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# Bottom-up rewriting for words and terms

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## 1.1-RECALLS-SEMI-THUE SYSTEMS

A **rewrite rule** over the alphabet  $A$  is a pair

$$l \rightarrow r$$

of words in  $A^*$ .

A **semi-Thue system** is a pair  $(S, A)$  where  $S$  a set of rewrite rules built upon the alphabet  $A$ .

For every  $f, g \in A^*$ ,

$$f \rightarrow_S g$$

iff there exists  $l \rightarrow r \in S$  and  $\alpha, \beta \in A^*$  such that

$$f = \alpha l \beta \ \& \ g = \alpha r \beta.$$

We call  $\rightarrow_S^*$  the **derivation** generated by  $S$ .

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## 1.2-RECALLS- TERM-REWRITING SYSTEMS

A **rewrite rule** is a pair

$$\ell \rightarrow r$$

of terms in  $\mathcal{T}(\mathcal{F}, \mathcal{V})$  which satisfy  $\text{Var}(r) \subseteq \text{Var}(\ell)$ .

A **term rewriting system** is a pair  $(\mathcal{R}, \mathcal{F})$  where  $\mathcal{F}$  is a signature and  $\mathcal{R}$  a set of rewrite rules over the signature  $\mathcal{F}$ . For every  $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ ,

$$s \rightarrow_{\mathcal{R}} t$$

iff there exists  $\ell \rightarrow r \in \mathcal{R}$  a context  $C[\ ]$  and a substitution  $\sigma$  such that

$$s = C[\ell\sigma] \ \& \ t = C[r\sigma].$$

We call  $\rightarrow_{\mathcal{R}}^*$  the **derivation** generated by  $\mathcal{R}$ .

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## 2.1-INTRODUCTION- PROBLEMS

Given a system  $\mathcal{R}$  and a set of terms  $T$ , we define

$$(\rightarrow_{\mathcal{R}}^*)[T] = \{s \in \mathcal{T}(\mathcal{F}) \mid s \rightarrow_{\mathcal{R}}^* t \text{ for some } t \in T\}$$

**Problem:** under which hypothesis over  $\mathcal{R}$  does it hold that, for every recognizable set  $T$ ,  $(\rightarrow_{\mathcal{R}}^*)[T]$  is recognizable too.

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## 2.2-INTRODUCTION- MOTIVATIONS

- decidability of the **accessibility** problem for  $\rightarrow_{\mathcal{R}}$
- decidability of the **termination** problem
- rational subsets of a monoid  $M = A^* / \leftrightarrow_S^*$   
application to resolution of equations with **rational** constraints in  $M$ .
- decidability of the **common-ancestor** problem for  $\rightarrow_{\mathcal{R}}$
- sequentiality problems:
  - computation of a **needed redex** in a term  $t$  w.r.t.  $\mathcal{R}$
  - decidability of the **sequentiality** property for  $\mathcal{R}$



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## 2.3-INTRODUCTION- KNOWN RESULTS

Two kinds of results.

First kind: **syntactical** condition over  $\mathcal{R}$

Generic theorem:

if  $\mathcal{R}$  has property  $P$ , then, for every recognizable set  $T$ ,

$(\rightarrow_{\mathcal{R}}^*)[T]$  is **recognizable** too.

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## 2.3-INTRODUCTION- KNOWN RESULTS

First kind references ([syntax](#)):

- cancellation rules [Benois-Sakarovitch, IPL 86](#)
- basic semi-Thue systems [Benois, RTA 87](#)
- left-basic semi-Thue systems [Sakarovitch, PHD, 79](#)
- ground term-rewriting systems  
[Dauchet-Heuillard-Lescanne-Tison, Inf. and Comput. 87](#)
- linear shallow term-rewriting systems [Comon, LICS 95](#)
- linear growing term-rewriting systems [Jacquemard, RTA 96](#)
- left-linear growing term-rewriting systems  
[Nagaya-Toyama, Inf. and Comput. 02](#)
- left-linear inverse finite-path overlapping TRS  
[Takai-Kaji-Seki, Sci. Math. Jap., 2006](#)

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## 2.3-INTRODUCTION- KNOWN RESULTS

Second kind: use a special **strategy** in derivations

Generic theorem (for the strategy  $\mathcal{S}$ ):

For every TRS  $\mathcal{R}$  and every recognizable set  $T$ ,  $(s \rightarrow_{\mathcal{R}}^*)[T]$  is **recognizable** too.

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## 2.3-INTRODUCTION- KNOWN RESULTS

Second kind references ([strategy](#)):

- one-pass term rewriting

[Fulop-Jurvanen-Steinby-Vagvolgyi, MFCS 98](#)

- concurrent term rewriting

[Seynhaeve-Tison-Tommasi, FCT 99](#)

- “bottom-up derivations”

[Rety-Vuotto, JSC 05.](#)

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## 2.4-INTRODUCTION- **NEW** RESULTS

We define a new notion of **k-bottom-up** derivation, denoted by  $k \rightarrow_{\mathcal{R}}^*$ .

**Theorem 1** *Durand-Sénizergues, RTA'07* Let  $\mathcal{R}$  be some linear rewriting system over the signature  $\mathcal{F}$ , let  $T$  be some **recognizable** subset of  $\mathcal{T}(\mathcal{F})$  and let  $k \geq 0$ . Then, the set  $(k \rightarrow_{\mathcal{R}}^*)[T]$  is **recognizable** too.

We then introduce the class of Bottom-Up systems as the class of all systems for which the above strategy is complete.

**Theorem 2** *Sylvestre, M.th., 2008* For every integer  $k \geq 0$ , the termination property is **decidable** for Term Rewriting Systems in  $SBU(k)$ .

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## 2.5-INTRODUCTION- METHODS

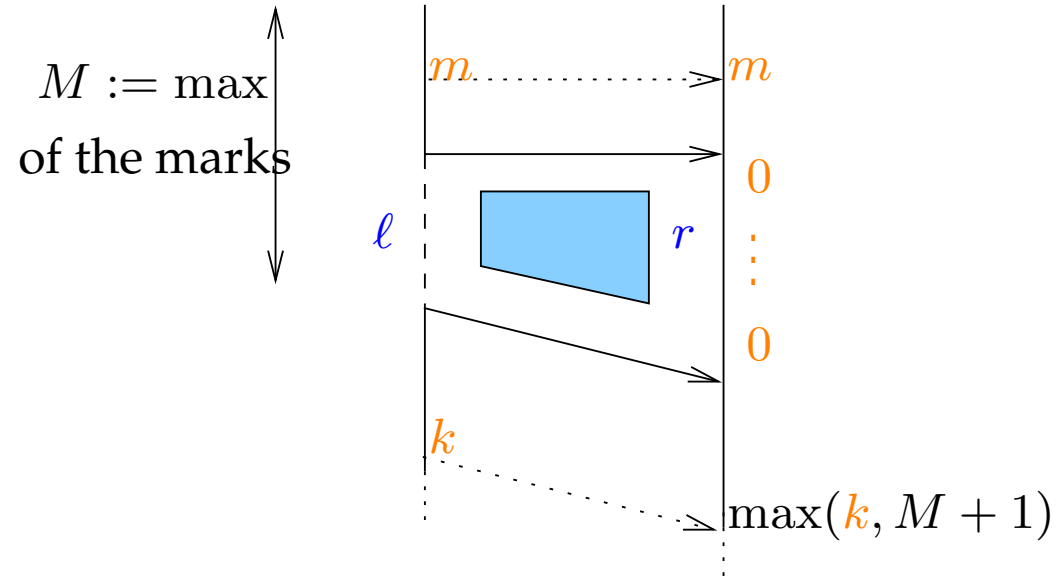
1- Define a **marking** operation: the marks are integers that measure the amount of top-down (i.e. “forbidden”) sequence of rules.

Bottom-up derivation are those derivations with small marks.

2- **Reduce** the preservation property to the same property for **ground** systems: a bottom-up derivation can be **simulated** by a ground derivation.

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### 3.1-MARKED DERIVATIONS- UNARY TERMS

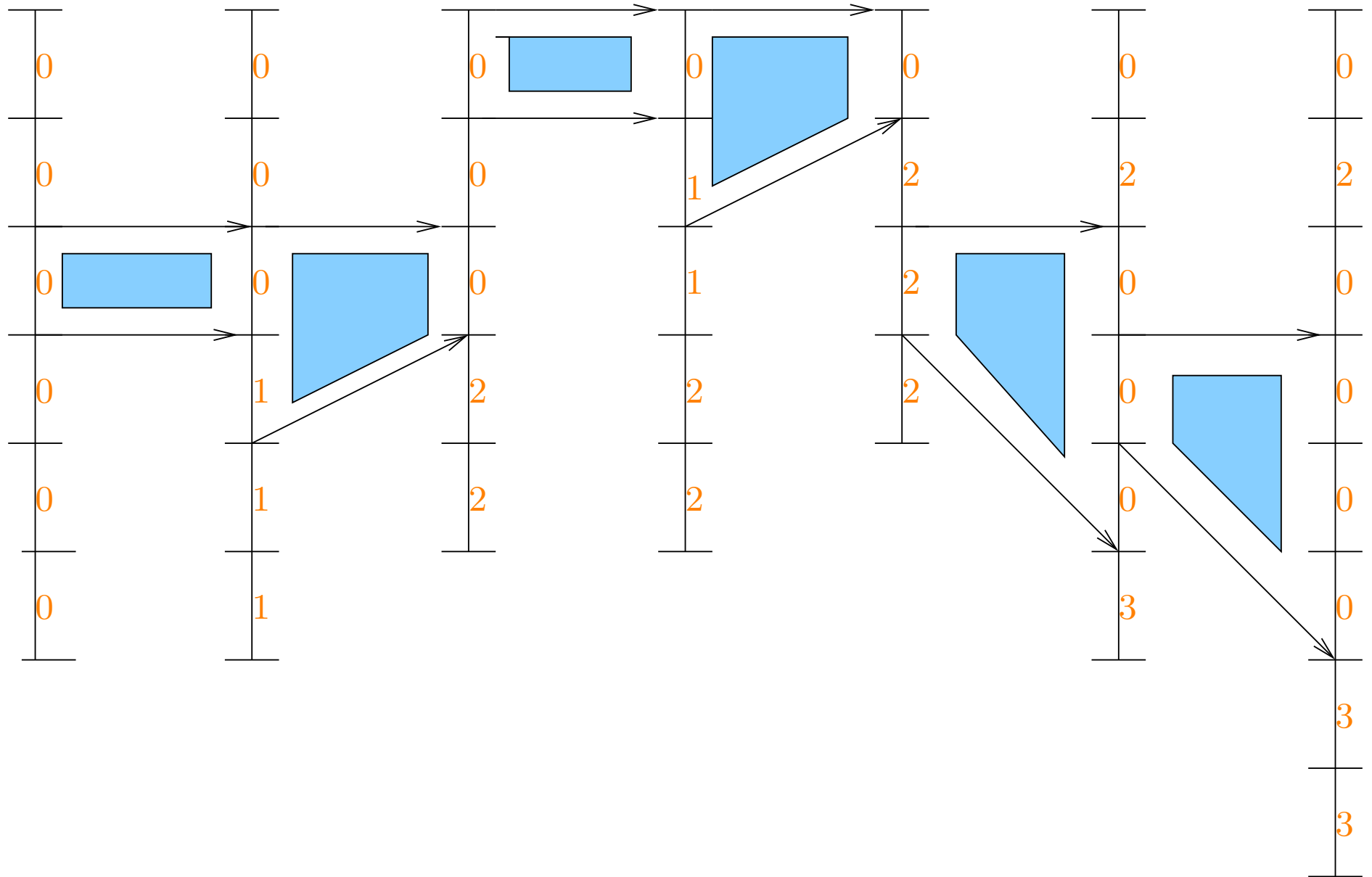


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## 3.1-MARKED DERIVATIONS- UNARY TERMS

A derivation graph.

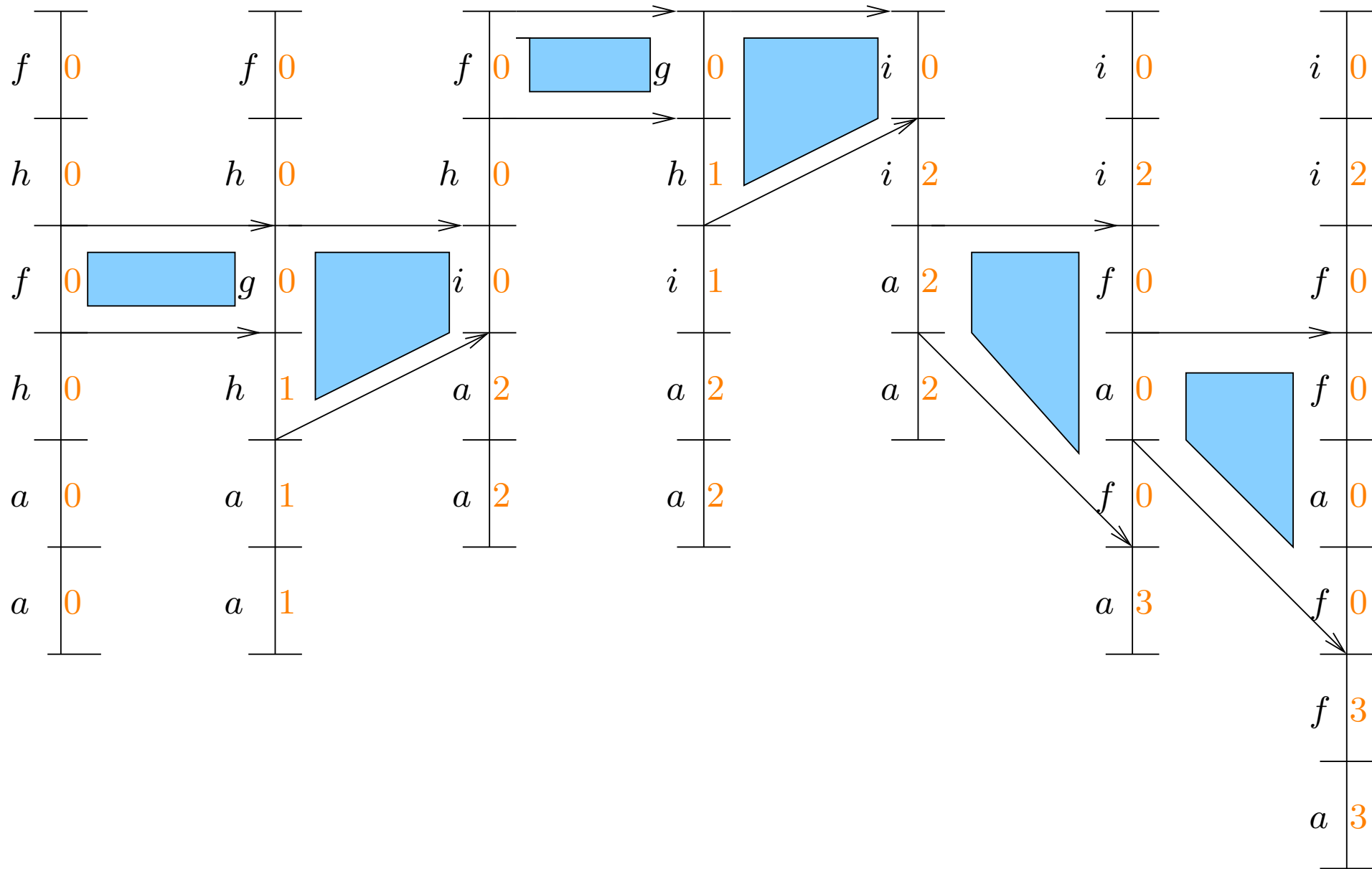




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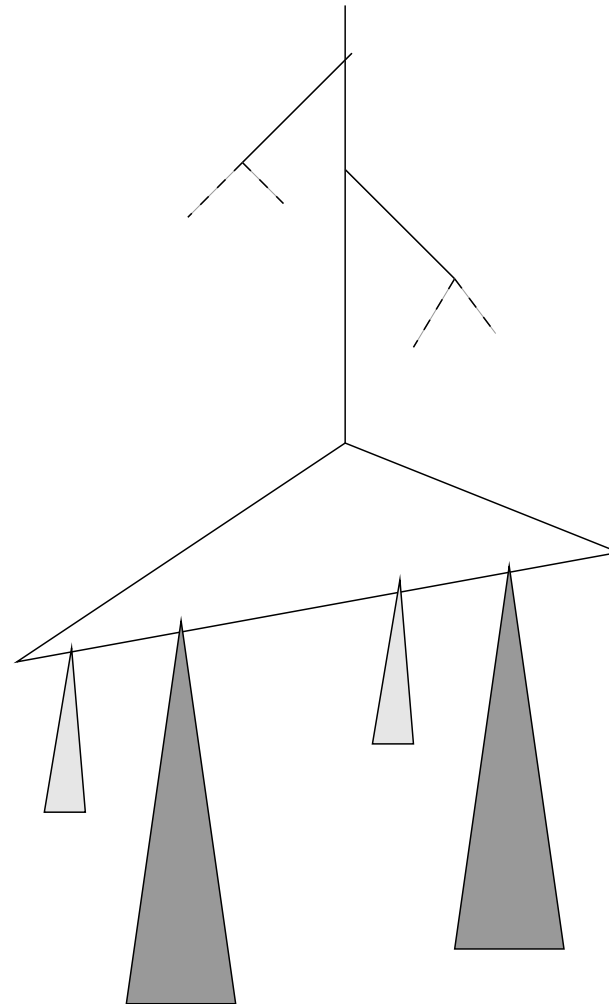
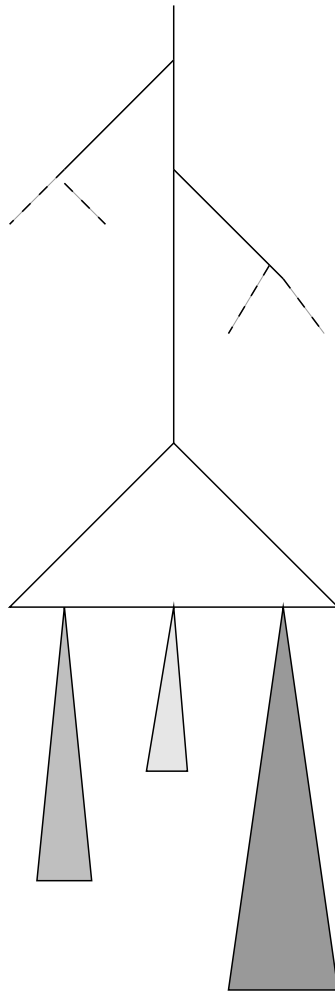
## 3.1-MARKED DERIVATIONS- UNARY TERMS

A derivation.



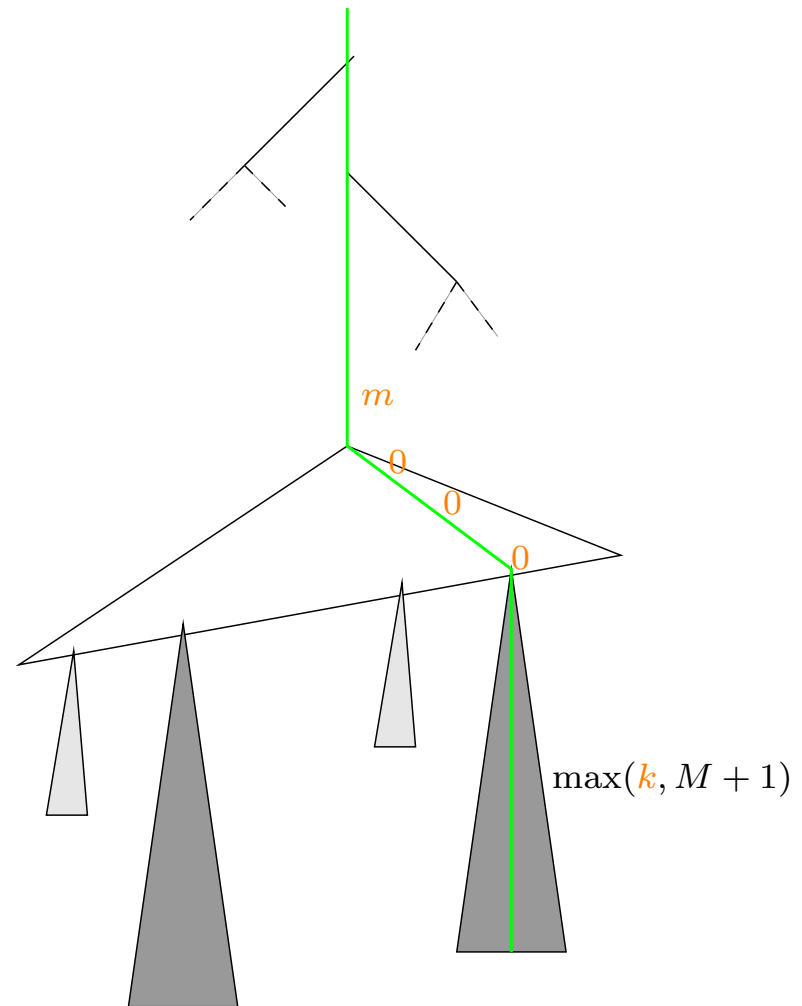
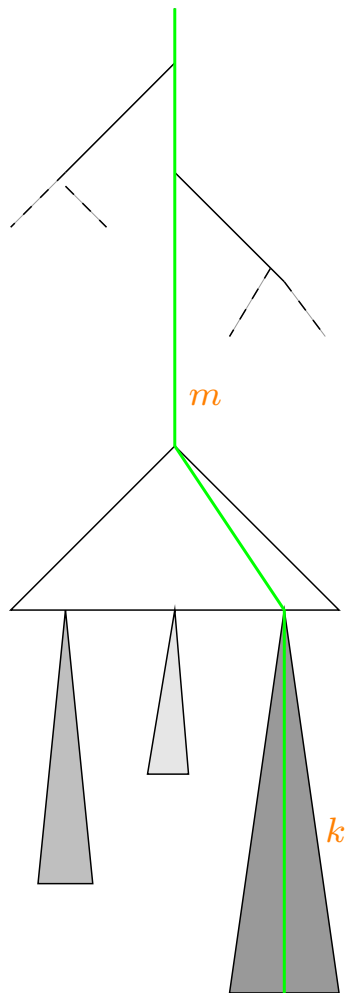
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## 3.2-MARKED DERIVATIONS- GENERAL TERMS



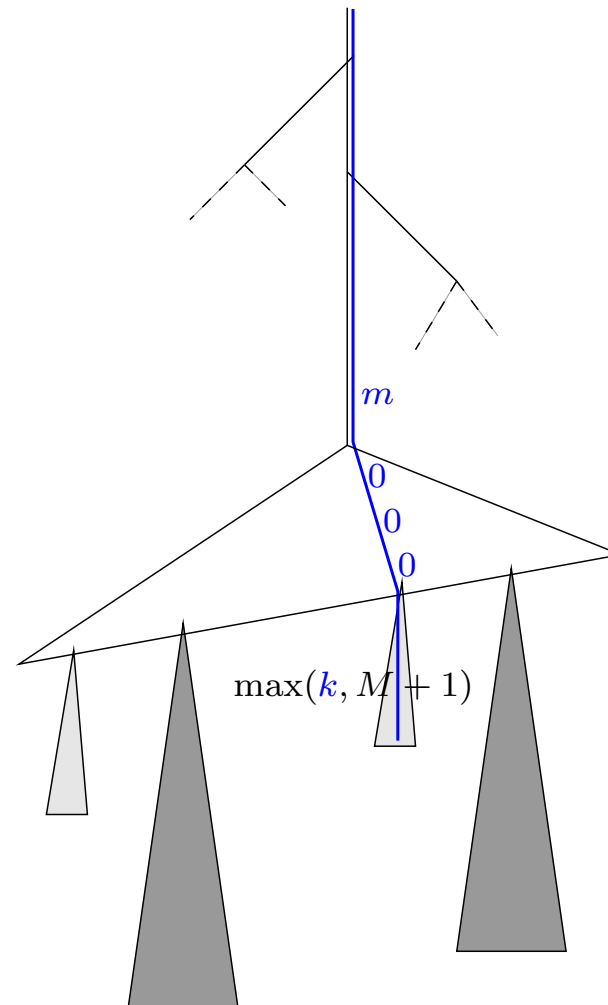
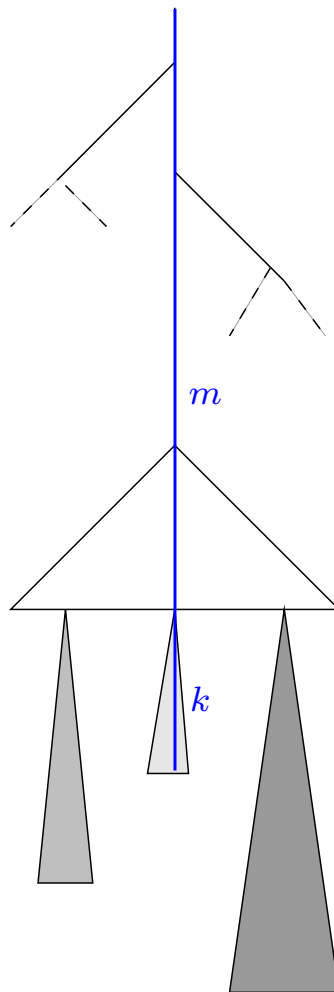
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## 3.2-MARKED DERIVATIONS- GENERAL TERMS



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## 3.2-MARKED DERIVATIONS- GENERAL TERMS



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## 4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

**Definition 3** A derivation  $s \rightarrow_{\mathcal{R}}^* t$  is *weakly bottom-up* iff, in the corresponding marked derivation, for every application of rule, the *minimum* mark of the lhs is 0.

Let  $k \geq 0$ .

**Definition 4** A derivation  $s \rightarrow_{\mathcal{R}}^* t$  is *k bottom-up* iff, in the corresponding marked derivation, for every application of rule, the *minimum* mark of the lhs is 0 and the *maximum* mark of the term is  $\leq k$ .

Notation:

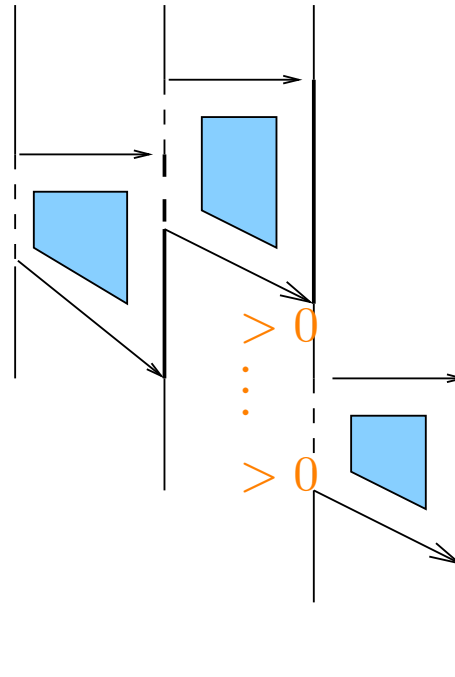
$$s \xrightarrow{k}_{\mathcal{R}}^* t$$

means that there exists a *k bottom-up* derivation from  $s$  to  $t$ .

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## 4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A generic non **weakly bottom-up** derivation.

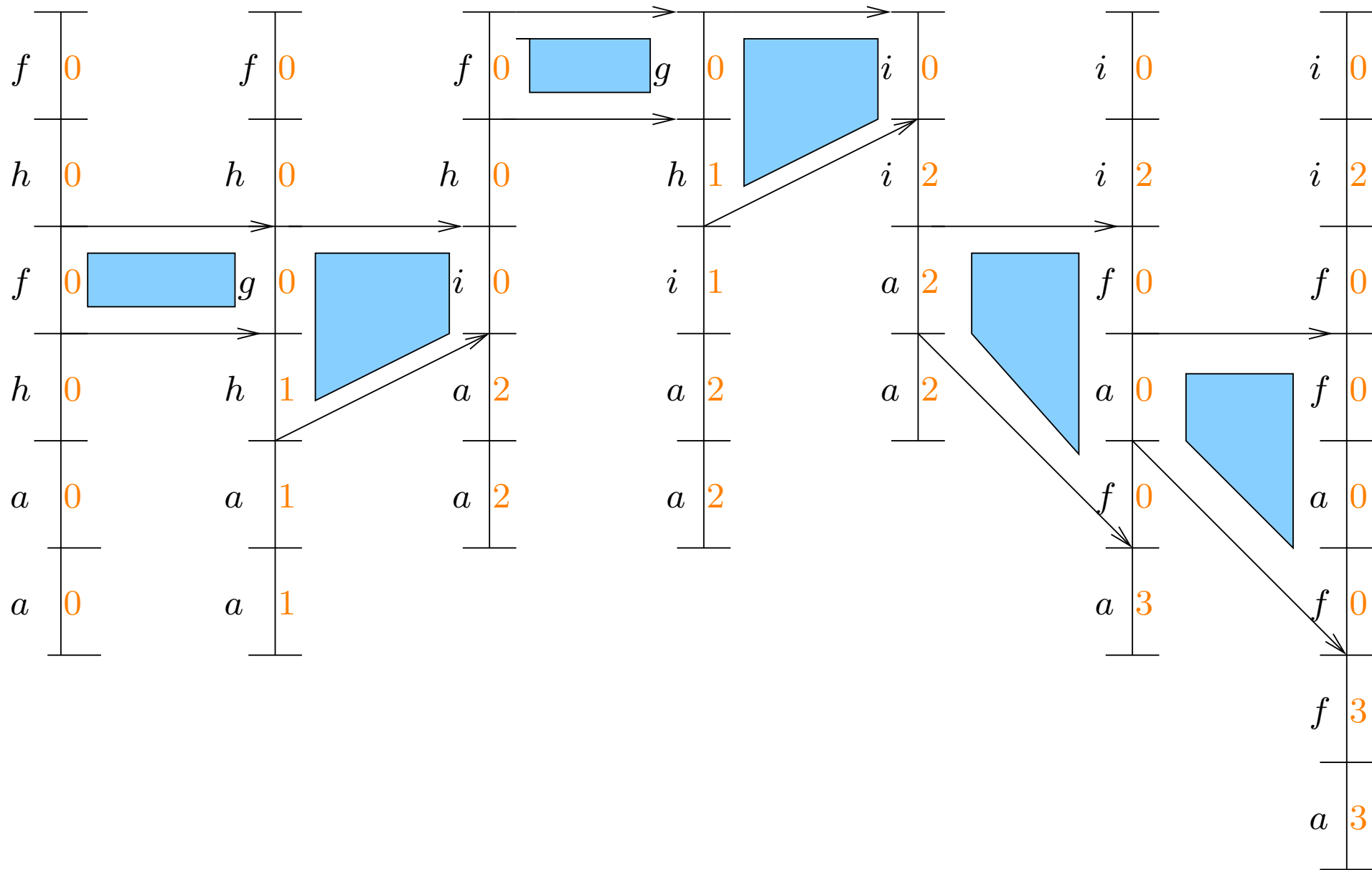




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## 4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A concrete non [weakly bottom-up](#) derivation.

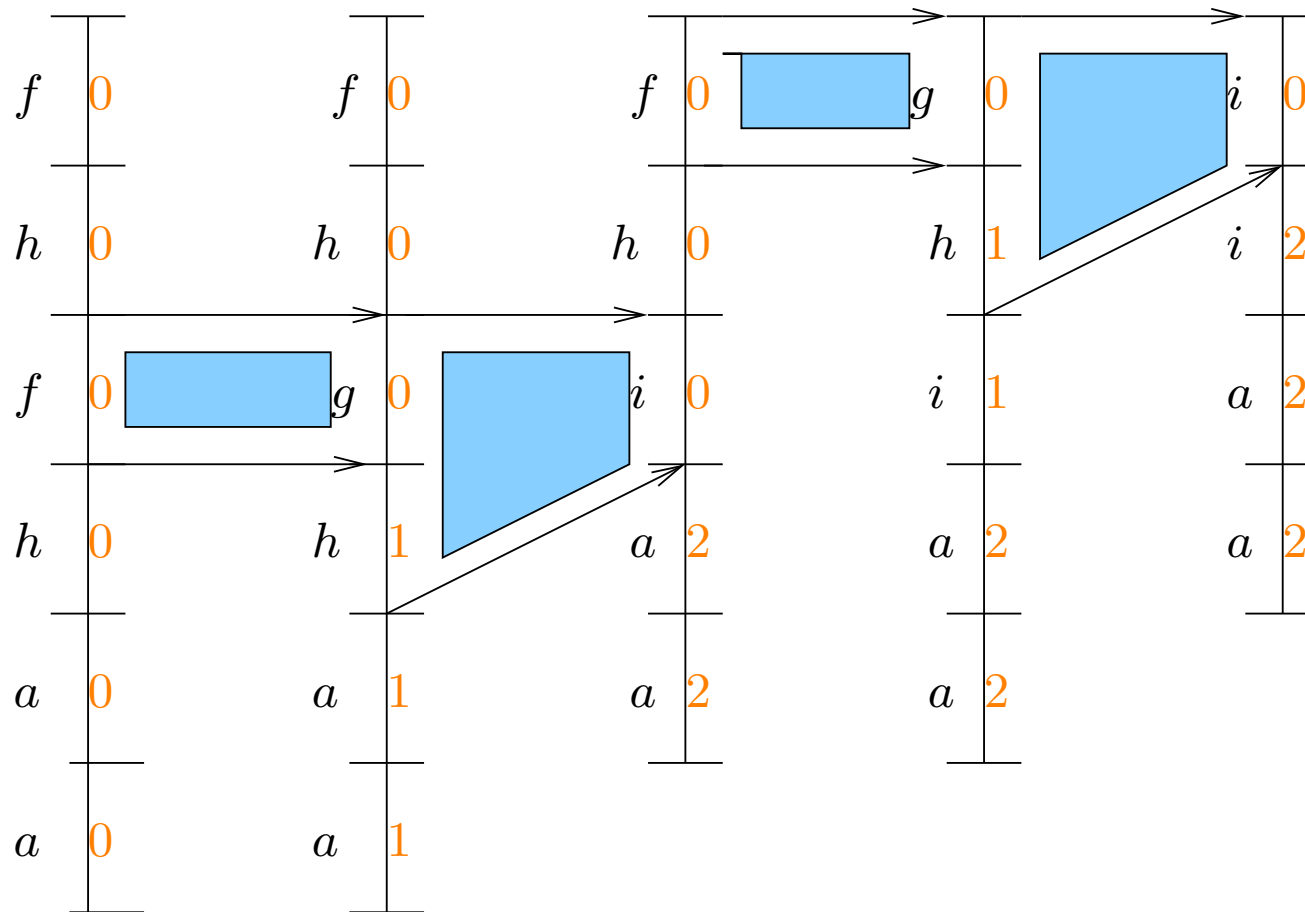


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## 4.1 -BOTTOM-UP DERIVATIONS- BOTTOM-UP DERIVATIONS

A concrete derivation:

- it is weakly bottom-up,
- it is not BU(1),
- it is BU(2),



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## 4.2 - BOTTOM-UP DERIVATIONS- BOTTOM-UP SYSTEMS

**Definition 5** Let  $k \geq 0$ . A system  $(\mathcal{R}, \mathcal{F})$  is called  $k$ -Bottom-Up iff, it is *linear* and for *every*  $s, t \in \mathcal{T}(\mathcal{F})$

$$s \rightarrow_{\mathcal{R}}^* t \Rightarrow s \xrightarrow{k}_{\mathcal{R}}^* t$$

i.e. if  $\mathcal{R}$  is linear and the  $k$ -BU strategy is complete for  $\mathcal{R}$ .

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## 4.3 -BOTTOM-UP DERIVATIONS- KNOWN SUBCLASSES

The following classes of systems are bottom-up:

- every **left-basic** semi-Thue system is BU(1)
- every **linear growing** Term Rewriting system is BU(1)
- every **linear FPO<sup>-1</sup>** Term Rewriting system is in  $\bigcup_{k \geq 0} \text{BU}(k)$ .

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## 5.1-PRESERVATION OF RATIONALITY-**THE** RESULT

### **Theorem 1** DS-RTA'07

Let  $\mathcal{R}$  be some linear rewriting system over the signature  $\mathcal{F}$ , let  $T$  be some **recognizable** subset of  $\mathcal{T}(\mathcal{F})$  and let  $k \geq 0$ . Then, the set  $(\xrightarrow{*}_{\mathcal{R}})^k[T]$  is **recognizable** too.

**Theorem 6** *Let  $k \geq 0$ , let  $\mathcal{R}$  be some  $\text{BU}(k)$  rewriting system over the signature  $\mathcal{F}$  and let  $T$  be some **recognizable** subset of  $\mathcal{T}(\mathcal{F})$ . Then, the set  $(\xrightarrow{*}_{\mathcal{R}})^k[T]$  is **recognizable** too.*

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## 5.2.2-PRESERVATION OF RATIONALITY-CONSTRUCTION OF $\mathcal{S}$

General idea:

- Simulate a bottom-up derivation, by a derivation where the substitutions used have a **bounded depth**
- The deeper part of the substitution is replaced by a **state** of the finite automaton recognizing  $T$ .



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## 5.2.2-PRESERVATION OF RATIONALITY-CONSTRUCTION OF $\mathcal{S}$

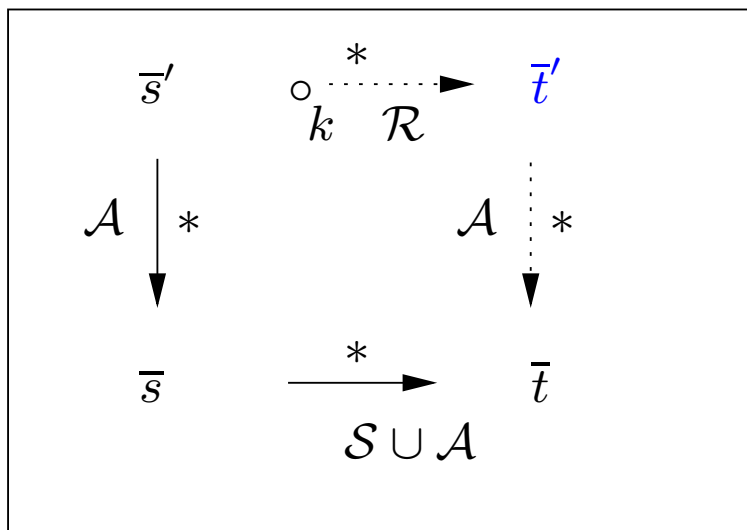
Let  $d := \max\{dpt(\ell) \mid \ell \rightarrow r \in \mathcal{R}\}$ .

The system  $\mathcal{S}$ : consists of all the rules

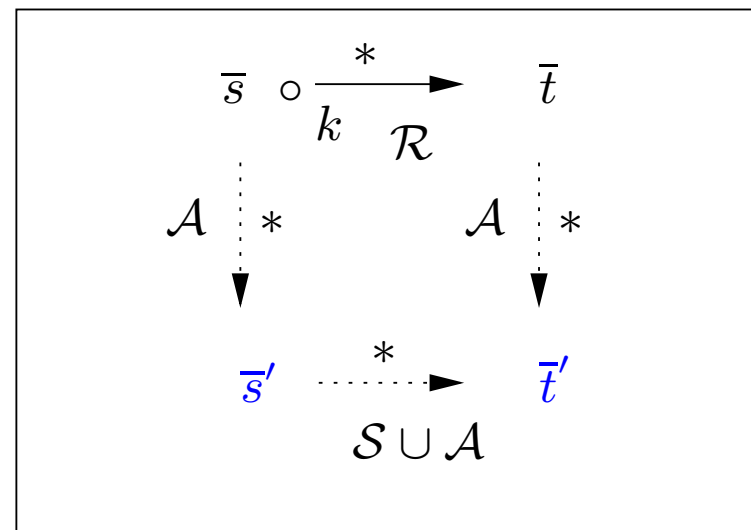
$$\bar{\ell}\bar{\tau} \rightarrow r\bar{\tau}$$

where  $\ell \rightarrow r$  is a rule of  $\mathcal{R}$ ,  $\bar{\tau}, \bar{\tau}$  are **marked** substitutions, with marks  $\leq k$ ,  $dpt(\bar{\tau}) \leq k \cdot d$  and  $\bar{\ell}\bar{\tau} \rightarrow r\bar{\tau}$  is a one-step,  $k$ -bu, marked derivation.

## 5.2.3-PRESERVATION OF RATIONALITY- SIMULATION LEMMAS



Lifting  $S \cup A$



Projecting  $\mathcal{R}$

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## 5.2.4-PRESERVATION OF RATIONALITY- CONCLUSION

Since  $\mathcal{A}$  and  $\mathcal{S}$  are *ground* rewriting-systems, it is known that  $\rightarrow_{\mathcal{S} \cup \mathcal{A}}^*$  inverse-preserves recognizability. By the simulation lemmas:

$$({}_k \rightarrow_{\mathcal{R}}^*)[T] = (\rightarrow_{\mathcal{S} \cup \mathcal{A}}^*)[Q_f^{\leq k}] \cap \mathcal{T}(\mathcal{F})$$

Hence  $({}_k \rightarrow_{\mathcal{R}}^*)[T]$  is *recognizable*.

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## 6.1-COMPLEXITY/DECIDABILITY- BOTTOM-UP

**Theorem 7** *The BU(1) property is **undecidable** for semi-Thue systems.*

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## 6.2-COMPLEXITY/DECIDABILITY- STRONG BOTTOM-UP

**Definition 8** Let  $k \geq 0$ . A system  $(\mathcal{R}, \mathcal{F})$  is called *strongly*  $k$ -Bottom-Up iff, it is *linear* and for *every* weakly bottom-up marked derivation

$$s = s_0 \rightarrow_{\mathcal{R}} s_1 \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} s_i \rightarrow_{\mathcal{R}} \dots \rightarrow_{\mathcal{R}} s_n = t$$

if  $s_0$  has only *null* marks, then all the  $s_i$  have all their marks  $\leq k$ .

**Theorem 9** The SBU( $k$ ) property is *decidable* for Term Rewriting systems.

Follows easily from theorem 1.

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## 6.3 - COMPLEXITY/DECIDABILITY- STRONG BOTTOM-UP

- every **left-basic** semi-Thue system is **SBU(1)**
- every **linear growing** Term Rewriting system is **SBU(1)**
- every **linear FPO<sup>-1</sup>** Term Rewriting system is in  $\bigcup_{k \geq 0} \mathbf{SBU}(k)$ .

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## 7- TERMINATION

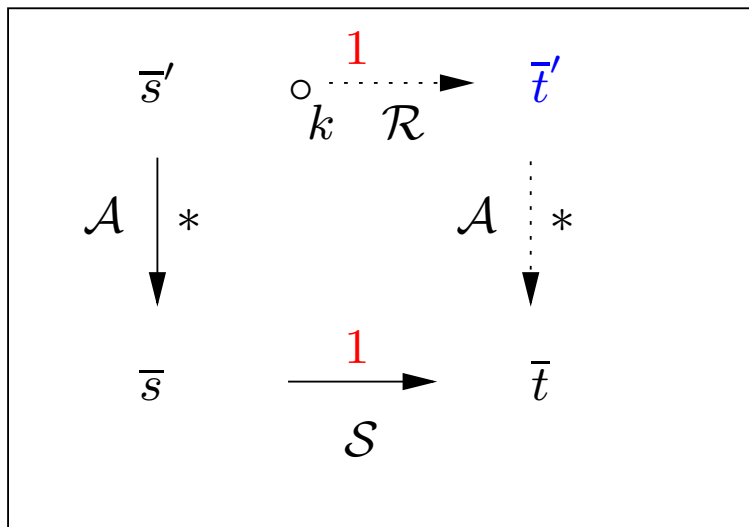
**Theorem 2** Sylvestre, M.Th. 2008

For every integer  $k \geq 0$ , the termination property is **decidable** for Term Rewriting Systems in SBU(k).

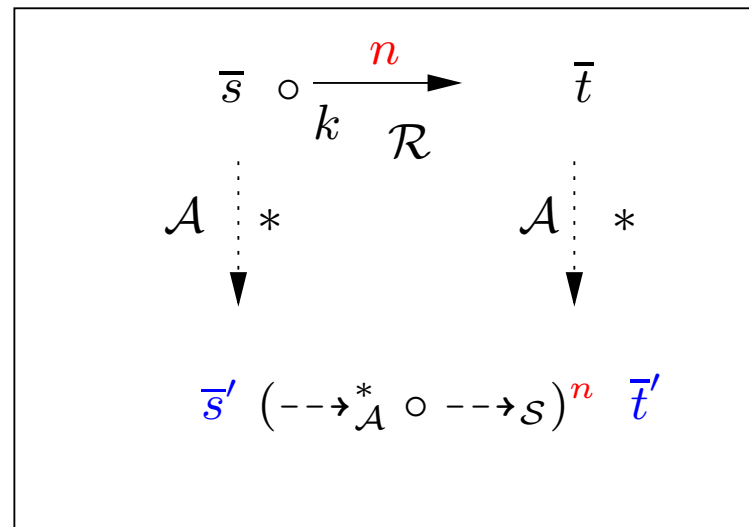
**Lemma 10** Let  $(\mathcal{R}, \mathcal{F})$  be **strongly**  $k$ -Bottom-Up. There exists an infinite derivation for  $\rightarrow_{\mathcal{R}}$  iff there exists an infinite derivation for  $k^{\circ} \rightarrow_{\mathcal{R}}$ .

## 7- TERMINATION

Strengthened lifting and projecting lemma.



Lifting  $\mathcal{S}$



Projecting  $\mathcal{R}$



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## 8- PERSPECTIVES

- Extend the notion and the results to left-linear, **non right-linear** systems  
(Current work of Marc Sylvestre in his PHD).
- Study the **direct images** of rational sets (or context-free sets) by bottom-up rewriting.
- Study a dual notion of **top-down** derivations