
Rational subsets in HNN-extensions

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0.1 INTRODUCTION-PROBLEMS

Algorithmic problems.

$$P = \langle X, \equiv \rangle \text{ and } M = X^* / \equiv.$$

The Word Problem:

instance: $u, v \in X^*$

question: $u \equiv v$?

The membership problem for rational expressions:

instance: A rational expression E over the alphabet X and a word $u \in X^*$.

question: $u \in L(E, M)$?

0.1 INTRODUCTION-PROBLEMS

The **emptiness problem** for boolean combinations of rational expressions:

instance: A boolean expression B , whose arguments are rational expressions over the alphabet X and a word $u \in X^*$.

question: $L(B, M) = \emptyset$?

The **membership problem** for finitely generated subgroups of a group,

instance: A sequence $(u_1, \dots, u_i, \dots, u_n, u)$ of words $u_i, u \in X^*$.

question: $u \in \{u_1, \bar{u}_1, \dots, u_i, \bar{u}_i, \dots, u_n, \bar{u}_n\}^*$?

(also named the **Generalized Word Problem**).

0.1 INTRODUCTION-PROBLEMS

Structural problems.

Definition 1 A group G is said to have the *Howson property* iff, for every two finitely generated subgroups K_1, K_2 of G , the subgroup $K_1 \cap K_2$ is finitely generated too.

Definition 2 A group G is said to have the *subgroup separation property* iff, for every finitely generated subgroup $K < G$, and every element $g \in G - K$, there exists a normal subgroup K' which has finite index in G and such that $K \subseteq K'$ and $g \in G - K'$.

0.2 MOTIVATIONS

1- Investigate in which monoids it is possible to **decide** the satisfiability problem for:

- equations with **constants**
- equations and **inequations**
- equations with **rational** constraints

Works by: (Makanin,1977,1983), (Schulz 1991),
(Razborov,1985), (Diekert-Matiyasevich-Muscholl 1999),
(Diekert-Hagenah-Gutierrez 2001),(Diekert-Lohrey
2004)(graph products),...

0.2 MOTIVATIONS

2- Investigate the properties of rational subsets of monoids:

- in which monoids does **Kleene theorem** hold ?
- can we decide whether a rational subset is **recognizable** ?
- which **operations** are preserving rational subsets ?
- do the **boolean** operations preserve rational sets ?
- links between rationality and **logical** definability ?

Works by:(Sakarovitch 1984)(free group), (Reutenauer 1985)(free product of semi-groups), (Pelletier-Sakarovitch 1986)(rational monoids),(Senizergues 1996)(virtually free groups), (Da Silva 2004) (free groups),

(Eilenberg-Schutzenberger 1969)(commutative monoids),

0.2 MOTIVATIONS

3- Investigate the properties of finitely generated subgroups of groups:

- which groups have a decidable [G. Word Problem](#) ?
- which groups have the [Howson](#) property?
- which groups have the [Subgroup Separation](#) property?

Works by: (Mikhailova 1958)(free product), (Howson 1954), (Benois 1970), (Anisimov-Seifert 1975), (Hall 1949)(free group), (Romanovskii 1970), (Burns 1970)(free product), (Allenby-Gregorac 1972) (free product with finite amalgamation), (Scott 1978)(surface groups), (Rips 1990)(counter-example for amalgamation over \mathbb{Z}), (Brunner-Burns-Solitar 1984)(free product with cyclic

amalgamation of free groups)

0.3 RESULTS

Let H be a monoid and

$$G = \langle H, t; t^{-1}at = \varphi(a)(a \in A) \rangle,$$

where $A < H, B < H$, finite subgroups, $\varphi : A \rightarrow B$ isomorphism.

Theorem 3

- 1- If $\text{Rat}(H)$ is closed under intersection , *so is $\text{Rat}(G)$*
- 2- If $\text{Rat}(H)$ is closed under complement , *so is $\text{Rat}(G)$*
- 3- If the membership problem is decidable for $\text{Rat}(H)$, *so is it for $\text{Rat}(G)$*
- 4- If the emptiness problem is decidable for the boolean closure of $\text{Rat}(H)$, *so is it for the boolean closure of $\text{Rat}(G)$.*

0.3 RESULTS

Let H be a group and G the extension defined above.

Theorem 4

- 1- If H has a decidable Generalized Word Problem, *so has G*
- 2- If H has the subgroup separation property, *so has G*
- 3- If H has the Howson property, *so has G* .

0.4 TOOLS

Finite automata.

Semi-Thue systems.

Fundamental group of a graph.

Virtually free groups.

1.1 HNN EXTENSIONS- PRESENTATION

Let X be an alphabet, \equiv be some congruence over X^* .

Suppose that

$$H = X^* / \equiv_H$$

Let $A, B < H$ be subgroups, and $\varphi : A \rightarrow B$ an isomorphism.

The **HNN**-extension of H with stable letter t and associated subgroups A, B is:

$$G = (X \cup \{t, \bar{t}\})^* / \equiv$$

where

$$u \equiv v, \quad \text{for all } u, v \in X^* \text{ such that } u \equiv_H v, \quad (1)$$

$$t\bar{t} \equiv \bar{t}t \equiv \epsilon, \quad \text{and} \quad (2)$$

$$\bar{t}at \equiv \varphi(a), \quad \text{for all } a \in A. \quad (3)$$

1.2 T-SEQUENCES

We name *t*-sequence an element s of $H * \{t, \bar{t}\}^*$:

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n$$

where $n \in \mathbb{N}$, $\alpha_i \in \{+1, -1\}$, t^{-1} means the letter \bar{t} and $h_i \in H$.
 s is said *reduced* iff it does not contain any factor of the form $\bar{t}at$ (with $a \in A$) nor $tb\bar{t}$ (with $b \in B$).

1.2 T-SEQUENCES

The congruence \approx over $H * \{t, \bar{t}\}^*$ is generated by the set of rules

$$t\bar{t} \approx \bar{t}t \approx 1, \tag{4}$$

$$at \approx t\varphi(a), \quad \text{for all } a \in A \text{ and} \tag{5}$$

$$b\bar{t} \approx \bar{t}\varphi^{-1}(b), \quad \text{for all } b \in B. \tag{6}$$

Second presentation:

$$G = H * \{t, \bar{t}\}^* / \approx .$$

We denote by \sim the congruence over $H * \{t, \bar{t}\}^*$ generated by the rules (5),(6)

Lemma 5 (Britton's lemma) : *Let s, s' be **reduced** sequences. Then $s \approx s'$ if and only if $s \sim s'$.*

1.2 T-SEQUENCES

Presentations and homomorphisms

$$(X \cup \{t, \bar{t}\})^* \xrightarrow{\pi_t} H * \{t, \bar{t}\}^* \xrightarrow{\pi_G} G$$

We set

$$\pi = \pi_t \circ \pi_G : (X \cup \{t, \bar{t}\})^* \rightarrow G$$

and

$$\text{Ker}(\pi) = \equiv; \quad \text{Ker}(\pi_G) = \approx;$$

$$\text{Ker}(\pi_G) \cap (\text{Red}(H * \{t, \bar{t}\}^*) \times \text{Red}(H * \{t, \bar{t}\}^*)) = \sim$$

1.3 BASIC TRANSFER

Corollary 6 *If H has a decidable Word Problem, so has G*

2.2- FREE MONOIDS

A finite automaton over the free monoid X^* , with labelling alphabet \mathcal{X} is a 5-tuple:

$$\mathcal{A} = \langle \mathcal{X}, Q, \delta, I, T \rangle$$

where, \mathcal{X} is a finite subset of X , Q is the finite set of states, $I \subseteq Q$ is the set of initial states, $T \subseteq Q$ is the set of terminal states and δ , is a subset of $Q \times \mathcal{X} \times Q$.

2.2- FREE MONOIDS

$$\mu_{\mathcal{A}} : (X^*, \cdot, \varepsilon) \rightarrow (\mathcal{P}(\mathbf{Q} \times \mathbf{Q}), \circ, \text{Id}_{\mathbf{Q}})$$

is the unique **monoid-homomorphism** such that: for every $x \in X$

$$\mu_{\mathcal{A}}(x) = \{(q, r) \in \mathbf{Q} \times \mathbf{Q} \mid (q, x, r) \in \delta\}.$$

$$\mathbf{L}(\mathcal{A}) = \{u \in X^* \mid \exists q \in \mathbf{I}, \exists r \in \mathbf{T}, (q, r) \in \mu_{\mathcal{A}}(u)\}.$$

2.3-GENERAL MONOIDS

Let $(M, \cdot, 1_M)$ be a monoid. Suppose $M = X^* / \equiv$.

The set

$$\text{Rat}(M) \in \mathcal{P}(\mathcal{P}(M))$$

is the smallest element of $\mathcal{P}(\mathcal{P}(M))$ which possesses the finite subsets of M and which is closed under union, product and $*$.

Finite automaton: as before.

For every $m \in M$,

$$\mu_{\mathcal{A}, M}(m) = \bigcup_{[u] \equiv m} \mu_{\mathcal{A}}(u).$$

$$\text{L}(\mathcal{A}, M) = \{m \in M \mid \exists q \in \mathbf{I}, \exists r \in \mathbf{T}, (q, r) \in \mu_{\mathcal{A}, M}(m)\}.$$

2.3-GENERAL MONOIDS

Note that:

$$\mu_{\mathcal{A},M}(1) \supseteq \text{Id}_Q; \quad \mu_{\mathcal{A},M}(m \cdot m') \supseteq \mu(m) \circ \mu(m'),$$

but in general the map μ is **not** a monoid homomorphism.

2.3 GROUPS

Theorem 7 (Anisimov-Seifert75)

Let G be some finitely generated group. The *rational* subgroups of G are exactly the *finitely generated* subgroups of G .

THE trick:

$$\pi : X^* \rightarrow G$$

$$P_1(\mathcal{A}, q) = \{u \in X^* \mid q \xrightarrow{\delta \cup \delta^{-1}}^u q\}$$

If q_0 is the unique initial and final state of \mathcal{A} recognizing a subgroup K , then

$$K = \pi(P_1(\mathcal{A}, q_0)).$$

One can then extract a *finite set of generators* of P_1 seen as a fundamental group of a graph.

3.1-HNN EXTENSION-F-AUTOMATA

Let \mathcal{F} be a set of subsets of H :

$$\mathcal{F} \in \mathcal{P}(\mathcal{P}(H)).$$

A f.a. over $H * \{t, \bar{t}\}$, with labelling set \mathcal{F} is a 5-tuple:

$$\mathcal{A} = \langle \mathcal{L}, Q, \delta, I, T \rangle$$

where, \mathcal{L} is finite and

$$\mathcal{L} \subseteq \mathcal{F} \cup \mathcal{P}(A) \cup \mathcal{P}(B) \cup \{\{t\}, \{\bar{t}\}\},$$

Q , I , T are as usual and $\delta \subseteq Q \times \mathcal{L} \times Q$.

3.1-HNN EXTENSION-F-AUTOMATA

$$\mu_0(1) = \{(q, q) \mid q \in Q\} \cup \{(q, r) \in Q \times Q \mid \exists (q, L, r) \in \delta, h \in L\}$$

for every $h \in H - \{1\} \cup \{t, \bar{t}\}$

$$\mu_0(h) = \{(q, r) \in Q \times Q \mid \exists (q, L, r) \in \delta, h \in L\}$$

for every

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n$$

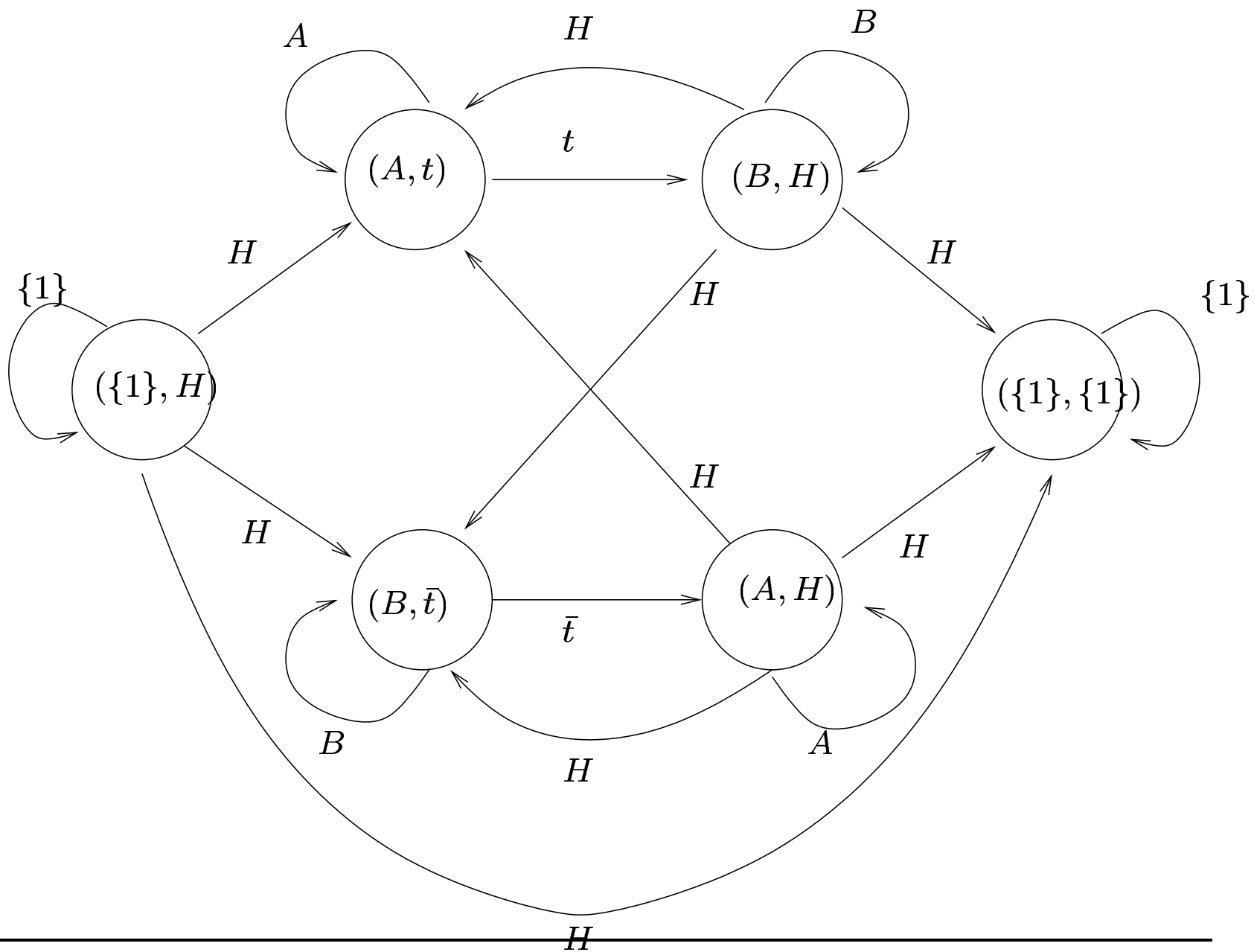
$$\mu(s) = \mu_0(h_0) \circ \mu_0(t^{\alpha_1}) \circ \mu_0(h_1) \cdots \mu_0(t^{\alpha_i}) \circ \mu_0(h_i) \cdots \mu_0(t^{\alpha_n}) \circ \mu_0(h_n)$$

$$L(\mathcal{A}) = \{u \in H * \{t, \bar{t}\}^* \mid \exists q \in I, \exists r \in T, (q, r) \in \mu_{\mathcal{A}}(u)\}.$$

3.2-HNN EXTENSION-PARTITIONNED AUTOMATA

Finite set of 6 types:

$$\mathcal{T}_6 = \{(A, t), (B, \bar{t}), (\{1\}, H), (\{1\}, \{1\}), (A, H), (B, H)\}$$



3.2-HNN EXTENSION-PARTITIONNED AUTOMATA

We name *partitionned automaton* over \mathcal{F} , a 6-tuple

$$\mathcal{A} = \langle \mathcal{L}, Q, \tau, \delta, I, T \rangle$$

where $\tau : Q \rightarrow \mathcal{T}_6$ is such that, for every $q \in Q$, $(q, L, r) \in \delta$

$$\{q\} \times L \times \{r\} \subseteq \mathcal{E}_6,$$

$$q \in I \Rightarrow \tau(q) = (\{1\}, H); \quad q \in T \Rightarrow \tau(q) = (\{1\}, \{1\}).$$

3.3 HNN EXTENSION-SATURATION

A partitionned finite automaton \mathcal{A} over a labelling set \mathcal{F} is said to be \sim -saturated if and only if, for every $s, s' \in H * \{t, \bar{t}\}^*$,

$$s \sim s' \Rightarrow \mu(s) = \mu(s').$$

It is said \approx -compatible if and only if,

$$[\mathbf{L}(\mathcal{A})]_{\approx} = [\mathbf{L}(\mathcal{A}) \cap \text{Red}(H * \{t, \bar{t}\}^*)]_{\approx}.$$

3.3 HNN EXTENSION-SATURATION

It is said **deterministic** iff

$$Card(I) = 1$$

and for every $q, r, r' \in Q, F, F' \in \mathcal{L}$,

$$((q, F, r), (q, F', r') \in \delta \wedge F \cap F' \neq \emptyset \wedge \tau(r) = \tau(r')) \Rightarrow (q, F, r) = (q, F', r').$$

It is said **complete** iff...

(local condition ensuring that the automaton can read every reduced sequence)

3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

Proposition 8 *Let $K \subseteq G$. The following are equivalent*

- 1- *K is a rational subset of G*
- 2- *There exists some $R \in \text{Rat}((X \cup \{t, \bar{t}\})^*)$ such that $K = \pi(R)$*
- 3- *$\pi_G^{-1}(K) \cap \text{Red}(H * \{t, \bar{t}\}^*) = \text{L}(\mathcal{A}) \cap \text{Red}(H * \{t, \bar{t}\}^*)$ for some partitionned, \approx -compatible, \sim -saturated, finite automaton \mathcal{A} over $\text{Rat}(H)$.*

We start with \mathcal{A}_0 , f.a. over the free monoid $(X \cup \{t, \bar{t}\})^*$, such that

$$K = \pi(\text{L}(\mathcal{A}_0)).$$

3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

step 1: semi-Thue system \mathcal{R}_r :

$$\bar{t}ut \rightarrow \varphi(u) \text{ (for } [u]_{\equiv_H} \in A); tut\bar{t} \rightarrow \varphi^{-1}(u) \text{ (for } [u]_{\equiv_H} \in B)$$

It is **monadic**, thus we construct \mathcal{A}_1 such that:

$$L(\mathcal{A}_1) = L(\mathcal{A}_0) \rightarrow_{\mathcal{R}_r}^*$$

step 2: Collecting H-transitions, we construct f.a., \mathcal{A}_2 , with labels in $\text{Rat}(H)$, which is \approx -compatible and such that

$$L(\mathcal{A}_2) = \pi_t(L(\mathcal{A}_1)).$$

step 3:

$$\mathcal{A}_3 = \mathcal{A}_2 \times \mathcal{G}_6$$

\mathcal{A}_3 is partitionned, \approx -compatible.

3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

step 4: Saturation. From \mathcal{A}_3 we construct

$$\mathcal{B} = \langle \mathcal{L}_{\mathcal{B}}, Q_{\mathcal{B}}, \tau_{\mathcal{B}}, \delta_{\mathcal{B}}, I, T \rangle$$

Idea: for every $q \in Q_3$,

$$\gamma(q) = \pi_1(\tau(q))$$

is one of the three subgroups $\{1\}, A, B$;

3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

$$Q_{\mathcal{B}} = \{(q, g) \in Q_3 \times (A \cup B) \mid g \in \gamma(q)\}$$

the set $\delta_{\mathcal{B}}$ consists of all the following transitions:

$$\begin{aligned} (qg, g', q(g \cdot g')) & \quad \text{for all} \quad q \in Q_3, g \in \gamma(q), g' \in \gamma(q) \\ (qg, g^{-1}Lg', rg') & \quad \text{for all} \quad q \in Q_3, g \in \gamma(q), g' \in \gamma(r), L \in \mathcal{L}_3, (q, L, r) \in \delta_3 \\ (qa, t, r\varphi(a)) & \quad \text{for all} \quad q \in Q_3, a \in A, (q, t, r) \in \delta_3 \\ (qb, \bar{t}, r\varphi^{-1}(b)) & \quad \text{for all} \quad q \in Q_3, b \in B, (q, \bar{t}, r) \in \delta_3 \end{aligned}$$

3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

Proposition 9

Let $R_t \subseteq H * \{t, \bar{t}\}^*$ be recognized by some partitionned, \approx -compatible, finite automaton \mathcal{A} over the labelling set \mathcal{F} .

1- Then, $\pi_G^{-1}(\pi_G(R_t)) \cap \text{Red}(H * \{t, \bar{t}\}^*) = L(\mathcal{B}) \cap \text{Red}(H * \{t, \bar{t}\}^*)$ for some partitionned, \approx -compatible, \sim -saturated, finite automaton \mathcal{B} over \mathcal{F}' , where

$$\mathcal{F}' = \{gFg' \mid F \in \mathcal{F}, g, g' \in A \cup B\}$$

2- $\pi_G^{-1}(\pi_G(R_t)) \cap \text{Red}(H * \{t, \bar{t}\}^*)$ is recognized by some partitionned, \sim -saturated, deterministic, complete finite automaton \mathcal{D} over \mathcal{F}'' , where \mathcal{F}'' is the boolean closure of \mathcal{F}' .

Idea: for point 1, construction of \mathcal{B} above

for point 2, classical idea of **subset-construction**; but take into account the 6 types.

3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

Proposition 10 *Let K be a subgroup of G . The following are equivalent*

0- K is finitely generated

1- K is rational

2- There exists some $R \in \text{Rat}((X \cup \{t, \bar{t}\})^*)$ such that $K = \pi(R)$

3- $\pi_G^{-1}(K) \cap \text{Red}(H * \{t, \bar{t}\}^*) = L(\mathcal{B}) \cap \text{Red}(H * \{t, \bar{t}\}^*)$ for some partitionned, \approx -compatible, \sim -saturated, finite automaton \mathcal{B} over S .

Idea: same constructions as before, but combined with

THE P_1 -trick.

4.1 -TRANSFER FOR RATIONAL SUBSETS

Theorem 11

- 1- If $\text{Rat}(H)$ is closed under intersection , *so is $\text{Rat}(G)$*
- 2- If $\text{Rat}(H)$ is closed under complement , *so is $\text{Rat}(G)$*
- 3- If the membership problem is decidable for $\text{Rat}(H)$, *so is it for $\text{Rat}(G)$*
- 4- If the emptiness problem is decidable for the boolean closure of $\text{Rat}(H)$, *so is it for the boolean closure of $\text{Rat}(G)$.*

4.2- TRANSFER FOR FINITELY GENERATED SUBGROUPS

Let H be a group and G the extension defined above.

Theorem 12

- 1- If H has a decidable Generalized Word Problem, *so has G*
- 2- If H has the subgroup separation property, *so has G*
- 3- If H has the Howson property, *so has G* .

4.3-TRANSFER TOWARDS AN AMALGAMATED PRODUCT

Let H_1, H_2 , be monoids, $A_1 \leq H_1, A_2 \leq H_2$, finite subgroups and $\varphi : A_1 \rightarrow A_2$ isomorphism.

$$G = \langle H_1, H_2; a = \varphi(a) (a \in A_1) \rangle.$$

G is embedded into the HNN-extension

$$\hat{G} = \langle H_1 * H_2, t; t^{-1}at = \varphi(a) (a \in A_1) \rangle$$

by the map

$$h_1 \in H_1 \mapsto t^{-1}h_1t; \quad h_2 \in H_2 \mapsto h_2$$

4.3-TRANSFER TOWARDS AN AMALGAMATED PRODUCT

Theorem 13

- 1-If $\text{Rat}(H_i)$ is closed under intersection (for every $i \in \{1, 2\}$),
so is $\text{Rat}(G)$
- 2-If $\text{Rat}(H_i)$ is closed under complement (for every $i \in \{1, 2\}$),
so is $\text{Rat}(G)$
- 3-If the membership problem is decidable for $\text{Rat}(H_i)$ (for every $i \in \{1, 2\}$), so is it for $\text{Rat}(G)$
- 4- If the emptiness problem is decidable for the boolean closure of $\text{Rat}(H_i)$ (for every $i \in \{1, 2\}$), so is it for the boolean closure of $\text{Rat}(G)$.

4.3-TRANSFER TOWARDS AN AMALGAMATED PRODUCT

Theorem 14

1-If H_i has a decidable Generalized Word Problem (for every $i \in \{1, 2\}$), *so has G*

2-If H_i has the Subgroup separation property (for every $i \in \{1, 2\}$), *so has G*

3-If H_i has the Howson property (for every $i \in \{1, 2\}$), *so has G .*

5- PERSPECTIVES

5.1 Equations in G : transfer of decidability.

5.2 More general extensions:

A, B infinite

A, B , finite submonoids.