

Groups acting on context-free graphs

Géraud Sénizergues

Armin Weiß

LaBRI, Bordeaux and FMI, Stuttgart.

26 March 2025

INTRODUCTION

Aims

We consider graphs which are labelled over a symmetric alphabet $X \cup \bar{X}$ and are symmetric.

Aims

We consider graphs which are labelled over a symmetric alphabet $X \cup \bar{X}$ and are symmetric.

A **spherical** end of (Γ, v_0) is a connected component of $V_\Gamma - B(v_0, R)$ for some $R > 0$.

Aims

We consider graphs which are labelled over a symmetric alphabet $X \cup \bar{X}$ and are symmetric.

A **spherical** end of (Γ, v_0) is a connected component of $V_\Gamma - B(v_0, R)$ for some $R > 0$.

A graph Γ is said **context-free** iff it is connected and it has finitely many spherical ends, up to end-isomorphism (this property is independent of the chosen v_0).

Aims

We consider graphs which are labelled over a symmetric alphabet $X \cup \bar{X}$ and are symmetric.

A **spherical** end of (Γ, v_0) is a connected component of $V_\Gamma - B(v_0, R)$ for some $R > 0$.

A graph Γ is said **context-free** iff it is connected and it has finitely many spherical ends, up to end-isomorphism (this property is independent of the chosen v_0).

A **p-structure** over some graph Γ is a pair (\mathcal{A}, φ) where \mathcal{A} is a pushdown-automaton and $\varphi : \mathcal{C}(\mathcal{A}) \rightarrow \Gamma$ is a graph isomorphism.

Aims

Theorem (Muller-Schupp TCS 1985)

*A graph is **context-free** iff it has some **p-structure**.*

Theorem (Muller-Schupp JCSS 1983)

*The Cayley graph of a group G is **context-free** iff G is **virtually free**.*

Aims

- 1- Prove Muller-Schupp's theorem (about c.f. groups) by **geometric** means (tilings)
- 2- Extend Muller-Schupp's theorem to the groups $G := \text{Aut}(\Gamma)$, for Γ **context-free** graphs.
- 3- Describe (part of) the landscape around this theorem.

contents

- 1 Introduction
- 2 (very) Old results
- 3 Cutting up graphs
- 4 Context-free Cayley graph
- 5 Context-free vertex-transitive graphs
- 6 Context-free graphs
- 7 Other problems

very OLD RESULTS

Tiling the plane : two old results

[H. Poincaré : Théorie des groupes Fuchsians. Acta Math. 1, 1882, p. 1-62]

[F. Klein : Neue Beiträge zur Riemann'schen Functionentheorie. Math. Ann. 21, 1883, p. 141-218]

[B. Maskit : Poincaré's theorem for fundamental polygons. Advances in Mathematics 7, 1971, p. 219-230]

the combination lemma

Lemma

Let G be a group, with generating set $\{g_x \mid x \in X \cup \bar{X}\}$ (with $g_{\bar{x}} = g_x^{-1}$) that acts on a set

$$\Omega = (\dot{\bigcup}_{x \in X \cup \bar{X}} \Omega_x) \dot{\cup} \Omega_0$$

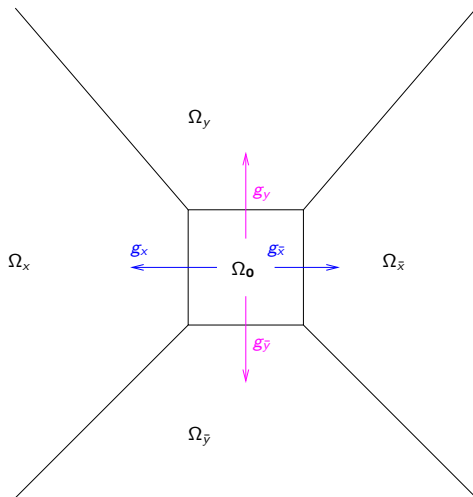
satisfying the two conditions :

- (i) all the sets Ω_α ($\alpha \in X \cup \bar{X} \cup \{0\}$) are non-empty
- (ii) for every $x \in X \cup \bar{X}$, $g_x \odot \bar{\Omega}_x \subseteq \Omega_x$.

Then

- (iii) G is **freely** generated by the family $(g_x)_{x \in X \cup \bar{X}}$.
- (iv) $\forall g \in G - \{1_G\}$, $(g \odot \Omega_0) \cap \Omega_0 = \emptyset$.

the combination lemma



Cutting up graphs

tree-sets

Definition

A set of subsets \mathcal{F} of a given set Ω is called a **tree set** if, for every elements $C, D \in \mathcal{F}$:

- 1 $\bar{C} \in \mathcal{F}$,
- 2 $C \subseteq D$ or $C \subseteq \bar{D}$ or $\bar{C} \subseteq D$ or $\bar{C} \subseteq \bar{D}$
- 3 the set $\{B \in \mathcal{F} \mid C \subseteq B \subseteq D\}$ is **finite**.

Theorem (Dunwoody 1979)

If \mathcal{F} is a tree set :

$$V = \{[C]_{\sim} \mid C \in \mathcal{C}\}, E = \{[C], [\bar{C}] \mid C \in \mathcal{F}\}$$

defines a **tree**, denoted by $T(\mathcal{F})$.

cuts

A **cut** is a subset $C \subseteq V$ such that C and \overline{C} are both connected and non-empty and δC is finite.

tree-sets of cuts

Theorem (Diekert-Weiss 2013)

*Suppose that a group G acts on a graph Γ which is accessible, locally finite and co-finite. Then Γ admits a **tree-set** \mathcal{C} , consisting of **cuts** of Γ , which is closed under the action of G and has only finitely many orbits under G .*

In the case of a context-free graph : every class $[C]_{\sim}$ is finite.
We note $\mathcal{S} := T(\mathcal{C})$.

Context-free Cayley graphs

equilibrated sets

For every subset P of $V(\mathcal{S})$ and edge $e \in E(\mathcal{S})$ we note

$$\#_i(e, P) := \text{Card}(\{ge \mid g \in G, gs(e) \notin P, gt(e) \in P\}),$$

$$\#_o(e, P) := \text{Card}(\{ge \mid g \in G, gs(e) \in P, gt(e) \notin P\}).$$

Definition

Let $P \subseteq V(\mathcal{S})$ and $e \in E(\mathcal{S})$.

The set P is called e -equilibrated iff $\#_i(e, P) = \#_o(e, P)$.

Definition

A subset $P \subseteq V(\mathcal{S})$ is called *equilibrated* iff

$$\forall e \in E(\mathcal{S}), P \text{ is } e\text{-equilibrated.}$$

equilibrated sets

It is called *connected* iff
the induced subgraph $\langle P \rangle$ of \mathcal{S} is connected.

It is called *complete* iff

$$\forall C \in \mathcal{C}_{\text{opt}}, \exists g \in G, [gC]_{\sim} \in P$$

(in other words : it is a set of representatives for $G \setminus (\mathcal{C}_{\text{opt}} / \sim)$).

Proposition

There exists a subset P of $V(\mathcal{S})$ which is *finite, equilibrated, connected and complete*.

equilibrated sets

Proof of the Proposition in the Cayley case :

Start with some P which is “partially” equilibrated i.e. for every e such that $Ge \cap P \neq \emptyset$, P is e -equilibrated.

- step 1 : $Gs(e) \cap P \neq \emptyset$ and $Gt(e) \cap P \neq \emptyset$

then P is e -equilibrated.

- step 2 : $Gs(e) \cap P \neq \emptyset$ and $Gt(e) \cap P = \emptyset$

Then $\exists Q \supset P$ which is partially-equilibrated and $Gt(e) \cap Q \neq \emptyset$

- step 3 : induction on $\text{Card}(G \setminus P)$.

equilibrated sets

Lemma

Let $e = ([C], [\overline{C}])$ be an edge of S and $P \subseteq V(S)$ finite. Let us denote $\mu([C]) := |(G \cdot [C]) \cap P|$, $\mu([\overline{C}]) := |(G \cdot [\overline{C}]) \cap P|$. Then P is *e-equilibrated* iff :

$$\mu([C]) \cdot [S_{[C]} : S_e] = \mu([\overline{C}]) \cdot [S_{[\overline{C}]} : S_e],$$

equilibrated sets

Let us consider a path in P connecting $s(e)$ with $t(e)$:

$$[C_0], \dots, [C_i], [C_{i+1}], \dots, [C_p]$$

where $s(e) = [C_0]$, $gt(e) = [C_p]$.

Let us note e_i the edge $([C_i], [C_{i+1}])$.

equilibrated sets

$$\begin{aligned}
\frac{\mu(C_0)}{\mu(C_p)} &= \prod_{i=0}^{p-1} \frac{\mu(C_i)}{\mu(C_{i+1})} \\
&= \prod_{i=0}^{p-1} \frac{[S_{[C_{i+1}]} : S_{\bar{e}_i}]}{[S_{[C_i]} : S_{e_i}]} \quad (\text{by the Lemma}) \\
&= \prod_{i=0}^{p-1} \frac{|S_{[C_{i+1}]}|}{|S_{[C_i]}|} \quad (\text{since } S_{e_i} = S_{\bar{e}_i} \text{ and the groups are finite}) \\
&= \frac{|S_{[C_p]}|}{|S_{[C_0]}|} \\
&= \frac{[S_{[C_p]} : S_{\bar{e}}]}{[S_{[C_0]} : S_e]}
\end{aligned}$$

equilibrated sets

- step 2 : $Gs(e) \cap P \neq \emptyset$ and $Gt(e) \cap P = \emptyset$

Then $\exists Q \supseteq P$, with $Ge \cap \langle P \rangle \neq \emptyset$ which is partially-equilibrated :

$$Q' := P \cup \{g \cdot [\bar{C}] \mid g \cdot [C] \in P\},$$

$$Q'' := \{[\bar{C}]\} \cup \bigcup \{g \cdot P \mid g \cdot [\bar{C}] = [\bar{C}]\}$$

and finally

$$Q := Q' \cup Q''.$$

- step 3 : induction on $\text{Card}(G \setminus P)$.

The tile

Notation :

$$\mathbb{D}[\textcolor{teal}{C}] := \bigcap_{D \sim C} C, \quad \mathbb{D}[\textcolor{teal}{P}] := \bigcup_{[C]_{\sim} \in \textcolor{teal}{P}} \mathbb{D}[\textcolor{teal}{C}].$$

The tile

Lemma

Let $T := \mathbb{D}[P]$ where P is a *finite, non-empty connected* subset of S . Then

1- There exist a set of cuts $\mathcal{D} = \{D_1, D_2, \dots, D_n\}$ such that

$$T = \bigcap_{i=1}^n D_i \quad \text{where } \forall i \neq j, \quad \bar{D}_i \cap \bar{D}_j = \emptyset$$

2- If P is *equilibrated* then $n = 2m$ (for some $m \geq 0$) and the set \mathcal{D} can be put under the form

$$\mathcal{D} = \{E_1^-, E_1^+, \dots, E_m^-, E_m^+\}$$

with $\bar{E}_i^+ \in G \cdot E_i^-$ (for every $i \in [1, n]$).

3- If P is *complete* then $\mathbb{D}[P] \neq \emptyset$.

Side-pairings

Theorem

Let T be a set of vertices of Γ , which is finite, non-empty and can be described by

$$T = \bigcap_{i=1}^n E_i^- \cap E_i^+$$

for some $n \geq 1$ and cuts E_i^-, E_i^+ with finite frontier, such that, for every $i, j \in [1, n], \varepsilon, \eta \in \{-, +\}$

$$(1) (i, \varepsilon) \neq (j, \eta) \Rightarrow \bar{E}_i^\varepsilon \cap \bar{E}_j^\eta = \emptyset$$

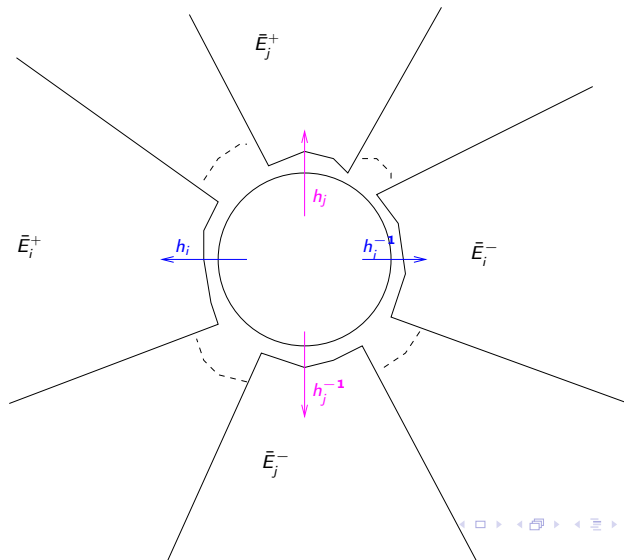
$$(2) \forall (i, \varepsilon), E_i^\varepsilon \in G \cdot \bar{E}_i^{-\varepsilon}$$

and let $(h_i)_{i \in [1, n]}$ be elements of G such that $h_i \cdot E_i^- = \bar{E}_i^+$. Then

(3) The group $H := \langle h_i \mid i \in [1, n] \rangle$ is **freely** generated by the $(h_i)_{i \in [1, n]}$

$$(4) V(\Gamma) = \dot{\bigcup}_{h \in H} h \cdot T.$$

Side-pairings



Side-pairings

Proof of the theorem :

(3) < combination-lemma

(4) < induction on the distance to the tile T .

Muller-Schupp theorem

Theorem (Muller-Schupp, 1983)

*Let G be a finitely generated group. G is context-free if and **only if** G is virtually-free.*

Free subgroup

Suppose the word-problem of G is context-free. Then :

- the Cayley-graph Γ of G is k -triangulated [Muller-Schupp 1983]
- the tree \mathcal{S} of classes of optimal cuts of Γ defines a subset P which is finite, connected, equilibrated and complete (Proposition 7);
- by Lemma 9 $T := \mathbb{D}[P]$ is a finite tile (for Γ) fulfilling the hypotheses of Theorem 10;
- by point 1 of Theorem 10 the subgroup H generated by the side-pairings of T is a finitely generated free subgroup;
- by point 2 of Theorem 10, T is a system of representatives of G for the right-classes modulo H , hence $[G : H] = |T| < \infty$.

Context-free vertex-transitive graphs

Adaptations

Stabilizers may be **infinite**.

Proof of Proposition 7 :

- step 1 : $Gs(e) \cap P \neq \emptyset$ and $Gt(e) \cap P \neq \emptyset$

Use **modified** telescoping product :

$$\frac{[S_{[C_{i+1}]} : S_{\bar{e}_i}]}{[S_{[C_i]} : S_{e_i}]} = \frac{[S_{[C_{i+1}]} : S_n]}{[S_{[C_i]} : S_n]}$$

where $S_n = \bigcap_{i=0}^p S_{[C_i]}$.

- step 2 : $Gs(e) \cap P \neq \emptyset$ and $Gt(e) \cap P = \emptyset$

Same argument.

- step 3 : induction on $\text{Card}(G \setminus P)$.

Same argument.

Proof of $V(\Gamma) = \bigcup_{h \in H} h \cdot T$:

Same argument.

Automorphisms

Decomposition as a 3-fold unambiguous product.

$$G = H \cdot R \cdot S$$

where H is a free group of finite rank, R is a finite subset of G , S is the stabilizer of vertex v_0 (S is profinite).

Question : is $\Pi : G \rightarrow \Gamma / \sim_G$ rational? (for some p-structure)?

Context-free graphs

Tiling a c.f. graph

Stabilizers may be **infinite**.

Action of $Aut(\Gamma)$ may be **non-transitive**.

Consider the orbit of some $v_0 \in V(\Gamma)$: by [L. Pelecq TCS 1985], the orbit of v_0 is κ -connected. It can be turned into a new context-free graph Γ' , on which $Aut(\Gamma)$ acts transitively.

Automorphisms

Decomposition as a 3-fold unambiguous product.

$$G = H \cdot R \cdot S$$

where H is a **free group of finite rank**, R is a **finite** subset of G , S is the stabilizer of vertex v_0 (S is **profinite**).

$G \backslash \Gamma$ is **context-free** [GS ICALP 96]

Question : is $\Pi : G \rightarrow G \backslash \Gamma$ rational? (for some p-structure)?

Other problems

Automorphisms and quotients

P1- Characterise the 3-tuples (F, R, S) that define the automorphism groups of a c.f. graph.

P2- Let \sim be the greatest bisimulation over Γ , for Γ context-free graph

2.1 Is Γ / \sim context-free?

2.2 Is the canonical projection rational?

Cayley-graphs

P3- Cayley graphs with unique geodesics : Shapiro's conjecture : the Cayley-graph of G over X has unique geodesics implies G is a free product of finitely many finite groups with a free group of finite rank.

Partial result : true if G has presentation which is finite, confluent, monadic and strictly reduces the length [Heisenberg and alii 2019]

P4- What are the Cayley graphs where the domino-problem is decidable? raised by [N. Aubrun JCB 24].

(MSO is decidable over $\Gamma \Leftrightarrow \Gamma$ is context-free [Kuske-Lohrey APAL 2005]).

Infinite degree

P5- Extension of the notion of c.f. graph to **arbitrary degree** (ends with finite frontier, finitely many ends under end-isomorphism). Is $\text{Aut}(\Gamma) \backslash \Gamma$ also “context-free” ?

Case of trees, with (at most denumerable)-degree studied by [Courcelle 2022] : one can decide whether $\text{Aut}(\Gamma) \backslash \Gamma$ is finite.

Cryptology

P6- The isomorphism problem for c.f. groups is decidable (in double-exponential time [GS-Weiss ICALP 2018]). Can we turn the difficulty of finding the isomorphism into a robust asymmetric cryptographic protocol?

Alice : knows a nice presentation P of G (confluent, noetherian) with set of normal forms T .

Others : know some (random) presentation Q of G with a transversal U

Encoding of $u \in U$ into u' (defining the same element of G).

Decoding : easy when you know P , difficult when you only know Q .

Internship subject, 1st year ENS student, June 2025.