

LOGICS, J1IN7M12

Test on 16/10/2014

All documents allowed ; duration : 1h 30.

The four exercises are independant one of each other.

Exercise 1 (4 pts)

Let us recall the rule \vee_ℓ of LK :

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

1- Is the following variant a *derived rule* of LK ? :

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}$$

2- Let us call LK' the formal system obtained as follows :

- its set of judgements is the same as the set of judgments of LK

- its set of rules is obtained from the set of rules of LK by removing the rule \vee_ℓ and adding the rule of question 1 (that we name \vee'_ℓ).

Compare the set of judgments of LK' with the set of judgments of LK. Are they equal ? Is one of them included in the other ?

Exercise 2 (4 pts)

We consider the formula

$$\Phi := \forall x \exists y_1 \exists y_2 \exists y_3 \exists y_4 EG(x, S(P(y_1, y_1), S(P(y_2, y_2), S(P(y_3, y_3), P(y_4, y_4))))))$$

and the structures :

$$\mathcal{A} := \langle \mathbb{N}; EG^{\mathcal{A}}; S^{\mathcal{A}}, P^{\mathcal{A}} \rangle$$

where $EG^{\mathcal{A}}$ is the equality relation over $\mathbb{N} \times \mathbb{N}$, $S^{\mathcal{A}}$ (resp. $P^{\mathcal{A}}$) is the sum (resp. product) of natural integers,

$$\mathcal{B} := \langle \mathbb{N}; EG^{\mathcal{B}}; S^{\mathcal{B}}, P^{\mathcal{B}} \rangle$$

where $EG^{\mathcal{B}}$ is the equality relation over $\mathbb{N} \times \mathbb{N}$, $S^{\mathcal{B}}$ (resp. $P^{\mathcal{B}}$) is the product (resp. sum) of natural integers,

$$\mathcal{C} := \langle \mathbb{Z}/3\mathbb{Z}; EG^{\mathcal{C}}; S^{\mathcal{C}}, P^{\mathcal{C}} \rangle$$

where $(\mathbb{Z}/3\mathbb{Z}, +, *)$ is the ring of integers modulo 3, $EG^{\mathcal{C}}$ is the equality relation over $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $S^{\mathcal{C}}$ (resp. $P^{\mathcal{C}}$) is the sum (resp. product) in this ring,

$$\mathcal{D} := \langle \mathbb{Z}/3\mathbb{Z}; EG^{\mathcal{D}}; S^{\mathcal{D}}, P^{\mathcal{D}} \rangle$$

where $EG^{\mathcal{D}}$ is the equality relation over $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $S^{\mathcal{D}}$ (resp. $P^{\mathcal{D}}$) is the product (resp. sum) in the ring $(\mathbb{Z}/3\mathbb{Z}, +, *)$.

The fact that

$$\mathcal{A} \models \Phi$$

is known as the "four-squares" theorem (due to Lagrange).

Determine whether :

$$\mathcal{B} \models \Phi? \quad \mathcal{C} \models \Phi? \quad \mathcal{D} \models \Phi?$$

For each structure give a precise argument.

Recall $\mathbb{Z}/3\mathbb{Z}$ is the structure with domain $\{0,1,2\}$, endowed with the two binary laws $+, *$ defined by the tables :

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$*$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Exercise 3 (8 pts)

Let consider the four formulas (that we name $F_{\forall\forall}, F_{\forall\exists}, F_{\exists\forall}, F_{\exists\exists}$) :

- $\forall x\forall y [(\neg R(x)) \vee R(y)]$
- $\forall x\exists y [(\neg R(x)) \vee R(y)]$
- $\exists x\forall y [(\neg R(x)) \vee R(y)]$
- $\exists x\exists y [(\neg R(x)) \vee R(y)]$

over the signature consisting of just one predicate symbol R , which has arity one.

- 1- Which of the 4 sequents $\vdash F_{\forall\forall}, \vdash F_{\forall\exists}, \vdash F_{\exists\forall}, \vdash F_{\exists\exists}$ are provable within LK? In case of a positive answer give a proof within LK. In case of a negative answer give a precise meta-argument.
- 2- Which of the 4 sequents $\vdash F_{\forall\forall}, \vdash F_{\forall\exists}, \vdash F_{\exists\forall}, \vdash F_{\exists\exists}$ are provable within LJ? In case of a positive answer give a proof within LJ. In case of a negative answer give a precise meta-argument.

Hint : remember the "constructivity" property of LJ.

Exercise 4 (4 pts)

Let us consider the following Kripke structure $\mathcal{K} := (K, \leq, \Vdash_0)$, over the propositional signature consisting of two predicate symbols A, B of arity 0 :

$K := \{0, 1, 2\}$, the partial ordering over K is defined by $0 \leq 1, 0 \leq 2$ and the initial forcing relation is $\Vdash_0 := \{(1, A), (2, B)\}$.

1- Determine whether, in this structure :

- $0 \Vdash \neg(A \wedge B)$?
- $0 \Vdash (\neg A) \vee (\neg B)$?
- $0 \Vdash [\neg(A \wedge B)] \rightarrow [(\neg A) \vee (\neg B)]$?

2- Determine whether :

$$\vdash_{\text{LJ}} [\neg(A \wedge B)] \rightarrow [(\neg A) \vee (\neg B)]?$$