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# Rational subsets in HNN-extensions

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## 0.1 INTRODUCTION-PROBLEMS

Algorithmic problems.

$$P = \langle X, \equiv \rangle \text{ and } M = X^*/\equiv.$$

The Word Problem:

**instance:**  $u, v \in X^*$

**question:**  $u \equiv v?$

The membership problem for rational expressions:

**instance:** A rational expression  $E$  over the alphabet  $X$  and a word  $u \in X^*$ .

**question:**  $u \in L(E, M)?$

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## 0.1 INTRODUCTION-PROBLEMS

The **emptiness problem** for boolean combinations of rational expressions:

**instance:** A boolean expression  $B$ , whose arguments are rational expressions over the alphabet  $X$  and a word  $u \in X^*$ .

**question:**  $L(B, M) = \emptyset$ ?

The **membership problem** for finitely generated subgroups of a group,

**instance:** A sequence  $(u_1, \dots, u_i, \dots, u_n, u)$  of words  $u_i, u \in X^*$ .

**question:**  $u \in \{u_1, \bar{u}_1, \dots, u_i, \bar{u}_i, \dots, u_n, \bar{u}_n\}^*$ ?

(also named the **Generalized Word Problem**).

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## 0.1 INTRODUCTION-PROBLEMS

Structural problems.

**Definition 1** A group  $G$  is said to have the *Howson property* iff, for every two finitely generated subgroups  $K_1, K_2$  of  $G$ , the subgroup  $K_1 \cap K_2$  is finitely generated too.

**Definition 2** A group  $G$  is said to have the *subgroup separation property* iff, for every finitely generated subgroup  $K < G$ , and every element  $g \in G - K$ , there exists a normal subgroup  $K'$  which has finite index in  $G$  and such that  $K \subseteq K'$  and  $g \in G - K'$ .

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## 0.2 MOTIVATIONS

1- Investigate in which monoids it is possible to **decide** the satisfiability problem for:

- equations with **constants**
- equations and **inequations**
- equations with **rational** constraints

Works by: (Makanin,1977,1983), (Schulz 1991),  
(Razborov,1985), (Diekert-Matiyasevich-Muscholl 1999),  
(Diekert-Hagenah-Gutierrez 2001),(Diekert-Lohrey  
2004)(graph products),...

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## 0.2 MOTIVATIONS

2- Investigate the properties of rational subsets of monoids:

- in which monoids does **Kleene theorem** hold ?
- can we decide whether a rational subset is **recognizable** ?
- which **operations** are preserving rational subsets ?
- do the **boolean** operations preserve rational sets ?
- links between rationality and **logical** definability ?

Works by: (Sakarovitch 1984) (free group), (Reutenauer 1985) (free product of semi-groups), (Pelletier-Sakarovitch 1986) (rational monoids), (Senizergues 1996) (virtually free groups), (Da Silva 2004) (free groups),

(Eilenberg-Schutzenberger 1969) (commutative monoids),

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## 0.2 MOTIVATIONS

3- Investigate the properties of finitely generated subgroups of groups:

- which groups have a decidable G. Word Problem ?
- which groups have the Howson property?
- which groups have the Subgroup Separation property?

Works by:(Mikhailova 1958)(free product), (Howson 1954), (Benois 1970), (Anisimov-Seifert 1975), (Hall 1949)(free group),(Romanovskii 1970),(Burns 1970)(free product),(Allenby-Gregorac 1972) (free product with finite amalgamation),(Scott 1978)(surface groups), (Rips 1990)(counter-example for amalgamation over  $\mathbb{Z}$ ), (Brunner-Burns-Solitar 1984)(free product with cyclic

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amalgamation of free groups)

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## 0.3 RESULTS

Let  $H$  be a monoid and

$$G = \langle H, t; t^{-1}at = \varphi(a) (a \in A) \rangle,$$

where  $A < H, B < H$ , finite subgroups,  $\varphi : A \rightarrow B$  isomorphism.

### Theorem 3

- 1- If  $\text{Rat}(H)$  is closed under intersection, so is  $\text{Rat}(G)$
- 2- If  $\text{Rat}(H)$  is closed under complement, so is  $\text{Rat}(G)$
- 3- If the membership problem is decidable for  $\text{Rat}(H)$ , so is it for  $\text{Rat}(G)$
- 4- If the emptiness problem is decidable for the boolean closure of  $\text{Rat}(H)$ , so is it for the boolean closure of  $\text{Rat}(G)$ .

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## 0.3 RESULTS

Let  $H$  be a group and  $G$  the extension defined above.

### **Theorem 4**

- 1- If  $H$  has a decidable Generalized Word Problem, so has  $G$
- 2- If  $H$  has the subgroup separation property, so has  $G$
- 3- If  $H$  has the Howson property, so has  $G$ .

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## 0.4 TOOLS

Finite automata.

Semi-Thue systems.

Fundamental group of a graph.

Virtually free groups.

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## 1.1 HNN EXTENSIONS- PRESENTATION

Let  $X$  be an alphabet,  $\equiv$  be some congruence over  $X^*$ .

Suppose that

$$H = X^* / \equiv_H$$

Let  $A, B < H$  be subgroups, and  $\varphi : A \rightarrow B$  an isomorphism.

The **HNN**-extension of  $H$  with stable letter  $t$  and associated subgroups  $A, B$  is:

$$G = (X \cup \{t, \bar{t}\})^* / \equiv$$

where

$$u \equiv v, \quad \text{for all } u, v \in X^* \text{ such that } u \equiv_H v, \quad (1)$$

$$t\bar{t} \equiv \bar{t}t \equiv \epsilon, \quad \text{and} \quad (2)$$

$$\bar{t}at \equiv \varphi(a), \quad \text{for all } a \in A. \quad (3)$$

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## 1.2 T-SEQUENCES

We name ***t*-sequence** an element  $s$  of  $H * \{t, \bar{t}\}^*$ :

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n$$

where  $n \in \mathbb{N}$ ,  $\alpha_i \in \{+1, -1\}$ ,  $t^{-1}$  means the letter  $\bar{t}$  and  $h_i \in H$ .  
 $s$  is said **reduced** iff it does not contain any factor of the form  $\bar{t}at$  (with  $a \in A$ ) nor  $tbt$  (with  $b \in B$ ).

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## 1.2 T-SEQUENCES

The congruence  $\approx$  over  $H * \{t, \bar{t}\}^*$  is generated by the set of rules

$$t\bar{t} \approx \bar{t}t \approx 1, \tag{4}$$

$$at \approx t\varphi(a), \quad \text{for all } a \in A \text{ and} \tag{5}$$

$$b\bar{t} \approx \bar{t}\varphi^{-1}(b), \quad \text{for all } b \in B. \tag{6}$$

Second presentation:

$$G = H * \{t, \bar{t}\}^* / \approx.$$

We denote by  $\sim$  the congruence over  $H * \{t, \bar{t}\}^*$  generated by the rules (5),(6)

**Lemma 5 (Britton's lemma)** : Let  $s, s'$  be reduced sequences. Then  $s \approx s'$  if and only if  $s \sim s'$ .

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## 1.2 T-SEQUENCES

Presentations and homomorphisms

$$(X \cup \{t, \bar{t}\})^* \xrightarrow{\pi_t} H * \{t, \bar{t}\}^* \xrightarrow{\pi_G} G$$

We set

$$\pi = \pi_t \circ \pi_G : (X \cup \{t, \bar{t}\})^* \rightarrow G$$

and

$$\text{Ker}(\pi) = \equiv; \quad \text{Ker}(\pi_G) = \approx;$$

$$\text{Ker}(\pi_G) \cap (\text{Red}(H * \{t, \bar{t}\}^*) \times \text{Red}(H * \{t, \bar{t}\}^*)) = \sim$$

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## 1.3 BASIC TRANSFER

**Corollary 6** *If  $H$  has a decidable Word Problem, so has  $G$*

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## 2.2- FREE MONOIDS

A finite automaton over the free monoid  $X^*$ , with labelling alphabet  $\mathcal{X}$  is a 5-tuple:

$$\mathcal{A} = \langle \mathcal{X}, Q, \delta, I, T \rangle$$

where,  $\mathcal{X}$  is a finite subset of  $X$ ,  $Q$  is the finite set of states,  $I \subseteq Q$  is the set of initial states,  $T \subseteq Q$  is the set of terminal states and  $\delta$ , is a subset of  $Q \times \mathcal{X} \times Q$ .

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## 2.2- FREE MONOIDS

$$\mu_{\mathcal{A}} : (X^*, \cdot, \varepsilon) \rightarrow (\mathcal{P}(Q \times Q), \circ, \text{Id}_Q)$$

is the unique monoid-homomorphism such that: for every  $x \in X$

$$\mu_{\mathcal{A}}(x) = \{(q, r) \in Q \times Q \mid (q, x, r) \in \delta\}.$$

$$L(\mathcal{A}) = \{u \in X^* \mid \exists q \in I, \exists r \in T, (q, r) \in \mu_{\mathcal{A}}(u)\}.$$

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## 2.3-GENERAL MONOIDS

Let  $(M, \cdot, 1_M)$  be a monoid. Suppose  $M = X^*/\equiv$ .

The set

$$\text{Rat}(M) \in \mathcal{P}(\mathcal{P}(M))$$

is the smallest element of  $\mathcal{P}(\mathcal{P}(M))$  which possesses the finite subsets of  $M$  and which is closed under union, product and  $*$ .

Finite automaton: as before.

For every  $m \in M$ ,

$$\mu_{\mathcal{A}, M}(m) = \bigcup_{[u]_{\equiv} = m} \mu_{\mathcal{A}}(u).$$

$$L(\mathcal{A}, M) = \{m \in M \mid \exists q \in I, \exists r \in T, (q, r) \in \mu_{\mathcal{A}, M}(m)\}.$$

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## 2.3-GENERAL MONOIDS

Note that:

$$\mu_{\mathcal{A}, M}(1) \supseteq \text{Id}_Q; \quad \mu_{\mathcal{A}, M}(m \cdot m') \supseteq \mu(m) \circ \mu(m'),$$

but in general the map  $\mu$  is **not** a monoid homomorphism.

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## 2.3 GROUPS

### Theorem 7 (Anisimov-Seifert75)

Let  $G$  be some *finitely generated* group. The *rational* subgroups of  $G$  are exactly the *finitely generated* subgroups of  $G$ .

THE trick:

$$\pi : X^* \rightarrow G$$

$$P_1(\mathcal{A}, q) = \{u \in X^* \mid q \xrightarrow{u} \delta \cup \delta^{-1} \quad q\}$$

If  $q_0$  is the unique initial and final state of  $\mathcal{A}$  recognizing a subgroup  $K$ , then

$$K = \pi(P_1(\mathcal{A}, q_0)).$$

One can then extract a *finite set of generators* of  $P_1$  seen as a fundamental group of a graph.

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### 3.1-HNN EXTENSION-F-AUTOMATA

Let  $\mathcal{F}$  be a set of subsets of  $H$ :

$$\mathcal{F} \in \mathcal{P}(\mathcal{P}(H)).$$

A f.a. over  $H * \{t, \bar{t}\}$ , with labelling set  $\mathcal{F}$  is a 5-tuple:

$$\mathcal{A} = \langle \mathcal{L}, Q, \delta, I, T \rangle$$

where,  $\mathcal{L}$  is finite and

$$\mathcal{L} \subseteq \mathcal{F} \cup \mathcal{P}(A) \cup \mathcal{P}(B) \cup \{\{t\}, \{\bar{t}\}\},$$

$Q, I, T$  are as usual and  $\delta \subseteq Q \times \mathcal{L} \times Q$ .

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### 3.1-HNN EXTENSION-F-AUTOMATA

$$\mu_0(1) = \{(q, q) \mid q \in Q\} \cup \{(q, r) \in Q \times Q \mid \exists (q, L, r) \in \delta, h \in L\}$$

for every  $h \in H - \{1\} \cup \{t, \bar{t}\}$

$$\mu_0(h) = \{(q, r) \in Q \times Q \mid \exists (q, L, r) \in \delta, h \in L\}$$

for every

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n$$

$$\mu(s) = \mu_0(h_0) \circ \mu_0(t^{\alpha_1}) \circ \mu_0(h_1) \cdots \mu_0(t^{\alpha_i}) \circ \mu_0(h_i) \cdots \mu_0(t^{\alpha_n}) \circ \mu_0(h_n)$$

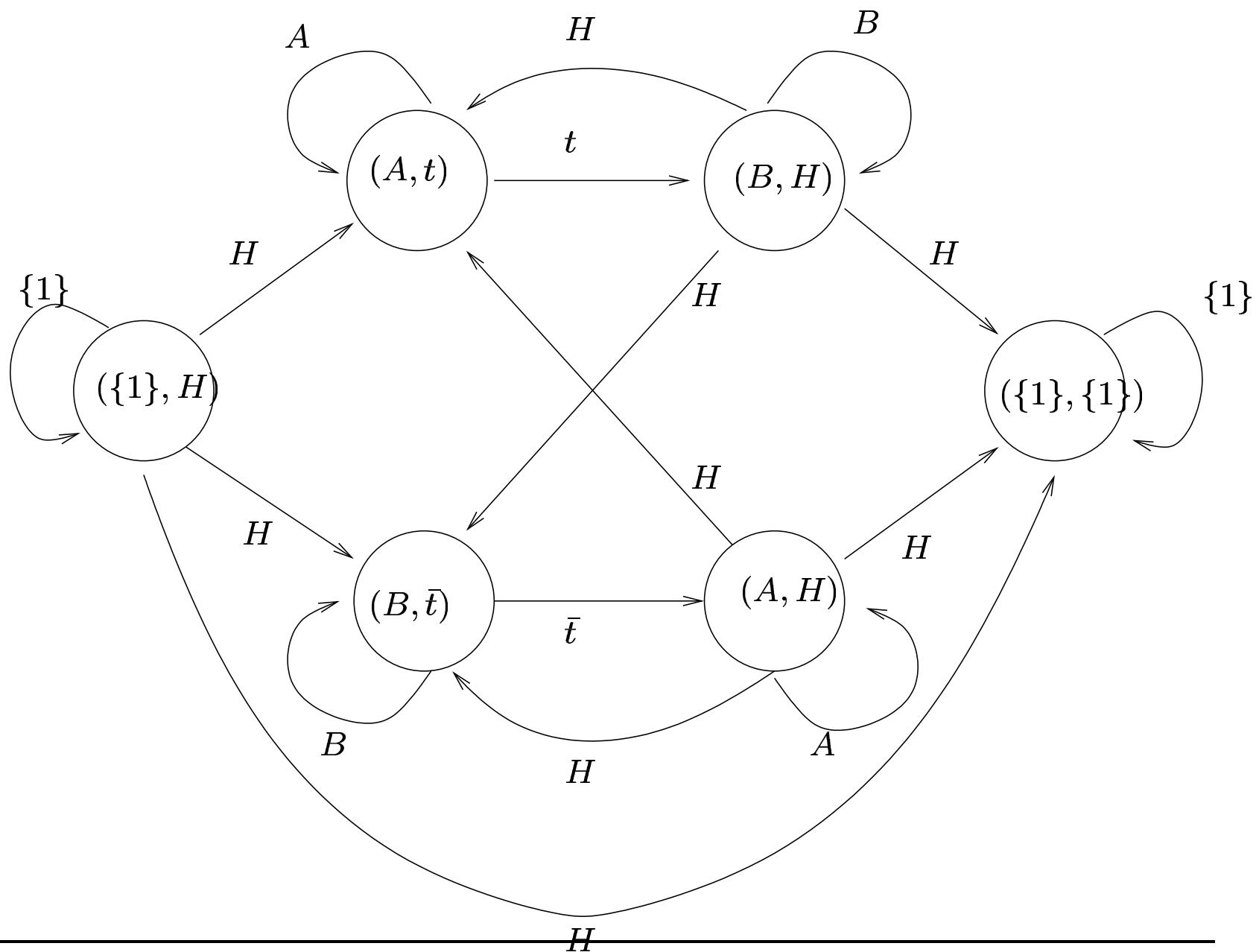
$$L(\mathcal{A}) = \{u \in H * \{t, \bar{t}\}^* \mid \exists q \in I, \exists r \in T, (q, r) \in \mu_{\mathcal{A}}(u)\}.$$

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## 3.2-HNN EXTENSION-PARTITIONNED AUTOMATA

Finite set of 6 types:

$$\mathcal{T}_6 = \{(A, t), (B, \bar{t}), (\{1\}, H), (\{1\}, \{1\}), (A, H), (B, H)\}$$



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## 3.2-HNN EXTENSION-PARTITIONNED AUTOMATA

We name *partitionned automaton* over  $\mathcal{F}$ , a 6-tuple

$$\mathcal{A} = \langle \mathcal{L}, Q, \tau, \delta, I, T \rangle$$

where  $\tau : Q \rightarrow \mathcal{T}_6$  is such that, for every  $q \in Q, (q, L, r) \in \delta$

$$\{q\} \times L \times \{r\} \subseteq \mathcal{E}_6,$$

$$q \in I \Rightarrow \tau(q) = (\{1\}, H); \quad q \in T \Rightarrow \tau(q) = (\{1\}, \{1\}).$$

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### 3.3 HNN EXTENSION-SATURATION

A partitionned finite automaton  $\mathcal{A}$  over a labelling set  $\mathcal{F}$  is said to be  *$\sim$ -saturated* if and only if, for every  $s, s' \in H * \{t, \bar{t}\}^*$ ,

$$s \sim s' \Rightarrow \mu(s) = \mu(s').$$

It is said  *$\approx$ -compatible* if and only if,

$$[\mathbf{L}(\mathcal{A})]_{\approx} = [\mathbf{L}(\mathcal{A}) \cap \mathbf{Red}(H * \{t, \bar{t}\}^*)]_{\approx}.$$

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### 3.3 HNN EXTENSION-SATURATION

It is said **deterministic** iff

$$\text{Card}(I) = 1$$

and for every  $q, r, r' \in Q, F, F' \in \mathcal{L}$ ,

$$((q, F, r), (q, F', r') \in \delta \wedge F \cap F' \neq \emptyset \wedge \tau(r) = \tau(r')) \Rightarrow (q, F, r) = (q, F', r').$$

It is said **complete** iff...

(local condition ensuring that the automaton can read every reduced sequence)

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## 3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

**Proposition 8** *Let  $K \subseteq G$ . The following are equivalent*

- 1-  *$K$  is a rational subset of  $G$*
- 2- *There exists some  $R \in \text{Rat}((X \cup \{t, \bar{t}\})^*)$  such that  $K = \pi(R)$*
- 3-  *$\pi_G^{-1}(K) \cap \text{Red}(H * \{t, \bar{t}\}^*) = L(\mathcal{A}) \cap \text{Red}(H * \{t, \bar{t}\}^*)$  for some partitionned,  $\approx$ -compatible,  $\sim$ -saturated, finite automaton  $\mathcal{A}$  over  $\text{Rat}(H)$ .*

We start with  $\mathcal{A}_0$ , f.a. over the free monoid  $(X \cup \{t, \bar{t}\})^*$ , such that

$$K = \pi(L(\mathcal{A}_0)).$$

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## 3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

step 1: semi-Thue system  $\mathcal{R}_r$ :

$$\bar{t}ut \rightarrow \varphi(u) \text{ ( for } [u]_{\equiv_H} \in A); tut\bar{t} \rightarrow \varphi^{-1}(u) \text{ ( for } [u]_{\equiv_H} \in B)$$

It is **monadic**, thus we construct  $\mathcal{A}_1$  such that:

$$L(\mathcal{A}_1) = L(\mathcal{A}_0) \xrightarrow{*_{\mathcal{R}_r}}$$

step 2: Collecting H-transitions, we construct f.a.,  $\mathcal{A}_2$ , with labels in  $\text{Rat}(H)$ , which is  $\approx$ -compatible and such that

$$L(\mathcal{A}_2) = \pi_t(L(\mathcal{A}_1)).$$

step 3:

$$\mathcal{A}_3 = \mathcal{A}_2 \times \mathcal{G}_6$$

$\mathcal{A}_3$  is partitionned,  $\approx$ -compatible.

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## 3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

step 4: Saturation. From  $\mathcal{A}_3$  we construct

$$\mathcal{B} = \langle \mathcal{L}_{\mathcal{B}}, Q_{\mathcal{B}}, \tau_{\mathcal{B}}, \delta_{\mathcal{B}}, I, T \rangle$$

Idea: for every  $q \in Q_3$ ,

$$\gamma(q) = \pi_1(\tau(q))$$

is one of the three subgroups  $\{1\}, A, B$ ;

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## 3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

$$Q_B = \{(q, g) \in Q_3 \times (A \cup B) \mid g \in \gamma(q)\}$$

the set  $\delta_B$  consists of all the following transitions:

$$(qg, g', q(g \cdot g')) \quad \text{for all } q \in Q_3, g \in \gamma(q), g' \in \gamma(q)$$

$$(qg, g^{-1}Lg', rg') \quad \text{for all } q \in Q_3, g \in \gamma(q), g' \in \gamma(r), L \in \mathcal{L}_3, (q, L, r) \in \delta_3$$

$$(qa, t, r\varphi(a)) \quad \text{for all } q \in Q_3, a \in A, (q, t, r) \in \delta_3$$

$$(qb, \bar{t}, r\varphi^{-1}(b)) \quad \text{for all } q \in Q_3, b \in B, (q, \bar{t}, r) \in \delta_3$$

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## 3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

### Proposition 9

Let  $R_t \subseteq H * \{t, \bar{t}\}^*$  be recognized by some partitionned,  $\approx$ -compatible, finite automaton  $\mathcal{A}$  over the labelling set  $\mathcal{F}$ .

1- Then,  $\pi_G^{-1}(\pi_G(R_t)) \cap \text{Red}(H * \{t, \bar{t}\}^*) = L(\mathcal{B}) \cap \text{Red}(H * \{t, \bar{t}\}^*)$  for some partitionned,  $\approx$ -compatible,  $\sim$ -saturated, finite automaton  $\mathcal{B}$  over  $\mathcal{F}'$ , where

$$\mathcal{F}' = \{gFg' \mid F \in \mathcal{F}, g, g' \in A \cup B\}$$

2-  $\pi_G^{-1}(\pi_G(R_t)) \cap \text{Red}(H * \{t, \bar{t}\}^*)$  is recognized by some partitionned,  $\sim$ -saturated, deterministic, complete finite automaton  $\mathcal{D}$  over  $\mathcal{F}''$ , where  $\mathcal{F}''$  is the boolean closure of  $\mathcal{F}'$ .

**Idea:** for point 1, construction of  $\mathcal{B}$  above  
for point 2, classical idea of **subset-construction**; but take into account the 6 types.



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## 3.4 HNN EXTENSION-NORMALIZATION OF AUTOMATA

**Proposition 10** *Let  $K$  be a subgroup of  $G$ . The following are equivalent*

- 0-  $K$  is finitely generated
- 1-  $K$  is rational
- 2- There exists some  $R \in \text{Rat}((X \cup \{t, \bar{t}\})^*)$  such that  $K = \pi(R)$
- 3-  $\pi_G^{-1}(K) \cap \text{Red}(H * \{t, \bar{t}\}^*) = L(\mathcal{B}) \cap \text{Red}(H * \{t, \bar{t}\}^*)$  for some partitionned,  $\approx$ -compatible,  $\sim$ -saturated, finite automaton  $\mathcal{B}$  over  $\mathcal{S}$ .

Idea: same constructions as before, but combined with THE  $P_1$ -trick.

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## 4.1-TRANSFER FOR RATIONAL SUBSETS

### Theorem 11

- 1- If  $\text{Rat}(H)$  is closed under intersection , **so is**  $\text{Rat}(G)$
- 2- If  $\text{Rat}(H)$  is closed under complement , **so is**  $\text{Rat}(G)$
- 3- If the membership problem is decidable for  $\text{Rat}(H)$ , **so is it for**  $\text{Rat}(G)$
- 4- If the emptiness problem is decidable for the boolean closure of  $\text{Rat}(H)$ , **so is it for** the boolean closure of  $\text{Rat}(G)$ .

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## 4.2- TRANSFER FOR FINITELY GENERATED SUBGROUPS

Let  $H$  be a group and  $G$  the extension defined above.

### Theorem 12

- 1- If  $H$  has a decidable Generalized Word Problem, so has  $G$
- 2- If  $H$  has the subgroup separation property, so has  $G$
- 3- If  $H$  has the Howson property, so has  $G$ .

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## 4.3-TRANSFER TOWARDS AN AMALGAMATED PRODUCT

Let  $H_1, H_2$ , be tmonoids,  $A_1 \leq H_1, A_2 \leq H_2$ , finite subgroups and  $\varphi : A_1 \rightarrow A_2$  isomorphism.

$$G = \langle H_1, H_2; a = \varphi(a) (a \in A_1) \rangle.$$

$G$  is embedded into the HNN-extension

$$\hat{G} = \langle H_1 * H_2, t; t^{-1}at = \varphi(a) (a \in A_1) \rangle$$

by the map

$$h_1 \in H_1 \mapsto t^{-1}h_1t; \quad h_2 \in H_2 \mapsto h_2$$

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## 4.3-TRANSFER TOWARDS AN AMALGAMATED PRODUCT

### Theorem 13

- 1-*If  $\text{Rat}(H_i)$  is closed under intersection (for every  $i \in \{1, 2\}$ ), so is  $\text{Rat}(G)$*
- 2-*If  $\text{Rat}(H_i)$  is closed under complement (for every  $i \in \{1, 2\}$ ), so is  $\text{Rat}(G)$*
- 3-*If the membership problem is decidable for  $\text{Rat}(H_i)$  (for every  $i \in \{1, 2\}$ ), so is it for  $\text{Rat}(G)$*
- 4-*If the emptiness problem is decidable for the boolean closure of  $\text{Rat}(H_i)$  (for every  $i \in \{1, 2\}$ ), so is it for the boolean closure of  $\text{Rat}(G)$ .*

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## 4.3-TRANSFER TOWARDS AN AMALGAMATED PRODUCT

### Theorem 14

1-If  $H_i$  has a decidable Generalized Word Problem (for every  $i \in \{1, 2\}$ ), so has  $G$

2-If  $H_i$  has the Subgroup separation property (for every  $i \in \{1, 2\}$ ), so has  $G$

3-If  $H_i$  has the Howson property (for every  $i \in \{1, 2\}$ ), so has  $G$ .

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## 5- PERSPECTIVES

5.1 Equations in  $G$ : transfer of decidability.

5.2 More general extensions:

$A, B$  infinite

$A, B$ , finite submonoids.