L(A)=L(B)?

decidability results from complete formal systems

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The problem

Decision problem:

INPUT: A, B, two deterministic pushdown automata;

OUTPUT: 1 if $\mathrm{L}(A) = \mathrm{L}(B)$, 0 if $\mathrm{L}(A) \neq \mathrm{L}(B)$.

Raised in:

S. Ginsburg and S. Greibach, Deterministic Context-free languages,Information and Control,pages 620-648, 1966

Solution

G. Sénizergues, L(A) = L(B)?, *Technical report*, LaBRI, Université Bordeaux I, report nr1161-97, 1997, Pages 1-71.

G. Sénizergues. The Equivalence Problem for Deterministic Pushdown Automata is Decidable. In *Proceedings ICALP 97*, pages 671–681. Springer, LNCS 1256, 1997.

G. Sénizergues, L(A) = L(B)? In *Proceedings INFINITY 97*, pages 1–26. Electronic Notes in Theoretical Computer Science 9.

THE ARTICLE

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CONTENTS:
1- Decidability of the equivalence problem;
pages 9-82
2- Simplifications of the logical system:
pages 82-104
3- Generalization to transducers $X^* \to (\mathbb{Z}, +)$; pages 104-141

Plan

1 Before the article

2 Inside the article

  2.1 Main ideas of the basic proof

  2.2 A simple logical system

  2.3 Generalization to transducers

3 “After” the article, until now

4 Perspectives for future work.

1-Before the article

Some subclasses where the equivalence is decidable:

- dpda's with one state and no ϵ-transition (Hopcroft-Korenjac,66),
- one dpda has one state and no ϵ-transition and the other is general (Harrison-Havel-Yehudai,79),
- dpda's with only one stack symbol (Valiant-Paterson,75),
- finite-turn dpda's (Valiant 74, Beeri 76)),
- dpda's with no ϵ-transition, with empty final configurations (Oyamaguchi-Honda-Inagaki 80),(Romanovski 80),
- dpda's with no ϵ-transition, (Romanovski 86),(Oyamaguchi 87).

1-BEFORE

Some methods:

(Valiant 73):
parallel simulation of two dpda's, interspersed with
replacements

(Harrison-Havel-Yehudai,79):
strict-deterministic grammars

(Hopcroft-Korenjac 66),(Courcelle, 83) :
deduction systems

(Meitus, 89):
linear independence

1 - Before

(Harrison-Havel,JCSS 1973):

Definition 1 *Let $G = \langle X, V, P\rangle$ be some context-free grammar. G is said* **strict deterministic** *iff there exists some equivalence relation $\smile$ over $X \cup V$ such that*
1- X is some class mod $\smile$
2- $\forall v_1, v_1' \in V, \forall \alpha \in (X \cup V)^*, \forall w, w' \in V^*$,
if

$$v_1 \smile v_1', v_1 \to \alpha w_1, v_1' \to \alpha w_1'$$

then $(w_1 = w_1' = \epsilon$ *and* $v_1 = v_1')$ *or*
$\exists v_2, v_2' \in (X \cup V), w_2, w_2' \in (X \cup V)^*$,

$$v_2 \smile v_2', w_1 = v_2 w_2, w_1' = v_2' w_2'.$$

1-BEFORE

Theorem 2 *(Harrison-Havel,73) A language* $L \subseteq X^*$ *is recognized (by empty stack) by some dpda* **iff** *it is generated by some strict-deterministic context-free grammar.*

(Courcelle 83): **deduction systems**

Let us call *deduction system* any triple $\mathcal{D} = \langle \mathcal{A}, H, \vdash\!\!- \rangle$ where
- $\mathcal{A}$ is a denumerable set called the *set of assertions*
- $H : \mathcal{A} \to \mathbb{N} \cup \{\infty\}$ is the *cost-function*
- $\vdash\!\!-$, the *deduction relation* is a subset of $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$

such that the following axioms are satisfied:

(A 1) $\forall (P, A) \in \vdash\!\!-$, (inf $\{H(p), p \in P\} < H(A))$ or $(H(A) = \infty)$.
(A 2) $\vdash\!\!-$ is recursively enumerable.

We call *proof* in the system $\mathcal{D}$, any subset $P \subseteq \mathcal{A}$ fulfilling :

$$\forall p \in P, \exists Q \subseteq P, Q \vdash\!\!- p.$$

1-Before

Schema of a proof:

If P is a proof, then $\forall A \in P, H(A) = \infty$.

1 - BEFORE

(Meitus,89): **linear independence**

Definition 3 *A row-vector* $(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n) \in (\mathcal{P}(X^*))^n$ *is said* prefix-free *iff*

1-the α_i*'s are disjoints*

2-$\cup_{i=1}^{n} \alpha_i$ *is prefix-free.*

Let $(L_1, L_2, \ldots, L_n)$ be a family of prefix languages.

(1) Either for every two prefix vectors

$(\alpha_1, \alpha_2, \ldots, \alpha_n), (\beta_1, \beta_2, \ldots, \beta_n)$

$$\sum_{i=1}^{n} \alpha_i \cdot L_i = \sum_{i=1}^{n} \beta_i \cdot L_i \quad \Rightarrow \quad (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_n)$$

(2) or, there exists some $i_0 \in [1, n]$, and a prefix vector

$(\gamma_1, \gamma_2, \ldots, \gamma_n)$, such that

$$L_{i_0} = \sum_{i=1}^{n} \gamma_i \cdot L_i \quad \text{where} \quad \gamma_{i_0} = \emptyset$$

When (1) (resp. (2)) is true, the family $(L_1, L_2, \ldots, L_n)$ is said *linearly independent* (resp. *linearly dependent*).

2-INSIDE-1

Adaptation from (Harrison-Havel-Yehudai 1979).

Definition 4 *A series* $S \in \mathbb{B}\langle\langle\, V^* \,\rangle\rangle$ *is said* deterministic *iff, for every words* $\alpha, w_1, w_1' \in V^*$ *such that* $\alpha \cdot w_1, \alpha \cdot w_1' \in \mathrm{supp}(S)$
either $w_1 = w_1' = \epsilon$
or ($\exists v_2, v_2' \in V, w_2, w_2' \in V^*$, *such that*

$$v_2 \smile v_2', w_1 = v_2 w_2, w_1' = v_2' w_2'.$$

Intuition: a *deterministic* series is a generalized *configuration* of the initial pushdown automaton.

We define a σ-**right-action** $\odot$ of the monoid X^* over the semi-ring $\mathbb{B}\langle\langle\, V^* \,\rangle\rangle$ by:
for every $w \in V^*, u \in X^*$,

$$w \odot u = \sum_{\substack{w' \in V^* \\ w \longrightarrow^*_{lm} u \cdot w'}} w'$$

for every $S \in \mathbb{B}\langle\langle\, V^* \,\rangle\rangle, u \in X^*$,

$$\Big(\sum_{w \in V^*} S_w \cdot w\Big) \odot u = \sum_{w \in V^*} S_w \cdot (w \odot u).$$

2-INSIDE-1

A map $\varphi : V \to \mathbb{B}\langle\langle\ X^*\ \rangle\rangle$ is defined by:

$$\varphi(v) = \sum_{v \odot u = \epsilon} u$$

This map is extended in as σ-additive **semi-ring homomorphism** $\varphi : \mathbb{B}\langle\langle\ V^*\ \rangle\rangle \to \mathbb{B}\langle\langle\ X^*\ \rangle\rangle$ by

$$\varphi(\sum_{w \in V^*} S_w \cdot w) = \sum_{w \in V^*} S_w \cdot \varphi(w).$$

We set

$$S \equiv T \text{ iff } \varphi(S) = \varphi(T).$$

Deterministic matrices

Definition 5 *A row-vector* $(S_1, \ldots, S_i, \ldots, S_n)$ *is said* deterministic *iff*
1-The supports of the S_i*'s are disjoints*
2-$\sum_{i=1}^{n} S_i$ *is deterministic.*

Definition 6 *A matrix of dimension* (m, n)*,* $(S_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ *is said* deterministic *iff every one of its row-vectors is admissible.*

Proposition 7
1-If S *is a deterministic series in* $\mathbb{B}\langle\langle\ V^*\ \rangle\rangle$ *and* $u \in X^*$ *then* $S \odot u$ *is deterministic too.*
2- If S, T *are deterministic series then* $S \cdot T$ *is deterministic*
3- If S, T *are deterministic matrices , then* $S \cdot T$ *is deterministic too.*

Linear combinations

Let $S_1, \ldots, S_j, \ldots, S_n$ be deterministic series (in $\mathbb{B}\langle\langle\, V^* \,\rangle\rangle$).
We call *linear combination* of the S_j's, any series of the form

$$\sum_{j=1}^{n} \alpha_j \cdot S_j$$

where $(\alpha_1, \ldots, \alpha_j, \ldots, \alpha_n)$ is a deterministic vector.

Linear independence

Lemma 8 *Let $S_1, \ldots, S_j, \ldots, S_m \in \mathrm{DR}\mathbb{B}\langle\langle\, V \,\rangle\rangle$. The following are equivalent*

1. $\exists \vec{\alpha}, \vec{\beta} \in \mathrm{DR}\mathbb{B}_{1,m}\langle\langle\, V \,\rangle\rangle, \vec{\alpha} \not\equiv \vec{\beta}$, *such that* $\sum_{1 \leq j \leq m} \alpha_j \cdot S_j \equiv \sum_{1 \leq j \leq m} \beta_j \cdot S_j$,
2. $\exists j_0 \in [1, m], \exists \vec{\gamma} \in \mathrm{DR}\mathbb{B}_{1,m}\langle\langle\, V \,\rangle\rangle, \gamma_{j_0} \equiv \emptyset$, *such that* $S_{j_0} \equiv \sum_{1 \leq j \leq m} \gamma_j \cdot S_j$,

2-INSIDE-1

A deduction system: $\mathcal{D}_0$.
Set of assertions:

$$\mathcal{A} = \mathbb{N} \times \mathsf{DRB}\langle\langle\ V\ \rangle\rangle \times \mathsf{DRB}\langle\langle\ V\ \rangle\rangle$$

The "cost-function":

$$H(n, S, S') = n + 2 \cdot \mathrm{Div}(S, S').$$

Where the *divergence* between S and S', is defined by:

$$\mathrm{Div}(S, S') = \inf\{|\ u\ ||\ u \in \varphi(S) \triangle \varphi(S'))\}.$$

Deduction system $\mathcal{D}_0$: the meta-rules.

(R0)

$$\{(p, S, T)\} \mid\!\vdash\!- (p+1, S, T)$$

for $p \in \mathbb{N}, S, T \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R1)

$$\{(p, S, T)\} \mid\!\vdash\!- (p, T, S)$$

for $p \in \mathbb{N}, S, T \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R2)

$$\{(p, S, S'), (p, S', S'')\} \mid\!\vdash\!- (p, S, S'')$$

for $p \in \mathbb{N}, S, S', S'' \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R3)

$$\emptyset \mid\!\vdash\!- (0, S, S)$$

for $S \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R'3)

$$\emptyset \mid\!\vdash\!\!- (0, S, T)$$

for $S \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle, T \in \{\emptyset, \epsilon\}, S \equiv T$,

(R4)

$$\{(p+1, S \odot x, T \odot x) \mid x \in X\} \mid\!\vdash\!\!- (p, S, T)$$

for $p \in \mathbb{N}, S, T \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle, (S \not\equiv \epsilon \wedge T \not\equiv \epsilon)$,

(R5)

$$\{(p, S, S')\} \mid\!\vdash\!\!- (p+2, S \odot x, S' \odot x)$$

for $p \in \mathbb{N}, S, T \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle, x \in X$,

(R6)

$$\{(p, S \cdot T' + S', T')\} \mid\!\vdash\!\!- (p, S^* \cdot S', T')$$

for $p \in \mathbb{N}, (S, S') \in \mathrm{DR}\mathbb{B}_{1,2}\langle\langle\ V\ \rangle\rangle, T' \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle, S \not\equiv \epsilon$,

(R7)

$$\{(p,S,S'),(p,T,T')\} \mid\!\vdash\!- (p, S+T, S'+T')$$

for $p \in \mathbb{N}, (S,T), (S',T') \in \mathrm{DR}\mathbb{B}_{1,2}\langle\langle\ V\ \rangle\rangle$,

(R8)

$$\{(p,S,S')\} \mid\!\vdash\!- (p, S \cdot T, S' \cdot T)$$

for $p \in \mathbb{N}, S, S', T \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R9)

$$\{(p,T,T')\} \mid\!\vdash\!- (p, S \cdot T, S \cdot T')$$

for $p \in \mathbb{N}, S, T, T' \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R10)

$$\emptyset \mid\!\vdash\!- (0, S, \rho_\epsilon(S))$$

for $S \in \mathrm{DR}\mathbb{B}\langle\langle\ V\ \rangle\rangle$.

2-INSIDE-1

Strategies for the deduction system $\mathcal{D}_0$.

Informal definition:
A strategy is a "method" allowing to enlarge every "partial proof" P into a partial proof Q with $P \subset Q$.

2-INSIDE-1

Strategies. We suppose $X = \{x_1, x_2\}$.

$$T_A$$

$$A_1$$

$$A_2$$

$$\vdots$$

$$A_{n-1}$$

$$(\pi, S, T)$$

$$\swarrow \quad \searrow$$

$$(\pi + \mathbf{1}, \mathbf{S} \odot \mathbf{x_1}, \mathbf{T} \odot \mathbf{x_1}) \quad (\pi + \mathbf{1}, \mathbf{S} \odot \mathbf{x_2}, \mathbf{T} \odot \mathbf{x_2})$$

$$T_B^+$$

$$\vdots$$

$$(\pi, S, \sum_{q \in Q} [\bar{p} z q] \cdot V_q$$

$$\vdots$$

$$\vdots$$

$$(\pi + k_1, S \odot u, \sum_{q \in Q} ([\bar{p} z q] \odot u) \cdot V_q$$

$$\downarrow$$

$$(\pi + \mathbf{k_1} - \mathbf{1}, \mathbf{S} \odot \mathbf{u}, \sum_{\mathbf{q} \in \mathbf{Q}'} ([\bar{\mathbf{p}}\mathbf{z}\mathbf{q}] \odot \mathbf{u}) \cdot [\mathbf{qeq}] \cdot (\mathbf{S} \odot \mathbf{u_q})$$

where $Q' = \{q \in Q \mid [\bar{p} z q] \not\equiv \emptyset\}$ and $u_q = \min(\varphi([\bar{p} z q]))$

Strategies.

$$T_C$$

Let us consider, in a partial proof, a sequence of d_0 assertions

$$(\pi_1, U_1, V_1), \ldots, (\pi_2, U_2, V_2), \ldots, (\pi_{d_0}, U_{d_0}, V_{d_0})$$

such that

1. all the U_i, V_i are linear combinations of the same set of d_0 series $\{S_1, S_2, \ldots, S_{d_0}\}$
2. the coefficients used in these linear combinations are **uniformly bounded**
3. the sequence of weights π_i is both uniformly bounded and also sufficiently increasing

By lemma 8, this sytem $\mathcal{S}$ of d_0 equations over the d_0 series $\{S_1, S_2, \ldots, S_{d_0}\}$ can always be reduced to a new equation

$$\mathcal{E}$$

over *the coefficients only!*.
$\mathcal{E}$ has uniformly bounded size.

Strategies

$$T_C$$

$$\vdots$$

$$(\pi_1, U_1, V_1)$$

$$\vdots$$

$$(\pi_2, U_2, V_2)$$

$$\vdots$$

$$(\pi_{d_0}, U_{d_0}, V_{d_0})$$

$$\downarrow$$

$$\mathcal{E}$$

2-INSIDE-1

Theorem 9 *The system $\mathcal{D}_0$ is complete.*

Theorem 10 *The equivalence problem for deterministic pushdown automata is decidable.*

2-INSIDE-2

A simple deduction system: $\mathcal{D}_5$.
Set of assertions:

$$\mathcal{A} = \mathrm{DRB}\langle\langle\ V\ \rangle\rangle \times \mathrm{DRB}\langle\langle\ V\ \rangle\rangle$$

The "cost-function":

$$H(S, S') = \mathrm{Div}(S, S').$$

2-INSIDE-2

Simple deduction system $\mathcal{D}_5$: the meta-rules.

(R51)

$$\{(S,T)\} \mid\!\vdash\!- (T,S)$$

for $S, T \in$ DR$\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R52)

$$\{(S,S'),(S',S'')\} \mid\!\vdash\!- (S,S'')$$

for $S, T \in$ DR$\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R53)

$$\emptyset \mid\!\vdash\!- (S,S)$$

for $S \in$ DR$\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R"53)

$$\emptyset \mid\!\vdash\!\!- ([qzr], \epsilon)$$

for $q, r \in Q, z \in Z, [qzr] \equiv \epsilon$,

(R54)

$$\{(S \odot x, T \odot x) \mid x \in X\} \mid\!\vdash\!\!- (S, T)$$

for $S, T \in$ DR$\mathbb{B}\langle\langle\ V\ \rangle\rangle, (S \not\equiv \epsilon \wedge T \not\equiv \epsilon)$,

(R55)

$$\{(S \cdot T' + S', T')\} \mid\!\vdash\!\!- (S^* \cdot S', T')$$

for $(S, S') \in$ DR$\mathbb{B}_{1,2}\langle\langle\ V\ \rangle\rangle, T' \in$ DR$\mathbb{B}\langle\langle\ V\ \rangle\rangle, S \not\equiv \epsilon$,

(R56)

$$\{(S, S'), (T, T')\} \mid\!\vdash\!\!- (S + T, S' + T')$$

for $(S, T), (S', T') \in$ DR$\mathbb{B}_{1,2}\langle\langle\ V\ \rangle\rangle$,

(R57)

$$\{(S, S')\} \mid\!\vdash\!\!- (S \cdot T, S' \cdot T)$$

for $S, S', T \in$ DR$\mathbb{B}\langle\langle\ V\ \rangle\rangle$,

(R58)

$$\{(T, T')\} \mid\!\vdash\!\!- (S \cdot T, S \cdot T')$$

for $S, T, T' \in$ DR$\mathbb{B}\langle\langle\ V\ \rangle\rangle$.

2-Inside-2

Theorem 11 *The system* $\mathcal{D}_5$ *is* complete.

2-Inside-3

The equivalence problem for transducers is the following decision problem:

INPUT: A, B, two deterministic pushdown transducers $(X^*, \cdot) \to (\mathbb{Q}, \times)$;
OUTPUT: 1 if $T(A) = T(B)$, 0 if $T(A) \neq T(B)$.

Theorem 12 *The equivalence problem for deterministic pushdown transducers from $(X^*, \cdot)$ into $(\mathbb{Q}, \times)$ is decidable.*

Main adaptation of the boolean case:
we must consider series in $\mathbb{Q}\langle\langle\, V^* \,\rangle\rangle$.
We can restrict ourselves to series which are

- $\smile$-deterministic, i.e. their support is deterministic in the sense of section 2.1
- V-rational-deterministic, i.e. they have a linear representation which is deterministic.

3-“AFTER”

Corollary 13 *The equivalence problem for monadic recursion schemes (with interpreted* `if-then-else`*), is decidable.*

This follows from theorem 10 and the reduction given in (Garland and Luckham, 73).

Corollary 14 *The equivalence problem for recursive polyadic program schemes (with completely uninterpreted function symbols) is decidable.*

This follows from theorem 10 and the reduction given in (Courcelle,78) or the reduction given in (Gallier,81).

3-AFTER

Theorem 15 *The bisimulation problem for* non-deterministic *pushdown automata, with deterministic, decreasing ϵ-moves, is decidable.*

Published in (Sénizergues, FOCS'98)
Proved in (Sénizergues, technical report 1183-97 of LaBRI)

Theorem 16 *The bisimulation problem for arbitrary pda's is* **undecidable**.

Proved in (SRBA, Concur'02).

AFTER

Theorem 17 *The equivalence problem for deterministic pushdown transducers, from X^*, into a linear group, is decidable.*

Published in (Sénizergues, ICALP'99)
Proved in (Sénizergues, technical report 1209-99 of LaBRI)

3-AFTER

Non-deterministic series.

Theorem 18 *The equivalence problem for deterministic pushdown transducers, from X^*, into $(\mathbb{Q}, +, \times)$, over two rational series S, T, which are supposed*
1- $\smile$-deterministic
2- but might be non-V-deterministic rational
is decidable.

Exposed in (Sénizergues, ICWL'00, Kyoto)
Work in progress.

3- After

1- Conceptual simplifications of the proof (deterministic, boolean case):
-(Stirling, technical report 99) (Stirling, TCS 01),
-translated back in the above framework in (Sénizergues, TCS 02)
2- Simplification of the complexity:

Theorem 19 *(Stirling, ICALP'02)The equivalence problem for dpda's is primitive recursive.*

For strict real-time deterministic pda's, this problem was raised in (Oyamaguchi, 81), and remained open until today. Realizes perspective 3 of the article.

4-FUTURE

Automata of *level* k have been defined by (Maslov 74),(Maslov 76),(Damm 82),(Engelfriet 83),(engelfriet-Vogler 86)

The level 1 corresponds to pushdown automata.

Open problem 1:

Is the *equivalence* problem decidable for deterministic automata of *level* k?

(Joint work with F. Carrere, S. Fratani)

Open problem 2:

Is the *isomorphism* problem decidable for deterministic automata of *level* k?

(Joint work with D. Caucal, C. Feuillade)

4-FUTURE

Open problem 3:
Is the bisimulation problem decidable for loop-free pda's?
(raised in (Stirling, TCS'98))
Open problem 4:
Is the bisimulation problem decidable for equational graphs?
(raised in (Caucal , CSLI lecture notes 95))
Open problem 5:
Is the quotient problem decidable for the following classes of infinite graphs
1-the deterministic equational graphs
(yes, (Sénizergues, MCU'01))
2-the non-det equational graphs of finite out-degree ?
3-the deterministic automatic graphs ?

4-FUTURE

Open problem 6:

Is there some link between:

- (A=B, Petkovsek-Wilf-Zeilberger,96)
- the "L(A)=L(B)?" problem?

(Current work with F. Carrere and S. Fratani).