Monogamy of highly symmetric states

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We introduce the notion of **graph-extendability**

A bipartite symmetric quantum state $\rho = \bullet \longrightarrow \bullet$ is $G = \bullet \longrightarrow \bullet$-extendible if there exists a global state $\sigma = \bullet \longrightarrow \bullet$ on $G$ such that for all edges $e = \bullet \longrightarrow \bullet \in G$, the reduced state $\sigma_e = \bullet \longrightarrow \bullet$ is equal to $\rho$.

For given $d$ and $n$, which highly symmetric states (such as Werner, Brauer, and isotropic states) on $\mathbb{C}^d \otimes \mathbb{C}^d$ are $G$-extendible?
Separability of quantum states
Quantum entanglement

- **Quantum states** are unit trace positive semidefinite matrices [NC10, Wat18]:
  \[ \rho \in \mathcal{M}_d^{sa}(\mathbb{C}) \] such that \( \rho \geq 0, \) \( \text{Tr} \rho = 1. \)

- A bipartite quantum state \( \rho \in \mathcal{M}_d^{sa}(\mathbb{C}) \otimes \mathcal{M}_d^{sa}(\mathbb{C}) \) is separable if it can be decomposed as a convex combination of product quantum states:
  \[ \rho = \sum_i \alpha_i \otimes \beta_i \quad \text{with} \quad \alpha_i, \beta_i \geq 0 \]

- A pure (i.e. unit rank) state \( \rho = |x\rangle \langle x| \) is separable iff it is product:
  \[ |x\rangle = |a\rangle \otimes |b\rangle \]

- The maximally entangled state
  \[ \omega := \frac{1}{d} \sum_{i,j=1}^{d} |ii\rangle \langle jj| = \frac{1}{d} \]

- Deciding whether a given state \( \rho \) is separable is an NP-hard problem [Gur03].
Detecting entanglement

- There exist various criteria to detect entanglement or separability

\[ \rho \in \text{SEP} \implies \rho^\Gamma := [\text{id} \otimes \text{transp}](\rho) = \sum_i \alpha_i \otimes \beta_i^\top \geq 0 \]

\[ \|\rho - \frac{I}{d} \otimes \frac{I}{d}\|_2 \leq \frac{1}{d\sqrt{d^2 - 1}} \implies \rho \in \text{SEP} \]

- The DPS hierarchy [DPS02, DPS04] can certify entanglement using a sequence of semidefinite programs

\[ \text{EXT}_k := \left\{ \rho_{AB} : \exists \sigma_{AB_1B_2\cdots B_k} \geq 0 \text{ s.t. } \sigma_{AB_i} = \rho_{AB} \quad \forall i \in [k] \right\} \]

all states = \text{EXT}_1 \supseteq \text{EXT}_2 \supseteq \cdots \supseteq \text{EXT}_k \supseteq \cdots \supseteq \text{EXT}_\infty = \text{SEP}

- Easy direction: if \( \rho \) is separable, \( \rho = \sum_i \alpha_i \otimes \beta_i \) \( \leadsto \) take \( \sigma = \sum_i \alpha_i \otimes \beta_i^{\otimes k} \)

- Quantitative version [CKMR07]:

\[ \rho \in \text{EXT}_k \implies \min_{\sigma \in \text{SEP}} \|\rho - \sigma\|_1 \leq \frac{4d^2}{k} \]
Graph extendability
Monogamy of entanglement & exchangeability

Monogamy is a fundamental property of quantum entanglement \([KW04]\). Informally, given 3 quantum parties Alice, Bob, and Charlie:

Alice cannot be maximally entangled with Bob and Charlie

\[ \nexists \rho_{ABC} \text{ s.t. } \rho_{AB} = \omega \text{ and } \rho_{AC} = \omega \]

Actually, we have more: given a quantum state \(\rho_{ABC}\),

\[ \rho_{AB} = \omega \implies \rho_{ABC} = \omega_{AB} \otimes \rho_{C} \]

A bipartite symmetric state \(\rho\) is called \(n\)-exchangeable if there exists a \(n\)-partite symmetric state \(\sigma\) such that \(\rho = \text{Tr}_{n-2} \sigma\)

The quantum de Finetti theorem \([HM76, CFS02, KR05, CKMR07]\): a bipartite state \(\rho\) is \(n\)-exchangable for every \(n\) iff

\[ \rho = \sum_i \alpha_i \otimes \alpha_i \]
A bipartite symmetric quantum state $\rho = \bullet \cdots \bullet$ is $G = \bullet \cdots \bullet$-extendible if there exists a global state $\sigma = \bullet \cdots \bullet$ on $G$ such that for all edges $e = \bullet \cdots \bullet \in G$, the reduced state $\sigma_e = \bullet \cdots \bullet$ is equal to $\rho$.

This notion generalizes the two previous ones:

- $n$-extendibility: $\exists \sigma_{AB_1B_2 \cdots B_n}$ s.t. $\sigma_{AB_i} = \rho_{AB} \iff K_{1,n}$-extendibility
- $n$-exchangeability: $\exists \sigma_{A_1A_2 \cdots A_n}$ s.t. $\sigma_{Ai Aj} = \rho_{AB} \iff K_n$-extendibility

The property above can be formulated as a semidefinite program.
Main result

- Consider isotropic states

\[ \rho_I(d) := p\omega + (1 - p)\frac{l}{d} \otimes \frac{l}{d} \]

The largest \( p \) for which the isotropic state \( \rho_I(d) \) is \( K_n \)-extendible is:

\[
p_I(n, d) = \begin{cases} 
\frac{1}{n-1+n \mod 2} & \text{if } d > n \text{ or either } d \text{ or } n \text{ is even} \\
\min \left\{ \frac{2d+1}{2dn+1}, \frac{1}{n-1} \right\} & \text{if } n \geq d \text{ and both } d \text{ and } n \text{ are odd}
\end{cases}
\]

- Compare with optimal \( p \) for \( K_{1,n} \)-extensibility (\( \iff \) quantum cloning [KW99])

\[
p_I(K_{1,n}, d) = \frac{d + n}{n(d + 1)}
\]

- Similar results for Werner states and for Brauer states

\[
\rho_W(d) := p\frac{\Pi_{\Box}}{\text{Tr } \Pi_{\Box}} + (1-p)\frac{\Pi_{\Box}}{\text{Tr } \Pi_{\Box}}, \quad \rho_B(d) := p\omega + q\frac{\Pi_{\Box}}{\text{Tr } \Pi_{\Box}} + (1-p-q)\left[ \frac{\Pi_{\Box}}{\text{Tr } \Pi_{\Box}} - \omega \right]
\]

\[
\Pi_{\Box} := \frac{l - F}{2}, \quad \Pi_{\Box} := \frac{l + F}{2}, \quad F := \sum_{i,j=1}^{d} |ij\rangle\langle ji| = \begin{array}{c}
i \bullet \\
|j\rangle
\end{array}
\]
Proof techniques
A perfect matching on a graph $G = (V, E)$ is a subset of edges from $E$, such that every vertex in $V$ is contained in exactly one of those edges.

There are $(2n - 1)!!$ perfect matchings on $K_{2n}$, and if $e$ is an edge on $K_{2n}$, then there are $(2n - 3)!!$ perfect matchings on $K_{2n}$ containing $e$.

Let $E_1, \ldots, E_{(2n-1)!!}$ be all the perfect matchings on $K_{2n}$, and for each perfect matching $E_k$, define the quantum state $\rho^{(k)}$ on $K_{2n}$ by

$$\rho^{(k)} := \bigotimes_{e \in E_k} \omega_e$$

and

$$\rho := \frac{1}{(2n - 1)!!} \sum_{k=1}^{(2n-1)!!} \rho^{(k)}$$

For any edge $e \in K_{2n}$, we have

$$\rho_e = \frac{1}{2n - 1} \omega + \left(1 - \frac{1}{2n - 1}\right) \frac{l}{d^2} \quad \implies \quad p_l(2n, d) \geq \frac{1}{2n - 1}.$$
UB for Werner states: symmetry

- Consider the simpler **Werner states** $p \cdot \Pi_\oplus / \text{Tr} \Pi_\oplus + (1 - p) \cdot \Pi_\otimes / \text{Tr} \Pi_\otimes$.
- We want to solve, for a graph $G$ with $n$ vertices

  $$p_W(G, d) := \max_{\rho, p} p \text{ s.t. } \text{Tr}[\Pi_e \rho] = p \quad \forall e \in E, \quad \text{Tr} \rho = 1, \quad \rho \geq 0$$

  where $\Pi_e$ acts like $\Pi_\oplus$ on the tensor factors associated to the vertices of $e$ and as the identity elsewhere; $\rho$ is a state on $(\mathbb{C}^d)^\otimes n$.
- Given an optimal $\rho$, we can assume wlog that it has symmetry:

  $$\forall U \in \mathcal{U}(d) \quad U^\otimes n \rho (U^\otimes n)^* = \rho$$

  $$\forall \pi \in S_n \quad \pi.\rho = \rho$$

  with $\pi.A_1 \otimes A_2 \otimes \cdots \otimes A_n := A_{\pi^{-1}(1)} \otimes A_{\pi^{-1}(2)} \otimes \cdots \otimes A_{\pi^{-1}(n)}$.
- By **Schur–Weyl duality** [Aub18, GO22, Bra37], we have

  $$\rho = \sum_{\lambda \vdash n} \beta_\lambda \rho_\lambda$$

  where $\beta_\lambda$ is a probability distribution $\{\beta_\lambda : \lambda \vdash n\}$ and $\rho_\lambda$ are the normalized isotypical projectors.
The groups $\mathcal{U}(d)$ and $\mathfrak{S}_n$ act on $(\mathbb{C}^d)^\otimes n$:

$$U. |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle := U |x_1\rangle \otimes U |x_2\rangle \otimes \cdots \otimes U |x_n\rangle$$

$$\pi. |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle := |x_{\pi^{-1}(1)}\rangle \otimes |x_{\pi^{-1}(2)}\rangle \otimes \cdots \otimes |x_{\pi^{-1}(n)}\rangle$$

**Schur–Weyl duality:** the algebras spanned by the matrices associated to these actions are mutual commutants of each other. Equivalently, the space $(\mathbb{C}^d)^\otimes n$ decomposes into isotypic sectors consisting of tensor products of irreps:

$$(\mathbb{C}^d)^\otimes n \simeq \bigoplus_{\lambda \vdash n \atop l(\lambda) \leq d} V^{(\mathcal{U})}_{\lambda} \otimes V^{(\mathfrak{S})}_{\lambda}.$$ 

Since an optimal $\rho$ commutes is invariant w.r.t. both actions, it must act like the identity on each tensor factor, for every term of the direct sum.

We have [CKMR07] $\text{Tr}_{[n]!} \rho_{\lambda} = \alpha_{\mathcal{U}}^\lambda \epsilon_{\mathcal{U}} + \alpha_{\mathfrak{S}}^\lambda \epsilon_{\mathfrak{S}}$, where

$$\alpha_{\mathcal{U}}^\lambda = \frac{s^*_\bigcirc(\lambda)}{m_d(\bigcirc)n(n - 1)},$$

where $s^*_\mu(\lambda)$ is the shifted Schur function [OO97] and $m_d(\lambda) = \dim V^{(\mathcal{U})}_{\lambda}$. 
Optimization

- Plugging the partial trace expression into the formula for $p_W$, in the case $G = K_n$, we obtain

$$p_W(\rho) = \sum_{\lambda \vdash n \atop l(\lambda) \leq d} \beta_\lambda \frac{d(\lambda)s^*_\lambda}{n(n - 1)}$$

- Since $\beta_\lambda$ are probability weights, we need to maximize the expression above over partitions $\lambda \vdash n$ with $l(\lambda) \leq d$.

- Using a formula for the shifted Schur function [OO97] we obtain

$$p_W(n, d) = \max_{\lambda \vdash n \atop l(\lambda) \leq d} \sum_{d \geq i > j \geq 1} \lambda_i(\lambda_j + 1)$$

- The optimal $\lambda$ is the tallest approximate rectangle possible, and gives

$$p_W(n, d) = \frac{d - 1}{2d} \cdot \frac{(n + k + d)(n - k)}{n(n - 1)} + \frac{k(k - 1)}{n(n - 1)}$$

where $k = n \mod d$

- Clearly, if $d \geq n$, $p_W = 1$ is achieved by $\lambda = 1^n$, and $\rho$ is the normalized projection on the anti-symmetric subspace $\Lambda^n(\mathbb{C}^d) \subseteq (\mathbb{C}^d)^{\otimes n}$. 
Take home slide
Monogamy of highly symmetric states

A bipartite symmetric quantum state $\rho = \bullet----\bullet$ is $G = \bullet----\bullet$-extendible if there exists a global state $\sigma = \bullet----\bullet$ on $G$ such that for all edges $e = \bullet----\bullet \in G$, the reduced state $\sigma_e = \bullet----\bullet$ is equal to $\rho$.

- For $G = K_{1,n}$ or $G = K_{m,n}$, we obtain the standard DPS hierarchy.
- For given $d$ and $n$, we compute the value noise parameter $p$ for which highly symmetric states (Werner, Brauer, isotropic) on $\mathbb{C}^d \otimes \mathbb{C}^d$ are $K_n$-extendible

$$\rho_I = p \cdot \frac{1}{d} \sum_{ij} |ii\rangle\langle jj| + (1 - p) \cdot \frac{l}{d} \otimes \frac{l}{d}$$

- $G$-extendibility of isotropic states for all $n$: separability vs. $K_n$-extendibility

<table>
<thead>
<tr>
<th>Graph family</th>
<th>Form of $\infty$-extendible states</th>
<th>Range of $p$</th>
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<tbody>
<tr>
<td>$K_{1,n}$ or $K_{m,n}$</td>
<td>$\rho = \sum_i \alpha_i \otimes \beta_i$</td>
<td>$[\frac{-1}{d^2-1}, \frac{1}{d+1}]$</td>
</tr>
<tr>
<td>$K_n$</td>
<td>$\rho = \sum_i \alpha_i \otimes \alpha_i$</td>
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