

LOGICS, J1IN7M12

Test on 17/10/2013. Some solutions.

Exercise 1 (5 pts)

Yes, the rule \forall_ℓ of LK is reversible. Here are the proofs with hypotheses that witness the derived rules:

$$\frac{\frac{\frac{\overline{A \vdash A}^{\text{ax}'}}{A \vdash A, B}^{\text{wkn}_r}}{A \vdash A \vee B}^{\forall_r} \quad \frac{\overline{\Gamma, A \vee B \vdash \Delta}^{\text{Hyp}}}{\Gamma, A \vee B \vdash \Delta}^{\text{cut}}}{\Gamma, A \vdash \Delta}^{\text{cut}} \quad \frac{\frac{\frac{\overline{B \vdash B}^{\text{ax}'}}{B \vdash A, B}^{\text{wkn}_r}}{B \vdash A \vee B}^{\forall_r} \quad \frac{\overline{\Gamma, A \vee B \vdash \Delta}^{\text{Hyp}}}{\Gamma, A \vee B \vdash \Delta}^{\text{cut}}}{\Gamma, B \vdash \Delta}^{\text{cut}}$$

Exercise 2 (6 pts)

1- Let us give the required proofs:

$$\frac{\frac{\frac{\overline{A, B \vdash A}^{\text{ax}'}}{A, B \vdash A \wedge (B \vee C)}^{\wedge_r} \quad \frac{\frac{\overline{A, B \vdash B, C}^{\text{ax}'}}{A, B \vdash B \vee C}^{\vee_r}}{B \vdash A \wedge (B \vee C), \neg A}^{\neg_r}}{\vdash A \wedge (B \vee C), \neg A, \neg B}^{\neg_r} \quad \frac{\frac{\frac{\overline{A, C \vdash A}^{\text{ax}'}}{A, C \vdash A \wedge (B \vee C)}^{\wedge_r} \quad \frac{\frac{\overline{A, C \vdash B, C}^{\text{ax}'}}{A, C \vdash B \vee C}^{\vee_r}}{C \vdash A \wedge (B \vee C), \neg A}^{\neg_r}}{\vdash A \wedge (B \vee C), \neg A, \neg C}^{\neg_r}}{\vdash A \wedge (B \vee C), \neg A, (\neg B) \wedge \neg C}^{\wedge_r} \quad \frac{\frac{\overline{\neg(A \wedge (B \vee C)) \vdash \neg A, (\neg B) \wedge \neg C}^{\neg_i}}{\neg(A \wedge (B \vee C)) \vdash \neg A \vee ((\neg B) \wedge \neg C)}^{\vee_r}}{\neg(A \wedge (B \vee C)) \vdash \neg A \vee ((\neg B) \wedge \neg C)}^{\neg_i}$$

$$\frac{\frac{\frac{\frac{\overline{A, Q(x) \vdash Q(x)}^{\text{ax}'}}{A, Q(x) \vdash \exists x Q(x)}^{\exists_r}}{A \vdash \exists x Q(x), \neg Q(x)}^{\neg_r}}{A \vdash A, \forall x \neg Q(x)}^{\text{ax}'}}{A \vdash \exists x Q(x), \forall x \neg Q(x)}^{\forall_r} \quad \frac{\frac{\overline{A \vdash A \wedge \exists x Q(x), \forall x \neg Q(x)}^{\wedge_r}}{\vdash A \wedge \exists x Q(x), \neg A, \forall x \neg Q(x)}^{\neg_r}}{\vdash A \wedge \exists x Q(x), (\neg A) \vee \forall x \neg Q(x)}^{\vee_r} \quad \frac{\overline{\neg(A \wedge \exists x Q(x)) \vdash (\neg A) \vee \forall x \neg Q(x)}^{\neg_i}}{\neg(A \wedge \exists x Q(x)) \vdash (\neg A) \vee \forall x \neg Q(x)}^{\neg_i}$$

2- Let us give the required proof:

$$\frac{\frac{\frac{\frac{\frac{\frac{P(y), Q(y) \vdash P(y)}{\text{ax}'}}{P(y), Q(y) \vdash Q(y)}{\text{ax}'}}{\wedge_r}}{P(y), Q(y) \vdash P(y) \wedge Q(y)}{\exists_r}}{P(y), Q(y) \vdash \exists y(P(y) \wedge Q(y))}{\forall_i}}{\forall x P(x), Q(y) \vdash \exists y(P(y) \wedge Q(y))}{\exists_l}}{\forall x P(x), \exists y Q(y) \vdash \exists y(P(y) \wedge Q(y))}{\exists_l}}{\wedge_l}}{(\forall x P(x)) \wedge \exists y Q(y) \vdash \exists y(P(y) \wedge Q(y))}$$

Exercise 3 (5 pts)

1- The signature \mathcal{S} used by MO and the sequent S is $\mathcal{S} := \langle =, *, e \rangle$ where both symbols $=, *$ have arity 2 and e has arity 0.

A structure over \mathcal{S} , fulfilling all the axioms of MO is a t-uple

$$\mathcal{M} := \langle M; =^{\mathcal{M}}, *^{\mathcal{M}}, e^{\mathcal{M}} \rangle$$

such that M is non-empty, $*$ is an associative law over M and e is a neutral element for the law $*$. If every element m of M has a right-inverse m^{-1} i.e. such that $m * m^{-1} = e$ then the formula $\forall x \forall y \exists z x = y * z$ is true in this structure: let $x, y \in M$, let us define $z := y^{-1} * x$. We then have:

$$\begin{aligned} y * z &= y * (y^{-1} * x) && \text{by our choice of } z \\ &= (y * y^{-1}) * x && \text{associativity} \\ &= e * x && \text{right-inverse} \\ &= x && \text{neutral element} \end{aligned}$$

It follows that $\mathcal{M} \models \text{MO}$ and

$$\mathcal{M} \models \forall x \forall y \exists z x = y * z$$

It thus suffices to choose a structure \mathcal{M} which is a group: for example the trivial group, with only one element $\{1\}$, is such a model. (But any other group is also a model).

2- Let us consider the set of booleans, $\{0, 1\}$, endowed with the law $*$ defined by

$$0 * 0 = 0, 0 * 1 = 1 * 0 = 1, 1 * 1 = 1$$

(this law is usually denoted by $+$ and defines the semantics of the disjunction).

The boolean 0 is a neutral element for $*$.

But, for $x := 0$ and $y := 1$, there does not exist any $z \in \{0, 1\}$ such that $x = y * z$ because $1 + 0 = 1 + 1 = 1 \neq 0$. Thus, if we choose

$$\mathcal{M}_2 := \langle \{0, 1\}; =^{\mathcal{M}_2}; +, 0 \rangle$$

$$\mathcal{M}_2 \models \text{MO} \text{ and } \mathcal{M}_2 \not\models \forall x \forall y \exists z x = y * z$$

3- The sequent S has a counter-model; hence, by the soundness theorem, it cannot be proved within LK.

Since every sequent provable in LJ is also provable in LK, it cannot, a fortiori, be proved in LJ.

Exercise 4 (5 pts)

1- Since $0 \Vdash_0 A$ we are sure that $0 \Vdash A$, hence that

$$0 \Vdash A \vee \neg A$$

We know that $0 \Vdash (\neg\neg B)$ means that,

$$\forall k \geq 0, \exists k' \geq k, k' \Vdash B.$$

Since, for $k' \in \{1, 2\}$, $k' \Vdash B$ and $\{1, 2\}$ are the two maximal elements of K , we conclude that

$$0 \Vdash (\neg\neg B) \tag{1}$$

But B is atomic and $0 \not\Vdash_0 B$, hence

$$0 \not\Vdash B \tag{2}$$

It follows from (1),(2) that

$$0 \not\Vdash (\neg\neg B) \rightarrow B$$

2- Let us consider the following Kripke structure $\mathcal{K}' := (K', \leq', \Vdash'_0)$, over the propositional signature consisting of one predicate symbol A of arity 0 :

$K' := \{0', 1'\}$, the partial ordering over K' is defined by $0' \leq 1'$ and the initial forcing relation is $\Vdash'_0 := \{(1', A)\}$.

Since A is atomic and $0' \not\Vdash'_0 A$, we conclude that $0' \not\Vdash A$.

Since $1' \geq 0'$ and $1' \Vdash A$, we conclude that $0' \not\Vdash \neg A$.

Hence $0' \not\Vdash A \vee \neg A$. The sequent $\vdash A \vee \neg A$ thus admits the Kripke counter-model \mathcal{K}' . Hence it cannot be proved within LJ.

By question 1, the sequent $\vdash (\neg\neg B) \rightarrow B$ admits the Kripke counter-model \mathcal{K} . Hence it cannot be proved within LJ.