

Correlation constraints and the Bloch geometry of two-party systems

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Outline

- ▶ Geometry of the quantum state space – old and some new stuff
- ▶ Correlation constraints
 - ▶ Quantum marginal problem
 - ▶ Bloch representation
 - ▶ Two-party systems

Geometry of the quantum state space

- ▶ quest for image/model of quantum-mechanical state space
→ to visualize dynamics, decoherence . . .

- ▶ pure states $|\psi\rangle$ $|\psi\rangle \in \mathcal{H}_d$

- ▶ mixed states ρ $\rho \in \mathcal{B}(\mathcal{H}), \rho = \rho^\dagger$

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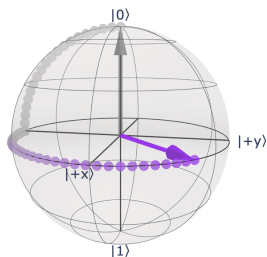
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the qubit Bloch ball



$$\rho = \frac{1}{2} (\mathbb{1}_2 + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z)$$

Bloch vector $\mathbf{r} = (r_x, r_y, r_z)$

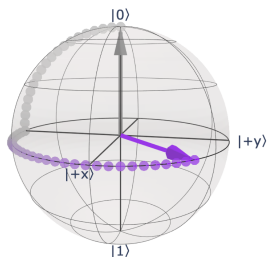
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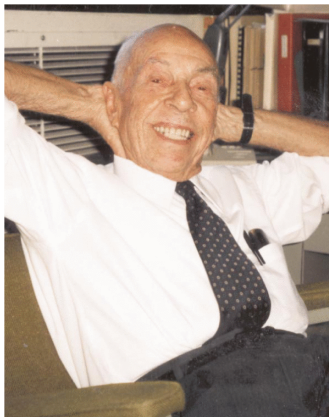
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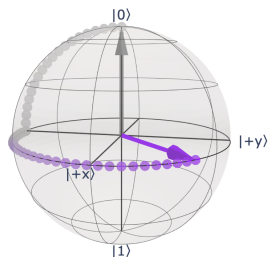
Bloch 1946 ?



Ugo Fano (1912 – 2001)

Geometry of the quantum state space

the qubit Bloch ball $d = 2$



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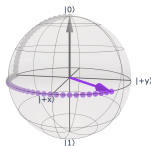
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Fano 1954, Fano 1957, Feynman *et al.* 1957 !

Geometry of the quantum state space

unfortunately . . . the qubit Bloch ball

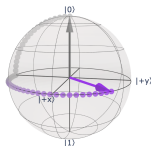
- is not representative for $d > 2$
- provides some misleading intuition



Geometry of the quantum state space

the $d = 2$ Bloch ball

- not representative for $d > 2$
- misleading intuition



$d = 2$

$d > 2$

surface

pure states ψ

mostly mixed states

arbitrary rotations



flat surface parts



Geometry of the quantum state space: the qutrit

► ...then try: a qutrit $d = 3$

►
$$\rho = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01}^* & a_{11} & a_{12} \\ a_{02}^* & a_{12}^* & a_{22} \end{bmatrix}$$

► 8 real parameters

→ look for 3d model of 8d object

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Geometry of the quantum state space: the qutrit

What are the "basis vectors" we use?

$$d = 2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad Z$$

$$d = 3$$

$$\sqrt{\frac{3}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad Z_1, Z_2$$

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Geometry of the quantum state space: the qutrit

$$\rho = \frac{1}{3} [\mathbb{1}_3 + z_1 Z_1 + z_2 Z_2 + \\ + x_1 X_1 + x_2 X_2 + x_3 X_3 \\ + y_1 Y_1 + y_2 Y_2 + y_3 Y_3]$$

Geometry of the quantum state space: the qutrit

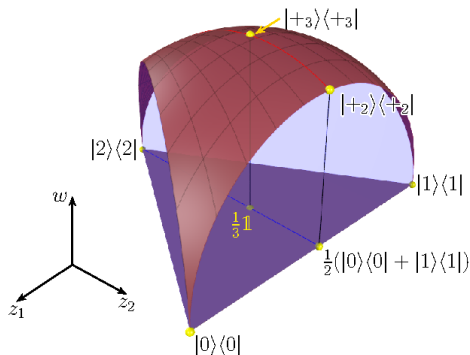
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only one coordinate left to choose: z_1 , z_2 , and

$$w = \sqrt{\sum_j (x_j^2 + y_j^2)}$$

Geometry of the quantum state space: the qutrit

coordinates z_1, z_2 , and $w = \sqrt{\sum_j (x_j^2 + y_j^2)}$

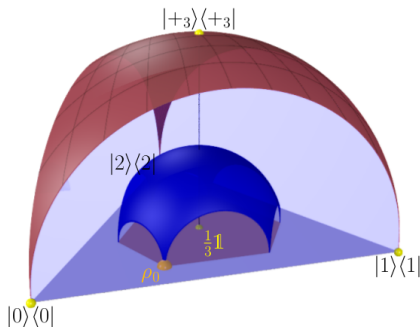


- ▶ convex ✓
- ▶ simply connected ✓
- ▶ pure states @surface, max radius $\sqrt{2}$ ✓
- ▶ pure states simply connected ✓
- ▶ neither polytope nor smooth object ✓

⋮

Geometry of the quantum state space: the qutrit

application (I) unitary orbits of mixed states



- ▶ BIRKHOFF polytope ✓
- ▶ HORN's lemma ✓

Geometry of the quantum state space: the qutrit

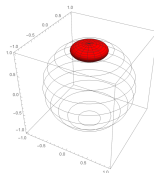
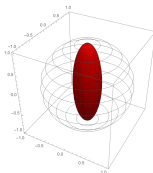
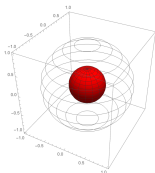
application (II) action of paradigmatic quantum channels

depolarizing

phase-damping

amplitude-damping

$d = 2$



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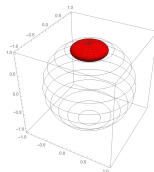
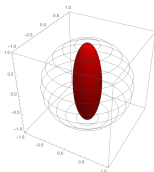
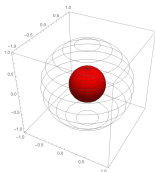
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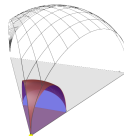
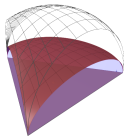
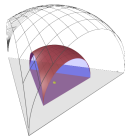
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$d = 2$



$d = 3$



Intermediate conclusions

- ▶ Bloch representation has built-in Hilbert-Schmidt (= Euclidean) geometry
- ▶ provides appropriate geometric intuition for non-trivial algebra
- ▶ method: find effective parameters for the characterization of high-dimensional objects

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Correlation constraints

Correlation constraints: composite systems

we want to study **correlations** between many **quantum** particles

→ recall description of **composite** systems

$$|\psi\rangle \in \mathcal{H} = \mathcal{H}_{d_1}^{(1)} \otimes \mathcal{H}_{d_2}^{(2)} \otimes \dots \otimes \mathcal{H}_{d_N}^{(N)}$$

→ again, we are also interested in the **mixed** states

$$\rho \in \mathcal{B}(\mathcal{H})$$

→ for example, consider **$N = 2$** $\mathcal{H}_{AB} = \mathcal{H}_d^{(A)} \otimes \mathcal{H}_d^{(B)}$

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Correlation constraints: Schmidt decomposition

→ recall Schmidt decomposition

pure state $|\psi_{AB}\rangle \in \mathcal{H}_{AB}$, reduced states

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$$\iff \rho_B = U \rho_A U^\dagger$$

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how pure can ρ_{AB} be if $\rho_B \neq U \rho_A U^\dagger$?

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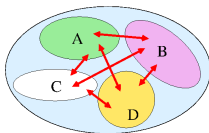
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Correlation constraints and many-body physics



global requirements



local properties

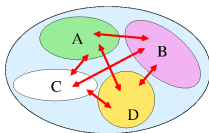
- ▶ symmetries
- ▶ boundary conditions, ...
- ▶ $\mathcal{H} |\Psi\rangle = E |\Psi\rangle$

- ▶ $\rho_A, \rho_B, \rho_C, \dots$
- ▶ $\rho_{AB}, \rho_{AC}, \dots$
- ▶ observables (specific heat, conductivity, susceptibilities, ...) determined by local states

→ which global state Ψ (or ρ) is compatible with local states?

→ “quantum marginal problem” aka “ N representability”

Correlation constraints and many-body physics



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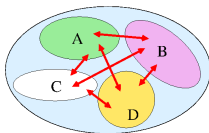
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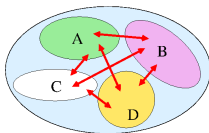
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Correlation constraints: the quantum marginal problem

- ▶ back to the **simplest** problem: $N = 2$, $d = 2$ – **two qubits**
- ▶ was solved 20 years ago: Bravyi (2004)

marginals compatible with a global state with spectrum

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ iff

$$\lambda^A \geq \lambda_3 + \lambda_4$$

$$\lambda^A + \lambda^B \geq \lambda_2 + \lambda_3 + 2\lambda_4$$

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well ...

Correlation constraints: the quantum marginal problem

- ▶ back to the **simplest** problem: $N = 2$, $d = 2$ – **two qubits**
- ▶ was solved 20 years ago: **Bravyi (2004)**

marginals compatible with a global state with spectrum

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ iff

$$\lambda^A \geq \lambda_3 + \lambda_4$$

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gets worse for more complex systems

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$$\begin{aligned}\lambda^A &\geq \lambda_3 + \lambda_4 \\ \lambda^A + \lambda^B &\geq \lambda_2 + \lambda_3 + 2\lambda_4 \\ \lambda^B - \lambda^A &\leq \lambda_1 - \lambda_3 \\ \lambda^B - \lambda^A &\leq \lambda_2 - \lambda_4\end{aligned}$$

Klyachko (2004):

2×3	41 ineqs.
3×3	197 ineqs. + permut.

Bloch representation for two qudits

orthogonal basis $\{\mu_j\}$, $\mu_0 \equiv \mathbb{1}$, $\text{Tr } \mu_j = 0$ ($j \geq 1$), $\text{Tr } (\mu_j \mu_k^\dagger) = d \delta_{jk}$

$$\rho_{AB} = \frac{1}{d^2} \left[\mathbb{1} \otimes \mathbb{1} + \sum_{j=1}^{d^2-1} a_j \mu_j \otimes \mathbb{1} + \sum_{j=1}^{d^2-1} b_k \mathbb{1} \otimes \mu_k + \sum_{lm} t_{lm} \mu_l \otimes \mu_m \right]$$

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Bloch vector lengths

$$\|\mathbf{a}\|^2 = \sum_j |a_j|^2 = d \operatorname{Tr}(\rho_A^2) - 1$$

$$\|\mathbf{b}\|^2 = \sum_k |b_k|^2 = d \operatorname{Tr}(\rho_B^2) - 1$$

correlation strength

$$\|\mathbf{T}_{AB}\|^2 = \sum_{lm} |t_{lm}|^2$$

Correlation constraints: bounding two-party correlations

- ▶ how **pure** can **global** state get if **local** purities are different?
- ▶ important result: **tight** bound for maximal **global** eigenvalue

$$\lambda_{\max} \leq 1 - \frac{1}{\sqrt{2d}} | \| \mathbf{a} \| - \| \mathbf{b} \| |$$

- ▶ this immediately gives a **purity** bound

$$d \operatorname{Tr}(\rho_{AB}^2) \leq d - \sqrt{2d} | \| \mathbf{a} \| - \| \mathbf{b} \| | + | \| \mathbf{a} \| - \| \mathbf{b} \| |^2$$

→ can be saturated for all d

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Correlation constraints: tripartite 'monogamy' relation

- ▶ consider **pure** tripartite state $|\psi_{ABC}\rangle$
- ▶ since $\text{Tr}(\rho_C^2) = \text{Tr}(\rho_{AB}^2) \longrightarrow$ **monogamy**-type relation

$$\|\mathbf{c}\|^2 \leq \frac{d}{2} - 1 + \left[\sqrt{\frac{d}{2}} - \left| \|\mathbf{a}\| - \|\mathbf{b}\| \right| \right]^2$$

- ▶ reminiscent of Coffman-Kundu-Wootters monogamy relation
for **three qubits** Coffman et al PRA (2000)

$$2[1 - \text{Tr}(\rho_A^2)] \geq C^2(\rho_{AB}) + C^2(\rho_{AC})$$

\longrightarrow now $d \times d \times d$

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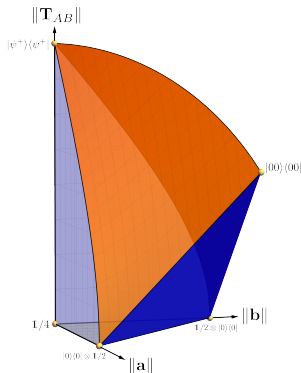
Correlation constraints: Two-qubit Bloch ball

- ▶ another bound for correlation strength

Morelli et al. Lin Alg Appl (2020)

$$\|\mathbf{T}_{AB}\| \geq \sqrt{d-1} (\|\mathbf{a}\| + \|\mathbf{b}\|) - (d-1)$$

- ▶ bounds give rise to Bloch-ball type picture for two qubits



Correlation constraints: Two-qubit Bloch ball

- ▶ again: lots of **physics information** in this picture
- ▶ in particular: characterize **entangled** vs. **separable** regions

entangled

$$\|\mathbf{T}_{AB}\|^2 > 1 - \left| \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2 \right|$$

separable ball

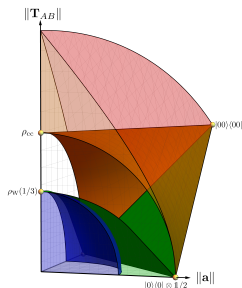
$$\|\mathbf{T}_{AB}\|^2 \leq \frac{1}{3} - \|\mathbf{a}\|^2 - \|\mathbf{b}\|^2$$

separable region

$$\|\mathbf{T}_{AB}\| \leq \begin{cases} \sqrt{\frac{1}{6} (2 - 3(\|\mathbf{a}\| + \|\mathbf{b}\|)^2)} & \|\mathbf{a}\| + \|\mathbf{b}\| \leq \frac{2}{\sqrt{3}} \\ 1 - \|\mathbf{a}\| - \|\mathbf{b}\| & \frac{2}{\sqrt{3}} < \|\mathbf{a}\| + \|\mathbf{b}\| \\ \|\mathbf{a}\| + \|\mathbf{b}\| - 1 & 1 < \|\mathbf{a}\| + \|\mathbf{b}\| \end{cases}$$

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Conclusions

- ▶ Bloch representation of quantum states facilitates discovery and geometric visualization of algebraic relations
- ▶ derived new correlation constraints for two-party systems
- ▶ method: study coarse-grained version of quantum marginal problem
- ▶ hopefully new geometric intuition for quantum correlations for systems with more than two parties
- ▶ method to visualize higher-dimensional state spaces
 - necessary loss of features
 - retain & emphasize relevant features

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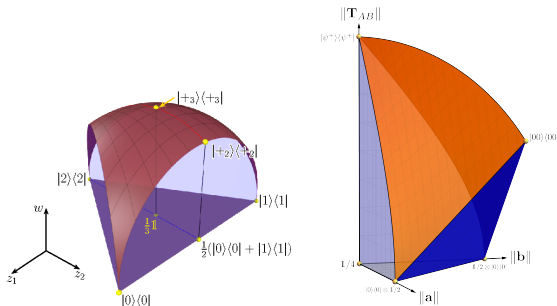
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Thank you

Eltschka, Huber, Morelli, Siewert, Quantum **5**, 485 (2021)

Morelli, Eltschka, Huber, Siewert, Phys. Rev. A **109**, 012423 (2024).