

Monogamy of highly symmetric states

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Outline

We introduce the notion of **graph-extendability**

A bipartite symmetric quantum state $\rho = \bullet \text{---} \bullet$ is $G =$  -extendible if

there exists a global state $\sigma =$  on G such that

for all edges $e =$  $\in G$, the reduced state $\sigma_e =$  is equal to ρ .

For given d and n , which *highly symmetric states* (such as Werner, Brauer, and isotropic states) on $\mathbb{C}^d \otimes \mathbb{C}^d$ are G -extendible?

Separability of quantum states

Quantum entanglement

- **Quantum states** are unit trace positive semidefinite matrices [NC10, Wat18]:

$$\rho \in \mathcal{M}_d^{\text{sa}}(\mathbb{C}) \text{ such that } \rho \geq 0, \text{Tr } \rho = 1.$$

- A bipartite quantum state $\rho \in \mathcal{M}_d^{\text{sa}}(\mathbb{C}) \otimes \mathcal{M}_d^{\text{sa}}(\mathbb{C})$ is **separable** if it can be decomposed as a convex combination of product quantum states:

$$\rho = \sum_i \alpha_i \otimes \beta_i \quad \text{with } \alpha_i, \beta_i \geq 0$$

- A pure (i.e. unit rank) state $\rho = |x\rangle\langle x|$ is separable iff it is product:

$$|x\rangle = |a\rangle \otimes |b\rangle$$

- The **maximally entangled state**

$$\omega := \frac{1}{d} \sum_{i,j=1}^d |ii\rangle\langle jj| = \frac{1}{d} \begin{array}{c} \bullet^i \quad \bullet^j \\ \text{) } \quad \text{(} \\ \bullet^i \quad \bullet^j \end{array}$$

- Deciding whether a given state ρ is separable is an NP-hard problem [Gur03].

Detecting entanglement

- There exist various **criteria** to detect entanglement or separability

$$\rho \in \text{SEP} \implies \rho^\Gamma := [\text{id} \otimes \text{transp}](\rho) = \sum_i \alpha_i \otimes \beta_i^\top \geq 0$$

$$\left\| \rho - \frac{I}{d} \otimes \frac{I}{d} \right\|_2 \leq \frac{1}{d\sqrt{d^2 - 1}} \implies \rho \in \text{SEP}$$

- The **DPS hierarchy** [DPS02, DPS04] can certify entanglement using a sequence of semidefinite programs

$$\text{EXT}_k := \left\{ \rho_{AB} : \exists \sigma_{AB_1 B_2 \dots B_k} \geq 0 \text{ s.t. } \sigma_{AB_i} = \rho_{AB} \quad \forall i \in [k] \right\}$$

$$\text{all states} = \text{EXT}_1 \supseteq \text{EXT}_2 \supseteq \dots \supseteq \text{EXT}_k \supseteq \dots \supseteq \text{EXT}_\infty = \text{SEP}$$

- Easy direction: if ρ is separable, $\rho = \sum_i \alpha_i \otimes \beta_i \rightsquigarrow$ take $\sigma = \sum_i \alpha_i \otimes \beta_i^{\otimes k}$
- Quantitative version [CKMR07]:

$$\rho \in \text{EXT}_k \implies \min_{\sigma \in \text{SEP}} \|\rho - \sigma\|_1 \leq \frac{4d^2}{k}$$

Graph extendability

Monogamy of entanglement & exchangeability

- **Monogamy** is a fundamental property of quantum entanglement [KW04]. Informally, given 3 quantum parties Alice, Bob, and Charlie:

Alice cannot be maximally entangled with Bob **and** Charlie

$$\nexists \rho_{ABC} \quad \text{s.t.} \quad \rho_{AB} = \omega \quad \text{and} \quad \rho_{AC} = \omega$$


- Actually, we have more: given a quantum state ρ_{ABC} ,


$$\rho_{AB} = \omega \implies \rho_{ABC} = \omega_{AB} \otimes \rho_C$$



- A bipartite symmetric state ρ is called **n -exchangeable** if there exists a n -partite symmetric state σ such that $\rho = \text{Tr}_{n-2} \sigma$
- The **quantum de Finetti theorem** [HM76, CFS02, KR05, CKMR07]: a bipartite state ρ is n -exchangeable for every n iff

$$\rho = \sum_i \alpha_i \otimes \alpha_i$$

Main definition

A bipartite symmetric quantum state $\rho = \bullet \text{---} \bullet$ is $G =$  -extendible if

there exists a global state $\sigma =$  on G such that

for all edges $e =$  $\in G$, the reduced state $\sigma_e =$  is equal to ρ .

- This notion generalizes the two previous ones:

n -extendibility : $\exists \sigma_{AB_1 B_2 \dots B_n}$ s.t. $\sigma_{AB_i} = \rho_{AB} \iff K_{1,n}$ -extendibility

n -exchangeability : $\exists \sigma_{A_1 A_2 \dots A_n}$ s.t. $\sigma_{A_i A_j} = \rho_{AB} \iff K_n$ -extendibility

- The property above can be formulated as a **semidefinite program**.

Main result

- Consider isotropic states

$$\rho_I(d) := p\omega + (1-p)\frac{I}{d} \otimes \frac{I}{d}$$

The largest p for which the isotropic state $\rho_I(d)$ is K_n -extendible is:


$$p_I(n, d) = \begin{cases} \frac{1}{n-1+n \bmod 2} & \text{if } d > n \text{ or either } d \text{ or } n \text{ is even} \\ \min \left\{ \frac{2d+1}{2dn+1}, \frac{1}{n-1} \right\} & \text{if } n \geq d \text{ and both } d \text{ and } n \text{ are odd} \end{cases}$$

- Compare with optimal p for $K_{1,n}$ -extendibility (\iff quantum cloning [KW99])

$$p_I(K_{1,n}, d) = \frac{d+n}{n(d+1)}$$

- Similar results for Werner states and for Brauer states

$$\rho_W(d) := p \frac{\Pi_{\square}}{\text{Tr} \Pi_{\square}} + (1-p) \frac{\Pi_{\square\square}}{\text{Tr} \Pi_{\square\square}}, \quad \rho_B(d) := p\omega + q \frac{\Pi_{\square}}{\text{Tr} \Pi_{\square}} + (1-p-q) \left[\frac{\Pi_{\square\square}}{\text{Tr} \Pi_{\square\square}} - \omega \right]$$

$$\Pi_{\square} := \frac{I - F}{2}, \quad \Pi_{\square\square} := \frac{I + F}{2}, \quad F := \sum_{i,j=1}^d |ij\rangle\langle ji| =$$


Proof techniques

LB for isotropic states: perfect matchings

- A **perfect matching** on a graph $G = (V, E)$ is a subset of edges from E , such that every vertex in V is contained in exactly one of those edges.
- There are $(2n - 1)!!$ perfect matchings on K_{2n} , and if e is an edge on K_{2n} , then there are $(2n - 3)!!$ perfect matchings on K_{2n} containing e .



- Let $E_1, \dots, E_{(2n-1)!!}$ be all the perfect matchings on K_{2n} , and for each perfect matching E_k , define the quantum state $\rho^{(k)}$ on K_{2n} by

$$\rho^{(k)} := \bigotimes_{e \in E_k} \omega_e \quad \text{and} \quad \rho := \frac{1}{(2n-1)!!} \sum_{k=1}^{(2n-1)!!} \rho^{(k)}$$

- For any edge $e \in K_{2n}$, we have

$$\rho_e = \frac{1}{2n-1} \omega + \left(1 - \frac{1}{2n-1}\right) \frac{I}{d^2} \quad \implies \quad p_I(2n, d) \geq \frac{1}{2n-1}.$$

UB for Werner states: symmetry

- Consider the simpler **Werner states** $\rho \cdot \Pi_{\boxplus} / \text{Tr} \Pi_{\boxplus} + (1 - \rho) \cdot \Pi_{\boxminus} / \text{Tr} \Pi_{\boxminus}$.
- We want to solve, for a graph G with n vertices

$$\rho_W(G, d) := \max_{\rho: \rho} \rho \quad \text{s.t.} \quad \text{Tr}[\Pi_e \rho] = \rho \quad \forall e \in E, \quad \text{Tr} \rho = 1, \quad \rho \geq 0$$

where Π_e acts like Π_{\boxplus} on the tensor factors associated to the vertices of e and as the identity elsewhere; ρ is a state on $(\mathbb{C}^d)^{\otimes n}$.

- Given an optimal ρ , we can assume wlog that it has **symmetry**:

$$\begin{aligned} \forall U \in \mathcal{U}(d) \quad U^{\otimes n} \rho (U^{\otimes n})^* &= \rho \\ \forall \pi \in \mathfrak{S}_n \quad \pi \cdot \rho &= \rho \end{aligned}$$

with $\pi \cdot A_1 \otimes A_2 \otimes \cdots \otimes A_n := A_{\pi^{-1}(1)} \otimes A_{\pi^{-1}(2)} \otimes \cdots \otimes A_{\pi^{-1}(n)}$.

- By **Schur–Weyl duality** [Aub18, GO22, Bra37], we have

$$\rho = \sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} \beta_\lambda \rho_\lambda$$

where β_λ is a probability distribution $\{\beta_\lambda : \lambda \vdash n\}$ and ρ_λ are the normalized **isotypical projectors**.

Representation theory

- The groups $U(d)$ and \mathfrak{S}_n act on $(\mathbb{C}^d)^{\otimes n}$:

$$U \cdot |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle := U|x_1\rangle \otimes U|x_2\rangle \otimes \cdots \otimes U|x_n\rangle$$

$$\pi \cdot |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle := |x_{\pi^{-1}(1)}\rangle \otimes |x_{\pi^{-1}(2)}\rangle \otimes \cdots \otimes |x_{\pi^{-1}(n)}\rangle$$

- Schur–Weyl duality**: the algebras spanned by the matrices associated to these actions are mutual commutants of each other. Equivalently, the space $(\mathbb{C}^d)^{\otimes n}$ decomposes into isotypic sectors consisting of tensor products of irreps:

$$(\mathbb{C}^d)^{\otimes n} \simeq \bigoplus_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} V_\lambda^{(U)} \otimes V_\lambda^{(\mathfrak{S})}.$$

- Since an optimal ρ commutes is invariant w.r.t. both actions, it must act like the identity on each tensor factor, for every term of the direct sum.
- We have [CKMR07] $\text{Tr}_{[n] \setminus e} \rho_\lambda = \alpha_{\boxplus}^\lambda \varepsilon_{\boxplus} + \alpha_{\boxminus}^\lambda \varepsilon_{\boxminus}$, where

$$\alpha_{\boxplus}^\lambda = \frac{s_{\boxplus}^*(\lambda)}{m_d(\boxplus)n(n-1)},$$

where $s_{\mu}^*(\lambda)$ is the **shifted Schur function** [OO97] and $m_d(\lambda) = \dim V_\lambda^{(U)}$.

Optimization

- Plugging the partial trace expression into the formula for p_W , in the case $G = K_n$, we obtain

$$p_W(\rho) = \sum_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} \beta_\lambda \frac{d(\boxplus) s_{\boxplus}^*(\lambda)}{n(n-1)}$$

- Since β_λ are probability weights, we need to maximize the expression above over partitions $\lambda \vdash n$ with $l(\lambda) \leq d$.
- Using a formula for the shifted Schur function [OO97] we obtain

$$p_W(n, d) = \max_{\substack{\lambda \vdash n \\ l(\lambda) \leq d}} \frac{\sum_{d \geq i > j \geq 1} \lambda_i (\lambda_j + 1)}{n(n-1)}$$


- The optimal λ is the **tallest approximate rectangle** possible, and gives


$$p_W(n, d) = \frac{d-1}{2d} \cdot \frac{(n+k+d)(n-k)}{n(n-1)} + \frac{k(k-1)}{n(n-1)} \quad \text{where } k = n \bmod d$$



- Clearly, if $d \geq n$, $p_W = 1$ is achieved by $\lambda = 1^n$, and ρ is the normalized projection on the anti-symmetric subspace $\Lambda^n(\mathbb{C}^d) \subseteq (\mathbb{C}^d)^{\otimes n}$.

Take home slide

Monogamy of highly symmetric states

A bipartite symmetric quantum state $\rho = \bullet \text{---} \bullet$ is $G =$ -extendible if

there exists a global state $\sigma =$  on G such that

for all edges $e =$  $\in G$, the reduced state $\sigma_e =$  is equal to ρ .

- For $G = K_{1,n}$ or $G = K_{m,n}$, we obtain the standard **DPS hierarchy**.
- For given d and n , we compute the value noise parameter p for which **highly symmetric states** (Werner, Brauer, isotropic) on $\mathbb{C}^d \otimes \mathbb{C}^d$ are **K_n -extendible**

$$\rho_I = p \cdot \frac{1}{d} \sum_{ij} |ii\rangle\langle jj| + (1-p) \cdot \frac{1}{d} \otimes \frac{1}{d}$$

- G -extendibility of **isotropic states** for all n : separability vs. K_n -extendibility

Graph family	Form of ∞ -extendible states	Range of p
$K_{1,n}$ or $K_{m,n}$	$\rho = \sum_i \alpha_i \otimes \beta_i$	$\left[\frac{-1}{d^2-1}, \frac{1}{d+1} \right]$
K_n	$\rho = \sum_i \alpha_i \otimes \alpha_i$	$\{0\}$

References

- [Aub18] Guillaume Aubrun.
Schur-weyl duality, 2018.
- [Bra37] Richard Brauer.
On algebras which are connected with the semisimple continuous groups.
Annals of Mathematics, pages 857–872, 1937.
- [CFS02] Carlton M Caves, Christopher A Fuchs, and Rüdiger Schack.
Unknown quantum states: the quantum de finetti representation.
Journal of Mathematical Physics, 43(9):4537–4559, 2002.
- [CKMR07] Matthias Christandl, Robert König, Graeme Mitchison, and Renato Renner.
One-and-a-half quantum de finetti theorems.
Communications in Mathematical Physics, 273(2):473–498, 2007.
- [DPS02] Andrew C Doherty, Pablo A Parrilo, and Federico M Spedalieri.
Distinguishing separable and entangled states.
Physical Review Letters, 88(18):187904, 2002.
- [DPS04] Andrew C Doherty, Pablo A Parrilo, and Federico M Spedalieri.
Complete family of separability criteria.
Physical Review A, 69(2):022308, 2004.
- [GO22] Dmitry Grinko and Maris Ozols.
Linear programming with unitary-equivariant constraints.
2022.
- [Gur03] Leonid Gurvits.
Classical deterministic complexity of edmonds' problem and quantum entanglement.
In *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*, pages 10–19, 2003.
- [HM76] Robin L Hudson and Graham R Moody.
Locally normal symmetric states and an analogue of de finetti's theorem.
Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, 33(4):343–351, 1976.
- [KR05] Robert König and Renato Renner.
A de finetti representation for finite symmetric quantum states.
Journal of Mathematical physics, 46(12), 2005.
- [KW99] Michael Keyl and Reinhard F Werner.
Optimal cloning of pure states, testing single clones.
Journal of Mathematical Physics, 40(7):3283–3299, 1999.
- [KW04] Masato Koashi and Andreas Winter.
Monogamy of quantum entanglement and other correlations.
Physical Review A, 69(2):022309, 2004.
- [NC10] Michael A Nielsen and Isaac L Chuang.
Quantum computation and quantum information.
Cambridge University Press, 2010.
- [OO97] Andrei Okounkov and Grigori Olshanski.
Shifted schur functions.
Algebra i Analiz, 9(2):73–146, 1997.
- [Wat18] John Watrous.
The Theory of Quantum Information.
Cambridge University Press, 2018.