

LOGICS

TD0 : Natural deduction

Propositional calculus

Exercise 0.1 NJ

Give a proof within system NJ for the formulas :

- $\vdash (B \rightarrow \perp) \leftrightarrow \neg B$
- $\vdash A \rightarrow \neg\neg A$
- $\vdash \neg\neg\neg A \rightarrow \neg A$
- $\vdash (\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$
- $\vdash \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$
- $\vdash \neg\neg(A \vee \neg A)$
- $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

Exercise 0.2 NK

Give a proof within system NK for the formulas :

- $\vdash \neg\neg A \rightarrow A$
- $\vdash A \vee \neg A$
- $\vdash \neg(A \wedge B) \rightarrow \neg A \vee \neg B$
- $\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

Exercise 0.3 De Bruijn's numbering

Let Φ be some first-order formula and p a leaf of the planar tree $P(\Phi)$ that represents this formula. If p is labeled by a variable $v \in \mathcal{V}$, we define its *De Bruijn's number* by :

$$\begin{aligned}
 N(p) &:= -1 \text{ if } v \text{ is free at } p \\
 N(p) &:= \text{Card}\{q \in \text{Dom}(P(\Phi)) \mid r \prec q \prec p, P(\Phi)(q) \in \mathcal{QV}, P(\Phi)(q) \notin \mathcal{Q}\{v\}\} \\
 &\quad \text{if } P(\Phi)(r) \in \mathcal{Q}\{v\} \text{ and } \forall q, (r \prec q \prec p \Rightarrow P(\Phi)(q) \notin \mathcal{Q}\{v\}).
 \end{aligned}$$

In words : $N(p)$ is the number of the first quantification of the variable v that is encountered when “climbing” the tree from position p up to the root. One defines then the planar tree $\text{DB}(\Phi)$ by

$$\begin{aligned}
 \text{Dom}(\text{DB}(\Phi)) &:= \text{Dom}(P(\Phi)) \\
 \text{DB}(\Phi)(p) &:= (P(\Phi))(p) && \text{if } (P(\Phi))(p) \text{ is a connective or a symbol of the signature} \\
 \text{DB}(\Phi)(p) &:= Q && \text{if } (P(\Phi))(p) \in \mathcal{QV} \text{ where } Q \text{ is a quantifier} \\
 \text{DB}(\Phi)(p) &:= N(p) && \text{if } (P(\Phi))(p) \in \mathcal{V} \text{ and } N(p) \geq 0 \\
 \text{DB}(\Phi)(p) &:= v && \text{if } (P(\Phi))(p) \in \mathcal{V} \text{ and } N(p) = -1
 \end{aligned}$$

- 1- Show that, if Φ, Φ' are first-order formulas, $\Phi \equiv_{\alpha} \Phi'$ if and only if $\text{DB}(\Phi) = \text{DB}(\Phi')$.
- 2- Show that $\Phi \mapsto \text{DB}(\Phi)$ can be computed in linear time.

3- Construct an algorithm that takes in input (Φ, v, t) where Φ is a first-order formula, v is a variable and t is a term and returns a representative of $\Phi[v := t]$.

4- Describe an algorithmic method allowing, given a first-order formula Ψ to compute all the formulas Φ , up to α -equivalence, such that there exists some variable v and some term t such that

$$\Phi[v := t] \equiv_{\alpha} \Psi.$$

Exercise 0.4 variables that are linked several times

The “four-squares theorem” asserts that, every natural integer can be written as a sum of four squares. Using the signature $\mathcal{S} = \{E; P, M\}$, this theorem can be expressed by :

$$\forall x \cdot \exists y_1 \cdot \exists y_2 \cdot \exists y_3 \cdot \exists y_4 \cdot E(x, P(M(y_1, y_1), P(M(y_2, y_2), P(M(y_3, y_3), M(y_4, y_4))))))$$

Could you express the same theorem by a formula that uses *three* variables only?

Exercise 0.5

What do you think about the following “proofs”?

π_1 :

- 1 – $P(x) \vdash P(x)$ (axiom)
- 2 – $P(x) \vdash \forall x \cdot P(x)$ (1, \forall_{intro})
- 3 – $P(x) \vdash P(y)$ (2, \forall_{elim})
- 4 – $\vdash P(x) \rightarrow P(y)$ (3, $\rightarrow_{\text{elim}}$)
- 5 – $\vdash \forall y \cdot P(x) \rightarrow P(y)$ (4, \forall_{intro})
- 6 – $\vdash \forall x \cdot \forall y \cdot P(x) \rightarrow P(y)$ (5, \forall_{intro})

π_2 :

- 1 – $\forall z \cdot z = z \vdash \forall z \cdot z = z$ (axiom)
- 2 – $\forall z \cdot z = z \vdash x = x$ (1, \forall_{elim})
- 3 – $\forall z \cdot z = z \vdash \exists y \cdot x = y$ (2, \forall_{intro})
- 4 – $\forall z \cdot z = z \vdash \forall x \cdot \exists y \cdot x = y$ (3, \forall_{intro})
- 5 – $\forall z \cdot z = z \vdash \exists y \cdot S(y) = y$ (4, \forall_{elim})

Exercise 0.6 NJ

Give a NJ proof for the following formulas :

- $\vdash (\forall x P(x) \wedge \forall y Q(y)) \rightarrow \forall z (P(z) \wedge Q(z))$
- $\vdash (\forall x P(x) \wedge \exists y Q(y)) \rightarrow \exists z (P(z) \wedge Q(z))$
- $\vdash \neg \exists x P(x) \leftrightarrow \forall x \neg P(x)$
- $\vdash \exists x P(x) \rightarrow \neg \forall x \neg P(x)$
- $\vdash \exists x \neg P(x) \rightarrow \neg \forall x P(x)$

Exercise 0.7 NK

Give a NK proof for the following formulas :

- $\vdash [\neg \forall x \cdot \neg P(x)] \rightarrow [\exists x \cdot P(x)]$
- $\vdash [\forall x \cdot (R \vee R'(x))] \rightarrow [R \vee \forall x \cdot R'(x)]$