ON THE RATIONAL SUBSETS OF THE FREE GROUP

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Abstract

We show that every rational subset of a free group of finite rank is either disjunctive or has finite index. We then generalise this result to virtually free groups: if G is virtually free and R is a rational subset of G, then the syntactic normal subgroup N of R is either finite or has finite index.

Keywords: formal languages-free group-Thue systems-finite automata-pushdown automata

1 INTRODUCTION

The purpose of this paper is to prove that the following theorem is true as was conjectured by J. Sakarovitch in [26] and [28].

Theorem 1.1: *Every rational subset R of a finitely generated free group F either is disjunctive or has finite index.*

This theorem was already proved in the particular cases where R is a subgroup or where R has a finite image in its syntactical monoid ([26],[28]).

As remarked in [28], this theorem can be regarded as a kind of Kleene's theorem (recall that a group fulfills Kleene's theorem iff it is finite, hence it is natural to try a weak version of Kleene's theorem for infinite groups).

When we generalise theorem 1.1 to *virtually free* groups (a group is said virtually free when it contains a free subgroup of finite index) we get the following statement.

Theorem 4.3: *Let G be a finitely generated virtually free group. If R is a rational subset of G and N is the syntactic normal subgroup of R then either N is finite or G/N is finite.*

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These two theorems bring some light on the general question of "what is the relationship between the notion of *rational* subset and the notion of *recognizable* subset?" which can be asked in various algebraic structures (see the surveys of [5] for monoids , [9] for algebras of graphs and [10] for general algebras). Though the condition "virtually-free" is purely algebraic, it is related in several ways to some familiar objects of theoretical computer science:

- a finitely generated group G is virtually free iff its word problem can be generated by some *context-free* grammar; this characterisation follows from the theorem of Muller and Schupp ([25]) together with the general result of [13] (see in section 4 a detailed presentation of that subject)
- every finitely generated group G which is presented by a strictly length-reducing, *Church-Rosser semi-Thue system*, is virtually free (see in [22] a survey of the results relating semi-Thue systems and virtually free groups and in [4, section 4] a survey of the results relating context-free grammars and virtually free groups).

Section 2 of this work presents our notation and recalls some definitions and results in the area of formal languages, semi-Thue systems and combinatorial group theory. For more information the reader can consult [3],[18],[8],[11],[21] and [24]. Section 3 is devoted to the proof of theorem 1.1 and section 4 to the proof of theorem 4.3. Section 5 presents some direct applications of our results as well as some perspectives.

2 PRELIMINARIES

2.1 Definitions, notation and classical results

2.1.1 Free monoid, rational subsets, finite automata

If X is an alphabet, we denote by X^* the *free monoid* on this alphabet. ϵ is the *empty word*. If f is a non-empty word we denote by $First(f)$ the first letter of f and by $Last(f)$ its last letter. A word $u \in X^*$ is a *prefix* of f if and only if $\exists v \in X^*$ such that $f = u.v$. By *Pref*(f) we denote the set of all prefixes of f. A word $u \in X^*$ is a *suffix* of f iff $\exists v \in X^*$ such that $f = v.u$. By *Suf*(f) we denote the set of all suffixes of f.

We say that two words f , g are *conjugates* and we write $f \approx g$, if and only if $f = u.v$ and $g = v.u$ for some words u, v. We denote by $\gamma(f)$ the set of all words g such that $f \approx g$.

The set of *rational subsets* of a monoid $(M, ., 1)$ is denoted by $\mathrm{Rat}(M)$ and defined by ([3],p.55): $\mathrm{Rat}(M)$ is the smallest subset of $\mathcal{P}(M)$ containing the finite subsets of M and closed by the operations: .(product of 2 subsets), $\cup$ (finite union) and $*$ (operation star).

We recall that a *finite deterministic automaton* is a 5-tuple $\mathcal{A} =< Q, X, \tau, q_0, Q_+ >$ where Q is the set of *states*, X is the *input alphabet*, q_0 the *initial state*, Q_+ is the set of *final* or *accepting* states. The *transition function* τ is a partial mapping from $Q \times X$ to Q. If f is a word on the alphabet X and q a state, $q.f$ denotes the state reached by $\mathcal{A}$ when starting in state q and reading the whole word f (if this is possible).

Q_{acc} is the set of *accessible* states of $\mathcal{A}$ i.e. the set of states q for which there exists $f \in X^*$ such that $q_0.f = q$.

Q_{coacc} is the set of *coaccessible* states of $\mathcal{A}$ i.e. the set of states q for which there exists $f \in X^*$ such that $q.f \in Q_+$.

2.1.2 Congruences

A *congruence* on a monoid $(M, ., 1)$ is an equivalence relation R such that, for every m_1, m_2, α, β in M , $m_1 R\, m_2 \Rightarrow (\alpha.m_1.\beta) R (\alpha.m_2.\beta)$.

The *syntactic congruence* $\equiv_L$ of a subset L of M is the largest congruence R which saturates L (i.e. such that L is a union of classes $\pmod R$, see [15] p.155). A characterisation of $\equiv_L$ is the following:
$m_1 \equiv_L m_2$ iff, for every α, β in M, $\alpha.m_1.\beta \in L \Leftrightarrow \alpha.m_2.\beta \in L$.
In the particular case where $(M, ., 1)$ is a *group*, we define the *syntactic normal subgroup* N of a subset $L \subseteq M$ by: N is the largest normal subgroup such that $LN \subseteq L$. Equivalently we can say that N is the normal subgroup associated to the syntactic congruence of L. We denote it by $Synt_M(L)$.

A subset L of M is *disjunctive* iff its syntactic congruence is the equality (i.e. $m_1 \equiv_L m_2$ iff $m_1 = m_2$). L is called *recognizable* iff its syntactic congruence has finite index (i.e. the quotient monoid $M/\equiv_L$ is finite). The so-called *Kleene's theorem* establishes that in a finitely generated free monoid X^*, a language (i.e. a subset) is rational if and only if it is recognizable.

2.1.3 Semi-Thue systems

Given an alphabet X, a *semi-Thue* system on X^* is a set of pairs $S \subseteq X^* \times X^*$. The relation $\longrightarrow_S$ is defined by: for all $f,g \in X^*$, $f \longrightarrow_S g$ iff there exists $(u, v) \in S$ and $\alpha, \beta \in X^*$ such that $f = \alpha u \beta, g = \alpha v \beta$. The relation $\stackrel{*}{\longrightarrow}_S$ is the reflexive and transitive closure of $\longrightarrow_S$. By $\longleftrightarrow_S$ we denote the symmetric closure of $\longrightarrow_S$ and $\stackrel{*}{\longleftrightarrow}_S$ is the reflexive and transitive closure of $\longleftrightarrow_S$. As $\stackrel{*}{\longleftrightarrow}_S$ is the smallest congruence containing S we call it the *congruence generated* by S. A semi-Thue system S is *confluent* iff

$$\forall f, g, h \in X^*, (f \stackrel{*}{\longrightarrow}_S g \ \wedge \ f \stackrel{*}{\longrightarrow}_S h) \Rightarrow (\exists k \in X^*, g \stackrel{*}{\longrightarrow}_S k \ \wedge \ h \stackrel{*}{\longrightarrow}_S k).$$

A semi-Thue system is *Noetherian* iff
there exists no infinite sequence $(f_n)_{n \geq 0}$ such that $\forall n \geq 0, f_n \longrightarrow_S f_{n+1}$.
If $f \in X^*$ we denote by $< f >_{\stackrel{*}{\longleftarrow}_S}$ the set $\{g \in X^*, g \stackrel{*}{\longrightarrow}_S f\}$ and by $[f]_{\stackrel{*}{\longleftrightarrow}_S}$

the set $\{g \in X^*, g \xleftrightarrow{*}_S f\}$. More generally, if A is a set, B a subset of A and $\sim$ an equivalence relation on A, by $[B]_\sim$ we denote the set $\{a \in A, \exists b \in B, a \sim b\}$. By $Domain(S)$ we denote the set $\{u \in X^* | \exists v \in X^*, (u,v) \in S\}$ and by $Range(S)$ we denote the set $\{v \in X^* | \exists u \in X^*, (u,v) \in S\}$. A word f is said *S-irreducible* iff there exists no g such that $f \longrightarrow_S g$ or, equivalently, iff f belongs to $X^* - (X^*.Domain(S).X^*)$. A semi-Thue system S is said *special* iff $Range(S) = \{\epsilon\}$. In [14] it is shown that, given a finite special system S, $< \epsilon >_{\xleftarrow{*}_S}$ is rational iff $Domain(S)$ is unavoidable (a subset W of X^* is *unavoidable* iff the complement of $X^*.W.X^*$ in X^* is finite). Here we shall only need the following easy lemma.

Lemma 2.1 *If S is a special semi-Thue system on a finite alphabet X such that $Domain(S)$ is unavoidable,then $\xleftrightarrow{*}_S$ has finite index.*

Proof:
The finite set $X^* - (X^* Domain(S) X^*)$ is a set of representatives $\pmod{\xleftrightarrow{*}_S}$.
□

2.1.4 Free groups

In order to define a free group of rank d (where $d \geq 1$) we consider an alphabet

$$\hat{X} = \{x_1, x_2, \ldots, x_d, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_d\}$$

and the semi-Thue system S over X defined by

$$S = \{(x_i \bar{x}_i, \epsilon)\}_{1 \leq i \leq d} \bigcup \{(\bar{x}_i x_i, \epsilon)\}_{1 \leq i \leq d}.$$

The *free group* with d generators F is the quotient $\hat{X}^* / \xleftrightarrow{*}_S$. We say that a word is *reduced* when it is S-irreducible. We denote by I the set of all S-irreducible words. As S is confluent and length-decreasing, every word f is equivalent $\pmod{\xleftrightarrow{*}_S}$ to a unique reduced word that we shall denote by $\delta(f)$.
Benois's theorem tells that for every rational set R in $\hat{X}^*$ the set $\delta(R)$ is again rational (prop.2.8 p.59 of [3]). A corollary of this theorem tells that $Rat(F)$ is closed under *complementation* and *intersection* (corollary 2.10 p.60 of [3]). Another way to express this result is given in the following

Lemma 2.2 *Let us denote by $Rat_S(\hat{X}^*)$ the family of all languages of the form $[R]_{\xleftrightarrow{*}_S}$ where R is any rational subset of $\hat{X}^*$.*
$Rat_S(\hat{X}^*)$ is closed under complementation and intersection.

Another closure property is given in the following

Lemma 2.3 *If L, R belong to $Rat_S(\hat{X}^*)$ then LR^{-1} and $R^{-1}L$ belong to $Rat_S(\hat{X}^*)$.*

Here, by LR^{-1} we mean

$$\{f \in X^* | \exists r \in R, fr \in L\}$$

and, as well

$$R^{-1}L = \{f \in X^* | \exists r \in R, rf \in L\}.$$

Proof:
To prove this lemma it suffices to use the fact that $\hat{X}^*/ \stackrel{*}{\longleftrightarrow}_S$ is a group.
□

For every letter $x \in X$, by $\bar{x}$ we denote the letter

- $\bar{x}_i$ if $x = x_i$
- x_i if $x = \bar{x}_i$.

We extend this notation to every word

$$f = y_1 y_2 \ldots y_n$$

(where for $i \in [1, n], y_i \in \hat{X}$)by setting

$$\bar{f} = \bar{y}_1 \bar{y}_2 \ldots \bar{y}_n.$$

By ${}^t f$ we understand the mirror image of f:

$${}^t f = y_n \ldots y_2 y_1.$$

Composing these transformations we have

$${}^t \bar{f} = \bar{y}_n \ldots \bar{y}_2 \bar{y}_1.$$

By $\Pi : \hat{X}^* \to F$ we denote the canonical projection. The inverse image by Π of the conjugacy relation in F is the following equivalence relation $\sim$:

$$f \sim g \Longleftrightarrow \exists u \in \hat{X}^*, uf{}^t\bar{u} \stackrel{*}{\longleftrightarrow}_S g.$$

A word f is cyclically reduced iff

- either $f = \epsilon$
- or $f \neq \epsilon$, f is reduced, and $First(f) \neq \overline{Last(f)}$

which is equivalent to saying that $\gamma(f) \subseteq I$. Every word is equivalent (mod $\sim$) to at least one cyclically reduced word and two equivalent (for $\sim$) cyclically reduced words must be conjugates:

$$f \sim g \text{ and } f, g \text{ cyclically reduced } \Longrightarrow f \approx g$$

(theorem 1.3 p.36 of [24]). Let us denote by C the set of cyclically reduced words. For every f in $\hat{X}^*$ we set

$$\theta(f) = [f]_\sim \cap C.$$

Lemma 2.4 *If R is rational then $\theta(R)$ is rational.*

Proof:
One can check that,

$$\forall f \in \hat{X}^*, \theta(f) = (\delta \circ \gamma \circ \delta(f)) \cap C.$$

By Benois' theorem δ preserves rationality. It is well-known that γ preserves rationality too. As C is rational, the intersection with C preserves rationality too. Hence, for every $R \in Rat(\hat{X}^*)$

$$(\delta \circ \gamma \circ \delta(R)) \cap C \text{ is rational .}$$

□
The following is straightforward.

Lemma 2.5 *If L is a language of $\hat{X}^*$ which is saturated by $\sim$, then $\theta(\hat{X}^* - L) = C - \theta(L)$.*

2.2 Link between saturated languages of $\hat{X}^*$ and languages of F

Let $\equiv$ be a congruence on $\hat{X}^*$ such that $\stackrel{*}{\longleftrightarrow}_S \subseteq \equiv$.
We can define a congruence $\Pi(\equiv)$ on F by:

$$\forall u, v \in F, u\Pi(\equiv)v \stackrel{def}{\Longleftrightarrow} \exists u_1, v_1 \in \hat{X}^* | \Pi(u_1) = u, \Pi(v_1) = v \text{ and } u_1 \equiv v_1.$$

The following facts can be easily established

- the mapping $\equiv \mapsto \Pi(\equiv)$ is a bijection between the congruences on $\hat{X}^*$ containing $\stackrel{*}{\longleftrightarrow}_S$ and the congruences of F.
- The image of $\stackrel{*}{\longleftrightarrow}_S$ by this mapping is the equality relation in F.
- The mapping $[f]_\equiv \mapsto [\Pi(f)]_{\Pi(\equiv)}$ is well-defined and is an isomorphism between $\hat{X}^*/\equiv$ and $F/\Pi(\equiv)$.
- If L is a language of $\hat{X}^*$ which is saturated by $\stackrel{*}{\longleftrightarrow}_S$, then the image of its syntactic congruence is the syntactic congruence of $\Pi(L)$, i.e.

$$\Pi(\equiv_L) = \equiv_{\Pi(L)} .$$

From these facts it is clear that, if L is a language of $\hat{X}^*$ which is saturated by $\stackrel{*}{\longleftrightarrow}_S$ then:

- $\equiv_L$ is equal to $\stackrel{*}{\longleftrightarrow}_S$ if and only if $\equiv_{\Pi(L)}$ is the equality in F

- $\equiv_L$ has finite index if and only if $\equiv_{\Pi(L)}$ has finite index.

Moreover, a subset K of F is rational iff there exists a rational subset R of $\hat{X}^*$ such that $K = \Pi([R]_{\overset{*}{\longleftrightarrow}_S})$. Hence theorem 1.1 can be translated as follows

Theorem 2.6 *If R is a rational subset of $\hat{X}^*$ and $L = [R]_{\overset{*}{\longleftrightarrow}_S}$ either $\equiv_L = \overset{*}{\longleftrightarrow}_S$ or $\equiv_L$ has finite index.*

By Benois's theorem, without loss of generality we can suppose that $R \subseteq I$.

3 PROOF OF THE CONJECTURE

Let R be a rational subset of $\hat{X}^*$, $L{=}[R]_{\overset{*}{\longleftrightarrow}_S}$ and $L_0 = [\epsilon]_{\equiv}$ (where $R \subseteq I$ and $\equiv$ is the syntactic congruence of L). Our aim is to show that $\equiv$ is either the congruence $\overset{*}{\longleftrightarrow}_S$ itself or is a congruence of finite index.
Our *first step* will be to show that the set of cyclically reduced words associated to L_0, $\theta(L_0)$, is rational.
The *second step* will consist in exhibiting a finite special semi-Thue system T such that $L_0 = [\epsilon]_{\overset{*}{\longleftrightarrow}_T}$.
The *third step* will consist in proving that if $\theta(L_0) \neq \{\epsilon\}$, then the set W of lefthand sides of the system T is unavoidable. It follows that L_0 is rational and that $\equiv$ has finite index.

STEP 1

Lemma 3.1 *For every $f \in \hat{X}^*$, $f \in \hat{X}^* - [\epsilon]_{\equiv}$ iff $\exists r_1, r_2, u \in I$ such that one of the following cases occurs*

case 1 $r_1 r_2 \in R \ \wedge \ r_1 u f^t \bar{u} r_2 \in [I - R]_{\overset{*}{\longleftrightarrow}_S}$

case 2 $r_1 r_2 \in I - R \ \wedge \ r_1 u f^t \bar{u} r_2 \in [R]_{\overset{*}{\longleftrightarrow}_S}$.

Proof:

- The only-if part of this statement is clear because $(r_1 u_!^t \bar{u} r_2)$ would be a context sending ϵ into L (resp. into $\hat{X}^* - L$) while sending f into $\hat{X}^* - L$ (resp. into L).

- Let us suppose that $f \in \hat{X}^* - [\epsilon]_{\equiv}$.
  Then there exists a context $(m_1, m_2) \in \hat{X}^* \times \hat{X}^*$ such that one of the following cases occurs

  case 1 $m_1 m_2 \in L$ and $m_1 f m_2 \in \hat{X}^* - L$

  case 2 $m_1 m_2 \in \hat{X}^* - L$ and $m_1 f m_2 \in L$.

Let us treat case 1:
As L is saturated by $\xleftrightarrow{*}_S$, we can suppose that m_1, m_2 are reduced.
$m_1 m_2 \xrightarrow{*}_S r \in R$ and $m_1, m_2 \in I$ imply that there exist r_1, r_2, u such that $m_1 = r_1 u, m_2 = {}^t\bar{u} r_2$ and $r_1, r_2, u, r_1 r_2 \in I$.
Moreover $m_1 m_2 \xrightarrow{*}_S r_1 r_2 \in I$ hence $r_1 r_2 = r \in R$.
Case 2 can be treated in the same way.

□

Lemma 3.2 $\theta(L_0)$ *is rational.*

Proof: As L_0 is saturated by $\sim$, according to lemma 2.5, it is sufficient to prove that $\theta(\hat{X}^* - L_0)$ is rational.
Let $\asymp$ be a congruence of $\hat{X}^*$ included in $\equiv_R$ and in $\equiv_{I-R}$ and having finite index. Let $\overset{\asymp}{\Pi}$ be the canonical projection $\hat{X}^* \rightarrow \hat{X}^*/\asymp$ and $\odot$ be the product in $\hat{X}^*/\asymp$.
Let $\{(K_i, H_i)\}_{i \in [1,n]}$ be the finite set of all pairs (K, H) of classes $\pmod{\asymp}$ such that $K \odot H$ is in $\overset{\asymp}{\Pi}(R)$.
As well $\{(K'_j, H'_j)\}_{j \in [1,p]}$ is the set of all pairs (K', H') of classes $\pmod{\asymp}$ such that $K' \odot H'$ is in $\overset{\asymp}{\Pi}(I - R)$.
We claim that a word f is in $\hat{X}^* - L_0$ iff there exists $u \in I$ such that

$$u f {}^t\bar{u} \in \bigcup_{i \in [1,n]} K_i^{-1} [I - R]_{\xleftrightarrow{*}_S} H_i^{-1} \bigcup_{j \in [1,p]} K_j'^{-1} [R]_{\xleftrightarrow{*}_S} H_j'^{-1} \qquad (1)$$

The first term of this union corresponds to those f which fulfill case 1 of lemma 3.1, the second term to those fulfilling case 2 of lemma 3.1.
But the family of languages $Rat_S(X^*)$ is closed under union, left-quotient by a regular set, right-quotient by a regular set (lemma 2.3). Hence the righthand-side of relation (1) is of the form $[H]_{\xleftrightarrow{*}_S}$ where H is a rational set. It follows that $\hat{X}^* - L_0 = [H]_\sim$. Hence $\theta(\hat{X}^* - L_0) = \theta(H)$ which is rational by lemma 2.4. □

STEP 2

Let $\mathcal{A} = < Q, X, s, q_0, Q_+ >$ be a finite deterministic automaton recognizing the set $\theta(L_0)$. Let $p = |Q|$.
We define finite sets of words W_1, W_2, W_3, W by:
$W_1 = \{f \in \theta(L_0) | 0 < |f| \leq p - 1\}$
$W_2 = \{f \in I | 0 < |f| \leq p \text{ and } \exists q \in Q_{acc} \bigcap Q_{coacc}, q.f = q\}$
$W_3 = \{x\bar{x}\}_{x \in \hat{X}}$ and $W = \bigcup_{i \in [1,3]} W_i$.
Let us consider the semi-Thue system $T = \{(w, \epsilon)\}_{w \in W}$

Lemma 3.3 $L_0 = [\epsilon]_{\xleftrightarrow{*}_T}$.

Proof:
1- Let us observe that every $w \in W$ is in L_0:

- this is clear if $w \in W_1$ or $w \in W_3$
- if $w \in W_2$, there exists $u, v \in \hat{X}^*$, such that $q_0.u = q$ and $q.v \in Q_+$; hence $q_0.uv \in Q_+$ and $q_0.uwv \in Q_+$, so that

$$uwv \equiv uv \tag{2}$$

As $\hat{X}^*/\equiv$ is cancellative, (2) implies that $w \equiv \epsilon$.
Hence $T \subseteq \equiv$ and $[\epsilon]_{\overset{*}{\longleftrightarrow}_T} \subseteq L_0$.

2- In order to prove that $L_0 \subseteq [\epsilon]_{\overset{*}{\longleftrightarrow}_T}$, we prove the stronger property:
$L_0 \subseteq <\epsilon>_{\overset{*}{\longleftarrow}_T}$.
It suffices to show that $f \in L_0 - \{\epsilon\}$ implies that there exists $g \in \hat{X}^*$ such that $f \longrightarrow_T g$ (and by point 1 of this proof it follows that $g \in L_0$).
Let $f \in L_0 - \{\epsilon\}$.

case 1: $f \in \hat{X}^* - I$
then f has a factor in W_3, hence $f \longrightarrow_T g$ (for some word g).

case 2: $f \in I$
then $f = uf'^t\bar{u}$ where f' is cyclically reduced.
$f' \in \theta(L_0) - \{\epsilon\}$. Either $|f'| \leq p-1$ and then $f' \in W_1$ or $|f'| \geq p$ and by the classical pumping lemma (see for example [3] p.21) f' has a factor in W_2; in both subcases $f \longrightarrow_T g$ (for some word g).
□

STEP 3
We remind that $[\epsilon]_\equiv = L_0 = [\epsilon]_{\overset{*}{\longleftrightarrow}_T}$. As $\hat{X}^*/\equiv$ is a group, and since all group-congruences are determined by their ϵ-classes, the above equalities imply that

$$\equiv = \overset{*}{\longleftrightarrow}_T .$$

If $\theta(L_0) = \{\epsilon\}$, then $W = W_3$, so $\equiv = \overset{*}{\longleftrightarrow}_S$ and $\Pi(L)$ is disjunctive.
We deal now with the remaining case.

Lemma 3.4 *If $\theta(L_0) \neq \{\epsilon\}$ then the set of factors of $\theta(L_0)$ is exactly I.*

Proof: It suffices to prove that every reduced word is a left-factor of $\theta(L_0)$.

case 1: $|\hat{X}| = 2$ (i.e. $F = Z$)
Then, if $\theta(L_0) \neq \{\epsilon\}$, there exists some integer $p > 0$ such that $\theta(L_0) \supseteq \{x^p\}^* \cup \{\bar{x}^p\}^*$. Hence every reduced word is a factor of $\theta(L_0)$.

case 2: $|\hat{X}| \geq 4$

Let $f = x_1 x_2 \ldots x_n \in I$.

Let $u \in \theta(L_0) - \{\epsilon\}$ such that $First(u) \neq \bar{x}_n$. Such a u exists because, otherwise, as $\theta(L_0)$ is saturated by $\approx$, we should have $\theta(L_0) \subseteq \bar{x}_n^*$. But $\theta(L_0)$ is also saturated by the mapping $g \mapsto {}^t\bar{g}$, hence a word x_n^p (with $p > 0$) would belong to $\theta(L_0)$, contradicting the hypothesis.

Let $x \in \hat{X} - \{x_1, Last(u)\}$ (such an x exists because $|\hat{X}| \geq 4$) and

$$g = fu^3 \bar{x}_n u^3 \bar{x}_{n-1} \ldots \bar{x}_1 u^3 x u^3 \bar{x}.$$

For every letter $y \in \hat{X}, \delta(uyu)$ has one of the two following forms:

- $u_1 u_2$ where $u_1 \in$ *Pref*$(u) - \{\epsilon\}$ and $u_2 \in$ *Suf*$(u) - \{\epsilon\}$
- or z where $z \in \{First(u), Last(u)\}$

It follows that $u\delta(uyu)u$ is irreducible.

Moreover $\delta(u\bar{x}) = u\bar{x}$ (because $x \neq Last(u)$).

Hence

$$\delta(g) = fu^2 \delta(u\bar{x}_n u) u \delta(u\bar{x}_{n-1}u) \ldots \delta(u\bar{x}_1 u) u \delta(uxu) u^2 \bar{x} \qquad (3)$$

$\delta(g)$ is cyclically reduced because $x \neq x_1$.

From the relations

$$u \equiv \epsilon, f\bar{x}_n \ldots \bar{x}_2 \bar{x}_1 \equiv \epsilon, x\bar{x} \equiv \epsilon$$

we deduce that

$$g \equiv \epsilon$$

and as $\xleftrightarrow{*}_S \subseteq \equiv$, we have

$$\delta(g) \equiv \epsilon$$

Hence $\delta(g) \in \theta(L_0)$ and, by formula (3), f is a left-factor of $\theta(L_0)$.

□

Lemma 3.5 *W is unavoidable.*

Proof:

Let $f \in \hat{X}^*$ such that $|f| \geq p$.

case 1: $f \in \hat{X}^* - I$

then $f \in \hat{X}^* W_3 \hat{X}^*$

case 2: $f \in I$

then by lemma 3.4, there exists $s \in \hat{X}^*$ such that $fs \in \theta(L_0)$ and by the pumping lemma $f \in \hat{X}^* W_2 \hat{X}^*$.

$\square$

Now we have proved that if $\theta(L_0) \neq \{\epsilon\}$ then the set W of lefthand sides of the system T is unavoidable. By lemma 2.1 it follows that $\equiv = \overset{*}{\longleftrightarrow}_T$ has finite index. Hence the proof of theorem 1.1 is achieved.

We remark that our proof is *constructive*.

In terms of language theory:
Given a rational subset R of $\hat{X}^*$ (defined for example by a regular expression or a finite automaton), we are able to test whether the syntactic congruence of the language $[R]_{\overset{*}{\longleftrightarrow}_S}$ is equal to $\overset{*}{\longleftrightarrow}_S$ or has finite index (it suffices to compute a f.a. $\mathcal{A}$ recognizing $\theta([\epsilon]_{\equiv})$ and test whether $L(\mathcal{A}) = \{\epsilon\}$).
In the case where $\equiv$ has finite index we can compute a finite semi-Thue system T generating $\equiv$ and a finite set of representatives $\pmod{\equiv}$ which is $\{f \in I \mid |f| \leq p-1\}$. The word-problem for $\equiv$ is solved by the following linear algorithm:
given a pair (f, g) of words, we determine an h such that $f^t\bar{g} \overset{*}{\longrightarrow}_S h$ and h is T-irreducible. $f \equiv g$ iff $h = \epsilon$ (point 2 of the proof of lemma 3.3).

In terms of group-theory:
Given a rational subset R of a free group F (defined for example by a rational expression on the generators of F) we can test whether R is disjunctive or has finite index. In this last case we can compute a finite set of generators for the largest normal subgroup N such that $RN \subseteq R$: it suffices to compute a regular expression for $N = \Pi([\epsilon]_{\equiv})$ (by lemma 3.2 we get a f.a. $\mathcal{A}$ recognizing $\theta([\epsilon]_{\equiv})$, from which we deduce W and then a f.a. $\mathcal{B}$ recognizing $[\epsilon]_{\equiv}$, which we can convert into a regular expression by a classical algorithm) and deduce from this expression a set of generators of N (a method for this is given in the proof of theorem 2.7 p.58 of [3]). The set $P = \{\Pi(f) | f \in I \wedge |f| \leq p-1\}$ is a set of representatives $\pmod{N}$.

4 GENERALISATION TO VIRTUALLY FREE GROUPS

A group is said virtually free iff it has a free subgroup of finite index. It is shown in [25] that this algebraic notion is closely related to a classical notion in the area of formal language theory: the notion of context-free language. A group G is said context-free if and only if it admits a finite presentation $< X, E >$ such that $[\epsilon]_{\overset{*}{\longleftrightarrow}_E}$ is a context-free language over X^* ([1]). An easy lemma shows that in fact this property does not depend on the particular chosen presentation but on the group G only. The main theorem of [25] is the following

Theorem 4.1 *Let G be a finitely generated group. Then G is virtually free if and only if G is context-free and accessible.*

The notion of *accessible* group is a technical condition which is explained in detail in [25]. Dunwoody's theorem ([13]) asserts that every finitely presented group is accessible. As every context-free group is finitely presented ([1]), we get the following theorem [1].

Theorem 4.2 *A finitely generated group G is virtually free if and only if it is context-free.*

We shall prove here the

Theorem 4.3 *Let G be a group which is finitely generated and virtually free. Let R be a rational subset of G and N be its syntactic normal subgroup. Then either N is finite or G/N is finite.*

As every finite subgroup of a free group is trivial, theorem 4.3 is a generalisation of theorem 1.1.
Taking into account the language-theoretic characterisation given in theorem 4.2, we can express theorem 4.3 in the following way:

Theorem 4.4 *Let E be a finite semi-Thue system over a finite alphabet X such that $X^*/\xleftrightarrow{*}_E$ is a group. If $[\epsilon]_{\xleftrightarrow{*}_E}$ is context-free then, for every rational language $R \subseteq X^*$, the syntactic congruence $\equiv$ of the language $L = [R]_{\xleftrightarrow{*}_E}$ is such that:*

- *either every class $[f]_\equiv$ is a finite union of classes* $\pmod{\xleftrightarrow{*}_E}$
- *or $\equiv$ has finite index.*

Let us now prove theorem 4.3. We assume the reader is familiar with reference [25].
Let G be a finitely generated, virtually free group. By definition, G admits a free subgroup H which has finite index. As remarked in the proof of lemma 3 p.299 of [25], we can suppose that H is normal in G. Let $n = [G : H]$ and $p = rank(H)$.
We shall first give a particular presentation $< Y, E >$ of G and show that it fulfills some properties analogous to the properties of the presentation $< \hat{X}, S >$ of the free group considered in section 2 (these properties are stated in lemma 4.11 and 4.12).
Let us consider

- a rational subset R of G
- a base of H : $\{h_i\}_{i\in[1,p]}$

[1] [29] contains another proof of the fact that every c.f. group is accessible; moreover an explicit upper-bound on the accessiblity-length is given.

- a system of representatives $\pmod H$: $\{r_i\}_{i\in[0,n-1]}$ where $r_0 = 1_G$
- for every $i \in [0, n-1]$ the subset $R_i = (R \cap Hr_i)r_i^{-1}$.

We express in lemma 4.15 a relation between the syntactic normal subgroup $Synt_G(R)$ and the syntactic normal subgroups $(Synt_H(R_i))_{0\leq i\leq n-1}$. Putting all these lemmas and theorem 1.1 together, we give a proof of theorem 4.3.

4.1 Presentation of G

As was shown in lemma 3 p.299 of [25], a presentation of G can be built as follows: let us consider two disjoint alphabets

$$\hat{X} = \{x_1, x_2, \ldots, x_p, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p\}\ ,\ D = \{d_1, d_2, \ldots, d_{n-1}\}$$

and define an alphabet Y by $Y = \hat{X} \cup D$.
We define a surjective homomorphism $\varphi : Y^* \to G$ by

$$\begin{aligned} \forall i \in [1,p], \varphi(x_i) &= h_i \\ \forall i \in [1,p], \varphi(\bar{x}_i) &= h_i^{-1} \\ \forall j \in [1,n-1], \varphi(d_j) &= r_j \end{aligned}$$

The congruence $\ker(\varphi)$ (i.e. the relation on Y^* defined by $(f,g) \in \ker(\varphi) \Leftrightarrow \varphi(f) = \varphi(g)$) is generated by the finite semi-Thue system E defined below (definition 4.5); some properties of the system E are then established (lemma 4.6 to 4.11).

Definition 4.5 *E is the semi-Thue system over Y^* consisting of all the following rules*

(E_1) $\forall i \in [1,p],\ x_i\bar{x}_i \longrightarrow \epsilon$ *and* $\bar{x}_i x_i \longrightarrow \epsilon$

(E_2) $\forall j \in [1,n-1], \forall k \in [1,n-1], d_j d_k \longrightarrow z_{j,k} d_l$
where $l \in [0,n-1], r_j r_k \equiv r_l \pmod H$ *and* $z_{j,k}$ *is the unique reduced word over* $\hat{X}^*$ *such that* $r_j r_k = \varphi(z_{j,k}) r_l$

(E_3) $\forall j \in [1,n-1], \forall x \in \hat{X}, d_j x \longrightarrow u_{j,x} d_j$
where $u_{j,x}$ *is the unique reduced word over* $\hat{X}^*$ *such that* $r_j\varphi(x) = \varphi(u_{j,x})r_j$.

Lemma 4.6 $\longrightarrow_E \subseteq \ker(\varphi)$.

Proof: One can check that every rule given in definition 4.5 is in $\ker(\varphi)$.
□

Lemma 4.7 $Irr(E) = I.(D \cup \{\epsilon\})$.

Proof: obvious from definition 4.5.
□

Lemma 4.8 *Let $w, w' \in Irr(E)$.*
$\varphi(w) = \varphi(w') \Leftrightarrow w = w'$.

Proof: Let $u, u' \in I, d, d' \in D \cup \{\epsilon\}$ such that $\varphi(ud) = \varphi(u'd')$.
As $D \cup \{\epsilon\}$ is a system of representatives $\pmod H$:
$\varphi(u) = \varphi(u')$ and $\varphi(d) = \varphi(d')$.
As $\varphi_{|D}$ is a bijection between $\{d_i\}_{i\in[0,n-1]}$ and $\{r_i\}_{i\in[0,n-1]}$, $d = d'$.
As H is free on the base $\{\varphi(x_i)\}_{i\in[1,p]}$ and u, u' are reduced words, $u = u'$.
□

Lemma 4.9 *E is noetherian.*

Proof: Let us set

$$l = \max(\{|z_{j,k}|\}_{(j,k)\in[1,n-1]^2} \bigcup \{|u_{j,x}|\}_{j\in[1,n-1],x\in\hat{X}}) + 1$$

For every $f \in Y^*$ we define the size of f, $||f||$, by

if $f \in X^*$ then $||f|| = |f|$

else let us consider the decomposition

$$f = v_0 d_{i,1} v_1 \dots d_{i,j} v_j \dots d_{i,q} v_q \ (\ where\ q \in N - \{0\}, v_j \in \hat{X}^*\ and\ d_{i,j} \in D)$$

and set

$$||f|| = ql + \sum_{j=0}^{q} |v_j| l^j$$

One can check that $f \longrightarrow_E g$ implies $||f|| > ||g|| \geq 0$.
Hence every chain

$$f = f_1 \longrightarrow_E f_2 \longrightarrow_E \dots \longrightarrow_E f_s$$

has a length $s \leq ||f||$, which proves that E is noetherian.
□

Lemma 4.10 $\overset{*}{\longleftrightarrow}_E = \ker(\varphi)$.

Proof: By lemma 4.6, it is sufficient to prove that $\ker(\varphi) \subseteq \overset{*}{\longleftrightarrow}_E$.
Let $f, g \in Y^*$ such that $\varphi(f) = \varphi(g)$.
By lemma 4.9, there exist $u, v \in Irr(E)$ such that $f \overset{*}{\longrightarrow}_E u$ and $g \overset{*}{\longrightarrow}_E v$, and by lemma 4.6, $\varphi(u) = \varphi(v)$. By lemma 4.8, $u = v$, hence $f \overset{*}{\longrightarrow}_E {}_E\overset{*}{\longleftarrow} g$.
□

Let us notice that $< Y, E >$ is then a finite presentation of G.

Lemma 4.11 *E is confluent and noetherian.*

Proof: By lemma 4.9 E is noetherian. Let $f, g \in Y^*$ such that $f \overset{*}{\longleftrightarrow}_E g$. By lemma 4.10, $\varphi(f) = \varphi(g)$. The proof of lemma 4.10 then shows that $f \overset{*}{\longrightarrow}_E \, {}_E\overset{*}{\longleftarrow} g$. Hence E is Church-Rosser or, equivalently, confluent.
□

A consequence of lemma 4.11 is that for every $f \in Y^*$,

$$Card([f]_{\overset{*}{\longleftrightarrow}_E} \cap Irr(E)) = 1$$

We denote by $\rho(f)$ the unique element of the above set of cardinality one.
The map $\rho : Y^* \rightarrow Y^*$ has the following property

Lemma 4.12 *Let R be a rational subset of Y^*. Then, $\rho(R)$ is rational too.*

Proof: Let $\mathcal{M}$ be the deterministic pushdown automaton described on p. 300 of [25]. If $f \in Y^*$, after reading f, $\mathcal{M}$ has on its stack a word w and is in a state q_j such that $\rho(f) = wd_j$. In other words, ρ is *computed* by the d.p.d.a. $\mathcal{M}$. According to a result of S.Greibach ([17]) every relation computed by a p.d.a. preserves rationality. Hence ρ preserves rationality.
□

Remark:
The main result of [23] states that the validity problem in a given context-free group G for the so-called *linear sentences* (see [7]) is in P-space. The main tool used in [23] is a monadic finite semi-Thue system E' which is confluent on ϵ and which presents G (the existence of such a system for every context-free group G was established in [2]). One of the key properties shown in [23] is then that the set of *irreducible* words associated (mod E') to a given rational set R is rational too (lemma 5 of [23]). A variant of this proof would consist in considering system E instead of E' and then applying lemma 4.11 instead of lemma 5 of [23] . [2]

4.2 Properties of $Rat(G)$

Lemma 4.13 *The family $Rat(G)$ of rational subsets of G is closed under complementation and finite intersection.*

Proof: As E and ρ fulfill properties analogous to those of S and δ, the classical arguments proving that $Rat(X^*/\overset{*}{\longleftrightarrow}_S)$ is closed under complementation and finite intersection (see [3] p.60,corollary 2.10) are still valid here.
□

Remark: Lemma 4.12 (resp. 4.13) are also consequences of corollary 5.1 (resp.5.2) of [27] because G is a monoid *described* by a d.p.d.a.

Lemma 4.14 $Rat(G) \bigcap Subset(H) = Rat(H)$.

[2] We thank an anonymous referee for this remark

Proof: The inclusion $Rat(H) \subseteq Rat(G) \bigcap Subset\ (H)$ is obvious.
Let $R \in Rat(G) \bigcap Subset\ (H)$. Then there exists $K \in Rat(Y^*)$ such that $R = \varphi(K)$. We then have $R = \varphi(\rho(K))$.
But $K \subseteq \varphi^{-1}(H)$ hence $\rho(K) \subseteq \rho(\varphi^{-1}(H)) = I \subseteq \hat{X}^*$. So, $\rho(K)$ is definable by a rational expression on the alphabet $\hat{X}$. Hence R is defined by the same expression where the letters $x \in \hat{X}$ are interpreted as the elements $\varphi(x) \in H$, the product is interpreted as the product in H and the star operation is also interpreted in the monoid H.
Hence $R \in Rat(H)$.
□

We recall that for every subset K of a group A, $Synt_A(K)$ denotes the syntactic normal subgroup of K in A. Let us consider the subsets R and R_i defined just before section 4.1.

Lemma 4.15 $Synt_G(R) \cap H = \bigcap_{i=0}^{n-1} Synt_H(R_i)$.

Proof: We observe the following two facts

Fact 4.16 *For every $i \in [0, n-1]$, $Synt_G(R) \cap H$ saturates R_i.*

For, let $h \in Synt_G(R) \cap H$ and $g_i \in R_i$.
Then $g_i r_i \in R \cap H r_i$, so

- $h g_i r_i \in R$ (because $h \in Synt_G(R)$)
- and $h g_i r_i \in H r_i$ (because h and g_i are in H)

Hence $h g_i r_i \in R \cap H r_i$ and $h g_i \in R_i$.

Fact 4.17 *$\bigcap_{i=0}^{n-1} Synt_H(R_i)$ saturates R.*

For, let $h \in \bigcap_{i=0}^{n-1} Synt_H(R_i)$ and $g \in R$.
There exists some $i_0 \in [0, n-1]$ such that $g \equiv r_{i_0} \pmod{H}$.
Hence $g \in R_{i_0} r_{i_0}$.
As $h \in Synt_H(R_{i_0})$, $hg \in R_{i_0} r_{i_0}$, hence $hg \in R$.
From Fact 4.16, it follows that $Synt_G(R) \cap H \subseteq \bigcap_{i=0}^{n-1} Synt_H(R_i)$.
From Fact 4.17, it follows that $\bigcap_{i=0}^{n-1} Synt_H(R_i) \subseteq Synt_G(R) \cap H$.
□

4.3 Proof of theorem 4.3

Let us suppose that R is in $Rat(G)$. As H is a rational subset too, by lemma 4.13 each R_i is rational and by lemma 4.14 is a rational subset of H. By theorem 1.1, every subgroup $Synt_H(R_i)$ is either trivial or has finite index.

case 1: there exists some $i \in [0, n-1]$, such that $Synt_H(R_i) = \{1_G\}$.
Then, by lemma 4.15 $Synt_G(R) \cap H = \{1_G\}$. Hence the canonical projection $\pi : G \to G/H$ induces an isomorphism between $Synt_G(R)$ and a subgroup of G/H.
Hence $|Synt_G(R)| \leq n$.

case 2: for every $i \in [0, n-1]$, $Synt_H(R_i)$ has finite index in H.
By lemma 4.15, $Synt_G(R) \cap H$ is a finite intersection of subgroups having finite index in H, hence it has a finite index k in H.
Let us note $N = Synt_G(R)$.
$[G : N \cap H] = [G : H] \times [H : N \cap H] = nk$ and
$[G : N \cap H] = [G : N] \times [N : N \cap H]$, hence $[G : N] \leq nk$.

□

5 Applications and perspectives

5.1 Applications

The results obtained in sections 3 and 4 yield new proofs of the following theorems.

Theorem 5.1 *(Greenberg [16], see [21, proposition 3.11 p.17]) If a finitely generated subgroup H of a free group F contains a non-trivial normal subgroup of F, then it has finite index in F.*

Proof: Let N be the syntactic normal subgroup of H and let N' be some non-trivial normal subgroup of G which is included in H. We have

$$N' \subseteq N \subseteq H. \tag{4}$$

As H is finitely generated it is a rational subset of F. By inclusion (4), N is non-trivial and by theorem 1.1 N has finite index, which shows that H has finite index. □

Theorem 5.2 *(Karass and Solitar, [19]): If G is a finitely generated virtually free group and if H is a finitely generated subgroup of G containing an infinite normal subgroup of G, then H is of finite index in G.*

Proof: Let N be the syntactic normal subgroup of H and let N' be some infinite normal subgroup of G which is included in H. We have

$$N' \subseteq N \subseteq H \tag{5}$$

As H is finitely generated it is a rational subset of F. By inclusion (5), N is infinite and by theorem 4.3 N has finite index, which shows that H has finite index. □

5.2 Perspectives and related work

It would be interesting to analyse the *complexity of the algorithm* underlying our proof of theorem 1.1 (this algorithm is sketched at the end of section 3) and to study the *inherent complexity* of the problem itself. Some algorithmic ideas developed in [20] might be helpful for this purpose.

In [6], the authors present several decidability and complexity results concerning *finitely generated subgroups* of a free group of finite rank F. In view of [6] and of section 5.1 we are lead to ask which results (still making sense for *rational subsets*) could be adapted to arbirary *rational subsets* of F?

In view of the wide use we make of the particular system E given in definition 4.5, one might ask the question: "how difficult is it to construct from some set of generators X of G and some p.d.a. $\mathcal{A}$ recognizing $[\epsilon]_{\equiv}$ (where $< X, \equiv>$ is a presentation of G), a new presentation $< Y, E >$ of G *of the form described in definition 4.5*?"

It follows from [30] and [12, theorem 7.4 p.33] that this construction is *primitive recursive*. It is also known from [29] and [12, theorem 7.4 p.33], that there exists some presentation $< Y, E >$ of the required form, the *size* of which is an *elementary* function of the size of $< X, \mathcal{A} >$.

But it remains unknown whether the *construction* of $< Y, E >$ from $< X, \mathcal{A} >$ can be achieved in *elementary* time.

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