# Complete Formal Systems for Equivalence Problems

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**Abstract**

We describe four complete and recursively enumerable formal systems $\mathcal{S}_0, \mathcal{D}_0, \mathcal{H}_0, \mathcal{B}_0$. Each one of them proves the decidability of some *equivalence problem* for some class of automata: namely the *language equivalence* problem for simple automata, the *language equivalence* problem for deterministic pushdown automata, the *function equivalence* problem for deterministic pushdown transducers with outputs in an abelian group, the *bisimulation equivalence* problem for loop-free pushdown automata.

**Keywords**: complete formal systems; equivalence problems; deterministic pushdown automata; deterministic pushdown transducers; bisimulation.

## 1 Introduction

The equivalence problem for deterministic pushdown automata is the following decision problem:

INSTANCE: two dpda $A$, $B$.  QUESTION: $\mathrm{L}(A) = \mathrm{L}(B)$?

where $\mathrm{L}(A)$ (resp. $\mathrm{L}(B)$) is the language recognized by $A$ (resp. $B$). The decidability of this decision problem has been established in [Sén97b, Sén97d, Sén97c] by the following method:
the equivalence of the automata $A, B$ is reduced to the study of an equivalence relation $\equiv$ over some set $E$ associated with $A, B$. Roughly speaking:

- $E$ is a set which contains (up to some computable bijection) the set of configurations of the given automata $A, B$; in particular it contains an element $e_A$ (resp. $e_B$) corresponding to the initial configuration of $A$ (resp. $B$).
- there is a map $\varphi$ associating with every $e \in E$ a language $\varphi(e)$; in particular $\varphi(e_A) = \mathrm{L}(A)$, $\varphi(e_B) = \mathrm{L}(B)$.
- for every $e, e' \in E$, $e \equiv e'$ means that $\varphi(e) = \varphi(e')$.

It is then clear that $\mathrm{L}(A) = \mathrm{L}(B)$ if and only if $e_A \equiv e_B$. The fact that $\equiv$ has a recursively enumerable complement is an easy remark and the remaining difficulty is to show that $\equiv$ is recursively enumerable. This is shown by exhibiting



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a *recursively enumerable* formal system $\mathcal{D}_0$ which is *complete* in the sense that: for every $(e, e') \in E \times E$, $e \equiv e'$ if and only if there exists some finite proof (w.r.t. $\mathcal{D}_0$) of the fact that $e \equiv e'$.
This way of solving an equivalence problem goes back to [Cou83] which was a systematization of the analysis and improvements ([Cou74, HHY79]) of [HK66]. Therefore we restate, in the precise framework of [Sén97b], the formal system $\mathcal{S}_0$ establishing the decidability of the equivalence problem for simple deterministic pushdown automata.
In the opposite direction, we show how $\mathcal{D}_0$ can be extended to other formal systems $\mathcal{H}_0, \mathcal{B}_0$. The system $\mathcal{H}_0$ allows to decide the equivalence problem for deterministic pushdown automata with *multiplicities* in an abelian group, equivalently, $\mathcal{H}_0$ allows to decide the function equivalence problem for deterministic pushdown *transducers* with outputs in an abelian group ([Sén98b], see [Sén98c] for a short exposition). The system $\mathcal{B}_0$ allows to decide the *bisimulation equivalence* problem for loop-free (non-deterministic) pushdown automata ([Sén97a], see [Sén98a] for a short exposition).

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# 2 General formal systems

We define a general notion of formal system which follows the general philosophy of [HHY79, Cou83].
Let us call *formal system* any triple $\mathcal{D} =< \mathcal{A}, H, \vdash\!\!- >$ where $\mathcal{A}$ is a denumerable set called the *set of assertions*, $H$, the *cost function* is a mapping $\mathcal{A} \rightarrow \mathsf{N} \cup \{\infty\}$ and $\vdash\!\!-$ , the *deduction relation* is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$ ; $\mathcal{A}$ is given with a fixed bijection with $\mathsf{N}$ (an “encoding” or “Gödel numbering”) so that the notions of recursive subset, recursively enumerable subset, recursive function, ... over $\mathcal{A}, \mathcal{P}_f(\mathcal{A})$, ... are defined, up to this fixed bijection ; we assume that $\mathcal{D}$ satisfies the following axiom:
(**A 1**) $\forall (P, A) \in \vdash\!\!-$ , (min $\{H(p), p \in P\} < H(A)$) or ($H(A) = \infty$).
(We let $\min(\emptyset) = \infty$).
We call $\mathcal{D}$ a *deduction system* iff $\mathcal{D}$ is a formal system satisfying the additional axiom:
(**A 2**) $\vdash\!\!-$ is recursively enumerable.

In the sequel we use the notation $P \vdash\!\!- A$ for $(P, A) \in \vdash\!\!-$ . We call *proof* in the system $\mathcal{D}$, *relative to the set of hypotheses* $\mathcal{H} \subseteq \mathcal{A}$, any subset $P \subseteq \mathcal{A}$ fulfilling :

$$\forall p \in P, (\exists Q \subseteq P, Q \vdash\!\!- p) \text{ or } (p \in \mathcal{H}).$$

We call $P$ a *proof* iff

$$\forall p \in P, (\exists Q \subseteq P, Q \vdash\!\!- p)$$

(i.e. iff $P$ is a proof relative to $\emptyset$).
Let us define the total map $\chi : \mathcal{A} \rightarrow \{0, 1\}$ and the partial map $\overline{\chi} : \mathcal{A} \rightarrow \{0, 1\}$ by :

$$\chi(A) = 1 \text{ if } H(A) = \infty, \chi(A) = 0 \text{ if } H(A) < \infty,$$

$$\overline{\chi}(A) = 1 \text{ if } H(A) = \infty, \overline{\chi} \text{ is undefined if } H(A) < \infty.$$

($\chi$ is the "truth-value function", $\overline{\chi}$ is the "1-value function").

**Lemma 2.1** *Let $P$ be a proof relative to $\mathcal{H} \subseteq H^{-1}(\infty)$ and $A \in P$. Then $\chi(A) = 1$.*

In other words : if an assertion is provable from true hypotheses, then it is true.

**Proof**: Let $P$ be a proof. We prove by induction on $n$ that,

$$\mathcal{P}(n) : \forall p \in P, H(p) \geq n.$$

It is clear that, $\forall p \in P, H(p) \geq 0$. Suppose that $\mathcal{P}(n)$ is true. Let $p \in P - \mathcal{H} : \exists Q \subseteq P, Q \vdash\!\!- p$. By induction hypothesis, $\forall q \in Q, H(q) \geq n$ and by (A1),$H(p) \geq n+1$. It follows that : $\forall p \in P - \mathcal{H}, H(p) = \infty$. But by hypothesis, $\forall p \in \mathcal{H}, H(p) = \infty$. □

A formal system $\mathcal{D}$ will be said *complete* iff, conversely, $\forall A \in \mathcal{A}, \chi(A) = 1 \Longrightarrow$ there exists some *finite* proof $P$ such that $A \in P$. (In other words, $\mathcal{D}$ is complete iff every true assertion is "finitely" provable).

**Lemma 2.2** : *If $\mathcal{D}$ is a complete deduction system, $\overline{\chi}$ is a recursive partial map.*

**Sketch of proof**: This can be easily deduced from the fact that, as the relation $\vdash\!\!-$ is recursively enumerable, the set of finite $\mathcal{D}$-proofs is recursively enumerable too. A semi-algorithm computing $\overline{\chi}(A)$ simply consists in enumerating all finite $\mathcal{D}$-proofs until it reaches a finite proof containing the assertion $A$. □

In order to define deduction relations from more elementary ones, we set the following definitions.

Let $\vdash\!\!- \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$. For every $P, Q \in \mathcal{P}_f(\mathcal{A}), n \geq 0$, we set :

$$P \overset{[0]}{\vdash\!\!-} Q \text{ iff } P \supseteq Q; \quad P \overset{[1]}{\vdash\!\!-} Q \text{ iff } \forall q \in Q, \exists R \subseteq P, R \vdash\!\!- q$$

$$P \overset{<0>}{\vdash\!\!-} Q \text{ iff } P \overset{[0]}{\vdash\!\!-} Q; \quad P \overset{<1>}{\vdash\!\!-} Q \text{ iff } \forall q \in Q, (\exists R \subseteq P, R \vdash\!\!- q) \text{ or } (q \in P)$$

$$P \overset{<n+1>}{\vdash\!\!-} Q \text{ iff } \exists R \in \mathcal{P}_f(A), P \overset{<1>}{\vdash\!\!-} R \text{ and } R \overset{<n>}{\vdash\!\!-} Q.$$

$$\overset{<*>}{\vdash\!\!-} = \bigcup_{n \geq 0} \overset{<n>}{\vdash\!\!-}.$$

Given $\vdash\!\!-_1, \vdash\!\!-_2 \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(\mathcal{A})$, for every $P, Q \in \mathcal{P}_f(\mathcal{A})$ we set :

$$P(\vdash\!\!-_1 \circ \vdash\!\!-_2)Q \text{ iff } \exists R \subseteq \mathcal{A}, (P \vdash\!\!-_1 R) \wedge (R \vdash\!\!-_2 Q).$$

The particular deduction systems $\mathcal{S}_0, \mathcal{D}_0, \mathcal{H}_0, \mathcal{B}_0$, that we shall introduce later will always be defined as $< \mathcal{A}, H, \vdash\!\!- >$ where $\vdash\!\!-$ is defined from a simpler binary relation $|\!\vdash\!\!-$ by means of the above constructions. Therefore, relation $|\!\vdash\!\!-$ will be named the *elementary* deduction relation.

# 3 Simple automata

## 3.1 Simple automata and grammars

### 3.1.1 Definitions

A *simple* deterministic pushdown automaton is a deterministic pushdown automaton (see section 4.1) which fulfills the following strong restriction:

$$\mathrm{Card}(Q) = 1 \text{ and } \forall q \in Q, \forall z \in Z, \mathrm{Card}(\delta(qz, \epsilon)) = 0.$$

The class of simple deterministic context-free grammars is defined in such a way that , for every alphabet $X$ and language $L \subseteq X^*$, $L$ is recognized by some simple d.p.d.a. iff $L$ is generated by some simple deterministic c.f. grammar. A *simple* deterministic context-free grammar is a 3-tuple

$$G =< X, V, P >$$

where $X$ is the terminal alphabet, $V$ is the variable alphabet and $P$ is a finite subset of $V \times (X \cdot V^*)$ fulfilling the restriction: for every $v \in V, x \in X, m, m' \in V^*$,

$$((v, x \cdot m) \in P \text{ and } (v, x \cdot m') \in P) \Rightarrow m = m'.$$

### 3.1.2 Monoid $V^{(*)}$

Let us denote by $(V^{(*)}, \cdot, \epsilon)$ the submonoid of $(\mathsf{B}\langle\langle V \rangle\rangle, \cdot, \epsilon)$ (see subsection 4.2) consisting of all the monomials $m \in V^*$ and the null series $\emptyset$. $(V^{(*)}, \cdot, \epsilon)$ can be seen as the monoid obtained by "adjoining a zero" to the free monoid $(V^*, \cdot, \epsilon)$.

### 3.1.3 Actions of monoids

Given a set $\mathsf{S}$ and a monoid $(\mathsf{M}, \cdot, 1_M)$, a map $\circ : \mathsf{S} \times \mathsf{M} \to \mathsf{S}$ is called a *right-action* of the monoid $\mathsf{M}$ over the set $\mathsf{S}$ iff, for every $S, T \in \mathsf{S}, m, m' \in \mathsf{M}$:

$$S \circ 1_M = S, \quad S \circ (m \cdot m') = (S \circ m) \circ m'. \tag{1}$$

### 3.1.4 The action of $X^*$ on $V^{(*)}$

We define $\odot$ as the unique right-action of the monoid $X^*$ over the set $V^{(*)}$ such that: $\forall v \in V, \forall \beta \in V^*, \forall x \in X$,

$$(v \cdot \beta) \odot x = m \cdot \beta \text{ iff } (v, x \cdot m) \in P, \tag{2}$$

$$(v \cdot \beta) \odot x = \emptyset \text{ iff } (\{v\} \times x \cdot V^*) \cap P = \emptyset, \tag{3}$$

$$\epsilon \odot x = \emptyset, \quad \emptyset \odot x = \emptyset. \tag{4}$$

Let us consider the unique monoid-homomorphism $\varphi : V^{(*)} \longrightarrow \mathsf{B}\langle\langle X \rangle\rangle$ such that, for every $v \in V$,

$$\varphi(v) = \{u \in X^* \mid v \odot u = \epsilon\},$$

(in other words, $\varphi$ maps every word $m \in V^*$ on the language generated by the grammar $G$ from the axiom $m$). Let us denote by $\bullet$ the action of $X^*$ over $\mathsf{B}\langle\langle V \rangle\rangle$ by "left-quotient" (see section 4.2).

**Lemma 3.1** *For every $S \in V^{(*)}, u \in X^*$, $\varphi(S \odot u) = \varphi(S) \bullet u$ ( i.e. $\varphi$ is a morphism of right-actions).*

We denote by $\equiv$ the kernel of $\varphi$ i.e.: for every $S, T \in V^{(*)}$,

$$S \equiv T \Leftrightarrow \varphi(S) = \varphi(T).$$

### 3.2 Formal system $\mathcal{S}_0$

We define here a deduction system $\mathcal{S}_0$ which solves the equivalence problem for simple d.c.f. grammars.

Let us fix some simple d.c.f. grammar $< X, V, P >$. The set of assertions is defined by :

$$\mathcal{A} = V^{(*)} \times V^{(*)}.$$

The "cost-function" $H : \mathcal{A} \to \mathsf{N} \cup \{\infty\}$ is defined by :

$$H(S, S') = \mathrm{Div}(S, S'),$$

where $\mathrm{Div}(S, S')$, the *divergence* between $S$ and $S'$, is defined by :

$$\mathrm{Div}(S, S') = \inf\{\mid u \mid \mid u \in \varphi(S) \triangle \varphi(S')\}.$$

Let us notice that here :

$$\chi(S, S') = 1 \Longleftrightarrow S \equiv S'.$$

We define a binary relation $||\!\!-\!\!-\ \subset \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

**(S1)**

$$\{(S, T)\} \,||\!\!-\!\!-\, (T, S)$$

for $S, T \in V^{(*)}$,

**(S2)**

$$\{(S, S'), (S', S'')\} \,||\!\!-\!\!-\, (S, S'')$$

for $S, S', S'' \in V^{(*)}$,

**(S3)**

$$\emptyset \mid\!\vdash\!\!- (S,S)$$

for $S \in V^{(*)}$,

**(S4)**

$$\{(S \odot x, T \odot x) \mid x \in X\} \mid\!\vdash\!\!- (S,T)$$

for $S,T \in V^{(*)}, S \neq \epsilon, T \neq \epsilon$,

**(S5)**

$$\{(S,S')\} \mid\!\vdash\!\!- (S \cdot T, S' \cdot T)$$

for $S,S',T \in V^{(*)}$,

**(S6)**

$$\{(T,T')\} \mid\!\vdash\!\!- (S \cdot T, S \cdot T')$$

for $S,T,T' \in V^{(*)}$.

We define $\vdash\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$,

$$P \vdash\!\!- A \Longleftrightarrow P \overset{<*>}{\mid\!\vdash\!\!-} \circ \overset{[1]}{\mid\!\vdash\!\!-}_{3,4} \circ \overset{<*>}{\mid\!\vdash\!\!-} \{A\}.$$

where $\mid\!\vdash\!\!-_{3,4}$ is the relation defined by $S3, S4$ only.

**Theorem 3.2** *$\mathcal{S}_0$ is a complete deduction system.*

**Corollary 3.3** *The equivalence problem for simple deterministic grammars ( or simple deterministic pushdown automata) is decidable.*

Let us recall that this result has been proved first in [HK66] by a more direct algorithm. The system exhibited here is a reformulation of the proof of [Cou74]. A similar treatment of the more general case of "stateless deterministic pushdown automata" has been given in [Cau93]. At end, the equivalence problem for simple dpda's has been shown solvable in *polynomial time* by [HJM96].

# 4 Deterministic pushdown automata

## 4.1 Pushdown automata

A *pushdown automaton* on the alphabet $X$ is a 6-tuple $\mathcal{M} = < X, Z, Q, \delta, q_0, z_0 >$ where $Z$ is the finite stack-alphabet, $Q$ is the finite set of states, $q_0 \in Q$ is the initial state, $z_0$ is the initial stack-symbol and $\delta : QZ \times (X \cup \{\epsilon\}) \rightarrow \mathcal{P}_f(QZ^*)$, is the transition mapping.

Let $q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, f \in X^*$ and $a \in X \cup \{\epsilon\}$ ; we note $(qz\omega, af) \longmapsto_{\mathcal{M}} (q'\omega'\omega, f)$ if $q'\omega' \in \delta(qz, a)$. $\overset{*}{\longmapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{M}}$ . For every $q\omega, q'\omega' \in QZ^*$ and $f \in X^*$, we note $q\omega \xrightarrow{f}_{\mathcal{M}} q'\omega'$ iff $(q\omega, f) \overset{*}{\longmapsto}_{\mathcal{M}} (q'\omega', \epsilon)$. $\mathcal{M}$ is said *deterministic* iff, for every $z \in Z, q \in Q$:

$$\text{either } \mathrm{Card}(\delta(qz,\epsilon)) = 1 \text{ and for every } x \in X, \mathrm{Card}(\delta(qz,x)) = 0, \tag{5}$$

$$\text{or } \mathrm{Card}(\delta(qz,\epsilon)) = 0 \text{ and for every } x \in X, \mathrm{Card}(\delta(qz,x)) \leq 1. \tag{6}$$

$\mathcal{M}$ is said *real-time* iff, for every $qz \in QZ$, $\mathrm{Card}(\delta(qz,\epsilon)) = 0$. A pda $\mathcal{M}$ is said *normalized* iff, for every $qz \in QZ, x \in X$:

$$q'\omega' \in \delta(qz,x) \Rightarrow \mid \omega' \mid \leq 2, \text{ and } q'\omega' \in \delta(qz,\epsilon) \Rightarrow \mid \omega' \mid = 0. \tag{7}$$

Given some finite set $F \subseteq QZ^*$ of configurations, the *language recognized by $\mathcal{M}$ with final configurations $F$* is defined by

$$\mathrm{L}(\mathcal{M}, F) = \{w \in X^* \mid \exists c \in F, q_0 z_0 \xrightarrow{w}_{\mathcal{M}} c\}.$$

## 4.2 Deterministic context-free grammars

Let $\mathcal{M}$ be some deterministic pushdown automaton ( we suppose here that $\mathcal{M}$ is normalized). The *variable* alphabet $V_{\mathcal{M}}$ associated to $\mathcal{M}$ is defined as:

$$V_{\mathcal{M}} = \{[p,z,q] \mid p,q \in Q, z \in Z\}.$$

The *context-free* grammar $G_{\mathcal{M}}$ associated to $\mathcal{M}$ is then

$$G_{\mathcal{M}} = < X, V, P >$$

where
$V = V_{\mathcal{M}}$,
$P$ is the set of all the pairs of one of the following forms:

$$([p,z,q], x[p',z_1,p''][p'',z_2,q]) \tag{8}$$

where $p,q,p',p'' \in Q, x \in X, p'z_1z_2 \in \delta(pz,x)$

$$([p,z,q], x[p',z',q]) \tag{9}$$

where $p,q,p' \in Q, x \in X, p'z' \in \delta(pz,x)$

$$([p,z,q], a) \tag{10}$$

where $p,q, \in Q, a \in X \cup \{\epsilon\}, q \in \delta(pz,a)$. $G_{\mathcal{M}}$ is a *strict-deterministic* grammar. (A general theory of this class of grammars is exposed in [Har78] and used in [HHY79]).

We call *mode* every element of $QZ \cup \{\epsilon\}$. For every $q \in Q, z \in Z$, $qz$ is said *$\epsilon$-bound* (respectively $\epsilon$-free) iff condition (5) (resp. condition (6)) in the above definition of deterministic automata is realized. The mode $\epsilon$ is said $\epsilon$-free. We define a mapping $\mu : V^* \to QZ \cup \{\epsilon\}$ by

$$\mu(\epsilon) = \epsilon \text{ and } \mu([p, z, q] \cdot \beta) = pz,$$

for every $p, q \in Q, z \in Z, \beta \in V^*$. For every $w \in V^*$ we call $\mu(w)$ the *mode* of the word $w$. The reader is referred to [Har78] or [ABB96] for more information about pushdown automata and grammars.

### 4.2.1 Semi-ring $\mathsf{K}\langle\langle W\rangle\rangle$

Let us consider a semi-ring $(\mathsf{K}, +, \cdot, 0_K, 1_K)$ and an alphabet $W$. By $(\mathsf{K}\langle\langle W\rangle\rangle, +, \cdot, \emptyset, \epsilon)$ we denote the semi-ring of *series* over the set of non-commutative undeterminates $W$, with coefficients in $\mathsf{K}$:
the set $\mathsf{K}\langle\langle W\rangle\rangle$ is defined as $\mathsf{K}^{W^*}$; the sum and product are defined by: $\forall S, T \in \mathsf{K}^{W^*}, w \in W^*$,

$$(S + T)(w) = S(w) + T(w) \ \ ; \ \ (S \cdot T)(w) = \sum_{w_1 \cdot w_2 = w} S(w_1) \cdot T(w_2).$$

Each word $w \in W^*$ can be identified with the element of $\mathsf{K}^{W^*}$ mapping the word $w$ on $1_K$ and every other word $w' \neq w$ on $0_K$; each scallar $k \in \mathsf{K}$ can be identified with the element of $\mathsf{K}^{W^*}$ mapping the word $\epsilon$ on $k$ and every word $w' \neq \epsilon$ on $0_K$. Every series $S \in \mathsf{K}\langle\langle W\rangle\rangle$ can then be written in a unique way as:

$$S = \sum_{w \in W^*} S_w \cdot w,$$

where, for every $w \in W^*$, $S_w \in \mathsf{K}$.

The semi-rings $\mathsf{K}$ considered in the sequel are endowed with a notion of *sum*,

$$\sum_{i \in I} k_i,$$

for every denumerable family $(k_i)_{i \in I}$ of elements of $K$. In such a semi-ring, for every $k \in K$, the star of $k$, denoted $k^*$, is defined by:

$$k^* = \sum_{n \in \mathsf{N}} k^n. \tag{11}$$

Given two alphabets $W, W'$ and a semi-ring $\mathsf{K}$, a map $\psi : \mathsf{K}\langle\langle W\rangle\rangle \to \mathsf{K}\langle\langle W'\rangle\rangle$ is said *$\sigma$-additive* iff it fulfills: for every denumerable family $(S_i)_{i \in I}$ of elements of $\mathsf{K}\langle\langle W\rangle\rangle$,

$$\psi(\sum_{i \in I} S_i) = \sum_{i \in I} \psi(S_i). \tag{12}$$

A map $\psi : \mathsf{K}\langle\langle W\rangle\rangle \rightarrow \mathsf{K}\langle\langle W'\rangle\rangle$ which is a semi-ring homomorphism, a $\sigma$-additive map and which fixes every element of $K$, will be called a *substitution*. The *support* of $S$ is the language

$$\mathrm{supp}(S) = \{w \in W^* \mid S_w \neq 0_K\}.$$

#### 4.2.2 Semi-ring $\mathsf{B}\langle\langle W\rangle\rangle$

Let $(\mathsf{B}, +, \cdot, 0, 1)$, where $\mathsf{B} = \{0, 1\}$, denote the semi-ring of "booleans". In this particular case we sometimes identify a series $S$ with its support. The reader is referred to [BR88] or [Kui96] for more information about formal power series.

#### 4.2.3 Actions of monoids

Given a semi-ring $(\mathsf{S}, +, \cdot, 0, 1)$ and a monoid $(\mathsf{M}, \cdot, 1_M)$, a map $\circ : \mathsf{S} \times \mathsf{M} \rightarrow \mathsf{S}$ is called a *right-action* of the monoid $\mathsf{M}$ over the semi-ring $\mathsf{S}$ iff, for every $S, T \in \mathsf{S}, m, m' \in \mathsf{M}$:

$$0 \circ m = 0,\ \ S \circ 1_M = S,\ \ (S+T) \circ m = (S \circ m) + (T \circ m) \ \ \text{and}\ S \circ (m \cdot m') = (S \circ m) \circ m'. \tag{13}$$

A right-action $\circ$ is said to be a $\sigma$-right-action if it fulfills the additional property that, for every denumerable family $(S_i)_{i \in I}$ of elements of $\mathsf{S}$ and $m \in \mathsf{M}$:

$$(\sum_{i \in I} S_i) \circ m = \sum_{i \in I} (S_i \circ m). \tag{14}$$

#### 4.2.4 The action of $W^*$ on $\mathsf{B}\langle\langle W\rangle\rangle$

We recall the following classical $\sigma$-right-action $\bullet$ of the monoid $W^*$ over the semi-ring $\mathsf{B}\langle\langle W\rangle\rangle$ : for all $S, S' \in \mathsf{B}\langle\langle W\rangle\rangle, u \in W^*$

$$S \bullet u = S' \Leftrightarrow \forall w \in W^*, (S'_w = S_{u \cdot w}),$$

(i.e. $S \bullet u$ is the *left-quotient* of $S$ by $u$ , or the *residual* of $S$ by $u$ ).

For every $S \in \mathsf{B}\langle\langle W\rangle\rangle$ we denote by $\mathsf{Q}(S)$ the set of residuals of $S$:

$$\mathsf{Q}(S) = \{S \bullet u \mid u \in W^*\}.$$

We recall that $S$ is said *rational* iff the set $\mathsf{Q}(S)$ is *finite*.

#### 4.2.5 The action of $X^*$ on $\mathsf{B}\langle\langle V\rangle\rangle$

Let us fix now a deterministic (normalized) pda $\mathcal{M}$ and consider the associated grammar $G$. We define a $\sigma$-right-action $\otimes$ of the monoid $(X \cup \{e\})^*$ over the semi-ring $\mathsf{B}\langle\langle V\rangle\rangle$ by: for every $p, q \in Q, A \in Z, \beta \in V^*, x \in X$

$$[p, A, q] \cdot \beta \otimes x = ((\sum_{([p,A,q],m) \in P_{\mathcal{M}}} m) \bullet x) \cdot \beta, \tag{15}$$

$$[p, A, q] \cdot \beta \otimes e = \beta \text{ iff } ([p, A, q], \epsilon) \in P_{\mathcal{M}}, \tag{16}$$

$$[p, A, q] \cdot \beta \otimes e = \emptyset \text{ iff } , ([p, A, q], \epsilon) \notin P_{\mathcal{M}}, \tag{17}$$

$$\epsilon \otimes x = \emptyset, \quad \epsilon \otimes e = \emptyset. \tag{18}$$

A series $S \in \mathsf{B}\langle\langle V \rangle\rangle$ is said *$\epsilon$-free* iff $\forall w \in V^*, S_w = 1 \Rightarrow \mu(w)$ is $\epsilon -$ free. We define the map $\rho_\epsilon : \mathsf{B}\langle\langle V \rangle\rangle \to \mathsf{B}\langle\langle V \rangle\rangle$ as the unique $\sigma$-additive map such that,

$$\rho_\epsilon(\emptyset) = \emptyset, \quad \rho_\epsilon(\epsilon) = \epsilon,$$

and for every $p \in Q, z \in Z, q \in Q, \beta \in V^*$,

$$\rho_\epsilon([p, z, q] \cdot \beta) = \rho_\epsilon(([p, z, q] \otimes e) \cdot \beta) \text{ if } pz \text{ is } \epsilon - \text{bound and,}$$

$$\rho_\epsilon([p, z, q] \cdot \beta) = [p, z, q] \cdot \beta \text{ if } pz \text{ is } \epsilon - \text{free.}$$

The above definition is sound because, by hypothesis (7), every $[p, z, q] \otimes e$ is either the unit series $\epsilon$ or the empty series $\emptyset$. One can notice that for every $w \in V^*$, $\rho_\epsilon(w) \in V^{(*)}$. We call $\rho_\epsilon$ the $\epsilon$-reduction map. We then define $\odot$ as the unique $\sigma$-right-action of the monoid $X^*$ over the semi-ring $\mathsf{B}\langle\langle V \rangle\rangle$ such that: for every $S \in \mathsf{B}\langle\langle V \rangle\rangle, x \in X$,

$$S \odot x = \rho_\epsilon(\rho_\epsilon(S) \otimes x).$$

Let us consider the unique substitution $\varphi : \mathsf{B}\langle\langle V \rangle\rangle \to \mathsf{B}\langle\langle X \rangle\rangle$ fulfilling: for every $p, q \in Q, z \in Z$,

$$\varphi([p, z, q]) = \{u \in X^* \mid [p, z, q] \odot u = \epsilon\},$$

(in other words, $\varphi$ maps every subset $L \subseteq V^*$ on the language generated by the grammar $G$ from the set of axioms $L$).

**Lemma 4.1** *For every $S \in \mathsf{B}\langle\langle V \rangle\rangle, u \in X^*$,*

1. *$\varphi(S) = \varphi(\rho_\epsilon(S))$,*
2. *$\varphi(S \odot u) = \varphi(S) \bullet u$ ( i.e. $\varphi$ is a morphism of right-actions).*

We denote by $\equiv$ the kernel of $\varphi$ i.e.: for every $S, T \in \mathsf{B}\langle\langle V \rangle\rangle$,

$$S \equiv T \Leftrightarrow \varphi(S) = \varphi(T).$$

## 4.3 Deterministic series

We introduce here a notion of *deterministic* series which, in the case of the alphabet $V$ associated to a dpda $\mathcal{M}$, generalizes the classical notion of *configuration* of $\mathcal{M}$. Let us consider a pair $(W, \smile)$ where $W$ is an alphabet and $\smile$ is an equivalence relation over $W$. We call $(W, \smile)$ a *structured* alphabet. The two examples we have in mind are:

- the case where $W = V$, the variable alphabet associated to $\mathcal{M}$ and $[p, A, q] \smile [p', A', q']$ iff $p = p'$ and $A = A'$ (see [Har78])
- the case where $W = X$, the terminal alphabet of $\mathcal{M}$ and $x \smile y$ holds for every $x, y \in X$ (see [Har78]).

### 4.3.1 Definitions

**Definition 4.2** *Let $S \in \mathsf{B}\langle\langle W\rangle\rangle$. $S$ is said* left-deterministic *iff either*

*(1)* $S = \emptyset$ *or*

*(2)* $S = \epsilon$ *or*

*(3)* $\exists w_0 \in W^*, S_{w_0} \neq 0$ *and* $\forall w, w' \in W^*$,

$$S_w = S_{w'} = 1 \Rightarrow [\exists A, A' \in W, w_1, w_1' \in W^*, A \smile A', w = A{\cdot}w_1 \text{ and } w' = A'{\cdot}w_1'].$$

A left-deterministic series $S$ is said to have the type $\emptyset$ (resp. $\epsilon$, $[A]_\smile$) if case (1) (resp. (2), (3)) occurs.

**Definition 4.3** *Let $S \in \mathsf{B}\langle\langle W\rangle\rangle$. $S$ is said* deterministic *iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.*

This notion is the straighforward extension to the infinite case of the notion of (finite) *set of associates* defined in [HHY79].
We denote by $\mathsf{DB}\langle\langle W\rangle\rangle$ the subset of deterministic boolean series over $W$.

## 4.4 Deterministic vectors

Let us denote by $\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$ the set of $(n, m)$-matrices with entries in the semiring $\mathsf{B}\langle\langle W\rangle\rangle$.

**Definition 4.4** *Let $m \in \mathsf{N}, S \in \mathsf{B}_{1,m}\langle\langle W\rangle\rangle : S = (S_1, \cdots, S_m)$. $S$ is said* left-deterministic *iff either*

*(1)* $\forall i \in [1, m], S_i = \emptyset$ *or*

*(2)* $\exists i_0 \in [1, m], S_{i_0} = \epsilon$ *and* $\forall i \neq i_0, S_i = \emptyset$ *or*

*(3)* $\exists i_0 \in [1,m], S_{i_0} \neq \emptyset$ *and,* $\forall w, w' \in W^*, \forall i,j \in [1,m]$,
$(S_i)_w = (S_j)_{w'} = 1 \Rightarrow [\exists A, A' \in W, w_1, w'_1 \in V^*, A \smile A', w = A \cdot w_1$ *and* $w' = A' \cdot w'_1]$.

A left-deterministic row-vector $S$ is said to have the type $\emptyset$ (resp. $(\epsilon, i_0)$, $[A]_\smile$) if case (1) (resp. (2), (3)) occurs. The right-action $\bullet$ on $\mathsf{B}\langle\langle W\rangle\rangle$ is extended componentwise to $\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$: for every $S = (s_{i,j})$, $u \in W^*$, the matrix $T = S \bullet u$ is defined by

$$t_{i,j} = s_{i,j} \bullet u.$$

**Definition 4.5** *Let* $S \in \mathsf{B}_{1,m}\langle\langle W\rangle\rangle$*.* $S$ *is said* deterministic *iff, for every* $u \in W^*$*,* $S \bullet u$ *is left-deterministic.*

We denote by $\mathsf{DB}_{1,m}\langle\langle W\rangle\rangle$ the subset of deterministic row-vectors of dimension $m$ over $\mathsf{B}\langle\langle W\rangle\rangle$.

## 4.5 Formal system $\mathcal{D}_0$

We define here a deduction system $\mathcal{D}_0$ which solves the equivalence problem for dpda's.

Given a fixed dpda $\mathcal{M}$ over the terminal alphabet $X$, we consider the variable alphabet $V$ associated to $\mathcal{M}$ (see section 4.2) and the set $\mathsf{DRB}\langle\langle V\rangle\rangle$ (the set of Deterministic Rational Boolean series over $V^*$). The set of assertions is defined by :

$$\mathcal{A} = \mathsf{DRB}\langle\langle V\rangle\rangle \times \mathsf{DRB}\langle\langle V\rangle\rangle.$$

The "cost-function" $H : \mathcal{A} \to \mathsf{N} \cup \{\infty\}$ is defined by :

$$H(S, S') = \mathrm{Div}(S, S'),$$

where $\mathrm{Div}(S, S')$, the *divergence* between $S$ and $S'$, is defined by :

$$\mathrm{Div}(S, S') = \inf\{|\, u \,| \mid u \in \varphi(S) \triangle \varphi(S')\}.$$

Let us notice that here :

$$\chi(S, S') = 1 \Longleftrightarrow S \equiv S'.$$

We define a binary relation $||\!\!-\!\!- \;\subset \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

**(D1)**

$$\{(S, T)\} \;||\!\!-\!\!-\; (T, S)$$

for $S, T \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

**(D2)**

$$\{(S,S'),(S',S'')\} \mid\!\vdash\!\!- (S,S'')$$

for $S,S',S'' \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

**(D3)**

$$\emptyset \mid\!\vdash\!\!- (S,S)$$

for $S \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

**(D'3)**

$$\emptyset \mid\!\vdash\!\!- ([qzr],\epsilon)$$

for $q,r \in Q, z \in Z, [qzr] \equiv \epsilon$,

**(D4)**

$$\{(S \odot x, T \odot x) \mid x \in X\} \mid\!\vdash\!\!- (S,T)$$

for $S,T \in \mathsf{DRB}\langle\langle V\rangle\rangle, S \not\equiv \epsilon, T \not\equiv \epsilon$,

**(D5)**

$$\{(S \cdot T' + S', T')\} \mid\!\vdash\!\!- (S^* \cdot S', T')$$

for $(S,S') \in \mathsf{DRB}_{1,2}\langle\langle V\rangle\rangle, T' \in \mathsf{DRB}\langle\langle V\rangle\rangle, S \not\equiv \epsilon$,

**(D6)**

$$\{(S,S'),(T,T')\} \mid\!\vdash\!\!- (S+T, S'+T')$$

for $(S,T),(S',T') \in \mathsf{DRB}_{1,2}\langle\langle V\rangle\rangle$,

**(D7)**

$$\{(S,S')\} \mid\!\vdash\!\!- (S \cdot T, S' \cdot T)$$

for $S,S',T \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

**(D8)**

$$\{(T,T')\} \mid\!\vdash\!\!- (S \cdot T, S \cdot T')$$

for $S,T,T' \in \mathsf{DRB}\langle\langle V\rangle\rangle$.

We define $\vdash\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$,

$$P \vdash\!\!- A \Longleftrightarrow P \overset{<*>}{\mid\!\vdash\!\!-} \circ \overset{[1]}{\mid\!\vdash\!\!-}_{3,4} \circ \overset{<*>}{\mid\!\vdash\!\!-} \{A\}.$$

where $\mid\!\vdash\!\!-_{3,4}$ is the relation defined by $D3, D'3, D4$ only.

**Theorem 4.6** *$\mathcal{D}_0$ is a complete deduction system.*

**General ideas of the proof**: Beside the idea of formal system (see section 2) and the idea of deterministic vectors (see subsection 4.4), the following three ideas are used in the proof of theorem 4.6.

**Deterministic spaces** The notion of *linear independence* of languages (and also of configurations) appeared in ([Mei89]). Let us sketch this idea for prefix languages [1] . We recall that a language $L$ is said to have the *prefix* property if, for every $u, v \in L$, if $u$ is a prefix of $v$, then $u = v$. Similarly we shall say that a vector of languages $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a prefix vector iff $\cup_{i=1}^{n} \alpha_i$ is prefix and for every $i \neq j$, $\alpha_i \cap \alpha_j = \emptyset$. Let $(L_1, L_2, \ldots, L_n)$ be a family of prefix languages.

(1) Either for every two prefix vectors $(\alpha_1, \alpha_2, \ldots, \alpha_n), (\beta_1, \beta_2, \ldots, \beta_n)$

$$\sum_{i=1}^{n} \alpha_i \cdot L_i = \sum_{i=1}^{n} \beta_i \cdot L_i \quad \Rightarrow \quad (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_n)$$

(2) or, there exists some $i_0 \in [1, n]$ , and a prefix vector $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, such that

$$L_{i_0} = \sum_{i=1}^{n} \gamma_i \cdot L_i \quad \text{where} \quad \gamma_{i_0} = \emptyset.$$

When (1) (resp. (2)) is true, the family $(L_1, L_2, \ldots, L_n)$ is said linearly independent (resp. linearly dependent). In other words, if (1) is not true, then (2) must be true. The adaptation of this idea to *equivalence of configurations* (instead of equality of languages) was technically non-obvious because, even when (1) is shown to be untrue by a pair of vectors $\alpha, \beta$ defined by configurations, the vector $\gamma$ appearing in (2) needs not be still defined by a configuration . But we prove that it always corresponds to a deterministic rational boolean vector.
We are then naturally led to consider, for every given set of deterministic rational boolean series $\{U_i | i \in I\}$, the set of all deterministic rational linear combinations of these series. We call such a set the *deterministic space* generated by $\{U_i | i \in I\}$.

**Strategies** A *strategy* is a method allowing to *find* a proof of the fact that two configurations (or series) are equivalent. A basic step of all the usual strategies is to replace a pair

$$U \equiv V \tag{19}$$

by the finite set of all pairs obtained by letting one terminal letter $x \in X$ act on both sides:

$$\{U \odot x \equiv V \odot x, \quad x \in X\}.$$

Such a step in the construction of a proof is called a $T_A$ step.
In [Val73, p.68], is introduced a second kind of step called a *replacement*, which introduces , from a pair (19), another finite set of pairs

$$U' \equiv V' \quad , \quad U'' \equiv V'' \tag{20}$$

[1] One can notice that, when $W/\smile$ consists of only one element, the "deterministic series" $S \in \mathsf{DB}\langle\langle W \rangle\rangle$ are just the prefix languages over the alphabet $W$.

such that

$$U \equiv V \Leftrightarrow (\ U' \equiv V' \text{ and } U'' \equiv V''\ ).$$

The sequences of pairs $(U_i, V_i)$ obtained by a suitable alternation of $T_A$ steps and replacement steps are "smooth" in the sense that the lengths of both sides have similar variations.
We define here a kind of replacement called $T_B$ (because it is also analogous with transformation $T_B$ of [HK66]), which creates from two pairs

$$U \equiv V \quad , \quad U' \equiv V' \tag{21}$$

a new pair

$$U'' \equiv V'' \tag{22}$$

such that

$$(\ U \equiv V \text{ and } U' \equiv V'\ ) \Leftrightarrow (\ U \equiv V \text{ and } U'' \equiv V''\ ).$$

This transformation consists in replacing the pair $U' \equiv V'$ by the new pair $U'' \equiv V''$ *under the hypothesis* that $U \equiv V$. This type of replacements also leads to somewhat "smooth" sequences of pairs in an algebraic sense which is sketched below.

**N-stacking sequences** Let us call a $\mathcal{S}_{AB}$-tree the (possibly infinite) tree obtained from an initial true equation $U \equiv V$ by the above strategy. We show that this tree has "smooth branches" in the following sense: on every infinite branch $b = (x_i)_{0 \leq i}$, there exists

- a "short sequence" of nodes

  $$(x_i)_{i_0 \leq i \leq i_0 + L_{d_0} + k_1}$$

  (where $L_{d_0}, k_1$ are constants),

- a "small" generating set

  $$\mathcal{G}_1 = \{U_i | 1 \leq i \leq d_0\}$$

  (where $d_0$ is a constant),

- and $d_0$ integers

  $$\kappa_1, \kappa_2, \ldots, \kappa_{d_0} \in [i_0, i_0 + L_{d_0}]$$

such that all left-hand sides and right-hand sides of the equations at nodes $x_{\kappa_1}, x_{\kappa_2}, \ldots, x_{\kappa_{d_0}}$ belong to the deterministic space generated by $\mathcal{G}_1$ and have *small coefficients* on the generating set $\mathcal{G}_1$.
We are faced with a sytem of $d_0$ linear equations linking only $d_0$ different series. The "linear independence" idea (explained above) can then be applied to *cut*

the branch $b$ (we name $T_C$ the precise tranformation allowing to cut a branch containing such a system of equations). At end, we obtain from the initial $\mathcal{S}_{AB}$-tree a *finite* $\mathcal{S}_{ABC}$-tree which is a proof in the formal system $\mathcal{D}_0$. $\Box$
A much more precise survey of the proof of theorem 4.6 is given in [Sén97c]. A full proof of theorem 4.6, which corresponds to [Sén98b, theorem 10.25], is given in [Sén98b, pages 12-107].

**Corollary 4.7** *The equivalence problem for deterministic pushdown automata is decidable.*

(This decidability result has been proved first in [Sén97b] by means of a more complicated deduction system and finally in [Sén98b] by means of the above deduction system). Corollary 4.7 follows from lemma 2.2, theorem 4.6 and the fact that non-equivalence of two pda's is obviously semi-decidable.

# 5 Deterministic pushdown H-automata

We extend here the completeness result of section 4.5 to $H$-pushdown automata, where $H$ is any abelian group.

## 5.1 Pushdown $H$-automata

Let $H$ be some group. We call a *$H$-pushdown automaton* on the alphabet $X$ any 6-tuple

$$\mathcal{M} = < X, Z, Q, \delta, q_0, z_0 >$$

where $Z$ is the finite stack-alphabet, $Q$ is the finite set of states, $q_0 \in Q$ is the initial state, $z_0$ is the initial stack-symbol and $\delta : QZ \times (X \cup \{\epsilon\}) \rightarrow \mathcal{P}_f(H \times QZ^*)$, is the transition mapping. Let $q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, h \in H, f \in X^*$ and $a \in X \cup \{\epsilon\}$ ; we note $(qz\omega, h, af) \longmapsto_{\mathcal{M}} (q'\omega'\omega, h \cdot h', f)$ if $(h', q'\omega') \in \delta(qz, a)$. $\stackrel{*}{\longmapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\longmapsto_{\mathcal{M}}$ . For every $q\omega, q'\omega' \in QZ^*$ and $h \in H, f \in X^*$, we note $q\omega \xrightarrow{(h,f)}_{\mathcal{M}} q'\omega'$ iff

$$(q\omega, 1_H, f) \stackrel{*}{\longmapsto}_{\mathcal{M}} (q'\omega', h, \epsilon).$$

$\mathcal{M}$ is said *deterministic* iff it fulfills conditions (5,6) of section 4.1. A $H$-dpda $\mathcal{M}$ is said *normalized* iff, for every $qz \in QZ, x \in X$:

$$q'\omega' \in \delta_2(qz, x) \Rightarrow \mid \omega' \mid \leq 2, \text{ and } q'\omega' \in \delta_2(qz, \epsilon) \Rightarrow \mid \omega' \mid = 0, \qquad (23)$$

where $\delta_2 : QZ \times (X \cup \{\epsilon\}) \rightarrow \mathcal{P}_f(QZ^*)$, is the second component of the map $\delta$. Let us denote by $\mathsf{B}\langle\langle H \rangle\rangle$ the semi-ring of formal power series with boolean coefficients and undeterminates in $H$. It is isomorphic to the powerset of $H$, $\mathcal{P}(H)$, endowed with the set-product induced by the product in $H$. Given some finite set $F \subseteq QZ^*$ of configurations, the *series recognized by* $\mathcal{M}$ *with final configurations* $F$ is the element of $(\mathsf{B}\langle\langle H\rangle\rangle)\langle\langle V\rangle\rangle$ defined by:

$$\mathrm{S}(\mathcal{M}, F) = \sum_{c\in F} \sum_{q_0 z_0 \xrightarrow{(h,w)}_{\mathcal{M}} c} h \cdot w.$$

Intuitively, one can see the coefficient $S_w \in \mathsf{B}\langle\langle H\rangle\rangle$ of a word $w$ in the series $\mathrm{S}(\mathcal{M}, F)$ either as the "multiplicity" with which the word $w$ is recognized, or as the "output" of the automaton $\mathcal{M}$ on the "input" $w$. Notice that, from this last point of view, when $\mathcal{M}$ is deterministic and $(H, \cdot) = (\mathsf{Z}, +)$, the additive group of integers, $\mathcal{M}$ can be named a deterministic pushdown *transducer* from *words* to *integers*.

### 5.2 Right-actions

Similarly as in section 4.2 we fix some $H$-dpda $\mathcal{M}$ and consider the structured alphabet $(V, \smile)$ associated with $\mathcal{M}$.

**Action $\bullet$** A $\sigma$-right-action of the monoid $H \times V^*$ over $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle$ is defined by: $\forall S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle, \forall h \in H, \forall w \in V^*, T = S \bullet (h, w)$ is the series:

$$\forall v \in V^*, T_v = h^{-1} \cdot S_{w \cdot v}.$$

In words, $S \bullet (h, w)$ is the left-quotient of $S$ by the monomial $h \cdot w$. We denote by $\mathsf{Q}(S)$ the set of residuals of $S$ i.e.:

$$\mathsf{Q}(S) = \{S \bullet (h, u) \mid h \in H, u \in V^*\}.$$

**Action $\otimes$** Let us consider the set $P_{\mathcal{M}}$ of all the pairs of one of the following forms:

$$([p, z, q], h \cdot x \cdot [p', z_1, p''][p'', z_2, q]) \tag{24}$$

where $p, q, p', p'' \in Q, x \in X, (h, p'z_1z_2) \in \delta(pz, x)$

$$([p, z, q], h \cdot x \cdot [p', z', q]) \tag{25}$$

where $p, q, p' \in Q, x \in X, (h, p'z') \in \delta(pz, x)$

$$([p, z, q], h \cdot a) \tag{26}$$

where $p, q, \in Q, a \in X \cup \{\epsilon\}, (h, q) \in \delta(pz, a)$. We define a $\sigma$-right-action $\otimes$ of the monoid $H \times (X \cup \{e\})^*$ over the semi-ring $(\mathsf{B}\langle\langle H\rangle\rangle)\langle\langle V\rangle\rangle$by: for every $p, q \in Q, A \in Z, x \in X, h \in H, k \in \mathsf{B}\langle\langle H\rangle\rangle$:

$$[p, A, q] \otimes (1_H, x) = (\sum_{([p,A,q],m)\in P_{\mathcal{M}}} m) \bullet (1_H, x), \tag{27}$$

$$[p,A,q] \otimes (1_H, e) = h \text{ iff } ([p,A,q],h) \in P_{\mathcal{M}}, \tag{28}$$

$$[p,A,q] \otimes (1_H, e) = \emptyset \text{ iff } (\{[p,A,q]\} \times H) \cap P_{\mathcal{M}} = \emptyset, \tag{29}$$

$$k \otimes (1_H, x) = \emptyset, \quad k \otimes (1_H, e) = \emptyset. \tag{30}$$

The action is extended to all monomials by: for every $k \in \mathsf{B}\langle\langle H\rangle\rangle, \beta \in V^*, y \in X \cup \{e\}$,

$$(k \cdot [p,A,q] \cdot \beta) \otimes (1_H, y) = k \cdot ([p,A,q] \otimes y) \cdot \beta. \tag{31}$$

At end: for every $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle, h \in H$,

$$S \otimes (h, \epsilon) = h^{-1} \cdot S. \tag{32}$$

**Action $\odot$** We define a map $\rho_\epsilon : \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle \to \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle$ as the unique $\sigma$-additive map such that,

$$\rho_\epsilon(\emptyset) = \emptyset, \quad \rho_\epsilon(\epsilon) = \epsilon,$$

and for every $p \in Q, z \in Z, q \in Q, \beta \in V^*, k \in \mathsf{B}\langle\langle H\rangle\rangle, S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle$,

$$\rho_\epsilon([p,z,q] \cdot \beta) = \rho_\epsilon(([p,z,q] \otimes e) \cdot \beta) \text{ if } pz \text{ is } \epsilon-\text{bound}$$

$$\rho_\epsilon([p,z,q] \cdot \beta) = [p,z,q] \cdot \beta \text{ if } pz \text{ is } \epsilon-\text{free and,}$$

$$\rho_\epsilon(k \cdot S) = k \cdot \rho_\epsilon(S).$$

The right-action $\odot$ of the monoid $H \times X^*$ over the semi-ring $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle$ is then the unique monoid-action fulfilling: for every $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle, h \in H, x \in X$,

$$S \odot (h,x) = \rho_\epsilon(\rho_\epsilon(S) \otimes (h,x)).$$

**Case where $H$ is abelian** Let us consider the case where $H$ is abelian. Let $\varphi : \mathsf{B}\langle\langle H\rangle\rangle \cup V \to \mathsf{B}\langle\langle H\rangle\rangle\langle\langle X\rangle\rangle$ defined by:

$$\forall k \in \mathsf{B}\langle\langle H\rangle\rangle, \varphi(k) = k; \quad \forall v \in V, \varphi(v) = \sum_{v \odot (h \cdot u) = \epsilon} h \cdot u.$$

One can check that, as $H$ is supposed abelian, there exists a unique $\sigma$-additive semi-ring homomorphism $\tilde{\varphi} : \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle \to \mathsf{B}\langle\langle H\rangle\rangle\langle\langle X\rangle\rangle$ which extends $\varphi$. Let us denote by the same letter the original $\varphi$ and its extension $\tilde{\varphi}$.

**Lemma 5.1** *For every $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle, h \in H, u \in X^*$,*

1. $\varphi(S) = \varphi(\rho_\epsilon(S))$,
2. $\varphi(S \odot (h,u)) = \varphi(S) \bullet (h,u)$ *( i.e. $\varphi$ is a morphism of right-actions).*

We denote by $\equiv$ the kernel of $\varphi$ i.e.: for every $S, T \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle$,

$$S \equiv T \Leftrightarrow \varphi(S) = \varphi(T).$$

## 5.3 Deterministic rational series

**$W$-determinism** Let $H$ be a group, let $(W, \smile)$ be a structured alphabet. Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. We define an equivalence relation $\approx$ over $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ by: for every $S, T \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$,

$$S \approx T \Leftrightarrow \exists h \in H, S = h \cdot T.$$

Let us denote by $(H^0, \cdot, 1_H)$ the submonoid of $(\mathsf{B}\langle\langle H\rangle\rangle, \cdot, 1_H)$ consisting of the empty series and all the singletons $\{h\}$ for $h \in H$. $H^0$ can be seen as the monoid obtained by "adjoining a zero" to the group $H$.

**Definition 5.2** *Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. $S$ is said $W$-deterministic rational iff $\forall u \in W^*, (S_u \in H^0)$ and $\mathsf{Q}(S)/\approx$ is finite.*

**Definition 5.3** *Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. $S$ is said $\smile$-deterministic iff its support is deterministic (in the sense of definition 4.3).*

**Definition 5.4** *Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. $S$ is said deterministic rational iff it is both $W$-deterministic rational and $\smile$-deterministic.*

We denote by $\mathsf{DRH}^0\langle\langle W\rangle\rangle$ the set of all deterministic rational series in $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$.

## 5.4 Vectors

Let us denote by $\mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$ the set of $(n,m)$-matrices with entries in the semi-ring $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$.

**Definition 5.5** *Let $S \in \mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$. $S$ is said deterministic rational iff, for every $i \in [1,m], S_i$ is $W$-deterministic rational and the support of $S$ : $(\mathrm{supp}(S_1), \ldots, \mathrm{supp}(S_m))$, is a deterministic vector (in the sense of definition 4.5).*

We denote by $\mathsf{DRH}^0_{1,m}\langle\langle W\rangle\rangle$ the set of all deterministic rational row-vectors in $\mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$.

## 5.5 Formal system $\mathcal{H}_0$

We define here a deduction system $\mathcal{H}_0$ which solves the equivalence problem for $H$-dpda's, where $H$ is an abelian group.

Given an abelian group $H$ and a fixed $H$-dpda $\mathcal{M}$ over the terminal alphabet $X$, we consider the structured alphabet $(V, \smile)$ associated with $\mathcal{M}$ (see section 5.2) and the set $\mathsf{DRH}^0\langle\langle V\rangle\rangle$ (the set of Deterministic Rational series over $V^*$ with coefficients in $H^0$). The set of assertions is defined by :

$$\mathcal{A} = \mathsf{DRH}^0\langle\langle V\rangle\rangle \times \mathsf{DRH}^0\langle\langle V\rangle\rangle.$$

The "cost-function" $H : \mathcal{A} \to \mathsf{N} \cup \{\infty\}$ is defined by :

$$H(S,S') = \mathrm{Div}(S,S'),$$

where $\mathrm{Div}(S,S')$, the *divergence* between $S$ and $S'$, is defined by :

$$\mathrm{Div}(S,S') = \inf\{|\, u \,|\mid \varphi(S)_u \neq \varphi(S')_u\}.$$

Let us notice that here :

$$\chi(S,S') = 1 \Longleftrightarrow S \equiv S'.$$

We define a binary relation $|\!\!\vdash\!\!- \; \subset \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

**(H1)**

$$\{(S,T)\} \mid\!\!\vdash\!\!- (T,S)$$

for $S, T \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$,

**(H2)**

$$\{(S,S'),(S',S'')\} \mid\!\!\vdash\!\!- (S,S'')$$

for $S, S', S'' \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$,

**(H3)**

$$\emptyset \mid\!\!\vdash\!\!- (S,S)$$

for $S \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$,

**(H'3)**

$$\emptyset \mid\!\!\vdash\!\!- ([qzr],h)$$

for $q, r \in Q, z \in Z, h \in H, [qzr] \equiv h$,

**(H4)**

$$\{(S \odot x, T \odot x) \mid x \in X\} \mid\!\!\vdash\!\!- (S,T)$$

for $S, T \in \mathsf{DRH}^0\langle\langle V\rangle\rangle, (\forall h \in H, S \not\equiv h), (\forall h \in H, T \not\equiv h)$,

**(H5)**

$$\{(S \cdot T' + S', T')\} \mid\!\!\vdash\!\!- (S^* \cdot S', T')$$

for $(S,S') \in \mathsf{DRH}^0_{1,2}\langle\langle V\rangle\rangle, T' \in \mathsf{DRH}^0\langle\langle V\rangle\rangle, (\forall h \in H, S \not\equiv h)$,

**(H6)**

$$\{(S,S'),(T,T')\} \mid\!\!\vdash\!\!- (S+T, S'+T')$$

for $(S,T),(S',T') \in \mathsf{DRH}^0_{1,2}\langle\langle V\rangle\rangle$,

**(H7)**

$$\{(S,S')\} \mid\!\vdash\!\!- (S\cdot T, S'\cdot T)$$

for $S, S', T \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$,

**(H8)**

$$\{(T,T')\} \mid\!\vdash\!\!- (S\cdot T, S\cdot T')$$

for $S, T, T' \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$.

We define $\vdash\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$,

$$P \vdash\!\!- A \Longleftrightarrow P \overset{<*>}{\mid\!\vdash\!\!-} \circ \overset{[1]}{\mid\!\vdash\!\!-}_{3,4} \circ \overset{<*>}{\mid\!\vdash\!\!-} \{A\}.$$

where $\mid\!\vdash\!\!-_{3,4}$ is the relation defined by $H3, H'3, H4$ only.

**Theorem 5.6** *$\mathcal{H}_0$ is a complete deduction system.*

**Corollary 5.7** *The equivalence problem for $H$-dpda's ( i.e. deterministic pushdown automata with multiplicities in an abelian group $H$) is decidable.*

A reformulation of corollary 5.7 is that the function equivalence problem for deterministic pushdown transducers with outputs in an abelian group is decidable. This result is proved in [Sén98b, section 11,theorem 11.61]. Let us notice that the case where $H = \{1\}$ corresponds to the (classical) equivalence problem for dpda's (see corollary 4.7).

# 6 Non-deterministic pushdown automata

## 6.1 Bisimulation for graphs

Let $X$ be a finite alphabet. We call *graph over* $X$ any pair $\Gamma = (V_\Gamma, E_\Gamma)$ where $V_\Gamma$ is a set and $E_\Gamma$ is a subset of $V_\Gamma \times X \times V_\Gamma$. For every integer $n \in \mathsf{N}$, we call an $n$-graph every $(n+2)$-tuple $\Gamma = (V_\Gamma, E_\Gamma, v_1, \ldots, v_n)$ where $(V_\Gamma, E_\Gamma)$ is a graph and $(v_1, \ldots, v_n)$ is a sequence of distinguished vertices: they are called the *sources* of $\Gamma$.
Let us consider another finite alphabet $Y$, a strict litteral morphism $\psi : X^* \to Y^*$ ("strict litteral" means that , $\forall x \in X, \psi(x) \in Y$) and the equivalence relation $\eta$ over $X^*$ which is the kernel of $\psi$:

$$\forall u, u' \in X^*, (u,u') \in \eta \Leftrightarrow \psi(u) = \psi(u').$$

Let $\Gamma, \Gamma'$ be two $n$-graphs over $X$.

**Definition 6.1** *$\mathcal{R}$ is a $\eta$-*simulation *from $\Gamma$ to $\Gamma'$ iff*

*1.* $\mathrm{dom}(\mathcal{R}) = V_\Gamma$,

*2.* $\forall i \in [1,n], (v_i, v_i') \in \mathcal{R}$,

*3.* $\forall v, w \in V_\Gamma, v' \in V_{\Gamma'}, x \in X,$ *such that* $(v,x,w) \in E_\Gamma$ *and* $v\mathcal{R}v'$,

$$\exists w' \in V_{\Gamma'}, x' \in \eta(x) \textit{ such that } (v', x', w') \in E_{\Gamma'} \textit{ and } w\mathcal{R}w'.$$

$\mathcal{R}$ *is a* $\eta$-bisimulation *iff* $\mathcal{R}$ *is a* $\eta$*-simulation and* $\mathcal{R}^{-1}$ *is a* $\eta^{-1}$*- simulation.*

A relation $\mathcal{R}$ is a bisimulation iff it is a Id-bisimulation in the sense of definition 6.1 (where Id is just the equality relation). Notice that, when $\eta = \mathrm{Id}$ and $n = 0$, definition 6.1 coincides with the classical definition of [Par81], [Mil89].

## 6.2 Bisimulation for pushdown automata

We call *transition-graph* of a pda $\mathcal{M}$, denoted $\mathcal{T}(\mathcal{M})$, the 0-graph:
$\mathcal{T}(\mathcal{M}) = (V_{\mathcal{T}(\mathcal{M})}, E_{\mathcal{T}(\mathcal{M})})$ where

$$V_{\mathcal{T}(\mathcal{M})} = \{q\omega \mid q \in Q, \omega \in Z^*, q\omega \text{ is } \epsilon - \text{free}\}$$

and

$$E_{\mathcal{T}(\mathcal{A})} = \{(c, x, c') \in V_{\mathcal{T}(\mathcal{M})} \times X \times V_{\mathcal{T}(\mathcal{M})} \mid c \xrightarrow{x}_{\mathcal{M}} c'\}.$$

We call *computation 1-graph* of the pda $\mathcal{M}$, denoted $(\mathcal{C}(\mathcal{M}), v_{\mathcal{M}})$, the subgraph of $\mathcal{T}(\mathcal{M})$ induced by the set of vertices which are accessible from the vertex $q_0z_0$, together with the source $v_{\mathcal{M}} = q_0z_0$.
The *bisimulation-problem* for normalized pda's is the following decision problem:

INSTANCE: Two normalized dpda $A, B$, over the same terminal alphabet $X$.

QUESTION: $(\mathcal{C}(A), v_A) \sim (\mathcal{C}(B), v_B)$?, (where $\sim$ is the bisimulation relation for 1-graphs).

## 6.3 Bisimulation for deterministic vectors

For every integer $\lambda \in \mathsf{N} - \{0\}$, we define the special vectors $\epsilon_i^\lambda \in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$ by: for every $i \in [1, \lambda]$,

$$\epsilon_i^\lambda = (\delta_{i,j})_{j \in [1,\lambda]}, \quad \text{where } \delta_{i,j} = \epsilon \text{ ( if } i = j), \quad \delta_{i,j} = \emptyset \text{ ( if } i \neq j).$$

**Definition 6.2** *Let* $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$ *and* $\mathcal{R} \subseteq X^* \times X^*$. $\mathcal{R}$ *is a* $w - \eta$*-bisimulation with respect to* $(S, S')$ *iff* $\mathcal{R} \subseteq \eta$ *and*

**(1) totality:** $\mathrm{dom}(\mathcal{R}) = X^*, \mathrm{im}(\mathcal{R}) = X^*$,

**(2) extension:** $\forall (u, u') \in \mathcal{R}, \forall x \in X$,

$$\exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R} \textit{ and } \exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R},$$

**(3) coherence:** $\forall (u,u') \in \mathcal{R}, \forall i \in [1,\lambda], (\rho_\epsilon(S \odot u) = \epsilon_i^\lambda) \Leftrightarrow (\rho_\epsilon(S' \odot u') = \epsilon_i^\lambda)$,

**(4) prefix:** $\forall (u,u') \in X^* \times X^*, \forall (x,x') \in X \times X, (u \cdot x, u' \cdot x') \in \mathcal{R} \Rightarrow (u,u') \in \mathcal{R}$.

The letter $w$ in "$w - \eta$-bisimulation" stands for *word*, as $\mathcal{R}$ is a relation over *words*. Given an integer $n \in \mathsf{N}$, $\mathcal{R}$ is said to be a $w - \eta$-bisimulation of *order* $n$ with respect to $(S,S')$ iff it fulfills conditions (3-4) above and the modified conditions
**(1')**: $\mathrm{dom}(\mathcal{R}) = X^{\leq n}, \mathrm{im}(\mathcal{R}) = X^{\leq n}$,
**(2')**: $\forall (u,u') \in \mathcal{R} \cap (X^{\leq n-1} \times X^{\leq n-1}), \forall x \in X$,

$$\exists x' \in \eta(x), (u \cdot x, u' \cdot x') \in \mathcal{R} \text{ and } \exists x'' \in \eta^{-1}(x), (u \cdot x'', u' \cdot x) \in \mathcal{R}.$$

The $w - \eta$-bisimulations are also called $w - \eta$-bisimulations of *order* $\infty$. For every $n \in \mathsf{N}$, we define the set of binary relations:

$$\bar{\mathcal{B}}_n = \{\mathcal{R} \subseteq X^* \times X^* \mid \mathcal{R} \text{ fulfills conditions } (1)', (2)', (4)\}.$$

( For every $n$, this set is clearly computable).

**Definition 6.3** *Let $S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$ and $n \in \mathsf{N}$.*

**(1)** *$S, S'$ are said $\eta$-bisimilar, which is denoted by $S \sim S'$, iff there exists $\mathcal{R} \subseteq X^* \times X^*$ which is a $w - \eta$-bisimulation w.r.t. $(S,S')$*

**(2)** *$S, S'$ are said $\eta$-bisimilar at the order $n$, which is denoted by $S \sim_n S'$, iff there exists $\mathcal{R} \subseteq X^* \times X^*$ which is a $w - \eta$-bisimulation of order $n$ w.r.t. $(S,S')$*

Using Koenig's lemma, one can show that

$$S \sim S' \Leftrightarrow (\forall n \in \mathsf{N}, S \sim_n S').$$

The bisimulation-problem for normalized non-deterministic pda's reduces to the following decision problem (we call it the $\eta$-bisimulation problem for deterministic rational vectors):

INSTANCE: a normalized deterministic pda $\mathcal{M}$, its terminal alphabet $X$, a strict litteral morphism $\psi : X^* \to Y^*$ (we denote its kernel by $\eta$), and $\lambda \in \mathsf{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$ (where $V$ is the structured alphabet associated with $\mathcal{M}$).

QUESTION: $S \sim S'$? (where $\sim$ is the $\eta$-bisimulation relation).

## 6.4 Formal system $\mathcal{B}_0$

Let us fix some normalized deterministic pda $\mathcal{M}$ with terminal alphabet $X$ and associated structured alphabet $(V, \smile)$. Let $\psi : X^* \to Y^*$ be some strict litteral homomorphism and $\eta = Ker(\psi)$. We define here a deduction system $\mathcal{B}_0 =< \mathcal{A}, H, \mid\!\!-\!\!- >$ which solves the $\eta$-bisimulation problem for deterministic rational vectors. The set of assertions, $\mathcal{A}$, is defined by

$$\mathcal{A} = \bigcup_{\lambda \in \mathsf{N} - \{0\}} \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle \times \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle.$$

The cost function, $H : \mathcal{A} \to \mathsf{N} \cup \{\infty\}$ is defined by :

$$\forall (S, S') \in \mathcal{A}, H(S, S') = \mathrm{Div}(S, S'),$$

where $\mathrm{Div}(S, S') = \inf\{n \in \mathsf{N} \mid S \not\sim_n S'\}$.
Let us notice that here:

$$\chi(S, S') = 1 \Longleftrightarrow S \sim S'.$$

We define a binary relation $|\!\!\mid\!\!-\!\!- \subseteq \mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

**(B1)**

$$\{(S, T)\} \,|\!\!\mid\!\!-\!\!-\, (T, S)$$

for $\lambda \in \mathsf{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$,

**(B2)**

$$\{(S, S'), (S', S'')\} \,|\!\!\mid\!\!-\!\!-\, (S, S'')$$

for $\lambda \in \mathsf{N} - \{0\}, S, S', S'' \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$,

**(B3)**

$$\emptyset \,|\!\!\mid\!\!-\!\!-\, (S, S)$$

for $S \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$,

**(B'3)**

$$\emptyset \,|\!\!\mid\!\!-\!\!-\, ([qzr], \epsilon)$$

for $q, r \in Q, z \in Z, [qzr] \equiv \epsilon$,

**(B4)**

$$\{(S \odot x, T \odot x') \mid (x, x') \in \mathcal{R}_1\} \,|\!\!\mid\!\!-\!\!-\, (S, T)$$

for $\lambda \in \mathsf{N} - \{0\}, S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle, (S \not\equiv \epsilon \wedge T \not\equiv \epsilon)$ and $\mathcal{R}_1 \in \bar{\mathcal{B}}_1$,

**(B5)**

$$\{(S_1 \cdot T + S, T)\} \mid\!\vdash\!\!- (S_1^* \cdot S, T)$$

for $\lambda \in \mathsf{N} - \{0\}, S_1 \in \mathsf{DRB}_{1,1}\langle\langle V \rangle\rangle, S_1 \not\equiv \epsilon, (S_1, S) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle V \rangle\rangle, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$,

**(B6)**

$$\{(S, S'), (T, T')\} \mid\!\vdash\!\!- ((S, T), (S', T'))$$

for $\lambda, \mu \in \mathsf{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle, T, T' \in \mathsf{DRB}_{1,\mu}\langle\langle V \rangle\rangle, (S, T), (S', T') \in \mathsf{DRB}_{1,\lambda+\mu}\langle\langle V \rangle\rangle$, where ($\lambda = 1$ and $S \in \{\emptyset, \epsilon\}$) or ($\mu = 1$ and $T \in \{\emptyset, \epsilon\}$),

**(B7)**

$$\{(S, S')\} \mid\!\vdash\!\!- (S \cdot T, S' \cdot T)$$

for $\delta, \lambda \in \mathsf{N} - \{0\}, S, S' \in \mathsf{DRB}_{1,\delta}\langle\langle V \rangle\rangle, T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$,

**(B8)**

$$\{(T_{i,*}, T'_{i,*}) \mid 1 \leq i \leq \delta\} \mid\!\vdash\!\!- (S \cdot T, S \cdot T')$$

for $\delta, \lambda \in \mathsf{N} - \{0\}, S \in \mathsf{DRB}_{1,\delta}\langle\langle V \rangle\rangle, T, T' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$,

**(B9)**

$$\{(U_1, \epsilon)\} \mid\!\vdash\!\!- (U_1^* \cdot T, T)$$

for $\lambda \in \mathsf{N} - \{0\}, U_1 \in \mathsf{DRB}_{1,1}\langle\langle V \rangle\rangle, (U_1, T) \in \mathsf{DRB}_{1,\lambda+1}\langle\langle V \rangle\rangle$.

We define $\vdash\!\!-$ by : for every $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$,

$$P \vdash\!\!- A \Longleftrightarrow P \overset{<*>}{\mid\!\vdash\!\!-} \circ \overset{[1]}{\mid\!\vdash\!\!-}_{3,4} \circ \overset{<*>}{\mid\!\vdash\!\!-} \{A\}.$$

where $\mid\!\vdash\!\!-_{3,4}$ is the relation defined by $B3, B'3, B4$ only.

**Theorem 6.4** *: $\mathcal{B}_0$ is a complete deduction system.*

**Corollary 6.5** *The bisimulation problem for normalized non-deterministic pushdown automata is decidable.*

Let us notice that the case of *loop-free* non-deterministic pushdown automata can be easily reduced to the case of *normalized* pda's (a pda is called loop-free iff it has no infinite computation consisting of $\epsilon$-transitions only). This corollary is proved in [Sén97a]. It extends corollary 4.7 because, two *deterministic* pda's are bisimilar iff they recognize the same language (this was noticed in [BBK87]). It also extends the corresponding decidability results proved by [Sti96] for *strict real-time* non-deterministic pda's and by [Jan97] for *one-counter, real-time* non-deterministic pda's.

## 7 Perspectives

**More general storage types** The complete deduction system $\mathcal{D}_0$ given in section 4 raises the question whether such a construction could be generalized to other kinds of "storage type". In particular, we think it is plausible that the equivalence problem remains decidable for some kinds of deterministic "iterated pushdown automata" (various notions of pushdowns of pushdowns have been defined in [Mas76, Eng83, EV86, Vog86, Wei94]).

**Non-commutative groups** The complete deduction system $\mathcal{H}_0$ given in section 5 raises the question whether such a construction could be generalized to *non-commutative* groups. It turns out that the decidability of the equivalence problem for $H$-dpda's can fail for some groups $H$ even when $H$ is finitely generated and has decidable word-problem. Nevertheless, when $H$ is a free-group $F(Y)$, with finite basis $Y$ and when the transitions of the automata have only coefficients in the free monoid $Y^*$, we think the equivalence problem is still decidable (work in preparation). The more general case where $H = F(Y)$ but the transitions are arbitrary is open.

**Non-normalized non-deterministic pushdown automata** The complete deduction system $\mathcal{B}_0$ given in section 6 raises the question whether such a construction could be generalized to arbitrary non-deterministic pushdown automata (which might have *infinite $\epsilon$-computations*). This last open problem has been raised in [Cau95] and [Sti96].

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