n, p)$ has been established.
$\underline{n > 0, p = 0}$:$(\pi, S, S') \in P$.
As P is locally $\bar{\pi}$-consistent and $\pi \leq \pi + n - 1 < \bar{\pi}$, there exist $\mathcal{R}_1 \in \mathcal{B}_1(S, S'), q \in \mathbb{N}$ such that:

$$[\pi, S, S', \mathcal{R}_1] \subseteq \text{Cong}_q(P). \tag{210}$$

As $(n-1, q) < (n, 0)$, by induction hypothesis,$\forall (x, x') \in \mathcal{R}_1 \cap X \times X, \exists \mathcal{R}_{x,x',n-1} \in \mathcal{B}_{n-1}(S \odot x, S' \odot x')$ such that

$$[\pi + 1, S \odot x, S' \odot x', \mathcal{R}_{x,x',n-1}] \subseteq \text{Cong}(P). \tag{211}$$

Let us consider $\mathcal{R}_n = \{(\epsilon, \epsilon)\} \bigcup_{(x,x') \in \mathcal{R}_1 \cap X \times X} \{(x, x')\} \cdot \mathcal{R}_{x,x',n-1}$. One can check that $\mathcal{R}_n \in \mathcal{B}_n(S, S')$ and , by (210, 211) we obtain:

$$[\pi, S, S', \mathcal{R}_n] = \{(\pi, S, S')\} \bigcup_{(x,x') \in \mathcal{R}_1 \cap X \times X} [S \odot x, S' \odot x', \mathcal{R}_{x,x',n-1}] \subseteq \text{Cong}(P).$$

Let us define now an oracle $\mathcal{O} \in \Omega$ which is $\bar{\pi}$-consistent with P. For every $(S, S') \in \bigcup_{\lambda \geq 1} \mathsf{DRB}_{1,\lambda} < V >$ occuring in Cong(P) (i.e. as the projection on $\bigcup_{\lambda \geq 1} \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V >$ of an assertion in Cong(P)), let us note

$$W(S, S') = \min(\{\pi \in \mathbb{N} \mid (\pi, S, S') \in \text{Cong}(P)\}).$$

$$D(S, S') = \max\{\bar{\pi} - W(S, S'), 0\}.$$

$$C(S, S') = \min\{\mathcal{R} \in \mathcal{B}_{D(S,S')}(S, S') \mid [W(S, S'), S, S', \mathcal{R}] \subseteq \text{Cong}(P)\}.$$

Notice that $C(S,S')$ is well-defined, owing to property (206). We then define $\mathcal{O}$ by: for every (S,S') occuring in Cong(P),

$$\mathcal{O}(S,S') = \min\{\mathcal{R} \in \mathcal{B}_\infty(S,S') \mid C(S,S') = \mathcal{R} \cap (X^{\leq D(S,S')} \times X^{\leq D(S,S')})\}, \tag{212}$$

and for every (S,S') not occuring in Cong(P),

$$\mathcal{O}(S,S') = \min\{\mathcal{R} \in \mathcal{B}_\infty(S,S')\}(\text{ if } S \sim S'),\ \ \mathcal{O}(S,S') = \mathrm{Id}_{X^*}(\text{ if } S \not\sim S'). \tag{213}$$

One can check that, by the choice of $C(S,S')$, $\mathcal{O}$ is $\bar{\pi}$-consistent with P. □

Lemma 10.4 *Let $A_0 \in \mathcal{A}$ such that $H(A_0) = \infty$. Let us consider the sequence of trees $t_n = \hat{\mathcal{S}}^n_{ABC}(A_0)$. For every integer $n \geq 0$, t_n is consistent.*

Let us say that the strategy T "applies on" node x iff, x has exactly m sons $x \cdot 1, x \cdot 2, \ldots, x \cdot m$ and

$$\tau(x1) \cdot \tau(x \cdot 2) \cdots \tau(x \cdot m) \in T(\tau(x[0]) \cdot \tau(x[1]) \cdots \tau(x[|x|])),$$

i.e. the word consisting of the labels of the sons of x belongs to the image of the path from ϵ (included) to x (included) by the strategy T.
Proof: For every $k \in \mathbb{N}$ we define

$$\bar{\pi}_k = \bar{\Pi}(t_k).$$

We prove by induction on (n, π) the following property $\mathcal{R}(n, \pi)$:

$$\forall x \in \mathrm{dom}(t_n),\ \text{if } t_n(x) = (\pi, S, S') \text{ with } \pi < \bar{\pi}_n,\ \text{then} \tag{214}$$
$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S,S'), [\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_n)). \tag{215}$$

At every step of our proof by induction, we consider some node x of t_n fulfilling hypothesis (214) and we show that it must fulfill (215). Let us notice that , if x is not closed, then hypothesis (214) cannot be true, by minimality of $\bar{\pi}_n$. Let us notice also that, if x is closed , but there is some $x' \prec x$ such that $t_n(x') = t_n(x)$, then (215) on x is the same property as (215) for x'. Hence , in the sequel, we can suppose that x is closed and that it is minimal (w.r.t. to $\preceq$):

$$x = \min_{\preceq}\{y \in \mathrm{dom}(t_n) \mid t_n(y) = t_n(x)\}. \tag{216}$$

$\underline{n = 0, \pi = 0}$**:** $\mathrm{dom}(t_0) = \{\epsilon\}$, $t_0(\epsilon) = A_0$. If ϵ is not closed, then $\bar{\pi}_0 = \pi = 0$, hence there is no node x fulfilling hypothesis (214). Otherwise, $\bar{\pi}_0 = \infty$ and $x = \epsilon$ is closed: either $T_\emptyset(A_0) = \{\epsilon\}$ or $T_\varepsilon(A_0) = \{\epsilon\}$. Let us choose

$$\mathcal{R}_1 = \mathrm{Id}_{X^*} \cap X^{\leq 1} \times X^{\leq 1}. \tag{217}$$

If we note $A_0 = (\pi, S_0^-, S_0^+)$, then

$$[\pi, S_0^-, S_0^+, \mathcal{R}_1] = \{(\pi, S_0^-, S_0^+)\} \cup \{(\pi + 1, S_0^- \odot x, S_0^+ \odot x) \mid x \in X\},$$

where , $\forall x \in X, S_0^- \odot x \equiv S_0^+ \odot x \equiv \emptyset$. Using rule $R'3$, we see that

$$[\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\emptyset) \subseteq \mathrm{Cong}(\mathrm{im}(t_n)). \tag{218}$$

$n > 0, \pi = 0$: Let x be some node of t_n such that $\exists S, S', t_n(x) = (\pi, S, S')$ and $\pi < \bar{\pi}_n$. Let us denote by W_x the word labelling the path from the root of t_n (included) to x (included).

case 1: $\exists x' \in \mathrm{dom}(t_n), x'$ internal node , such that $t_n(x') = t_n(x)$.

As $\pi = 0$, the sons $x' \cdot 1, x' \cdot 2, \ldots, x' \cdot m$ of x' are such that $t_n(x' \cdot 1) \cdot t_n(x' \cdot 2) \cdots t_n(x' \cdot m) \in T_A^{(\mathcal{O})}(W_{x'})$, for some oracle $\mathcal{O}$. Let us choose

$$\mathcal{R}_1 = \mathcal{O}(S, S') \cap X^{\leq 1} \times X^{\leq 1}. \tag{219}$$

Then

$$[\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{im}(t_n). \tag{220}$$

case 2: $T_\emptyset(W_x) = \{\epsilon\}$ or $T_\varepsilon(W_x) = \{\epsilon\}$.

In this case the choice $\mathcal{R}_1 = \mathrm{Id}_{X^*} \cap X^{\leq 1} \times X^{\leq 1}$ satisfies again (218).

$\pi > 0$:

Let x fulfilling hypothesis (214). As t_n is a proof-tree for $\mathcal{S}_{ABC}$, and as we suppose x is closed and minimal (216), one of the following cases must occur.

case 1: T_{cut} applies on x.

There exists $x' \in \mathrm{dom}(t_n), \exists \pi' \in \mathbb{N}$, such that

$$t_n(x') = (\pi', S, S') \text{ and } \pi' < \pi.$$

By induction hypothesis

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), [\pi', S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_n)),$$

and by means of rule $R0$:

$$[\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}([\pi', S, S', \mathcal{R}_1]).$$

Hence (215) is true.

case 2: $T_\emptyset$ ot T_ϵ applies on x.

Here again, the choice (217) fufills property (218).

In the remaining cases we use the following notation: for every $k \in \mathbb{N}$ such that t_k is not closed,

$$x_k = \min\{x \in \mathrm{dom}(t_k), x \text{ is not closed for } \mathcal{S}_{ABC} \text{ and } \exists S, S', t(x) = (\bar{\pi}_k, S, S')\}.$$

If $\exists k < n \mid t_k$ is not consistent or is closed, then by (166), $t_k = t_{k+1} = \cdots = t_n$, hence $\mathcal{R}(n, \pi) \Leftrightarrow \mathcal{R}(k, \pi)$, and this last property is true by induction hypothesis. Let us suppose now that $\forall k < n$, t_k is consistent and unclosed. According to formula (165),

$$t_{k+1} = t_k[e_{k+1}/x_k],$$

for some tree of depth one, e_{k+1}.

Let $k \in [0, n-1], x = x_k, \pi = \bar{\pi}_k$ (such a k must exist because x is internal). Let $x \cdot 1, \ldots, x \cdot \mu$ be the sequence of sons of x.

case 3: T_A applies on x.
Hence there exists some oracle $\mathcal{O}$ such that $T_A^{(\mathcal{O})}$ applies on x. The choice (219) fufills property (220).
case 4: T_B^{α} applies on x (for some $\alpha \in \{-,+\}$).
Let us suppose $\alpha = +$. Let $x' = x(|x| - k_1)$ (the prefix of x having length $|x| - k_1$), $t_n(x') = (\pi', \bar{U}, U')$. By definition of $\hat{\mathcal{S}}_{ABC}$, there exists some oracle $\mathcal{O}$ which is $\bar{\pi}_k$-consistent with $\mathrm{im}(t_k)$ and such that:

$$\mu = 1 \text{ and } t_n(x \cdot 1) = T_B^{(\mathcal{O}),+}(W_x).$$

Let us look at the proof of lemma 7.2 in the particular case of this oracle $\mathcal{O}$: as the pairs (u_q, u'_q) belong to $\mathcal{O}(\bar{U}, U')$ (for every $q \in Q$) and $\pi' + |u_q| - 1 < \pi' + k_0 \leq \pi' + 2 \cdot k_0 < \bar{\pi}_k$, deduction (154) can be obtained just by using rules in $\mathcal{C}$. As deduction (154) is the only one (in the proof of lemma 7.2) using rules in $\mathcal{B}_0 - \mathcal{C}$ we conclude that deduction (153) can be replaced by:

$$\{t_n(x'), t_n(x \cdot 1)\} \cup \mathrm{im}(t_k) \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \tau_{-1}(t_n(x)). \tag{221}$$

(We recall τ_{-1} consists in replacing the weight of a given weighted equation into its predecessor). Deduction (221) implies that

$$\exists p \in \mathbb{N}, (\pi - 1, S, S') \in \mathrm{Cong}_p(\mathrm{im}(t_n)). \tag{222}$$

By induction hypothesis, as $\pi < \bar{\pi}_n$, $\mathrm{im}(t_n)$ is locally π-consistent, hence, by lemma 10.3, $\mathrm{im}(t_n)$ is π-consistent. Hypothesis (222) implies that

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), [\pi - 1, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_n)),$$

hence that

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), [\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_n)).$$

case 5: T_C applies on x.
By definition of $\hat{\mathcal{S}}_{ABC}$, there exists some oracle $\mathcal{O}$ which is $\bar{\pi}_k$-consistent with $\mathrm{im}(t_k)$ and such that:

$$\mu = 1 \text{ and } t_n(x \cdot 1) = T_C^{(\mathcal{O})}(W_x).$$

Let $W_x = A_1 \cdots A_\ell \cdots A_{|x|+1}$, $\kappa_1 < \cdots < \kappa_i < \kappa_{i+1} < \cdots \kappa_{D+1} = |x| + 1$, $\mathcal{S} = (\mathcal{E}_i)_{1 \leq i \leq D+1}$, where , for every $1 \leq i \leq d$,

$$\mathcal{E}_i = A_{\kappa_i} = (\pi_i, \sum_{j=1}^{d} \alpha_{i,j} S_j \ , \ \sum_{j=1}^{d} \beta_{i,j} S_j)$$

and

$$T_C^{(\mathcal{O})}(W_x) = \rho_e(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})), \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) \neq \bot, \mathrm{D}^{(\mathcal{O})}(\mathcal{S}) = D \leq d - 1.$$

Let us look at the proof of lemma 5.2 in the particular case of this oracle $\mathcal{O}$: the only place where a rule in $\mathcal{B}_0 - \mathcal{C}$ is used, is in deduction (124)(resp. deduction

(129)), when case 2, subcase1 (resp. case 2, subcase 2), of the recursive definition of $\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$ occurs . Let us recall that the pair (u, u') chosen by the oracle $\mathcal{O}$ is such that:

$$\mathcal{R} = \mathcal{O}(\sum_{j=1}^{d} \alpha_{1,j} S_j, \sum_{j=1}^{d} \beta_{1,j} S_j),$$

$$\nu = \mathrm{Div}(\alpha_{1,*}, \beta_{1,*}), \;\; \mathcal{R}_\nu = \mathcal{R} \cap X^{\leq \nu} \times X^{\leq \nu}, \;\; (u, u') \in \mathcal{R}_\nu.$$

Let us notice that $\pi_1 + \nu - 1 < \pi_1 + 2 \cdot \nu < \pi_2 \leq \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) + 1 = \pi = \bar{\pi}_k$. As $\mathcal{O}$ is $\bar{\pi}_k$-consistent with $\mathrm{im}(t_k)$, we conclude that

$$(\pi_1 + |u|, (\sum_{j=1}^{d} \alpha_{i,j} S_j) \odot u, (\sum_{j=1}^{d} \beta_{i,j} S_j) \odot u') \in [\pi_1, \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j, \mathcal{R}_\nu]$$
$$\subseteq \mathrm{Cong}(\mathrm{im}(t_k)).$$

Hence deduction (124) (resp.129) can be replaced by

$$\mathcal{E}'_1 \in \mathrm{Cong}(\mathrm{im}(t_k)). \tag{223}$$

Similarly, for every $i \in [2, D]$, as $\pi_i + 2 \cdot \mathrm{Div}(\alpha_{i,*}^{(i-1)}, \beta_{i,*}^{(i-1)}) < \pi_{i+1} \leq \mathrm{W}^{(\mathcal{O})}(\mathcal{S}) + 1 = \pi = \bar{\pi}_k$, and $\mathcal{E}_i^{(i-1)} \in \mathrm{Cong}(\mathrm{im}(t_k))$,

$$(\mathcal{E}_i^{(i-1)})' \in \mathrm{Cong}(\mathrm{im}(t_k)). \tag{224}$$

It follows that deduction (123) can be replaced by

$$\{\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})\} \cup \{A_{\kappa_i} \mid 1 \leq i \leq D\} \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \tau_{-1}(t_n(x)). \tag{225}$$

using the fact that $\rho_e(\mathrm{INV}^{(\mathcal{O})}(\mathcal{S})) \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \mathrm{INV}^{(\mathcal{O})}(\mathcal{S})$ we may conclude that

$$\{t_n(x \cdot 1)\} \cup \{A_{\kappa_i} \mid 1 \leq i \leq D\} \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \tau_{-1}(t_n(x)) = (\pi - 1, S, S'). \tag{226}$$

From (226) and the induction hypothesis, we can conclude, as in case 4, that

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), [\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_n)).$$

(End of the induction).
By the above induction, for every $n \in \mathbb{N}$, $\mathrm{im}(t_n)$ is $\bar{\pi}_n$-consistent i.e. t_n is consistent. □

Lemma 10.5 *$\hat{\mathcal{S}}_{ABC}$ is closed.*

Proof: Let $A_0 \in \mathcal{A}$. By lemma 10.4, $\forall n \in \mathbb{N}, \hat{\mathcal{S}}^n_{ABC}(A_0)$ is consistent.
If $\hat{\mathcal{S}}^n_{ABC}(A_0)$ is consistent and is not closed, then , by definition (165),

$$\hat{\mathcal{S}}^n_{ABC}(A_0) \neq \hat{\mathcal{S}}^{n+1}_{ABC}(A_0);$$

if $\hat{\mathcal{S}}^n_{ABC}(A_0)$ is consistent and is closed, then , by definition (166),

$$\hat{\mathcal{S}}^n_{ABC}(A_0) = \hat{\mathcal{S}}^{n+1}_{ABC}(A_0).$$

Hence the equivalence (118), which defines the notion of closed global strategy, is fulfilled by $\hat{\mathcal{S}}_{ABC}$. □

Theorem 10.6 : *$\mathcal{B}_0, \mathcal{B}_1$ are complete formal systems.*

Proof: By lemma 9.1 $\hat{\mathcal{S}}_{ABC}$ is terminating on every unmarked assertion and by lemma 10.5 $\hat{\mathcal{S}}_{ABC}$ is closed. Let A_0 be some unmarked true assertion. According to the proof of lemma 4.16, $\exists n_0 \in \mathbb{N}$ such that $t_\infty = \hat{\mathcal{S}}^{n_0}(A_0)$ is a proof-tree which is closed, hence such that $\Pi(t_\infty) = \infty$. By lemma 10.5, t_∞ is consistent, i.e. $\mathrm{im}(t_\infty)$ is ∞-consistent: $\forall (\pi, S, S') \in \mathrm{im}(t_\infty)$,

$$\exists \mathcal{R}_1 \in \mathcal{B}_1(S, S'), [\pi, S, S', \mathcal{R}_1] \subseteq \mathrm{Cong}(\mathrm{im}(t_\infty)),$$

hence,

$$\mathrm{im}(t_\infty) \;\; \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} [\pi, S, S', \mathcal{R}_1] \vdash\!\!-_{R4} (\pi, S, S'). \tag{227}$$

As the rules of $\mathcal{C}$ and $R4$ are rules of $\mathcal{B}_1$, deduction (227) shows that

$$\mathrm{im}(t_\infty) \vdash\!\!-_{\mathcal{B}_1} (\pi, S, S'). \tag{228}$$

i.e. $\mathrm{im}(t_\infty)$ is a $\mathcal{B}_1$-proof.
In the general case where $A_0 = (\pi_0, U_0^-, U_0^+)$ might be marked, we know by corollary 4.12 that

$$\emptyset \;\; \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{(\pi_0, U_0^-, \rho_e(U_0^-), (\pi_0, U_0^+, \rho_e(U_0^+)\}.$$

This deduction combined with some $\mathcal{B}_1$-proof of $\rho_e(A_0)$ gives a $\mathcal{B}_1$-proof of A_0. □

Theorem 10.7 *The bisimulation problem for rooted equational 1-graphs of finite out-degree is decidable.*

Proof: Let us consider the sequence of statements: lemma 2.7, lemma 2.8, corollary 2.6 and lemma 3.26. By means of the above statements, the bisimulation problem for rooted equational 1-graphs of finite out-degree reduces to the following decision problem (we call it the bisimulation problem for deterministic vectors):

INSTANCE: a bi-rooted, normalized dpda $\mathcal{M}$, its terminal alphabet X, a surjective litteral morphism $\psi : X^* \to Y^*$ (we denote its kernel by $\bar{\psi}$), and $\lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >$ (where V is the structured alphabet associated with $\mathcal{M}$).

QUESTION: $S \sim S'$? (where $\sim$ is the $\bar{\psi}$-bisimulation relation).

Let us consider $\mathcal{M}, X, V, \bar{\psi}$ given by some instance.
The equivalence relation $\sim$ on $\mathsf{DRB}_{1,\lambda} < V >$ has a recursively enumerable complement (this is well-known). By theorem 10.6 and lemma 4.2 $\sim$ is recursively enumerable too. Hence $\sim$ is recursive.
But the fuction associating to every $\mathcal{M}, X, V, \bar{\psi}$ the corresponding deduction sytem $\mathcal{B}_1$ is recursive. Hence the bisimulation problem for deterministic vectors is decidable. $\Box$

10.2 System $\mathcal{B}_2$

We exhibit here a deduction system $\mathcal{B}_2$ which is simpler than $\mathcal{B}_1$ and is still complete.

Elementary rules Let us *eliminate* the weights in the rules of $\mathcal{B}_1$: we define a new set of assertions, $\mathcal{A}_2$ by

$$\mathcal{A}_2 = \bigcup_{\lambda \in \mathbb{N} - \{0\}} \mathsf{DRB}_{1,\lambda} < V > \times \mathsf{DRB}_{1,\lambda} < V > .$$

We define a binary relation $||\!\!-\!\!- \subseteq \mathcal{P}_f(\mathcal{A}_2) \times \mathcal{A}_2$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

(R21)

$$\{(S,T)\} \;||\!\!-\!\!-\; (T,S)$$

for $\lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >$,

(R22)

$$\{(S,S'),(S',S'')\} \;||\!\!-\!\!-\; (S,S'')$$

for $\lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >$,

(R23)

$$\emptyset \;||\!\!-\!\!-\; (S,S)$$

for $S \in \mathsf{DRB}_{1,\lambda} < V >$,

(R'23)

$$\emptyset \;||\!\!-\!\!-\; (S,T)$$

for $S \in \mathsf{DRB}_{1,1} < V >, T \in \{\emptyset, \epsilon\}, S \sim T$,

(R24)

$$\{(S \odot x, T \odot x') \mid (x,x') \in \mathcal{R}_1\} \;||\!\!-\!\!-\; (S,T)$$

for $\lambda \in \mathbb{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda} < V >, (S \not\equiv \epsilon \wedge T \not\equiv \epsilon)$ and $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$,

(R25)

$$\{(S_1 \cdot T + S, T)\} \mid\!\!\vdash\!\!- (S_1^* \cdot S, T)$$

for $\lambda \in \mathbb{N} - \{0\}, S_1 \in \mathsf{DRB}_{1,1} < V >, S_1 \not\equiv \epsilon, (S_1, S) \in \mathsf{DRB}_{1,\lambda+1} < V >$ $, T \in \mathsf{DRB}_{1,\lambda} < V >$,

(R26)

$$\{(S, S'), (T, T')\} \mid\!\!\vdash\!\!- ((S, T), (S', T'))$$

for $\lambda, \mu \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda} < V >, T, T' \in \mathsf{DRB}_{1,\mu} < V >, (S, T), (S', T') \in \mathsf{DRB}_{1,\lambda+\mu} < V >$, where ($\lambda = 1$ and $S \in \{\emptyset, \epsilon\}$) or ($\mu = 1$ and $T \in \{\emptyset, \epsilon\}$),

(R27)

$$\{(S, S')\} \mid\!\!\vdash\!\!- (S \cdot T, S' \cdot T)$$

for $\delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\delta} < V >, T \in \mathsf{DRB}_{\delta,\lambda} < V >$,

(R28)

$$\{(T_{i,*}, T'_{i,*}) \mid 1 \leq i \leq \delta\} \mid\!\!\vdash\!\!- (S \cdot T, S \cdot T')$$

for $\delta, \lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\delta} < V >, T, T' \in \mathsf{DRB}_{\delta,\lambda} < V >$,

(R29)

$$\{(U_1, \epsilon)\} \mid\!\!\vdash\!\!- (U_1^* \cdot T, T)$$

for $\lambda \in \mathbb{N} - \{0\}, U_1 \in \mathsf{DRB}_{1,1} < V >, (U_1, T) \in \mathsf{DRB}_{1,\lambda+1} < V >$.

We define $\vdash\!\!-_{\mathcal{B}_2}$ by : for every $P \in \mathcal{P}_f(\mathcal{A}_2), A \in \mathcal{A}_2$,

$$P \vdash\!\!- A \iff P \overset{<*>}{\mid\!\!\vdash\!\!-} \circ \overset{[1]}{\mid\!\!\vdash\!\!-}_{23,24} \circ \overset{<*>}{\mid\!\!\vdash\!\!-} \{A\}.$$

where $\mid\!\!\vdash\!\!-_{23,24}$ is the relation defined by $R23, R'23, R24$ only.
We define a simpler cost function $H_2 : \mathcal{A}_2 \to \mathbb{N} \cup \{\infty\}$ by :

$$\forall (S, S') \in \mathcal{A}_2, H_2(S, S') = \mathrm{Div}(S, S').$$

We let

$$\mathcal{B}_2 =< \mathcal{A}_2, H_2, \vdash\!\!-_{\mathcal{B}_2} > .$$

Lemma 10.8 *: $\mathcal{B}_2$ is a deduction system.*

Completeness
Let us denote by $\mathcal{C}_2$ the subset of rules of $\mathcal{B}_2$ obtained by removing the weights in the rules of $\mathcal{C}$.

Definition 10.9 *Let $P \in \mathcal{P}_f(\mathcal{A}_2)$. P is said to be* self-generating *iff, for every $(S, S') \in P$,*

1. *either $S \equiv S' \equiv \epsilon$*
2. *or $\exists \mathcal{R}_1 \in \bar{\mathcal{B}}_1(S, S'), \forall (x, x') \in \mathcal{R}_1, P \overset{<*>}{\mid\!\!\vdash\!\!-}_{\mathcal{C}_2} (S \odot x, S' \odot x')$.*

Remark 10.10 *This notion of "self-generating set (of pairs)" is a straightforward adaptation to our d-space of vectors of the notion of "self-proving set of pairs" defined in [Cou83b, p.162] for the magma $M(F \cup \Phi, V)$.*

Lemma 10.11 *Let $A \in \mathcal{A}_2$. Then $H(A) = \infty$ iff there exists a finite self-generating set $P \subseteq \mathcal{A}_2$ such that $A \in P$.*

Proof: Owing to metarules $R'23, R24$ it is clear that every self-generating set $P \in \mathcal{P}_f(\mathcal{A}_2)$ is a $\mathcal{B}_2$-proof.
Let us suppose that $A = (U^-, U^+), H_2(A) = \infty$. Let $A_0 = (\rho_e(U^-), \rho_e(U^+))$. Let us consider the closed proof-tree t_∞ obtained by applying the global strategy $\hat{S}_{ABC}$ on the unmarked assertion $(0, A_0)$. By lemma 9.1 t_∞ is finite and by lemma 10.5, t_∞ is consistent, which means that $\mathrm{im}(t_\infty)$ is ∞-consistent. Moreover, by corollary 4.12,

$$\emptyset \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}} \{(0, U^-, \rho_e(U^-)), (0, U^+, \rho_e(U^+))\},$$

hence

$$(0, A) \in \mathrm{Cong}(\mathrm{im}(t_\infty)).$$

It follows that $\mathrm{im}(t_\infty) \cup \{(0, A)\}$ is locally ∞-consistent too, hence it is ∞-consistent (by lemma 10.3). Let us define

$$P = pr_{2,3}(\mathrm{im}(t_\infty)) \cup \{A\},$$

(where $pr_{2,3} : \mathcal{A} \to \mathcal{A}_2$ is the map erasing the weights).
As $\mathrm{im}(t_\infty) \cup \{(0, A)\}$ is ∞-consistent, P is self-generating. □

Theorem 10.12 *: $\mathcal{B}_2$ is a complete deduction system.*

Proof: It suffices to notice that every self-generating set is a $\mathcal{B}_2$-proof. Hence lemma 10.11 proves the completeness of $\mathcal{B}_2$. □

10.3 System $\mathcal{B}_3$

We exhibit here a deduction system $\mathcal{B}_3$ which is even simpler than $\mathcal{B}_2$ and is still complete. Let us consider

$$\mathcal{B}_3 = < \mathcal{A}_3, H_3, \vdash\!\!-_{\mathcal{B}_3} >,$$

where $\mathcal{A}_3 = \mathcal{A}_2, H_3 = H_2$ and $\vdash\!\!-_{\mathcal{B}_3}$ is defined below: we define the elementary deduction relation $|\!\!\vdash\!\!-_{\mathcal{B}_3}$ as the set of all the instances of the metarules $R21, R22, R23, R24, R25, R26, R27, R28, R29$ of $\mathcal{B}_2$ union all the instances of the new meta-rule

$$(R''23) \quad \emptyset \,|\!\!\vdash\!\!- ([qzr], \epsilon)$$

for $q, r \in Q, z \in Z, [qzr] \sim \epsilon$.
We then define $\models\!\!-_{\mathcal{B}_3}$ by : for every $P \in \mathcal{P}_f(\mathcal{A}_3), A \in \mathcal{A}_3$,

$$P \models\!\!-_{\mathcal{B}_3} A \Longleftrightarrow P \overset{<*>}{|\!\!\models\!\!-}_{\mathcal{B}_3} \circ \overset{[1]}{|\!\!\models\!\!-}_{23,24} \circ \overset{<*>}{|\!\!\models\!\!-}_{\mathcal{B}_3} \{A\}.$$

where $|\!\!\models\!\!-_{23,24}$ is now the relation defined by $R23, R''23, R24$ only.

As $\models\!\!-_{\mathcal{B}_3} \subseteq \models\!\!-_{\mathcal{B}_2}$, $H_3 = H_2$ and the new rule $(R''23)$ is recursively enumerable, it is clear that $\mathcal{B}_3$ is a deduction system.

Strategies
Let us define strategies for the system $\mathcal{B}_3$.
We shall define new auxiliary strategies $T^0_\emptyset, T_{A,\emptyset}$ and then derive some "compound" strategies from them.
Let us denote by $\mathcal{A}_\emptyset$ the set: $\mathcal{A}_\emptyset = \mathsf{DRB} < V > \times \{\emptyset\}$.

$T^0_\emptyset$:
$B_1 \cdots B_m \in T^0_\emptyset(A_1 A_2 \cdots A_n)$ iff

$$A_n = (\emptyset, \emptyset) \text{ and } m = 0$$

$T_{A,\emptyset}$:
$B_1 \cdots B_m \in T_{A,\emptyset}(A_1 \cdots A_n)$ iff

(e1) $A_n = ([pzq], \emptyset)$ for some $p, q \in Q, z \in Z$
(e2) $(B_1, B_2, \cdots, B_m)$ is an element of $\mathcal{A}_\emptyset{}^*$ fulfilling that: $\forall j \in [1, m], B_j = ([p_j, z_j, q_j], \emptyset), H_3(B_j) = \infty$ and
(e3)$\forall x \in X$, for every word $w \in \mathrm{supp}([pzq] \odot x), \exists j \in [1, m], [p_j z_j q_j]$ is a factor of w.

Lemma 10.13 *: $T^0_\emptyset, T_{A,\emptyset}$ are $\mathcal{B}_3$-strategies.*

Proof:
$T^0_\emptyset : \emptyset \models\!\!- (\emptyset, \emptyset)$ which proves (S1) and $\min\{H_3(B_j) \mid 1 \le j \le m\} = \min\{H_3(\emptyset, \emptyset)\} = \infty$, which proves (S2) .
$T_{A,\emptyset}$: By $(e3)$, $\forall x \in X$,

$$\{B_j, 1 \le j \le m\} \overset{<*>}{|\!\!\models\!\!-}_{\mathcal{B}_3} ([pzq] \odot x, \emptyset).$$

Using $R24$ we obtain that

$$\{B_j, 1 \le j \le m\} \models\!\!-_{\mathcal{B}_3} ([pzq], \emptyset),$$

hence (S1) is fulfilled.
By $(e2)$,

$$\min\{H(B_j) \mid 1 \le j \le m\} = \infty,$$

which establishes (S2).
□
Let us consider the following strategy $\mathcal{S}_\emptyset$: for every $W \in \mathcal{A}_3^+$,

(1) if $T^0_\emptyset(W) \neq \emptyset$ then, $\mathcal{S}_\emptyset(W) = T^0_\emptyset(W)$
(2) else $\mathcal{S}_\emptyset(W) = T_{A,\emptyset}(W)$

Completeness
In order to show the completeness of system $\mathcal{B}_3$ it remains to show that every rule in $R'23$ is provable in $\mathcal{B}_3$.

Lemma 10.14 *: Let $p, q \in Q, z \in Z$. $[pzq] \equiv \emptyset$ iff there exists a $\mathcal{B}_3$- proof of $([pzq], \emptyset)$.*

Proof: Let us suppose $[pzq] \equiv \emptyset$. Let $A = ([pzq], \emptyset)$. Let $t \in \mathcal{T}(\mathcal{S}_\emptyset, A)$ be a repetition-free proof-tree (w.r.t. $\mathcal{S}_\emptyset$) with maximum number of nodes. The definitions of $T^0_\emptyset, T_{A,\emptyset}$ show that the labels of t belong to the finite set $V \times \{\emptyset\}$. Hence every branch of t has a length $\leq \mathrm{Card}(V \times \{\emptyset\})$, showing that t is a finite tree.

Let us consider the label of a leaf x of t : $([p'z'q'], \emptyset)$. Let $W_x = A_1 A_2 \cdots A_n$ be the word associated to the branch ending at x. Suppose that $H_3(A) = \infty$. Then, by lemma 4.15, $H_3(A_n) = \infty$. It follows that

$$\forall x \in X, [p'z'q'] \odot x \neq \varepsilon$$

hence

$$\forall x \in X, \forall w \in \mathrm{supp}([p'z'q'] \odot x), w \text{ must contain at least one letter } [p"z"q"] \equiv \emptyset. \tag{229}$$

By (229) there exists some $B_1, B_2, \cdots, B_m$ fulfilling conditions $(e2, e3)$. Hence $T_{A,\emptyset}(W_x) \neq \emptyset$. This proves that, every leaf x of t is such that $\mathcal{S}_\emptyset(W_x) \neq \emptyset$. As t is supposed to have maximum possible cardinality, this implies that , for every leaf x of t, either $T^0_\emptyset(W_x) = \epsilon$ or $\exists y \prec x$ such that $t(y) = t(x)$. This shows that t is closed. By lemma 4.15, P is a $\mathcal{B}_3$-proof, containing A. □
Let us consider the unique substitution $\tau_{\emptyset,\epsilon} : \mathsf{B} < V > \rightarrow \mathsf{B} < V >$ such that: for every $p, q \in Q, z \in Z$,

$$\tau_{\emptyset,\epsilon}([pzq]) = \emptyset(\text{ if } [pzq] \equiv \emptyset), \;\; \tau_{\emptyset,\epsilon}([pzq]) = \epsilon(\text{ if } [pzq] \equiv \epsilon), \;\; \tau_{\emptyset,\epsilon}([pzq]) = [pzq](\text{ otherwise }).$$

Lemma 10.15 *$\tau_{\emptyset,\epsilon}$ is a deterministic rational substitution.*

Proof: Let $p \in Q, z \in Z$. If pz is an ϵ-free mode, then

$$\tau_{\emptyset,\epsilon}([pz]) = [pz].$$

If pz is an ϵ-bound mode, then

$$\tau_{\emptyset,\epsilon}([pz]) = \epsilon_j^\lambda,$$

where $\lambda = |Q|, \delta(pz, \epsilon) = q_j$. Hence, in every case, $\tau_{\emptyset,\epsilon}([pz])$ is a deterministic rational row-vector. □

Lemma 10.16 *Let $S \in \mathsf{B} < V >$.*

1. $S \equiv \emptyset \Leftrightarrow \tau_{\emptyset,\epsilon}(S) = \emptyset$.
2. $S \equiv \epsilon \Leftrightarrow \tau_{\emptyset,\epsilon}(S) = \epsilon$.

Proof: As for every $v \in V, \tau_{\emptyset,\epsilon}(v) \equiv v$, $\tau_{\emptyset,\epsilon}$ is a substitution and $\equiv$ is the kernel of a substitution, we have

$$\forall S \in \mathsf{B} < V >, S \equiv \tau_{\emptyset,\epsilon}(S). \tag{230}$$

Let us prove point (2). Let us suppose that $S \equiv \epsilon$. By (230), $\tau_{\emptyset,\epsilon}(S) \equiv \epsilon$. Let $w \in \mathrm{supp}(S)$ and let $w' = \tau_{\emptyset,\epsilon}(w)$. As $\varphi(\tau_{\emptyset,\epsilon}(S)) = \epsilon$ we must have $\varphi(w') \in \{\emptyset, \epsilon\}$. But the definition of $\tau_{\emptyset,\epsilon}$ implies that w' is written on the sub-alphabet $\bar{V}_{\emptyset,\epsilon} = \{v \in V \mid \varphi(v) \notin \{\emptyset, \epsilon\}\}$. Hence $w' \in \{\emptyset, \epsilon\}$. It follows that $\tau_{\emptyset,\epsilon}(S) = \epsilon$.
Point (1) can be proved by a similar argument. □

Lemma 10.17 *Let $S \in \mathsf{DRB} < V >$.*

1. *$S \equiv \emptyset$ if and only if there exists some $\mathcal{B}_3$-proof of $(S, \emptyset)$.*
2. *$S \equiv \epsilon$ if and only if there exists some $\mathcal{B}_3$-proof of (S, ϵ).*

Proof: Only the implication from left to right needs to be proved. Let $S \in \mathsf{DRB} < V >$, such that $S \equiv \emptyset$. By lemma 10.16, point (1),

$$\tau_{\emptyset,\epsilon}(S) = \emptyset. \tag{231}$$

As $\tau_{\emptyset,\epsilon}$ is a deterministic rational substitution, by lemma 4.11

$$\{(0, [pz], \tau_{\emptyset,\epsilon}([pz]) \mid p \in Q\} \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_0} (0, S, \tau_{\emptyset,\epsilon}(S)).$$

Let us denote by $\mathcal{C}_{03}$ the set of rules obtained from $\mathcal{C}_0$ by erasing the weights (i.e. the first component). One can check that, as the definition of $\mathcal{C}_0$ was not using metarules $R3, R'3$, $\mathcal{C}_{03}$ is a subset of the rules of $\mathcal{B}_3$. From the above equation we get:

$$\{([pz], \tau_{\emptyset,\epsilon}([pz])) \mid p \in Q\} \overset{<*>}{|\!\!\vdash\!\!-}_{\mathcal{C}_{03}} (S, \tau_{\emptyset,\epsilon}(S)). \tag{232}$$

By rules $R23, R''23, R26$ and lemma 10.14, every assertion $([pz], \tau_{\emptyset,\epsilon}([pz]))$ has a $\mathcal{B}_3$-proof $P_{[pz]}$. Using (231),(232), we obtain that

$$P = \bigcup_{p \in Q, z \in Z} P_{[pz]} \cup \{(S, \emptyset)\}$$

is a $\mathcal{B}_3$-proof.
The case where $S \equiv \epsilon$ can be treated in the same way. □

Theorem 10.18 *: $\mathcal{B}_3$ is a complete deduction system.*

Proof: By theorem 10.12, $\mathcal{B}_2$ is complete. $\mathcal{B}_3$ is a subsystem of $\mathcal{B}_2$ such that the only rules of $\mathcal{B}_2$ which do not belong to $\mathcal{B}_3$ are instances of $R'23$. But, by lemma 10.17, every instance of $R'23$ is provable in $\mathcal{B}_3$. Hence $\mathcal{B}_3$ is complete too. □

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