

LOGICS

Exercises on Chapter 2: Sequent Calculus

Symmetries in the system LK

Exercise 1.1 Reversibility

An inference rule $\frac{S_1}{S_2}$ in a formal system \mathcal{S} is called *reversible* iff $\frac{S_2}{S_1}$ is a derived rule of the system \mathcal{S} . Which rules of system LK are reversible ?

Exercise 1.2 Duality

Let us add a connector \top to the language of system LK i.e. the set of connectors is now $\{\perp, \wedge, \vee, \rightarrow, \neg\}$, the set of quantifiers is $\{\forall, \exists\}$, we have a denumerable set of relation symbols \mathcal{R} and a denumerable set of function symbols \mathcal{F} .

To every well-formed formula F , we associate a *dual* formula $D(F)$ by structural induction:

$$\begin{aligned} D(\perp) &:= \top, \quad D(\top) := \perp \\ D(R(t_1, \dots, t_k)) &:= R(t_1, \dots, t_k) \quad \text{pour tout } R \in \mathcal{R} \\ D(F \wedge G) &:= D(F) \vee D(G), \quad D(F \vee G) := D(F) \wedge D(G) \\ D(F \rightarrow G) &:= \neg(D(G) \rightarrow D(F)), \quad D(\neg F) := \neg D(F) \\ D(\forall x F) &:= \exists x D(F), \quad D(\exists x F) := \forall x D(F) \end{aligned}$$

1- Show that, for every formula F , with atomic subformulas A_1, \dots, A_n , the following equivalence holds:

$$F(\neg A_1, \dots, \neg A_n) \models \neg D(F)(A_1, \dots, A_n).$$

We extend the map $F \mapsto D(F)$ to sequents by:

$$D(F_1, \dots, F_n \vdash G_1, \dots, G_m) := D(G_1), \dots, D(G_m) \vdash D(F_1), \dots, D(F_n).$$

2- Show that, if for every structure \mathcal{T} we have:

$$\mathcal{T} \models S_1 \text{ implies } \mathcal{T} \models S_2$$

then, for every structure \mathcal{T} we also have:

$$\mathcal{T} \models D(S_1) \text{ implies } \mathcal{T} \models D(S_2).$$

3- What are the rules of LK, $\frac{S_1}{S_2}$ (where S_1, S_2 are sequents) such that $\frac{D(S_1)}{D(S_2)}$ is still a rule of LK?

4- Could you manage to add a rule to LK in such a way that it takes into account the extension to connector \top ? We would like that the sequent $\vdash (\neg \perp \rightarrow \top) \wedge (\top \rightarrow \neg \perp)$ be derivable in this extension of LK and we would like to keep (or even improve) the symmetries of system LK.

Examples of derivations

Exercise 1.3 Propositional LK

Find a derivation in LK for the sequents:

$$\begin{aligned} &\vdash A \rightarrow (B \rightarrow A) \\ &\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \\ &\vdash ((P \rightarrow Q) \rightarrow P) \rightarrow P \end{aligned}$$

Exercise 1.4 LK

Find a derivation in LK for the sequents:

$$\begin{aligned} &\neg \exists x R(x) \vdash \forall x \neg R(x) \\ &\neg \forall x R(x) \vdash \exists x \neg R(x) \\ &\vdash \forall x (Q \vee R(x)) \rightarrow Q \vee \forall x R(x) \\ &\vdash \exists x \forall y (R(y) \rightarrow R(x)) \end{aligned}$$

Exercise 1.5 Propositional LJ

Find a derivation in LJ for the sequents:

$$\begin{aligned} &\neg(A \vee B) \vdash \neg A \wedge \neg B \\ &\neg A \wedge \neg B \vdash \neg(A \vee B) \\ &A \vdash \neg \neg A \\ &\neg \neg \neg A \vdash \neg A \end{aligned}$$

Exercise 1.6 LJ

1- What do you think about the following “derivation”?

$$\begin{array}{ll} R(x) \vdash R(x) & \mathbf{ax} \\ \exists x R(x) \vdash R(x) & \exists_g \\ \neg R(x), \exists x R(x) \vdash & \neg_g \\ \forall x \neg R(x), \exists x R(x) \vdash & \forall_g \\ \forall x \neg R(x) \vdash \neg \exists x R(x) & \neg_d \end{array}$$

2- Does the sequent $\forall x \neg R(x) \vdash \neg \exists x R(x)$ admit a derivation in LJ ?

Properties of derivations

Exercise 1.7 Right-elimination

Let us consider the following right-elimination rule:

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}$$

(where Γ is a multi-set of formulas and A, B are formulas).

1- Is this rule a derived rule of LK ?

2- Is this rule a derived rule of LK $\setminus \{\mathbf{cut}\}$?

Exercise 1.8 Contractions

Let us consider the sequent $S := \exists x (R(a) \vee R(b) \rightarrow R(x))$.

1- Find a derivation of S in LK.

2- Find a cut-free derivation of S in LK.

3- Prove that there does not exist any derivation, without neither cut nor contraction, of S in LK.

4- Does there exist some derivation, without contraction, of S in LK ?