

LOGICS, J1IN7M12
 Test on 16/10/2014. Some solutions.

Exercise 1 (4 pts)

1- Yes, the rule \vee'_ℓ is a derived rule of LK. Here is the corresponding proof with hypotheses:

$$\frac{\frac{\overline{\Gamma, A \vdash \Delta}^{\text{Hyp}}}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}^{\text{wkn}_l^*, \text{wkn}_r^*} \quad \frac{\overline{\Gamma', B \vdash \Delta}^{\text{Hyp}}}{\Gamma, \Gamma', B \vdash \Delta, \Delta'}^{\text{wkn}_l^*, \text{wkn}_r^*}}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'}^{\vee_l}$$

2- The rule \vee_ℓ is a derived rule of LK'. Here is the corresponding proof with hypotheses:

$$\frac{\frac{\overline{\Gamma, A \vdash \Delta}^{\text{Hyp}} \quad \overline{\Gamma, B \vdash \Delta}^{\text{Hyp}}}{\Gamma, \Gamma, A \vee B \vdash \Delta, \Delta}^{\vee'_l}}{\Gamma, A \vee B \vdash \Delta}^{\text{contr}_l^*, \text{contr}_r^*}$$

It follows that both systems have exactly the same set of *provable* sequents.

But the question 2 was ill-posed: it asked only whether the two systems had the same judgments. Thus the answer: “yes, by definition” is considered a correct answer too. **Exercise 2** (4 pts)

Structure \mathcal{B} :

Formula Φ means that every natural integer is a product of four *even* numbers. But the integer 1 is not divisible by 16, hence is not decomposable as such a product. It follows that $\mathcal{B} \not\models \Phi$.

Structure \mathcal{C} :

With this interpretation, formula Φ means that every element of $\mathbb{Z}/3\mathbb{Z}$ is a sum of four squares. We check that:

$$0 = (0*0)+(0*0)+(0*0)+(0*0), \quad 1 = (1*1)+(0*0)+(0*0)+(0*0), \quad 2 = (1*1)+(1*1)+(0*0)+(0*0)$$

hence $\mathcal{C} \models \Phi$.

Structure \mathcal{D} :

With this interpretation, formula Φ means that every element of $\mathbb{Z}/3\mathbb{Z}$ is a product of four elements of the form $y + y$ (for some $y \in \mathbb{Z}/3\mathbb{Z}$). We check that:

$$0 = (0+0)*(0+0)*(0+0)*(0+0), \quad 1 = (1+1)*(1+1)*(2+2)*(2+2), \quad 2 = (1+1)*(2+2)*(2+2)*(2+2).$$

hence $\mathcal{D} \models \Phi$.

Exercise 3 (8 pts)

1- Let us build a counter-model \mathcal{M} for the formula:

$$\forall x \forall y [(\neg R(x)) \vee R(y)]$$

We define $D^{\mathcal{M}} := \{0, 1\}, R^{\mathcal{M}}(0) := \text{True}, R^{\mathcal{M}}(1) = \text{False}$. We define the valuation ν by $\nu(x) := 0, \nu(y) := 1$. One can check that $\mathcal{M}, \nu \not\models [(\neg R(x)) \vee R(y)]$. It follows that

$$\mathcal{M} \not\models \forall x \forall y [(\neg R(x)) \vee R(y)].$$

By the soundness theorem for LK, we conclude that

$$\not\vdash_{\text{LK}} \forall x \forall y [(\neg R(x)) \vee R(y)].$$

The three other sequents are provable within LK:

$$\frac{\frac{\frac{\overline{R(x) \vdash R(x)}^{\text{ax}}}{\vdash \neg R(x), R(x)}^{\neg_r}}{\vdash (\neg R(x)) \vee R(x)}^{\vee_r}}{\vdash \exists y [(\neg R(x)) \vee R(y)]}^{\exists_r}}{\vdash \forall x \exists y [(\neg R(x)) \vee R(x)]}^{\forall_r}$$

$$\frac{\frac{\frac{\frac{\frac{\overline{R(x) \vdash R(x)}^{\text{ax}}}{\vdash \neg R(x), R(x)}^{\neg_r}}{\vdash \neg R(x), R(y), \neg R(z), R(x)}^{\text{wkn}_r^*}}{\vdash (\neg R(x)) \vee R(y), (\neg R(z)) \vee R(x)}^{\vee_r^*}}{\vdash \forall y [(\neg R(x)) \vee R(y)], (\neg R(z)) \vee R(x)}^{\forall_r}}{\vdash \exists x \forall y [(\neg R(x)) \vee R(y)], (\neg R(z)) \vee R(x)}^{\exists_r}}{\vdash \exists x \forall y [(\neg R(x)) \vee R(y)], \forall x [(\neg R(z)) \vee R(x)]}^{\forall_r}}{\vdash \exists x \forall y [(\neg R(x)) \vee R(y)], \exists z \forall x [(\neg R(z)) \vee R(x)]}^{\exists_r}}{\vdash \exists x \forall y [(\neg R(x)) \vee R(y)]}^{\text{contr}_r}$$

$$\frac{\frac{\frac{\frac{\overline{R(x) \vdash R(x)}^{\text{ax}}}{\vdash \neg R(x), R(x)}^{\neg_r}}{\vdash \neg R(x) \vee R(x)}^{\vee_r}}{\vdash \exists y \neg R(x) \vee R(y)}^{\exists_r}}{\vdash \exists x \exists y \neg R(x) \vee R(y)}^{\exists_r}$$

2- Let us treat the provability within LJ now.

Suppose that, the following assumption (H) holds:

there exists some quantifiers $Q, Q' \in \{\forall, \exists\}$ such that the sequent $\vdash F_{QQ'}$ is provable within LJ.

By the cut-elimination theorem, this sequent would possess a *cut-free* proof. This proof would have the form:

$$\frac{\frac{\frac{\pi}{\vdash (\neg R(t)) \vee R(t')}^{\text{Q}'_r}}{\vdash Q' y (\neg R(t)) \vee R(y)}^{\text{Q}_r}}{\vdash Q x Q' y (\neg R(x)) \vee R(y)}$$

where t, t' are two terms (note that, if $Q = \forall$ then $t = x$ and if $Q' = \exists$, then $t' = y$). The last rule of π must be one of the two rules that introduce disjunction on the right. It follows that the sequent just above $\vdash (\neg R(t)) \vee R(t')$ in π must be either $\vdash \neg R(t)$ or $\vdash R(t')$. But, by theorem 3.2.2 none of these two sequents is derivable in LK, hence none is derivable in LJ. We have shown that assumption (H) cannot hold. Hence all the four sequents $\vdash F_{QQ'}$ are non-provable within LJ.

Exercise 4 (4 pts)

1- One can compute all the values of the assertion $k \Vdash \Phi$ for $k \in K$ and Φ subformula of $[\neg(A \wedge B)] \rightarrow [(\neg A) \vee (\neg B)]$, using the initial forcing relation and the inductive definition of \Vdash . We present these values in a table:

$k \Vdash \Phi$	A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$	$[\neg(A \wedge B)] \rightarrow [(\neg A) \vee (\neg B)]$
0	F	F	F	T	F	F	F	F
1	T	F	F	T	F	T	T	-
2	F	T	F	T	T	F	T	-

(the - entry means that this forcing value was unnecessary for solving the question, thus was not computed).

In particular

$$0 \Vdash \neg(A \wedge B), \quad 0 \not\Vdash (\neg A) \vee (\neg B)$$

$$0 \not\Vdash [\neg(A \wedge B)] \rightarrow [(\neg A) \vee (\neg B)]$$

2- By question 1, the sequent $[\neg(A \wedge B)] \rightarrow [(\neg A) \vee (\neg B)]$ admits the Kripke counter-model \mathcal{K} . Hence it cannot be proved within LJ.