

## LOGICS, J1IN7M21

Examination on 15/12/2014. Some solutions.

### Exercise 1 (4 pts)

1- Let us give the required proofs:

$$\frac{\frac{\frac{\overline{A \vdash A}^{\text{ax}}}{\vdash \neg A, A}^{\neg_r}}{\vdash \neg A, A, (\neg B) \rightarrow C}^{\text{wkn}_r} \quad \frac{\frac{\frac{\overline{B \vdash B}^{\text{ax}}}{B, \neg B \vdash}^{\neg_l}}{B, \neg B \vdash C}^{\text{wkn}_r}}{B \vdash (\neg B) \rightarrow C}^{\rightarrow_r} \quad \frac{\frac{\overline{C \vdash C}^{\text{ax}}}{C, \neg B \vdash C}^{\text{wkn}_l}}{C \vdash (\neg B) \rightarrow C}^{\rightarrow_r}}{B \vee C \vdash (\neg B) \rightarrow C}^{\vee_l}}{\frac{(\neg A) \rightarrow (B \vee C) \vdash A, (\neg B) \rightarrow C}{(\neg A) \rightarrow (B \vee C) \vdash A \vee ((\neg B) \rightarrow C)}^{\rightarrow_l}}^{\rightarrow_l}$$

$$\frac{\frac{\frac{\overline{P(x) \vdash Q(x), P(x)}^{\text{ax}'}}{\vdash P(x) \rightarrow Q(x), P(x)}^{\rightarrow_r}}{P(x) \vdash P(x)}^{\text{ax}}}{(P(x) \rightarrow Q(x)) \rightarrow P(x) \vdash P(x)}^{\rightarrow_l}}{\frac{\forall x((P(x) \rightarrow Q(x)) \rightarrow P(x)) \vdash P(x)}^{\forall_l}}{\forall x((P(x) \rightarrow Q(x)) \rightarrow P(x)) \vdash \forall y P(y)}^{\forall_r}}^{\rightarrow_r}$$

2- Let us give the required proof:

$$\frac{\frac{\frac{\overline{A \vdash A}^{\text{ax}}}{\neg A, A \vdash}^{\neg_l}}{\neg A, A \vdash B}^{\text{wkn}_r}}{\neg A \vdash A \rightarrow B}^{\rightarrow_r} \quad \frac{\overline{A \vdash A}^{\text{ax}}}{\neg A, A \vdash}^{\neg_l}}{(A \rightarrow B) \rightarrow A, \neg A \vdash}^{\rightarrow_l}}{(A \rightarrow B) \rightarrow A \vdash \neg \neg A}^{\neg_r}}{\vdash ((A \rightarrow B) \rightarrow A) \rightarrow (\neg \neg A)}^{\rightarrow_r}$$

### Exercise 2 (4 pts)

1- Let  $\mathcal{K} := (K, \leq, \Vdash)$  be a Kripke-structure over the signature  $\langle A, B; \rangle$  (where  $A, B$  have arity 0). Let  $k \in K$  be some node fulfilling:

$$k \Vdash A \rightarrow B \text{ and } k \Vdash (\neg A) \rightarrow B \text{ and } k \Vdash \neg B. \quad (1)$$

Let  $k' \in K$  such that  $k \leq k'$ :

**Case 1:**  $k' \Vdash \neg A$

Since  $k \Vdash (\neg A) \rightarrow B$ , we get that  $k' \Vdash B$ .

But since  $k \Vdash \neg B$ , we get that  $k' \Vdash \neg B$ , hence  $k' \Vdash \perp$ , which is impossible.

**Case 2:**  $k' \not\Vdash \neg A$

Hence, there exists some  $k'' \geq k'$  such that

$$k'' \Vdash A.$$

Since  $k \Vdash A \rightarrow B$  and  $k \leq k''$ , we get that  $k'' \Vdash B$ .

Since  $k \Vdash \neg B$  and  $k \leq k''$ , we get that  $k'' \Vdash \neg B$ , hence  $k'' \Vdash \perp$ , which is impossible.

We conclude that (1) is always false i.e. that

$$A \rightarrow B, (\neg A) \rightarrow B, \neg B \Vdash \perp. \quad (2)$$

2- By the Kripke-completeness theorem for intuitionistic propositional logics, from (2) we can infer that, also

$$A \rightarrow B, (\neg A) \rightarrow B, \neg B \vdash_{\text{LJ}} \perp.$$

3- Let us give the required proof:

$$\frac{\frac{\frac{\frac{\overline{A \vdash A}^{\text{ax}} \quad \overline{A, B \vdash B}^{\text{ax}'}}{A, A \rightarrow B \vdash B} \rightarrow_i}{A, A \rightarrow B, \neg B \vdash} \neg_i}{A \rightarrow B, \neg B \vdash \neg A} \neg_r}{A \rightarrow B, \neg B, B \vdash} \text{wkn}_l}{A \rightarrow B, (\neg A) \rightarrow B, \neg B \vdash} \rightarrow_i}{A \rightarrow B, (\neg A) \rightarrow B, \neg B \vdash \perp} \text{wkn}_r$$

**Exercise 3** (4 pts)

Let us consider the following Kripke structure  $\mathcal{K} := (K, \leq, \Vdash_0)$ , over the propositional signature consisting of one predicate symbol  $A$  of arity 0 :

$K := \{0, 1\}$ , the partial ordering over  $K$  is defined by  $0 \leq 1$  and the initial forcing relation is  $\Vdash_0 := \{(1, A)\}$ .

We know that  $0 \Vdash (\neg \neg A)$  means that,

$$\forall k \geq 0, \exists k' \geq k, k' \Vdash A.$$

The choice  $k' := 1$  satisfies this condition, thus:

$$0 \Vdash \neg \neg A.$$

On the other side, since  $(0, A)$  is not member of the initial forcing relation

$$0 \not\Vdash A.$$

By the adequation theorem, if  $\vdash_{\text{LJ}} (\neg \neg A) \rightarrow A$ , then, in every Kripke-structure we would have  $k \Vdash (\neg \neg A) \rightarrow A$ . But the above Kripke-structure is a counter-example, thus  $\vdash (\neg \neg A) \rightarrow A$  is not provable within LJ.

2- The left-introduction rule for negation is

$$\frac{\Gamma \Vdash A}{\Gamma, \neg A \Vdash} \neg_\ell$$

Its inverse (let us call it  $I(\neg_\ell)$ ) is the scheme:

$$\frac{\Gamma, \neg A \vdash \perp}{\Gamma \vdash A} I(\neg_\ell)$$

Using this additional rule we could construct the following proof:

$$\frac{\frac{\frac{\overline{\neg A \vdash \neg A}^{\text{ax}}}{\neg \neg A, \neg A \vdash \perp}^{\neg_i}}{\neg \neg A \vdash A}^{I(\neg_\ell)}}{\vdash (\neg \neg A) \rightarrow A}^{\rightarrow_r}$$

But, by question 1, the conclusion is a sequent which is *not* provable within LJ. hence the rule  $I(\neg_\ell)$  is *not* a derived rule of LJ.

3- The right-introduction rule for negation is

$$\frac{\Gamma, A \vdash \perp}{\Gamma \vdash \neg A} \neg_r$$

Its inverse (let us call it  $I(\neg_r)$ ) is the scheme:

$$\frac{\Gamma \vdash \neg A}{\Gamma, A \vdash \perp} I(\neg_r)$$

Here is a proof, with the hypothesis  $\Gamma \vdash \neg A$ , showing that  $I(\neg_r)$  is a derived rule of LJ.

$$\frac{\frac{\Gamma \vdash \neg A}{\Gamma, A \vdash \perp}^{\text{Hyp}} \frac{\overline{A \vdash A}^{\text{ax}}}{\neg A, A \vdash \perp}^{\neg_i}}{\Gamma, A \vdash \perp}^{\text{cut}}$$

#### Exercise 4 (6 pts)

1- Let us assume that

$$\mathcal{A} \models \Phi.$$

This means that , there exists some valuation  $\nu : \{x_1 \dots x_k\} \rightarrow A$ , such that

$$\mathcal{A}, \nu \models \forall y_1 \dots \forall y_\ell \psi(x_1 \dots x_k, y_1 \dots, y_\ell).$$

Let us define

$$A' := \{c_1^A, \dots, c_m^A\} \cup \{\nu(x_i) \mid 1 \leq i \leq k\} \quad (3)$$

and

$$A' := \langle A'; R_1^{A'}, R_2^{A'}, \dots, R_n^{A'}; c_1^A, c_2^A, \dots, c_m^A \rangle$$

where the predicate  $R_i^{A'}$  is the restriction of  $R_i^A$  to  $A'^{r_i}$ . For every values  $d_1, \dots, d_\ell \in A'$  given to the variables  $y_1, \dots, y_\ell \in A'$ , since  $A' \subseteq A$ , we are sure that

$$\psi(\nu(x_1) \dots \nu(x_k), d_1 \dots, d_\ell)$$

holds. Hence

$$\mathcal{A}', \nu \models \forall y_1 \dots \forall y_\ell \psi(x_1 \dots x_k, y_1 \dots, y_\ell).$$

This proves that

$$\mathcal{A}' \models \exists x_1 \dots \exists x_k \forall y_1 \dots \forall y_\ell \psi(x_1 \dots x_k, y_1 \dots, y_\ell).$$

Using formula (3), we get that  $\text{Card}(A') \leq k + m$ .

2-

By question 1, a formula  $\Phi$  of the above form, has a model iff it has a model with domain of cardinality  $\leq k + m$ . But, for each domain  $A$ , of finite cardinality  $N$ , there are only a finite number of possible interpretations of the predicate-symbols  $R_1, R_2, \dots, R_n$  and of the constant symbols  $c_1, c_2, \dots, c_m$ : namely at most  $2^{N^{r_i}}$  interpretations of  $R_i$  and  $N$  interpretations of  $c_j$ . Thus we can enumerate, in finite time, all the possible structures on the signature  $\mathcal{S}$  where the domain is an interval of the form  $[0, N - 1]$  with  $N \leq k + m$ . If some of them satisfies  $\Phi$  we answer “yes”, otherwise we answer “no”.

3.1 The formula  $\theta$  is false in some structure  $\mathcal{A}$  iff  $\mathcal{A} \models \Phi$ . Hence  $\models \theta$  iff  $\Phi$  is not satisfiable.

3.2 It suffices to apply the algorithm of question 2 on the input  $\Phi$ , and to negate its answer.

3.3 Following the computations of question 3.1:

let us note  $r := \max\{r_i \mid 1 \leq i \leq n\}$ ;

- there are at most  $O(2^{n \cdot (k+m)^{r+1}})$  different structures of size  $\leq k + m$

- given such a structure, there are  $O((k + m)^k)$  choices for the values of the  $x_i$

- for each  $\nu(\vec{x})$ , we have to test all the  $(k + m)^\ell$  choices for the values of the  $y_j$ , and in each case evaluate the truth of  $\psi$ : this takes time  $O((k + m)^{k+\ell}) \cdot |\psi|$ .

The total time taken by the algorithm is thus:  $O(2^{n \cdot (k+m)^{r+1}} \cdot (k + m)^{k+\ell} \cdot |\psi|)$  which is also

$$O(2^{2n \cdot (k+m)^{r+k+\ell+1}} \cdot |\psi|)$$

i.e. a *double-exponential* function of the size of the input.