

**LOGICS, J1IN7M21**  
Examination on 19/12/2013

Subject from M. Sénizergues ; all documents allowed ; duration : 1h 30.  
The mark for this half of the exam will be  $\min\{exo1 + exo2 + exo3, 10\}$ .

**Exercise 1** (4 pts)

In the following  $A, B, C$  are propositional variables and  $P, Q, R$  are unary predicate symbols.

1- Give a proof, within system LK, of the sequents :

$$(A \wedge B) \rightarrow C \vdash A \rightarrow (B \rightarrow C)$$
$$\neg \forall x (P(x) \rightarrow Q(x)) \vdash \exists x (P(x) \wedge \neg Q(x))$$

2- Give a proof, within system LJ, of the sequent :

$$\forall x [P(x) \rightarrow (Q(x) \rightarrow R(x))] \vdash \forall x [(P(x) \wedge Q(x)) \rightarrow R(x)]$$

**Exercise 2** (6 pts)

In the following  $A, B, C$  are propositional variables.

1- For the following sequent, either give a proof within LJ or show that such a proof does not exist :

$$\neg(A \wedge B) \vdash (\neg A) \vee (\neg B)$$

2- Show, by *semantical* arguments, that

$$\neg(A \wedge B) \Vdash \neg \neg((\neg A) \vee (\neg B))$$

Can you conclude that there exists a proof, within LJ, of the sequent

$$\neg(A \wedge B) \vdash \neg \neg((\neg A) \vee (\neg B))?$$

3- Give a proof, within LJ, of the sequent of question 2.

### Exercise 3 (10 pts)

We are interested here in the first-order theory of the structure :

$$\mathcal{M} := \langle \mathbb{N}, =, \mathbf{Q}, +, 0, 1 \rangle.$$

where  $\mathbf{Q}(x)$  means “ $x$  is a square”.

We denote by  $\mathcal{S}$  the signature  $\langle =, \mathbf{Q}; +, 0, 1 \rangle$  where  $=, \mathbf{Q}$  are predicate symbols of respective arities 2, 1 and  $+, 0, 1$  are function symbols of respective arities 2, 0, 0. We recall that  $\mathbb{N}$  is the set of natural integers. The signature is interpreted as follows on the structure  $\mathcal{M}$  :

the symbol  $=$  denotes the equality between natural integers

$\mathbf{Q}(x)$  is true when  $x$  is a square i.e when there exists some natural integer  $y \in \mathbb{N}$  such that  $x = y^2$ .

The symbol 0 denotes the null integer and 1 denotes the successor of 0 (number one).

0- Check briefly that  $\mathcal{M} \models \mathbf{Q}(1)$ ,  $\mathcal{M} \models \neg \mathbf{Q}(1+1)$ ,  $\mathcal{M} \models \neg \mathbf{Q}(1+1+1)$ ,  $\mathcal{M} \models \mathbf{Q}(1+1+1+1)$ ,  $\mathcal{M} \models \forall x \exists y \mathbf{Q}(x+y)$ ,  $\mathcal{M} \models \forall x \exists y \neg \mathbf{Q}(x+y)$ .

1- Give a valuation  $\mathbf{v} : \{x, y, z\} \rightarrow \mathbb{N}$  such that

$$\mathcal{M}, \mathbf{v} \models \mathbf{Q}(x) \wedge \mathbf{Q}(y) \wedge \mathbf{Q}(z) \wedge (z = x + y) \wedge \neg(x = 0) \wedge \neg(y = 0)$$

2- Recall a first-order formula  $\text{less}(x, y)$ , with two free variables  $x, y$  over  $\mathcal{S}$  such that, for every valuation  $\mathbf{v}$ ,

$$\mathcal{M}, \mathbf{v} \models \text{less}(x, y)$$

iff  $\mathbf{v}(y) \leq \mathbf{v}(x)$ . Hint : just notice that  $x \leq y \Leftrightarrow \exists z, y = x + z$ .

3- Construct a first-order formula  $\text{sqr}(x, y)$  over  $\mathcal{S}$ , with two free variables  $x, y$  such that, for every valuation  $\mathbf{v}$ ,

$$\mathcal{M}, \mathbf{v} \models \text{sqr}(x, y)$$

iff  $\mathbf{v}(y) = \mathbf{v}(x) \cdot \mathbf{v}(x)$ . i.e. the formula  $\text{sqr}(x, y)$  expresses the fact that  $y$  is the square of  $x$ .

Hint : note that  $y$  must be a square and that the previous smaller square must be at a distance  $x + x - 1$  of  $y$ .

4- Construct a first-order formula  $\text{prod}(x, y, z)$  over  $\mathcal{S}$ , with three free variables  $x, y, z$  such that, for every valuation  $\mathbf{v}$ ,

$$\mathcal{M}, \mathbf{v} \models \text{prod}(x, y, z)$$

iff  $\mathbf{v}(z) = \mathbf{v}(x) \cdot \mathbf{v}(y)$  i.e. the formula  $\text{prod}(x, y, z)$  expresses the fact that  $z$  is the product of  $x$  by  $y$ .

Hint : remember the identity  $(x + y)^2 = x^2 + y^2 + xy + xy$ .

5- Is the first order theory of the structure  $\mathcal{M}$  decidable? undecidable?

6- Let us consider the formal language

$$L := \{u \in \{0, 1\}^* \mid \text{val}(u) \text{ is a square}\}$$

where  $\text{val} : \{0, 1\}^* \rightarrow \mathbb{N}$  is the interpretation of the binary word  $u$  as a natural integer written in base 2 ( $\text{val}(101) = 5, \text{val}(001101) = 13$  for example). Is  $L$  a recognizable language? a recursive language?