EQUATIONS

in HNN-extensions

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4.2 Reduction

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0.1- INTRODUCTION-PROBLEMS

Algorithmic problems.

$P = \langle X, \equiv \rangle$ and $M = X^*/\equiv$.

The Word Problem:

instance: $u, v \in X^*$

question: $u \equiv v$?

The Emptiness Problem for boolean combinations of rational expressions:

instance: A boolean combination of rational expression E over the alphabet X.

question: $\mathrm{L}(E, M) = \emptyset$?

0.1- INTRODUCTION-PROBLEMS

The Satisfiability Problem for Equations:

instance: $u, v \in \mathcal{U}^* * M$

question: $\exists \sigma : \mathcal{U}^* \to M, \sigma(u) = \sigma(v)?$

The Satisfiability Problem for Equations with rational constraints:

instance: $u, v \in \mathcal{U}^*, \quad \mathsf{C} : \mathcal{U} \to \mathcal{B}(\mathrm{Rat}(M))$

question: $\exists \sigma : \mathcal{U}^* \to M,$

$$\sigma(u) = \sigma(v) \wedge \forall U \in \mathcal{U}, \sigma(U) \in \mathsf{C}(U)?$$

0.2- INTRODUCTION-MOTIVATIONS

Solving equation:

- is a more general problem than:

*conjugacy problem (case of groups)

*membership problem for submonoids or subgroups (even if non finitely generated)

- is a natural thema in mathematics

0.2- INTRODUCTION-MOTIVATIONS

Solving equations with rational constraints:

- is even more general than solving equations

- is a useful tool for solving equations in monoids

i.e. even for groups where rational constraints might be undecidable !

- is a useful tool for solving inequations in groups.

0.3- INTRODUCTION-RESULTS

Positive results:

(Lyndon 1960, Lorents 1968, Makanin 1977, Makanin 1983, Schulz 1991, Razborov,1985, Rips-Sela 1995, Diekert-Matiyasevich-Muscholl 1999, Plandowski 1999)

Diekert-Hagenah-Gutierrez 2001: equations with rational constraints in free groups,

Diekert-Lohrey 2004: transfer theorem for graph products,

Negative results:

(Rips 1982) (GWP in hyperbolic groups),(Rozenblatt 1986) (free inverse monoid),...

0.3- INTRODUCTION-RESULTS

New results.

Theorem 1 *Let $\mathbb{H}$ be a cancellative monoid and $\mathbb{G}$ an HNN-extension of $\mathbb{H}$ with finite associated subgroups. The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is decidable if and only if the satisfiability problem for systems of equations with rational constraints in $\mathbb{H}$ is decidable.*

Numerous variations around this result:
amalgamated product $\leftrightarrow$ HNN-extension
equations $\leftrightarrow$ equations and inequations $\leftrightarrow$ positive FO
rational constraints $\leftrightarrow$ constants.

0.4- INTRODUCTION-TOOLS

Finite automata: notion of t-automaton over $\mathbb{H} * \{t, \bar{t}\}^*$.

Word rewriting: monadic semi-Thue systems, normal form theorems

AB-algebras: new notion

Equations in free group, with rational constraints (Diekert-Hagenah-Gutierrez 2001)

Equations in free products (Diekert-Lohrey 2004)

1.1- HNN EXTENSIONS-PRESENTATION

Let $\mathbb{H}$ be some monoid, $A, B < \mathbb{H}$ be subgroups, and $\varphi : A \to B$ an isomorphism. The HNN-extension of $\mathbb{H}$ with stable letter t and associated subgroups A, B is:

$$\mathbb{G} = \mathbb{H} * \{t, \bar{t}\}^* / \approx$$

where $\approx$ is generated by the rules:

$$\begin{aligned} t\bar{t} &\approx \bar{t}t \approx \epsilon, & \\ at &\approx t\varphi(a), & \text{for all } a \in A, \\ b\bar{t} &\approx \bar{t}\varphi^{-1}(b), & \text{for all } b \in B, \end{aligned}$$

We define $\sim$ as the monoid-congruence generated by:

$$\begin{aligned} at &\sim t\varphi(a), & \text{for all } a \in A, \\ b\bar{t} &\sim \bar{t}\varphi^{-1}(b), & \text{for all } b \in B, \end{aligned}$$

1.2- HNN EXTENSIONS-T-SEQUENCES

We name t-sequence an element s of $H * \{t, \bar{t}\}^*$:

$$s = h_0 t^{\alpha_1} h_1 \cdots t^{\alpha_i} h_i \cdots t^{\alpha_n} h_n$$

where $n \in \mathbb{N}, \alpha_i \in \{+1, -1\}$, t^{-1} means the letter $\bar{t}$ and $h_i \in H$. s is said reduced iff it does not contain any factor of the form $\bar{t}at$ (with $a \in A$) nor $tb\bar{t}$ (with $b \in B$).

1.2- HNN EXTENSIONS-T-SEQUENCES

Homomorphisms

$$\mathbb{H} * \{t, \bar{t}\}^* \xrightarrow{\pi_\sim} \mathbb{H} * \{t, \bar{t}\}^* / \sim \xrightarrow{\bar{\pi}_{\mathbb{G}}} \mathbb{G}$$

We note:

$$\mathbb{H}_t := \mathbb{H} * \{t, \bar{t}\}^* / \sim .$$

1.3- HNN EXTENSIONS-BASIC TRANSFERS

Theorem 2 *If $\mathbb{H}$ has a decidable Word Problem, so has $\mathbb{G}$*

Theorem 3 *If $\mathbb{H}$ has a decidable Conjugacy Problem, so has $\mathbb{G}$*

1.4- HNN EXTENSIONS-FURTHER TRANSFERS

Theorem 4 *Suppose that $\mathbb{H}$ is a group.*
If $\mathbb{H}$ has a decidable Generalized Word Problem, so has $\mathbb{G}$

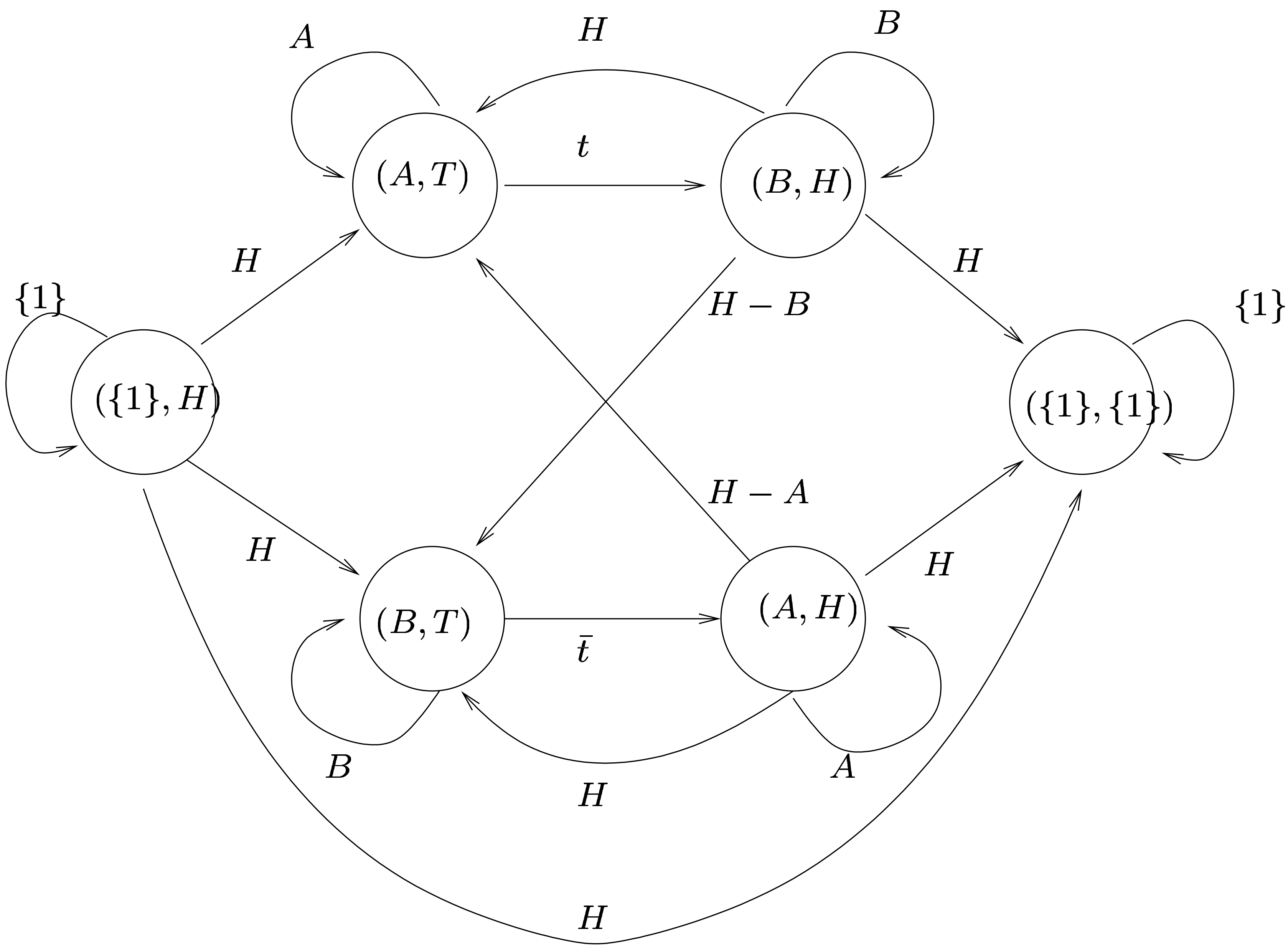
Theorem 5 *If the emptiness problem for $\mathcal{B}(\mathrm{Rat}(\mathbb{H}))$ is decidable, so is it for $\mathbb{G}$*

1.4- HNN EXTENSIONS-FURTHER TRANSFERS

Main tool: partitionned t-automata.

Graph: $\mathcal{R}_6$

Vertices: $\mathcal{T}_6$



A
H
B
(A, T)
t
(B, H)
H
H
$\{1\}$
$H - B$
$\{1\}$
$(\{1\}, H)$
$(\{1\}, \{1\})$
$H - A$
H
H
(B, T)
(A, H)
$\bar{t}$
B
A
H
H

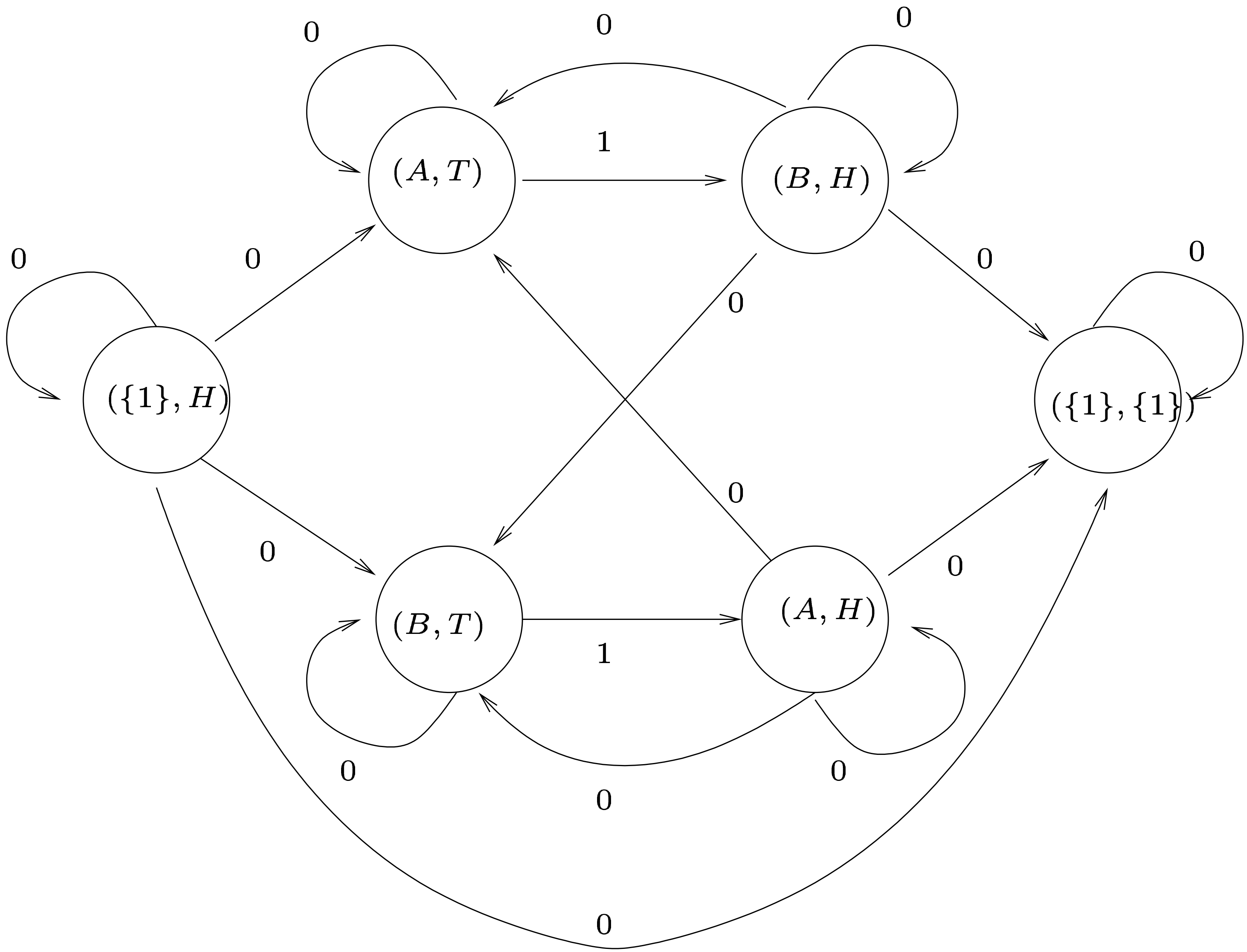
2.1-AB-ALGEBRAS-TYPES

$$\mathcal{T} = \mathcal{T}_6 \times \mathbb{B} \times \mathcal{T}_6$$

$\mathbb{B}$:booleans;
partial product: if $q = p'$ then

$$(p, b, q) \cdot (p', b', q') = (p, b \cdot b', q'),$$

otherwise the product is undefined.
$\langle \mathcal{P}(\mathcal{T}), \cdot \rangle$ is a semi-group.


(A, T)
(B, H)
$(\{1\}, H)$
$(\{1\}, \{1\})$
(B, T)
(A, H)
0
1

$\mathcal{TR}$: partial submonoid generated by the set of edges of $\mathcal{R}$.

2.2- AB-ALGEBRAS-AXIOMS

We call AB-algebra a structure of the form

$$\langle \mathbb{M}, \cdot, \mathbf{1}_{\mathbb{M}}, \mathbb{I}, \iota_A, \iota_B, \gamma, \mu, \delta \rangle$$

where $\iota_A : A \to \mathbb{M}, \iota_B : B \to \mathbb{M}$ are total maps,
$\mathbb{I} : \mathbb{M} \to \mathbb{M}$ is a partial map,
$\gamma : \mathbb{M} \to \mathcal{P}(\mathcal{T})$ is a total map,
$\mu : \mathcal{T} \times \mathbb{M} \to \mathcal{B}^2(\mathsf{Q}_{\mathcal{A}})$ is a total map,
$\delta : \mathcal{T} \times \mathbb{M} \to \mathrm{PGA}(A, B)$ is a total map ;

fulfilling the twelve axioms:

monoid:

$$(\mathbb{M}, \cdot, 1_{\mathbb{M}}) \text{ is a monoid ,} \tag{1}$$

embeddings:

$$\iota_A, \iota_B \text{ are injective monoid homomorphisms ,} \tag{2}$$

involution I:

$$\iota_A(A) \cup \iota_B(B) \subseteq \mathrm{dom}(\mathbb{I}) \subseteq \mathbb{M} - \gamma^{-1}(\emptyset) \tag{3}$$

for every $m, m' \in \mathbb{M}$,

$$[\gamma(m)\cdot\gamma(m') \neq \emptyset] \Rightarrow [m\cdot m' \in \mathrm{dom}(\mathbb{I}) \Leftrightarrow (m \in \mathrm{dom}(\mathbb{I}) \wedge m' \in \mathrm{dom}(\mathbb{I}))] \tag{4}$$

$$\mathbb{I} : (\mathrm{dom}(\mathbb{I}), \cdot, 1_{\mathbb{M}}) \to (\mathrm{dom}(\mathbb{I}), \cdot, 1_{\mathbb{M}}) \text{ is a monoid anti-isomorphism,} \tag{5}$$

$$\mathbb{I} \circ \mathbb{I} = \mathbb{I}, \tag{6}$$

almost homomorphisms:
for every $m, m' \in \mathbb{M}$,

$$\gamma(m \cdot m') \supseteq \gamma(m) \cdot \gamma(m'), \tag{7}$$

for every $m, m' \in \mathbb{M}, \theta \in \gamma(m), \theta' \in \gamma(m')$, such that $(\theta, \theta') \in D(\cdot)$,

$$\mu(\theta \cdot \theta', m \cdot m') = \mu(\theta, m) \cdot \mu(\theta', m'), \tag{8}$$

$$\mathrm{dom}(\delta(\theta, m)) \subseteq \mathrm{Gi}(\theta), \mathrm{im}(\delta(\theta, m)) \subseteq \mathrm{Ge}(\theta) \tag{9}$$

$$\delta(\theta \cdot \theta', m \cdot m') = \delta(\theta, m) \circ \delta(\theta', m'). \tag{10}$$

commutation with I:
for every $a \in A, b \in B, m \in \mathbb{M}, \theta \in \gamma(m)$,

$$\mathbb{I}(\iota_A(a)) = \iota_A(a^{-1}); \;\; \mathbb{I}(\iota_B(b)) = \iota_B(b^{-1}) \tag{11}$$

$$\gamma(\mathbb{I}(m)) = \mathbb{I}_{\mathcal{T}}(\gamma(m)); \;\; \mu(\mathbb{I}_{\mathcal{T}}(\theta), \mathbb{I}(m)) = \mathbb{I}_{\mathsf{Q}}(\mu(\theta, m)); \;\; \delta(\mathbb{I}_{\mathcal{T}}(\theta), \mathbb{I}(m)) = \delta(\theta, m)^{-1}. \tag{12}$$

2.3-AB-ALGEBRAS-AB-HOMOMORPHISMS

Let

$$\mathcal{M}_1 = \langle \mathbb{M}_1, \cdot, 1_{\mathbb{M}_1}, \iota_{A,1}, \iota_{B,1}, \mathbb{I}_1, \gamma_1, \mu_1, \delta_1 \rangle,$$

$$\mathcal{M}_2 = \langle \mathbb{M}_2, \cdot, 1_{\mathbb{M}_2}, \iota_{A,2}, \iota_{B,2}, \mathbb{I}_2, \gamma_2, \mu_2, \delta_2 \rangle$$

be two AB-algebras with the same underlying groups A, B and set Q. We call AB-homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ any map

$$\psi : \mathbb{M}_1 \to \mathbb{M}_2$$

fulfilling the seven properties below:

m-**homomorphism**:

$$\psi : (\mathbb{M}_1, \cdot, 1_{\mathbb{M}_1}) \to (\mathbb{M}_2, \cdot, 1_{\mathbb{M}_2}) \text{ is a monoid homomorphism} \tag{13}$$

ι-**preservation**:

$$\forall a \in A, \forall b \in B, \psi(\iota_{A,1}(a)) = \iota_{A,2}(a), \quad \psi(\iota_{B,1}(b)) = \iota_{B,2}(b) \tag{14}$$

I-preservation:

$$\forall m \in \mathbb{M}_1 - \gamma_1^{-1}(\emptyset), \quad m \in \mathrm{dom}(\mathbb{I}_1) \Leftrightarrow \psi(m) \in \mathrm{dom}(\mathbb{I}_2) \tag{15}$$

$$\forall m \in \hat{\mathbb{M}}_1, \quad \mathbb{I}_2(\psi(m)) = \psi(\mathbb{I}_1(m)) \tag{16}$$

γ-**compatibility**:

$$\forall m \in \mathbb{M}_1, \gamma_2(\psi(m)) \supseteq \gamma_1(m) \tag{17}$$

μ**-preservation**:

$$\forall m \in \mathbb{M}_1, \forall \theta \in \gamma_1(m), \mu_2(\psi(m)) = \mu_1(\theta, m), \tag{18}$$

δ**-preservation**:

$$\forall m \in \mathbb{M}_1, \forall \theta \in \gamma_1(m), \delta_2(\theta, \psi(m)) = \delta_1(\theta, m). \tag{19}$$

2.4-AB-ALGEBRA: $\mathbb{H}_t$

Given: an HNN-extension and a partitionned, $\sim$-saturated finite t-automaton $\mathcal{A}$, we define an AB-algebra with underlying monoid $\mathbb{H} * \{t, \bar{t}\}^*$ and set of states $\mathsf{Q}_{\mathcal{A}}$.

$$\langle \mathbb{H} * \{t, \bar{t}\}^*, \cdot, 1_{\mathbb{H}}, \iota_A, \iota_B, \mathbb{I}, \mu, \gamma, \delta \rangle$$

as follows:

$$\iota_A, \iota_B$$

are the natural injections from A (resp. B) into $\mathbb{H} * \{t, \bar{t}\}^*$,

$$\mathrm{dom}(\mathbb{I}) = (I(\mathbb{H}) \cup \{t, \bar{t}\})^*$$

where $I(\mathbb{H})$ is the set of invertible elements of $\mathbb{H}$;

$$\mathbb{I}(h) = h^{-1}; \;\; \mathbb{I}(t) = \bar{t}; \;\; \mathbb{I}(\bar{t}) = t.$$

$$\gamma(s) = \{(p, b, q) \in \mathcal{T}_6 \times \mathbb{B} \times \mathcal{T}_6 \mid (p, q) \in \mu_{\mathcal{R}_6}(s) \wedge b = (\|s\| \neq 0)\}$$

We define an auxiliary function $\mu_1 : \mathcal{T} \times \mathbb{H} * \{t, \bar{t}\}^* \to \mathcal{B}(\mathsf{Q})$ by:

$$\mu_1(\theta, s) = \mu_{\mathcal{A}}(s) \cap (\tau_{\mathcal{A}}^{-1}(\tau\mathrm{i}(\theta)) \times \tau_{\mathcal{A}}^{-1}(\tau\mathrm{e}(\theta)))$$

and then

$$\begin{aligned}
\mu(\theta, s) &= \quad (\mu_1(\theta, s), \mu_1(\mathbb{I}_{\mathcal{T}}(\theta), \mathbb{I}_t(s))^{-1}) && \text{if } s \in \mathrm{dom}(\mathbb{I}_t); \\
\mu(\theta, s) &= \quad (\mu_1(\theta, s), \mu_1(\theta, s)) && \text{if } s \notin \mathrm{dom}(\mathbb{I}_t). \\
\delta(\theta, s) &= \quad \{(g, g') \in \mathrm{Gi}(\theta) \times \mathrm{Ge}(\theta) \mid g \cdot s \sim s \cdot g'\}.
\end{aligned}$$

The monoid-congruence $\sim$ is compatible with $\mathbb{I}, \iota_A, \iota_B, \gamma, \mu, \delta$. (due to the special properties of the t-automaton $\mathcal{A}$). We can naturally endow $\mathbb{H}_t = \mathbb{H} * \{t, \bar{t}\}^* / \sim$ with a structure of AB-algebra:

$$\langle \mathbb{H}_t, \cdot, \mathbf{1}_{\mathbb{H}}, \iota_{A,\sim}, \iota_{B,\sim}, \mathbb{I}_{\sim}, \mu_{\sim}, \gamma_{\sim}, \delta_{\sim} \rangle$$

2.5-AB-ALGEBRA: $\mathbb{W}$

Underlying idea:

1- notion of generic solution of an equation over $\mathbb{H}_t$.

$\rightarrow$ solution in $\mathbb{W}$.

2- any concrete solution (i.e.in $\mathbb{H}_t$) is obtained by applying an AB-homomorphism on a generic solution.

2.5-AB-ALGEBRA: $\mathbb{W}$

$$\Omega := \mathcal{V}_0 \times \{-1, 0, 1\} \times \mathcal{TA} \times \mathcal{B}^2(\mathsf{Q}_{\mathcal{A}}) \times \mathrm{PGA}(A, B).$$

$$\mathcal{W} := \{(V, \epsilon, \theta, m, \varphi) \in \Omega \mid \varphi \in \mathrm{PIs}(\mathrm{Gi}(\theta), \mathrm{Ge}(\theta)), \forall (c, d) \in \varphi, \mu_{\mathcal{A}}(c) \cdot m = m \cdot \mu_{\mathcal{A}}(d)\}.$$

$$\check{\mathcal{W}} = \{W \in \mathcal{W} \mid p_2(W) = 0\}, \;\; \hat{\mathcal{W}} = \{W \in \mathcal{W} \mid p_2(W) \neq 0\}$$

$\iota_A : A \to \mathcal{W}^* * A * B$ the natural embedding.

We define the AB-algebra:

$$\langle \mathcal{W}^* * A * B, \cdot, 1, \iota_A, \iota_B, \mathbb{I}, \mu, \gamma, \delta \rangle$$

$$\iota_A : A \to \mathcal{W}^* * A * B, \quad \iota_B : B \to \mathcal{W}^* * A * B$$

are the natural embeddings.

$$\mathrm{dom}(\mathbb{I}) = \hat{\mathcal{W}}^* * A * B$$

$$\mathbb{I}(\iota_A(a)) = \iota_A(a^{-1}); \quad \mathbb{I}(\iota_B(b)) = \iota_B(b^{-1})$$

$$\mathbb{I}(V, \epsilon, \theta, m, \varphi) = (V, -\epsilon, \mathbb{I}_{\mathcal{T}}(\theta), \mathbb{I}_{\mathsf{Q}}(m), \varphi^{-1}).$$

$$\gamma(V, \epsilon, \theta, m, \varphi) = \{\theta\},$$

$$\gamma(\iota_A(a)) = \{(A, T, 0, A, T), (A, H, 0, A, H)\}, \text{ for } a \in A - \{1\}$$

$$\gamma(\iota_B(b)) = \{(B, T, 0, B, T), (B, H, 0, B, H)\}, \text{ for } b \in B - \{1\}$$

$$\begin{aligned} \gamma(1) &= \{(1, H, 0, 1, H), (1, 1, 0, 1, 1)\} \\ &\cup \{(A, T, 0, A, T), (A, H, 0, A, H), (B, T, 0, B, T), (B, H, 0, B, H)\} \end{aligned}$$

For $g_1, \ldots, g_i, \ldots, g_n \in \mathcal{W} \cup \iota_A(A) \cup \iota_B(B)$,

$$\gamma(\prod_{i=1}^{n} g_i) = \prod_{i=1}^{n} \gamma(g_i)$$

$$\mu(\theta, \iota_A(a)) = \mu_t(\theta, a), \;\; \mu(\theta, \iota_B(b)) = \mu_t(\theta, b),$$

$$\mu(\theta, (V, \epsilon, \theta, m, \varphi)) = m, \;\; \mu(\theta', (V, \epsilon, \theta, m, \varphi)) = \emptyset(\text{ if } \theta' \neq \theta).$$

$$\begin{aligned}
\delta(\theta, \iota_A(a)) = &\quad \{(\iota_A(c), \iota_A(d)) \mid (c, d) \in \mathrm{Gi}(\theta) \times \mathrm{Ge}(\theta), ca = ad\}, \text{ if } \theta \in \gamma(\iota_A(\\
\delta(\theta, \iota_A(a)) = &\quad \{(1,1)\}, \text{ if } \theta \notin \gamma(\iota_A(a)) \\
\delta(\theta, \iota_B(b)) = &\quad \{(\iota_B(c), \iota_B(d)) \mid (c, d) \in \mathrm{Gi}(\theta) \times \mathrm{Ge}(\theta), ca = ad\}, \text{ if } \theta \in \gamma(\iota_B \\
\delta(\theta, \iota_B(b)) = &\quad \{(1,1)\}, \text{ if } \theta \notin \gamma(\iota_B(b)), \\
\delta(\theta, (V, \epsilon, \theta, m, \varphi)) = &\quad \varphi, \\
\delta(\theta', (V, \epsilon, \theta, m, \varphi)) = &\quad \{(1,1)\}(\text{ if } \theta' \neq \theta).
\end{aligned}$$

2.5-AB-ALGEBRA: $\mathbb{W}$

Two sub-algebras:

$\mathcal{W}_t$ is the set of letters $W \in \mathcal{W}$ such that, $\exists s \in \mathbb{H}_t$, such that:

$$W \in \mathrm{dom}(\mathbb{I}_{\mathbb{W}}) \Leftrightarrow s \in \mathrm{dom}(\mathbb{I}_t)$$

$$\gamma(W) \subseteq \gamma(s), \;\; \mu(\theta, W) = \mu(\theta, s), \;\; \delta(\theta, W) = \delta(\theta, s).$$

$\mathcal{W}_H$ is the set of letters $W \in \mathcal{W}$ which have a H-type.

$$\mathbb{W}_t := \mathcal{W}_t^* * A * B/\equiv, \;\; \mathbb{W}_H := \mathcal{W}_H^* * A * B/\equiv .$$

3.1-EQUATIONS OVER $\mathbb{H}_t$-DEFINITION

A system of t-equations is a family of ordered pairs

$$\mathcal{S} = (w_i, w_i')_{i \in I}$$

where $w_i, w_i' \in \mathbb{W}_t, \gamma(w_i) = \gamma(w_i') \neq \emptyset$.
A *solution* of $\mathcal{S}$ is any

$$AB\text{-homomorphism } \psi_t : \mathbb{W}_t \to \mathbb{H}_t$$

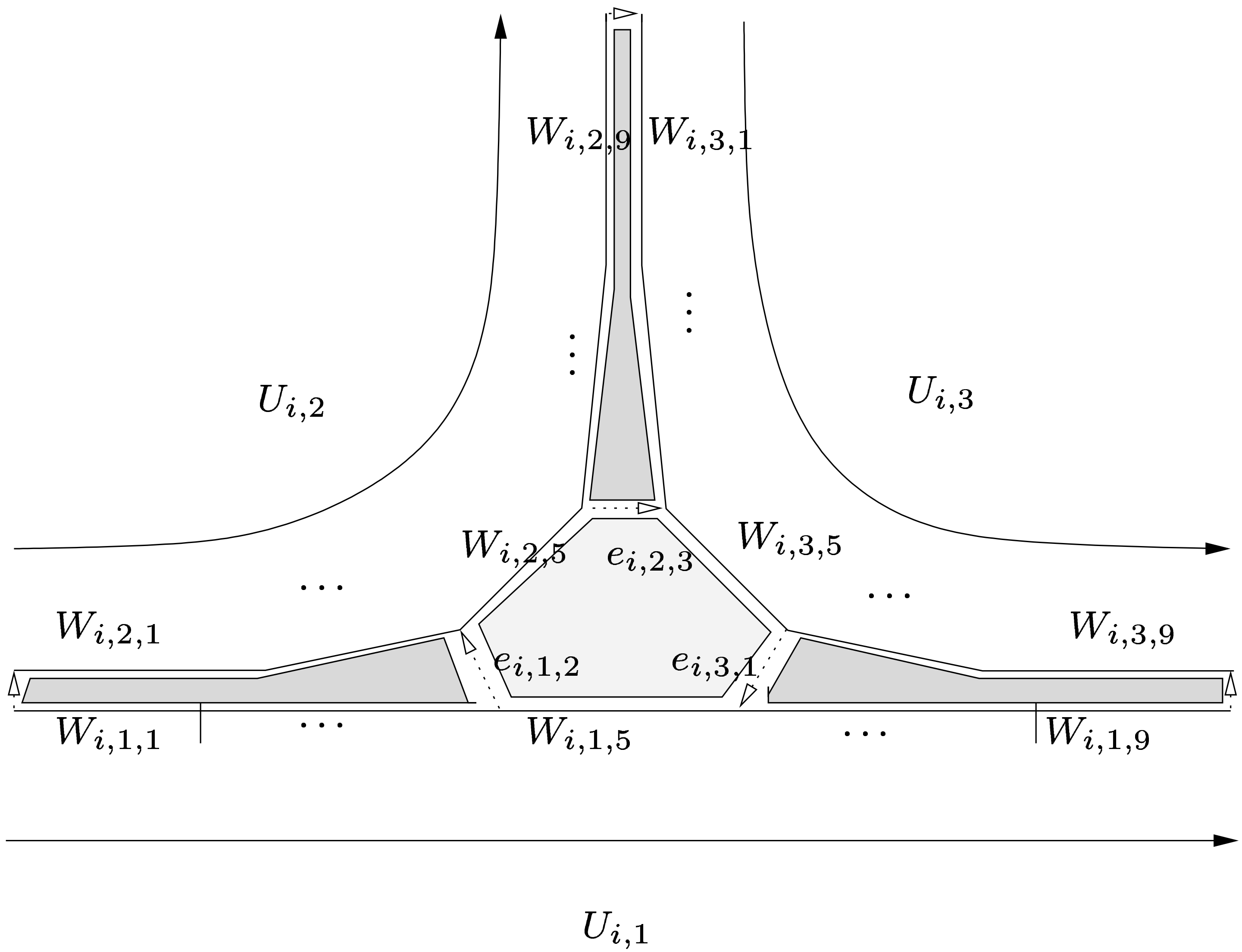
such that, for every $i \in I$

$$\psi_t(w_i) = \psi_t(w_i').$$

3.2-EQUATIONS OVER $\mathbb{H}_t$-REDUCTION

We start with equations over $\mathbb{G}$, with rational constraint C:

$$E_i : \quad (U_{i,1}, U_{i,2}U_{i,3}) \text{ for all } 1 \leq i \leq n$$


$W_{i,2,9}$
$W_{i,3,1}$
$U_{i,2}$
$U_{i,3}$
$W_{i,2,5}$
$e_{i,2,3}$
$W_{i,3,5}$
$W_{i,2,1}$
$W_{i,3,9}$
$e_{i,1,2}$
$e_{i,3,1}$
$W_{i,1,1}$
$W_{i,1,5}$
$W_{i,1,9}$
$U_{i,1}$

For every "admissible" vector $(\vec{W}, \vec{e})$ (i.e. with correct types and values of μ) we define:

t-equations $\mathcal{S}_t(\mathcal{S}, \vec{W}, \vec{e})$:

$$
\begin{array}{rll}
(\prod_{k=1}^{9} W_{i,j,k}, & \prod_{k=1}^{9} W_{i',j',k}) & \text{if } U_{i,j} = U_{i',j'} \\
(W_{i,1,1}W_{i,1,2}W_{i,1,3}W_{i,1,4}e_{i,1,2}, & W_{i,2,1}W_{i,2,2}W_{i,2,3}W_{i,2,4}) & \text{for } 1 \leq i \leq n \\
(W_{i,2,6}W_{i,2,7}W_{i,2,8}W_{i,2,9}, & e_{i,2,3}\overline{W}_{i,3,4}\overline{W}_{i,3,3}\overline{W}_{i,3,2}\overline{W}_{i,3,1}) & \text{for } 1 \leq i \leq n \\
(W_{i,1,5}W_{i,1,6}W_{i,1,7}W_{i,1,8}, & e_{i,1,3}W_{i,3,6}W_{i,3,7}W_{i,3,8}W_{i,3,9}) & \text{for } 1 \leq i \leq n
\end{array}
$$

$\mathbb{H}$-equations $\mathcal{S}_{\mathbb{H}}(\mathcal{S}, \vec{W}, \vec{e})$:

$$
(W_{i,1,5}, \quad e_{i,1,2}W_{i,2,5}e_{i,2,3}W_{i,3,5}e_{i,3,1}) \quad \text{for } 1 \leq i \leq n
$$

$$\begin{tikzcd}
\mathcal{U}^* \arrow[r, "\sigma_{\vec{W},\vec{e}}"] \arrow[d, "\sigma"'] & \mathbb{W} \arrow[d, "\sigma_t"'] \\
\mathbb{G} & \mathbb{H}_t \arrow[l, "\bar{\pi}_{\mathbb{G}}"]
\end{tikzcd}$$

4.1-EQUATIONS OVER $\mathbb{W}$-DEFINITION

A system of $\mathbb{W}$-equations is a family of ordered pairs together with an involution:

$$\mathcal{S} = ((w_i, w'_i)_{i \in I}, \mathbb{I}')$$

where $w_i, w'_i \in \mathbb{W}_t, \gamma(w_i) = \gamma(w'_i) \neq \emptyset, \mathbb{I}' \in \mathcal{I}$.
A *solution* of $\mathcal{S}$ is any AB-homomorphism
$\sigma_{\mathbb{W}} : (\mathbb{W}_t, \mathbb{I}) \to (\mathbb{W}_t, \mathbb{I}')$ such that, for every $i \in I$

$$\sigma_{\mathbb{W}}(w_i) = \sigma_{\mathbb{W}}(w'_i). \tag{20}$$

4.2-EQUATIONS OVER $\mathbb{W}$-REDUCTION

Lemma 6 (factorisation of t-solutions) *Let* $\mathcal{S} = ((w_i, w_i'))_{1 \leq i \leq n}$ *be a system of t-equations. Let us suppose that* $\sigma_t : \mathbb{W}_t \to \mathbb{H}_t$ *is an* AB*-homomorphism solving the system* $\mathcal{S}$*. Then there exists an involution* $\mathbb{I}' \in \mathcal{I}$ *and* AB *-homomorphisms*

$$\sigma_{\mathbb{W}} : (\mathbb{W}_t, \mathbb{I}) \to (\mathbb{W}_t, \mathbb{I}'), \quad \psi_t : (\mathbb{W}_t, \mathbb{I}') \to (\mathbb{H}_t, \mathbb{I}_t)$$

such that, $\sigma_t = \sigma_{\mathbb{W}} \circ \psi_t$ *and*

$$\sigma_{\mathbb{W}}(w_i) = \sigma_{\mathbb{W}}(w_i') \text{ for all } 1 \leq i \leq n.$$

$$
\begin{tikzcd}
\mathcal{U}^* \arrow[r, "\sigma_{\vec{W},\vec{e}}"] \arrow[d, "\sigma"'] & (\mathbb{W}_t, \mathbb{I}) \arrow[d, "\sigma_t"'] \arrow[dr, "\sigma_{\mathbb{W}}"] & \\
\mathbb{G} & (\mathbb{H}_t, \mathbb{I}_t) \arrow[l, "\bar{\pi}_{\mathbb{G}}"] & (\mathbb{W}_t, \mathbb{I}') \arrow[l, "\psi_t"]
\end{tikzcd}
$$

5.1-EQUATIONS OVER $\mathbb{U}$-DEFINITION

We define the group

$$\mathbb{U} := \langle A * B, \mathcal{W}'; \bar{W} e W = \delta(W)(e) \quad (e \in \mathrm{Gi}(W), W \in \mathcal{W}) \rangle$$

5.2-EQUATIONS OVER $\mathbb{U}$-REDUCTION

Let us consider a system of $\mathbb{W}$-equations, together with a morphism:

$$\mathcal{S}_{\mathbb{W}} = ((w_i, w_i')_{i \in I}, \mathbb{I}'); \quad \sigma_H \in \mathrm{Hom}_{AB}(\mathbb{W}_H, \mathbb{W}_H).$$

We can define a system of equations over $\mathbb{U}$ with rational constraint:

$$\mathcal{S}_{\mathbb{U}}(\sigma_H) := ((z_i, z_i')_{1 \leq i \leq n}, \mathsf{C}),$$

in such a way that

Lemma 7 *The map* $\Phi : \mathrm{Hom}_{AB}(\mathbb{W}_t, \mathbb{W}_t) \to \mathrm{Hom}(\mathcal{W}_t^* * A * B, \mathbb{U})$*,* $\sigma_{\mathbb{W}} \mapsto \pi_{\mathbb{W}} \circ \sigma_{\mathbb{W}} \circ \bar{\pi}_{\mathbb{U}}$ *induces a bijection from the set of solutions of* $\mathcal{S}_{\mathbb{W}}$ *which extend* σ_H*, into the set of solutions of* $\mathcal{S}_{\mathbb{U}}(\sigma_H)$*.*

$$
\begin{tikzcd}
 & \mathcal{W}_t^* * A * B \arrow[d, "\pi_{\equiv}"'] \arrow[dddrrr, "\sigma_{\mathbb{U}}"] & & & \\
\mathcal{U}^* \arrow[r, "\sigma_{\vec{W},\vec{e}}"] \arrow[dd, "\sigma"'] & (\mathbb{W}_t, \mathbb{I}) \arrow[dd, "\sigma_t"'] \arrow[ddrr, "\sigma_{\mathbb{W}}"'] & \mathbb{W}_H \arrow[l] \arrow[dr, "\sigma_H"] & & \\
 & & & \mathbb{W}_H \arrow[d] & \\
\mathbb{G} & (\mathbb{H}_t, \mathbb{I}_t) \arrow[l, "\bar{\pi}_{\mathbb{G}}"] & & (\mathbb{W}_t, \mathbb{I}') \arrow[ll, "\psi_t"] \arrow[r, "\bar{\pi}_{\mathbb{U}}"'] & \mathbb{U}
\end{tikzcd}
$$

6.1-TRANSFER-FROM $\mathbb{G}$ TO $(\mathbb{H}, \mathbb{U})$

The satisfiability problem for systems of equations with rational constraints in $\mathbb{G}$ is Turing-reducible to the pair of problems (Q_1, Q_2), where

1- Q_1 is the SAT-problem for systems of equations with rational constraints in $\mathbb{U}$

2- Q_2 is the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$

6.2-TRANSFER-STRUCTURE OF $\mathbb{U}$

$$\mathbb{K} \to \mathbb{U}_1 \to \ldots \to \mathbb{U}_i \to \mathbb{U}_{i+1} \ldots \to \mathbb{U},$$

where:

$$K = A \propto F(\mathcal{V})$$

$$\mathbb{U}_i \to \mathbb{U}_{i+1}$$

is an HNN-extension with associated subgroups strictly smaller than A.

Equations in $\mathbb{K}$: (Diekert-Gutierrez-Hagenah, 2001)

HNN-extensions with trivial associated subgroups: (Diekert-Lohrey,2004)

6.3-TRANSFER-FROM $\mathbb{G}$ TO $\mathbb{H}$

Theorem 8 *For cancellative monoids $\mathbb{H}$, the satisfiability problem for systems of equations with rational constraints in*

$$\langle \mathbb{H}, t, \bar{t}; t\bar{t} = \bar{t}t = \epsilon, \ \ \bar{t}at = \varphi(a) \ \ (a \in A)\rangle$$

is Turing-reducible to the SAT-problem for systems of equations with rational constraints in $\mathbb{H}$

6.4-TRANSFER-THEOREMS

Other transfer theorems (obtained by the same technique):

Equations with rational constraints in an amalgamated product

Equations with constants in an HNN-extension

Equations and inequations with rational constraints, in an HNN-extension (case of groups)

Equations and inequations with rat constraints in an amalgamated product (case of groups).

7-PERSPECTIVES

Equations and inequations with rational constraints, in an HNN-extension (case of cancellative monoids)
Equations and inequations with rat constraints in an amalgamated product (case of cancellative monoids).
Positive first order theory in an HNN-extension.