

# Integer sequences and automata of level $k$ .

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# INTRODUCTION

# Automata

A sequence  $u : \mathbb{N} \rightarrow \mathbb{N}$  has **level**  $k$  if there exists some pushdown automaton  $\mathcal{A}$  of level  $k$  such that:

- starting on the configuration

$$(q_0, \gamma_1[\gamma_2 \dots [\gamma_{k-1}[F^n]] \dots])$$

- there exists **exactly one** computation of  $\mathcal{A}$  that empties the stack
- this computation reads the word  $a^{u(n)}$ .

# Logics

Monadic Second Order logics over the structure:

$$\langle \mathbb{N}, S, P_1, \dots, P_n \rangle$$

where  $S$  denotes the successor predicate, and  $P_k$  are other predicates

First Order logics over

$$\langle \mathbb{N}, +, P_1, \dots, P_n \rangle$$

where  $+$  denotes the addition predicate and  $P_k$  are other predicates.

# Recurrences

A sequence  $u : \mathbb{N} \rightarrow \mathbb{N}$  is  **$\mathbb{N}$ -rational** iff  $u = f_1$ , where :

$$f_i(n+1) = L_i(f_1(n), \dots, f_d(n)) \text{ for all } i \in [1, d], n \in \mathbb{N}$$

and  $L_i$  are **linear forms** with coefficients in  $\mathbb{N}$ .

# Recurrences

A sequence  $u : \mathbb{N} \rightarrow \mathbb{N}$  is  $\mathbb{N}$ -polynomial-recurrent iff:

$u = f_1$ , where :

$$f_i(n+1) = P_i(f_1(n), \dots, f_d(n)) \text{ for all } i \in [1, d], n \in \mathbb{N}$$

where  $P_i$  are polynomials with  $d$  undeterminates and coefficients in  $\mathbb{N}$ .

# What is this talk about ?

It surveys results about:

- **decidability** of some logics, using automata.
- **links** between recurrent sequences and automata



# AUTOMATA

# Data structure

[J. Engelfriet, 1991] A *data-structure* is a tuple

$\mathbb{D} = \langle D, F, \text{OP}, \text{tops} \rangle$  consisting of

- a set  $D$ ,
- a finite set  $F$ ,
- a set OP of unary operations:  
 $\forall op \in \text{OP}, op$  is a total map :  $D \rightarrow D$
- a total map  $\text{tops} : D \rightarrow F$ .

# Automaton

[J. Engelfriet, 1991] An **automaton** over the data-structure  $\mathbb{D}$  and the terminal alphabet  $B$  is a 4-tuple

$$\mathcal{A} = (Q, B, \mathbb{D}, \delta)$$

where  $Q$  is a finite set of states,  
 $\mathbb{D} = \langle D, F, \text{OP}, \text{tops} \rangle$  is a data-structure,  
the transition function,  $\delta$ , is a map from  $Q \times (B \cup \{\varepsilon\}) \times F$  into  $\mathcal{P}(Q \times \text{OP}(\mathbb{D}))$ .

# Pushdown

[J. Engelfriet, 1991] Given a data-structure  $\mathbb{D}$  and an alphabet  $\Gamma \neq \emptyset$ , one constructs a new data-structure, denoted by

$$\mathbb{P}(\Gamma, \mathbb{D})$$

and called the set of *pushdowns over*  $(\Gamma, \mathbb{D})$ :

$$\mathbb{P}(\Gamma, \mathbb{D}) := \langle P(\Gamma, D), F', OP', \text{tops}' \rangle$$

where

$$\begin{aligned} P(\Gamma, D) &:= (\Gamma[D])^*, \quad F' := \Gamma \cdot F \cup \Gamma \cup \{\varepsilon\}, \\ OP' &:= OP \cup \{\text{pop}\} \cup \{\text{push}(h) \mid h \in \Gamma^+\}, \\ \text{tops}'(\gamma[d]w) &:= \gamma \cdot \text{tops}(d), \quad \text{tops}(\varepsilon) := \varepsilon. \end{aligned}$$

# Pushdown

$$op(\gamma[d]w) := \gamma[op(d)]w, \quad op(\varepsilon) = \varepsilon.$$

For every  $h \in \Gamma^+$ ,  $d \in D$ ,  $w \in D^*$ : if  $h = h_1 \cdots h_n$  with  $h_i \in \Gamma$

$$\text{push}(h)(\gamma[d]w) := h_1[d] \cdots h_n[d]w, \quad \text{pop}(\gamma[d]w) := w,$$

$$\text{push}(h)(\varepsilon) := \varepsilon, \quad \text{pop}(\varepsilon) := \varepsilon.$$

# Pushdown of level $k$

Data-structure  $\mathbb{D}_k$ :

$$\mathbb{D}_0 := \langle \{\emptyset\}, \{\emptyset\}, \emptyset, \{(\emptyset, \emptyset)\} \rangle \quad \text{and} \quad \forall i \in \mathbb{N}, \quad \mathbb{D}_{i+1} := \mathbb{P}(\Gamma, \mathbb{D}_i).$$

Finite automaton: automaton over  $\mathbb{D}_0$

Pushdown-automaton: automaton over  $\mathbb{D}_1$

Pushdown-automaton of level  $k$ : automaton over  $\mathbb{D}_k$ .

# COMPOSITION DECOMPOSITION

# Composition theorem

## Theorem

*If  $f \in \mathbb{S}_k(A^*, B^*)$  and  $g \in \mathbb{S}_\ell(B^*, C^*)$ , then  $g \circ f \in \mathbb{S}_{k+\ell-1}(A^*, C^*)$ .*

Proved in [Fratani, PHD 2005], [Fratani-GS, APAL 06] for natural integers (i.e.  $A^* = B^* = C^* = \mathbb{N}$ ). “Same” proof remains valid for general free monoids.



# Decomposition theorem

## Theorem

*If  $h \in \mathbb{S}_k(A^*, C^*)$  and  $k \geq 2$ , then there exists a finite alphabet  $B$  and maps  $f \in \mathbb{S}_{k-1}(A^*, B^*)$ ,  $g \in \mathbb{S}_2(B^*, C^*)$ , such that  $h = g \circ f$ .*

Asserted in [GS, invited talk at CSR'07].

Proved for  $k = 3$  in [GS, arxiv 2023].

Analogous proof for  $k \geq 4$  (ongoing work).

Beware:  $|A| = |C| = 1$  does **not** imply that there exists a decomposition with  $|B| = 1$  ([GS, unpublished draft]).

**Proved** for trees [Engelfriet-Vogler 1986].

# Rate of growth

Corollary 1: the inclusion  $\mathbb{S}_k(\mathbb{N}, \mathbb{N}) \subset \mathbb{S}_{k+1}(\mathbb{N}, \mathbb{N})$  is **strict**.

Proof: the decomposition theorem shows that, if  $f \in \mathbb{S}_k(\mathbb{N}, \mathbb{N})$ , then  $\exists C \in \mathbb{N}, \forall n \in \mathbb{N}, f(n) \leq \text{Exp}_{k-1}(C, n)$ .

But  $\text{Exp}_k(2, n) \in \mathbb{S}_{k+1}(\mathbb{N}, \mathbb{N})$ .

Corollary 2: there exists some sequence  $u$  that has the **RUP** property, but  $u \notin \bigcup_{k \geq 1} \mathbb{S}_k(\mathbb{N}, \mathbb{N})$ .

Proof: there exists some sequence  $u$  that has the RUP property and grows faster than any tower of exponentials [Carton, 2002]

# Extensions of SEQUENTIAL CALCULUS

# Previous results

[Elgot-Rabin 1966],[Siefkes 1971],[Hosch 1971],[Maes 1998]

Residual Ultimate Periodicity property [Carton-Thomas 2002]:

$u : \mathbb{N} \rightarrow \mathbb{N}$  is RUP iff for every rational subset  $R \subseteq \mathbb{N}$ , the subset  $f^{-1}(R)$  is rational too.

Theorem (Carton-Thomas 2002)

*If  $u$  is RUP then MSO of  $\langle \mathbb{N}, S, \{\Sigma u(n)\} \rangle$  is decidable*

# New results

Decidability of **MSO** of  $\langle \mathbb{N}, S, P_1, \dots, P_n \rangle$  where  $P_i$  are defined “through” automata of level  $k$ .

Proof 1: reduction to Residual Ultimate Periodicity property.

Proof 2: reduction to Muchnik-Semenov-Walukiewicz theorem.

# Meyer-rationality

A language  $L \subseteq A^{[2]}$  is said *Meyer-regular* iff, viewed as a language over  $(A \cup \{ "[", " ", " ", "]" \})^*$  it is regular.

(i.e. the metasymbols, square brackets and commas, are treated as ordinary symbols)

## First proof: Inverse-regular-preservation

## Theorem (Hague-Ong 2007)

*Let  $\mathcal{A}$  be some level  $k$  pushdown automaton. Given a Meyer-regular set of configurations  $C$ , the set of configurations  $\{c' \in Q \cdot A^{[2]} \mid \exists c \in C, c' \vdash_{\mathcal{A}}^* c\}$  is regular too.*

Corollary: If  $f \in \mathbb{S}_k(A^*, B^*)$ , then, for every rational subset  $R \subseteq B^*$ , the subset  $f^{-1}(R)$  is rational too.

Corollary: If  $u \in \mathbb{S}_k(\mathbb{N}, \mathbb{N})$ , then  $u$  has the RUP property.

## Second proof: Reduction to Muchnik's theorem

Let  $Sig = (r_1, \dots, r_n)$  be a relational signature.

For a structure  $M = \langle D_M, r_1, \dots \rangle$  over  $Sig$  one constructs the **tree-expansion**  $M^* = \langle D_M^*, son, clone, r_1^*, \dots \rangle$  over the **extended** signature  $Sig^* = Sig \cup \{son, clone\}$ , where:

$$son = \{(w, dw) : w \in D_M^*, d \in D_M\}$$

$$clone = \{ddw : w \in D_M^*, d \in D_M\}$$

$$r^* = \{(d_1 w, \dots, d_a w) : w \in D_M^*, (d_1, \dots, d_a) \in r_M\},$$

( for all  $r \in Sig$ , of arity  $a$ ).



## Second proof: Reduction to Muchnik's theorem

Theorem (Muchnik-Semenov 1992, Walukiewicz 2002)

For every MSO formula  $\varphi$  over the signature  $\text{Sig}^*$  one can *effectively find* a MSO formula  $\hat{\varphi}$  over the signature  $\text{Sig}$  such that, for every structure  $M$ :

$$M \models \hat{\varphi} \quad \text{iff} \quad M^* \models \varphi$$

## Second proof: Reduction to Muchnik's theorem

Every automaton  $\mathcal{A}$  of level  $k$ , computing  $u$ , gives an MSO-Interpretation of

$$\langle \mathbb{N}, S, \{u(n)\} \rangle$$

into the  $k$ -fold tree-expansion of  $\langle \Gamma, (P_\gamma)_{\gamma \in \Gamma} \rangle$  (the finite stack-alphabet)

$$\langle \Gamma^{[k]}, (\text{clone}_i)_{1 \leq i \leq k}, (\text{succ}_i)_{1 \leq i \leq k}, \rangle.$$

through the computation-graph of  $\mathcal{A}$ .

## Second proof:generalization

Every automaton  $\mathcal{A}$  of level  $k$ , computing  $u$ , with oracles  $P_1, \dots, P_q$  (over the innermost stack) gives an MSO-Interpretation of

$$\mathcal{S} = \langle \mathbb{N}, S, \{u(n)\}, \{u(n) \mid P_1(n)\}, \dots, \{u(n) \mid P_q(n)\} \rangle$$

into the  $(k-1)$ -fold tree-expansion of

$$\mathcal{G} = \langle \Gamma^*, (s_\gamma)_{\gamma \in \Gamma}, P_1 \circ \ell, \dots, P_q \circ \ell \rangle.$$

NB: here  $\ell$  denotes the length-map.

## Second proof:generalization

Proof:

$\mathcal{S} \hookrightarrow$  computation – graph – of  $\mathcal{A} \hookrightarrow$  tree – expansion – of  $\mathcal{G}$ .

q.e.d.

Moreover:

$\mathcal{G}$  is the unfolding of a graph, which is MSO-interpretable in  $\langle \mathbb{N}, S, P_1, \dots, P_q \rangle$ .

# Transfert for composition

## Theorem (Fratani, PHD 2005)

Suppose MSO of  $\langle \mathbb{N}, S, \{v_1(n)\}, \dots, \{v_q(n)\} \rangle$  is *decidable* and the sequence  $u(n)$  is computable at level  $k$  with oracles  $\{v_1(n)\}, \dots, \{v_q(n)\}$ .

Then MSO of  $\langle \mathbb{N}, S, \{\Sigma u(n)\}, \{\Sigma u \circ v_1(n)\}, \dots, \{\Sigma u \circ v_q(n)\} \rangle$  is *decidable*.

Proof: This structure has a MSO interpretation into a structure with decidable MSO (by MSO interpretation + Muchnik's theorem).

# Transfert for inversion

## Theorem (Fratani, PHD 2005)

*Suppose  $u : \mathbb{N} \rightarrow \mathbb{N}$  is strictly increasing and the MSO of  $\langle \mathbb{N}, S, \{u(n)\} \rangle$  is **decidable**. Then MSO of  $\langle \mathbb{N}, S, \{\Sigma u^{-1}(n)\} \rangle$  is **decidable**.*

Proof:

the sequence  $u^{-1}(n)$  is computable at level 2 with oracle  $\{u(n)\}$  over the innermost stack.

# Extensions of PRESBURGER arithmetics

# Previous results

$FO(\langle \mathbb{N}, + \rangle)$  is decidable

$FO(\langle \mathbb{N}, +, \{2^n\} \rangle)$  is decidable

$FO(\langle \mathbb{N}, +, \{n^2\} \rangle)$  is **undecidable**

$FO(\langle \mathbb{N}, +, \{(n, f(n))\} \rangle)$  is **undecidable**

provided that  $f$  is strictly increasing and  $\lim_{n \rightarrow \infty} f(n) - n = \infty$



## Previous results

Proofs:

$FO(\langle \mathbb{N}, + \rangle)$  is decidable

Many proofs.

Monadic **interpretation** into  $\langle \{0, 1\}^*, s_0, s_1 \rangle$  [Rabin 1977]

$FO(\langle \mathbb{N}, +, \{2^n\} \rangle)$  is decidable

**Finite automata** theory, [Buchi-Bruyere 1985]

$FO(\langle \mathbb{N}, +, \{n^2\} \rangle)$  is **undecidable**

**Product** is definable [Folklore]

$FO(\langle \mathbb{N}, +, \{(n, f(n))\} \rangle)$  is **undecidable**

**Grids** are definable [Siefkes 1971]

# New results

Method: Define **Monadic** interpretations [Colcombet-Loeding, LMCS 2007].

Variant of Rabin's interpretation, where we **add** to the successor predicate some unary predicates and maintain **decidability**:  
for example

$$MSO(\langle \mathbb{N}, S, \{n^2\}, \{n^6\} \rangle)$$

(decidable by previous theorem).

# New results

It follows that the following theory is decidable:

$$FO(\langle \mathbb{N}, +, \{2^n\}, \{2^{n^2}\}, \{2^{n^6}\} \rangle)$$

Using the [Fibonacci numeration](#) system [Zekendorf 1972], the following is also decidable:

$$FO(\langle \mathbb{N}, +, \{F(n)\}, \{F(n^2)\}, \{F(n^6)\} \rangle)$$

# Automata

versus

# RECURRENCES

## Level 2

## Theorem (Ferte-Marin-GS, TOCS, 2014)

*Let us consider a mapping  $f : \mathbb{N} \rightarrow \mathbb{N}$ . The following properties are equivalent:*

- 1-  $f \in \mathbb{S}_2(\mathbb{N}, \mathbb{N})$
- 2-  $f$  is  $\mathbb{N}$ -rational

Extensions:

- for  $\mathbb{S}_2(A^*, \mathbb{N})$ :  $\mathbb{N}$ -rational series
- for  $\mathbb{S}_2(\mathbb{N}, B^*)$ : HD0L (iterated homomorphism)
- for  $\mathbb{S}_2(A^*, B^*)$ : HDT0L

## Level 3

## Theorem (GS, arxiv, 2023)

*Let us consider a mapping  $f : \mathbb{N} \rightarrow \mathbb{N}$ . The following properties are equivalent:*

- 1-  $f \in \mathbb{S}_3(\mathbb{N}, \mathbb{N})$
- 2-  $f$  is  $\mathbb{N}$ -polynomial-recurrent

Extension:

- for  $\mathbb{S}_3(A^*, \mathbb{N})$ :  $\mathbb{N}$ -polynomial-recurrent sequences

Proof: consequence of the decomposition theorem.

# EXAMPLES

# Examples

$$u(n) = 2^n: \text{ level 2}$$

$$u(n) = F(n): \text{ level 2}$$

$$u(n) = n!: \text{ level 3}$$

$$u(w) = F(\nu(w)): \text{ level 3}$$

$$u(n) = \binom{2n}{n} = \frac{(2n)!}{n! \cdot n!} \text{ is not RUP [Carton-Thomas 2002],}$$

hence **not in**  $\mathbb{S}_k(\mathbb{N}, \mathbb{N})$  (for any  $n \in \mathbb{N}$ ).



# Examples

$u(n) = n^n: \in S_4(\mathbb{N}, \mathbb{N}) \setminus S_3(\mathbb{N}, \mathbb{N})$

- **in**  $S_4(\mathbb{N}, \mathbb{N})$ : by the composition theorem

- **not in**  $S_3(\mathbb{N}, \mathbb{N})$ : because, for every  $\ell \geq 1$

$n^n, (n+1)^{n+1}, \dots, (n+\ell)^{n+\ell}$  are **algebraically-independant** over  $\mathbb{Q}$ ,  
while every sequence of level 3 has an **annihilating**-polynomial.

[Cadilhac-Mazowiecki-Paperman-Pilipczuk-GS, ICALP'20].

# OPEN problems

# Hierarchy

Given  $u \in \mathbb{S}_{k+1}(\mathbb{N}, \mathbb{N})$ , can we decide whether  $u \in \mathbb{S}_k(\mathbb{N}, \mathbb{N})$  ?

Recent work by [Bojanczyk 2018],

[Colcombet-Douenneau-Talbot-Lopez 2023], [Kiefer-Tito-Pradic 2023] on subclasses of  $\mathbb{S}_{k+2}(\mathbb{N}, \mathbb{N})$  (the  $k$ -pebbles-transductions).

# Stability

Given  $u, v \in \mathbb{S}_k(\mathbb{N}, \mathbb{N})$ , does the convolution product  $u * v$  belong to  $\mathbb{S}_k(\mathbb{N}, \mathbb{N})$  ?

True for  $k = 2$

True with the stronger hypothesis  $u \in \mathbb{S}_{k-1}(\mathbb{N}, \mathbb{N})$ ,  $v \in \mathbb{S}_k(\mathbb{N}, \mathbb{N})$  for  $k \geq 3$

Open for  $k = 3$ .

# Logics

Suppose MSO of  $\langle \mathbb{N}, S, \{u(n)\} \rangle$  and MSO of  $\langle \mathbb{N}, S, \{v(n)\} \rangle$  are both **decidable**.

Does it imply that MSO of  $\langle \mathbb{N}, S, \{u(n)\}, \{u(v(n))\} \rangle$  is decidable ?

**True** when  $u$  is in  $\Sigma S_k(\mathbb{N}, \mathbb{N})$ .

**True** for  $u$  RUP-sequence ([Last minute sketch of proof by GS]).

**False** for MSO of  $\langle \mathbb{N}, S, \{u(n)\}, \{v(n)\} \rangle$  [Semenov].

# Multiple sequences

Define a useful notion of **multiple** sequence  $u(n_1, n_2, \dots, n_q)$  computed by pushdown automata of level  $k$ .

Some proposals are given in [Fratani-GS, APAL 2006, section 5], [E.Kien, internship at Bordeaux in 2017]

# Counting/representing

Given  $u, v : \mathbb{N} \rightarrow \mathbb{N}$  given by some kind of recurrence (linear,  $\mathbb{N}$ -algebraic, **P-recurrent**,  **$C^2$ -finite**, **polynomial**, ...), can we define some pda automaton of level  $k$  that generates words (or trees, or graphs) with **exactly  $u(n)$  objects** of size  **$v(n)$**  ?

N.B.  $v(n) = n$  is most traditionnal.

Ongoing work by [V.Ghigo, M2 internship at Bordeaux university], supervised by V.Penelle and GS.