Decidability of distributed complexity of locally checkable problems on paths

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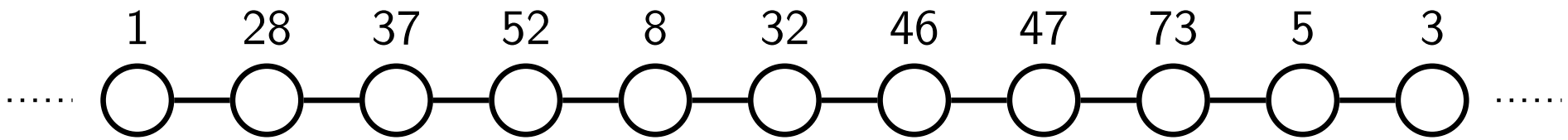
ANR DESCARTES/ESTATE

Tuesday, April 2

LCL Problems on Paths



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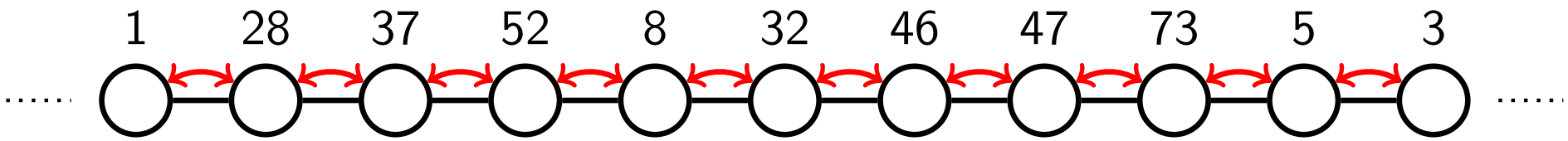

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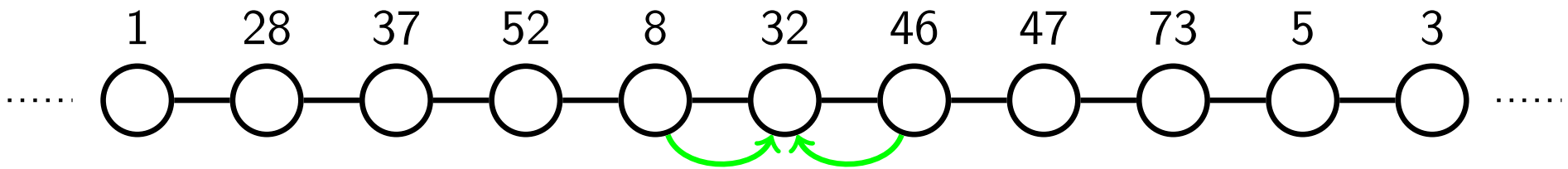

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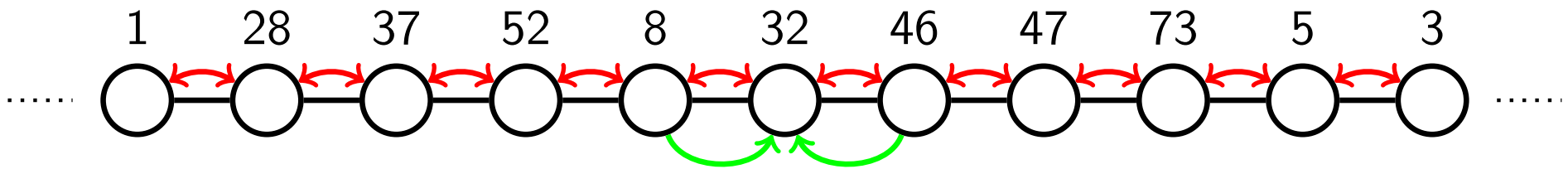

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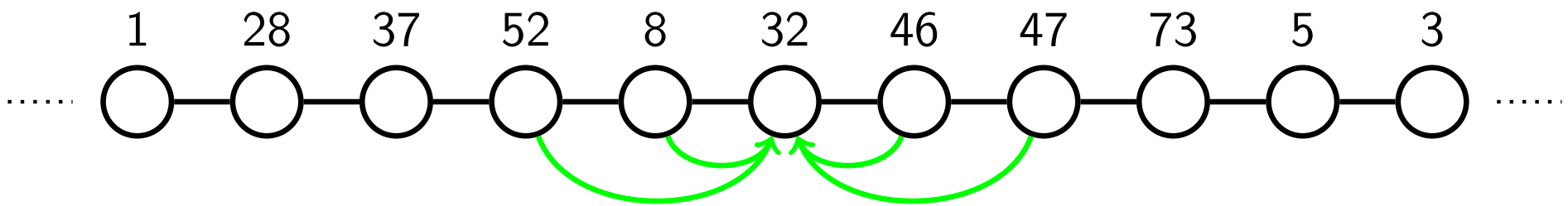

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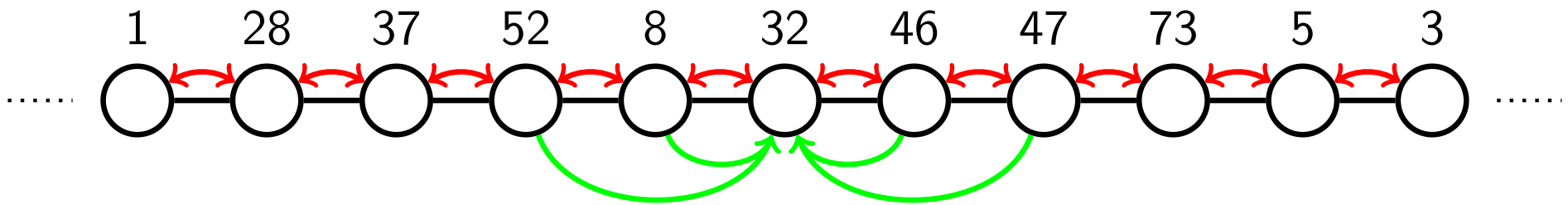

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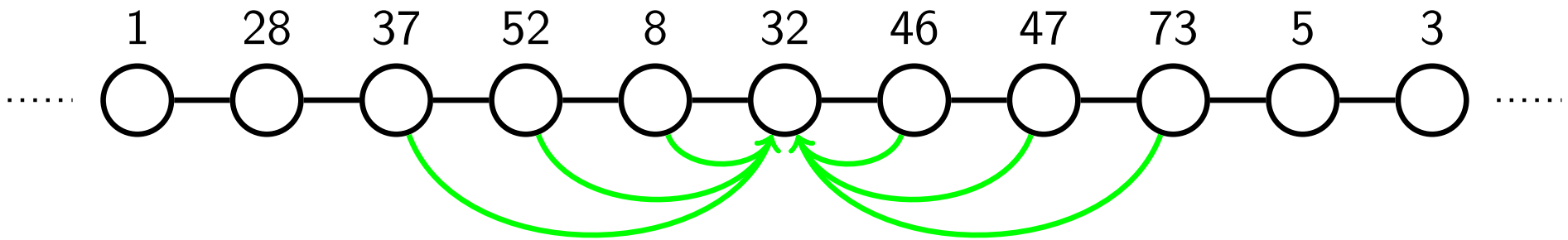

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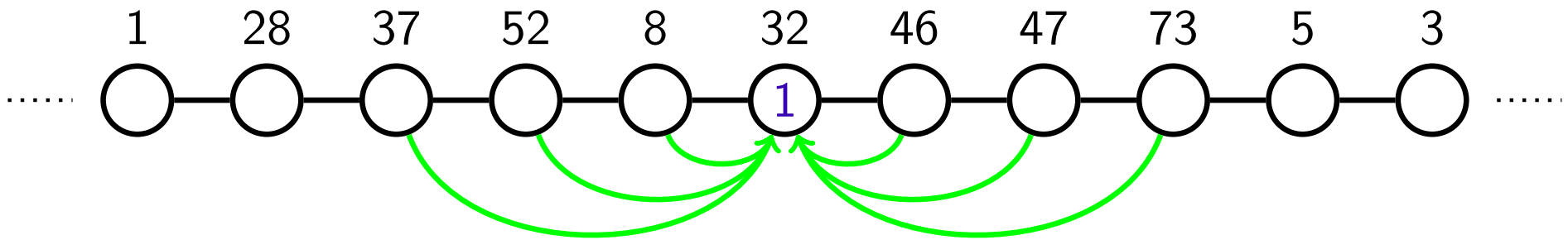

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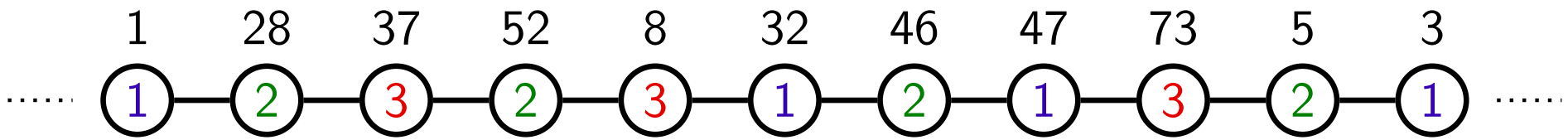

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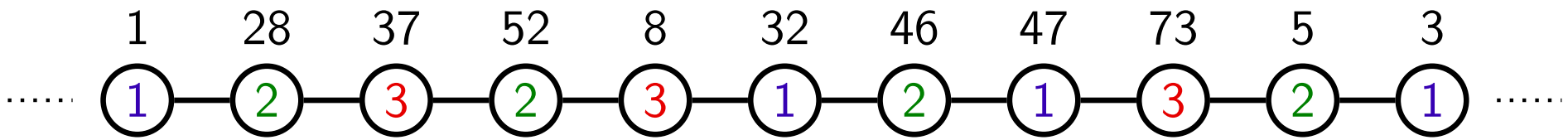

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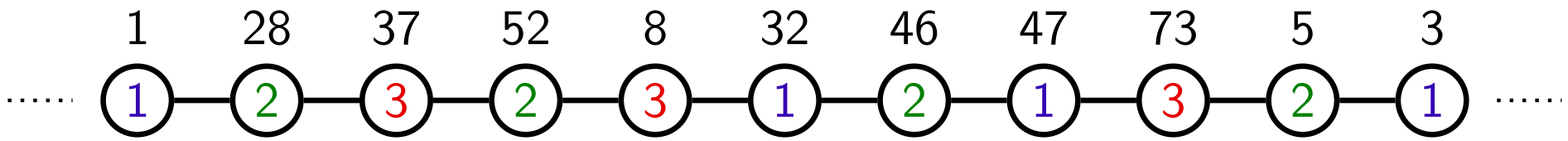
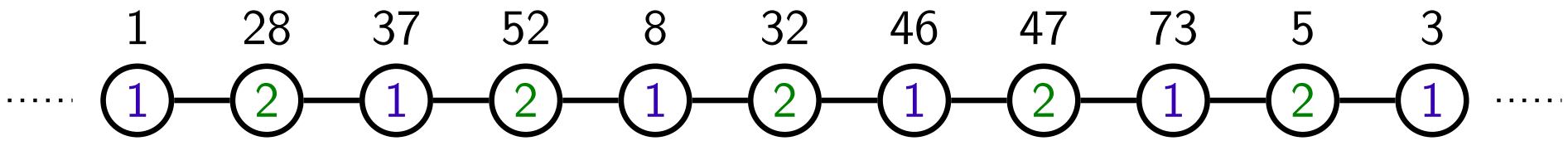


- 3 Coloring.

LCL Problems on Paths



- 3 Coloring.
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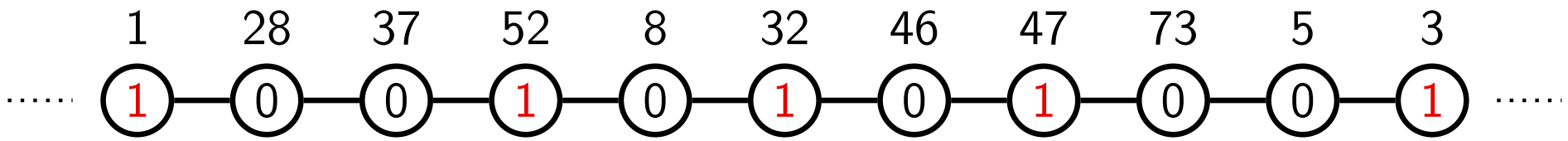
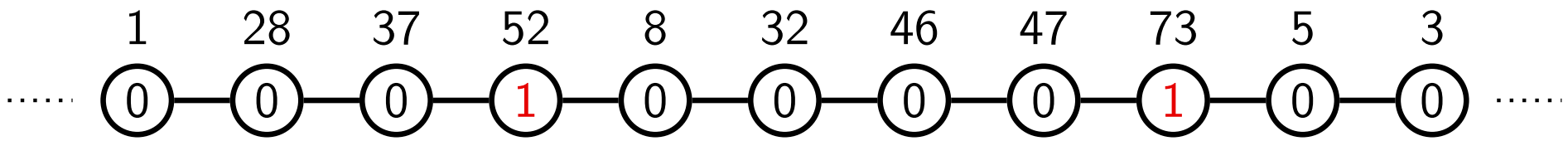


- 3 Coloring.
- 2 Coloring.
- Maximal Independent Set.

LCL Problems on Paths



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- Independent Set.

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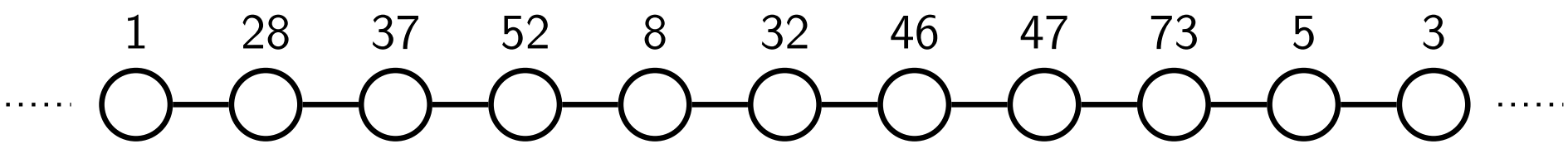

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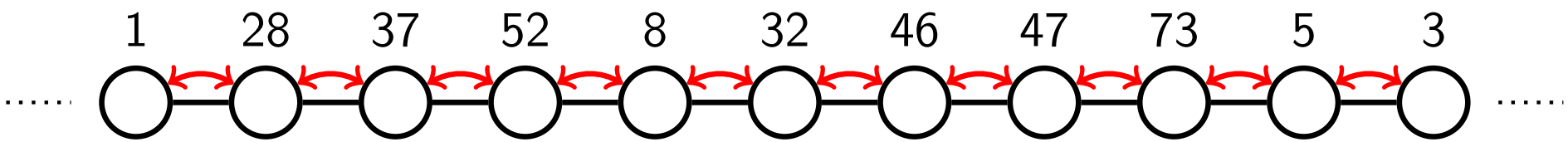

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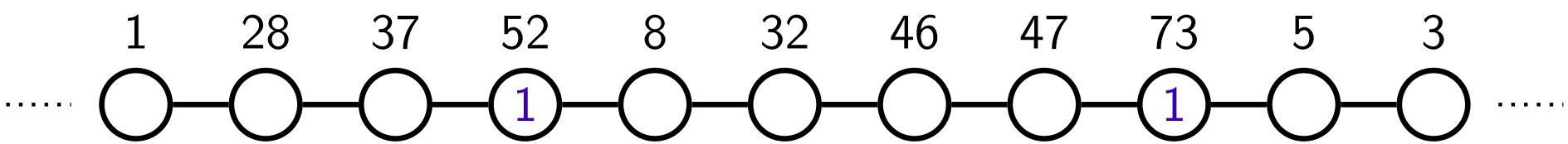

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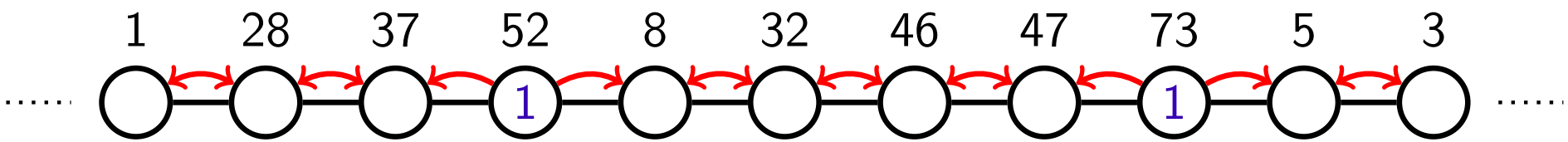

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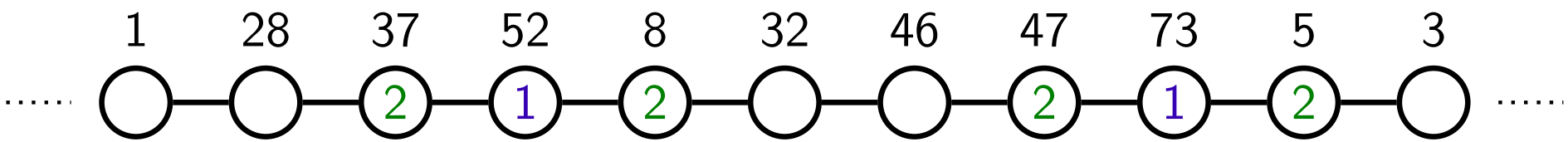

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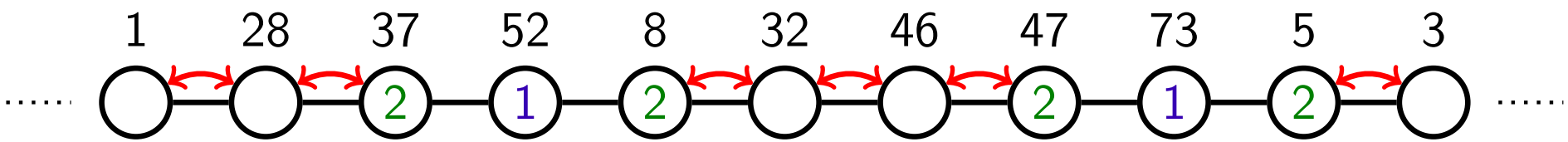

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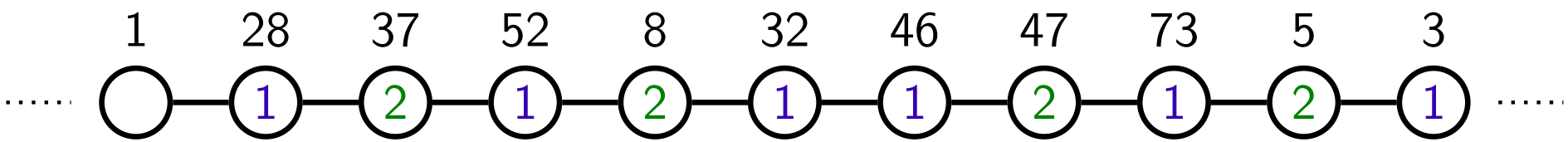

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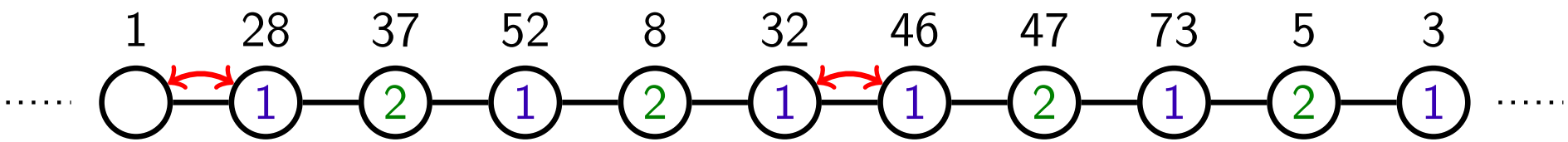

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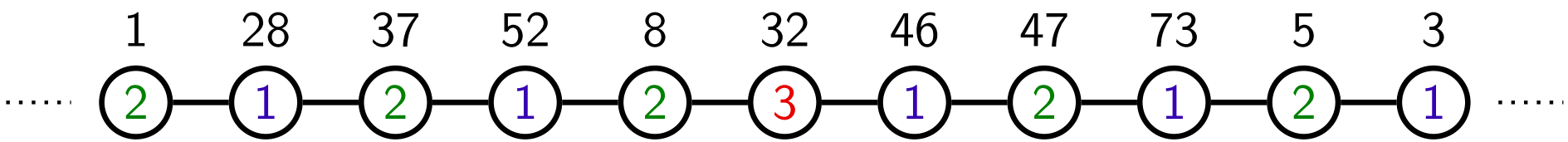

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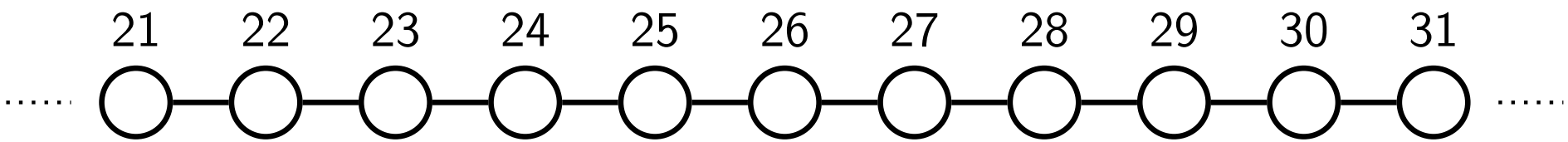

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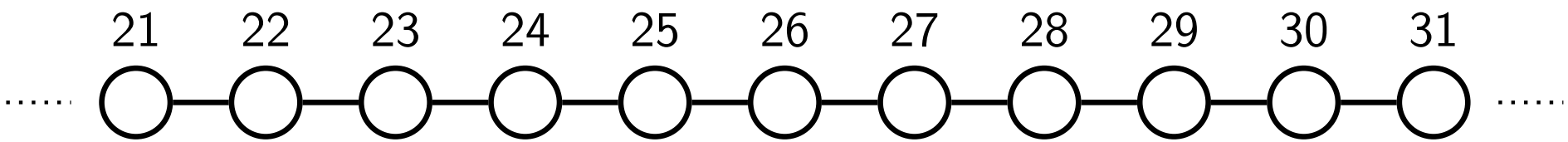


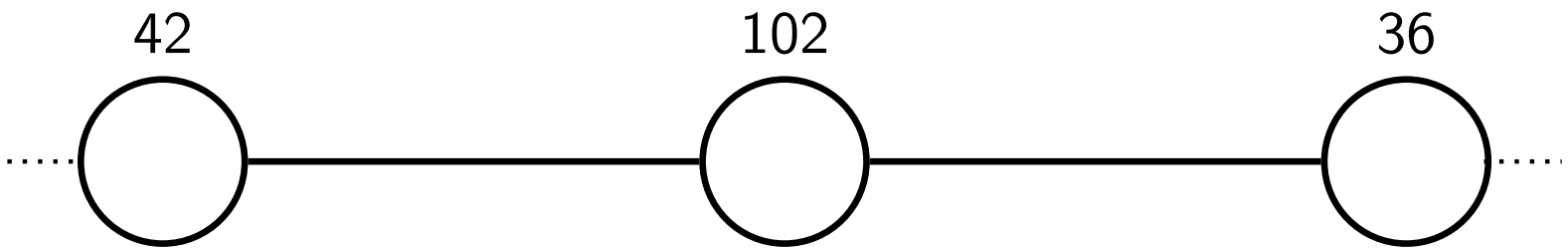
Worst case communication complexity : $\Theta(n)$.

3-coloring in $O(\log^* n)$ Communications

Cole, Vishkin (1986)

There exists a LCL algorithm to 3-color a path in $O(\log^* n)$ communications.

From n colors to $\log n$ colors



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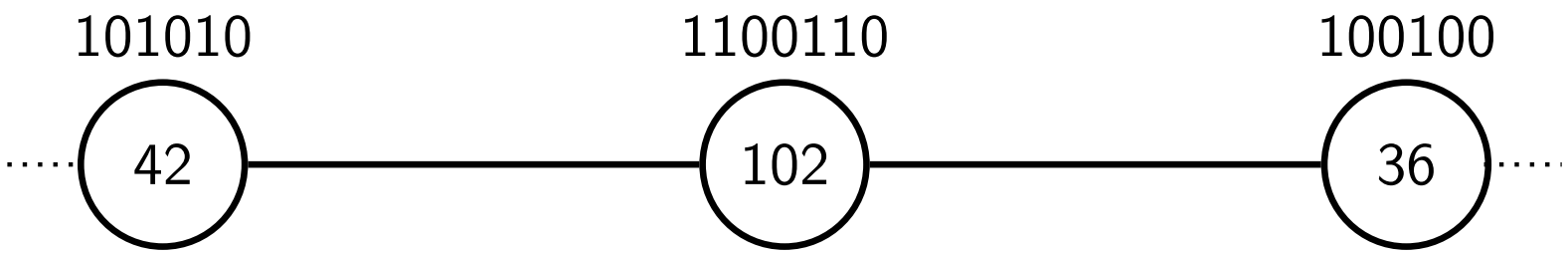

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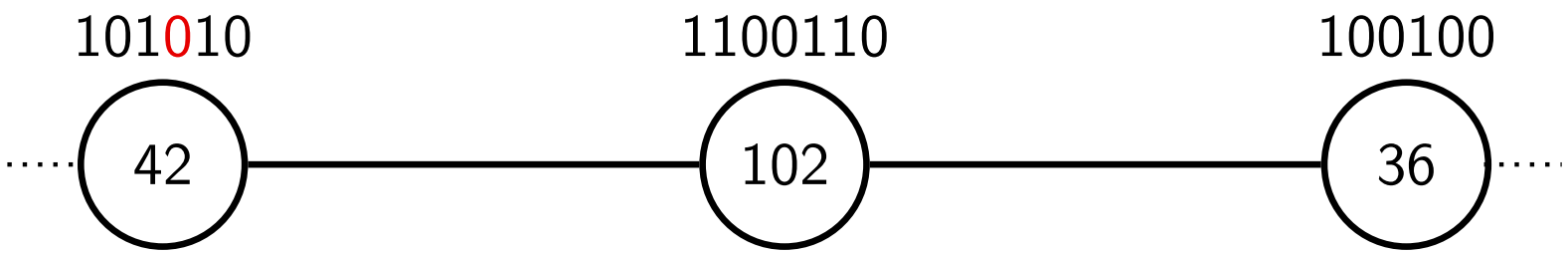

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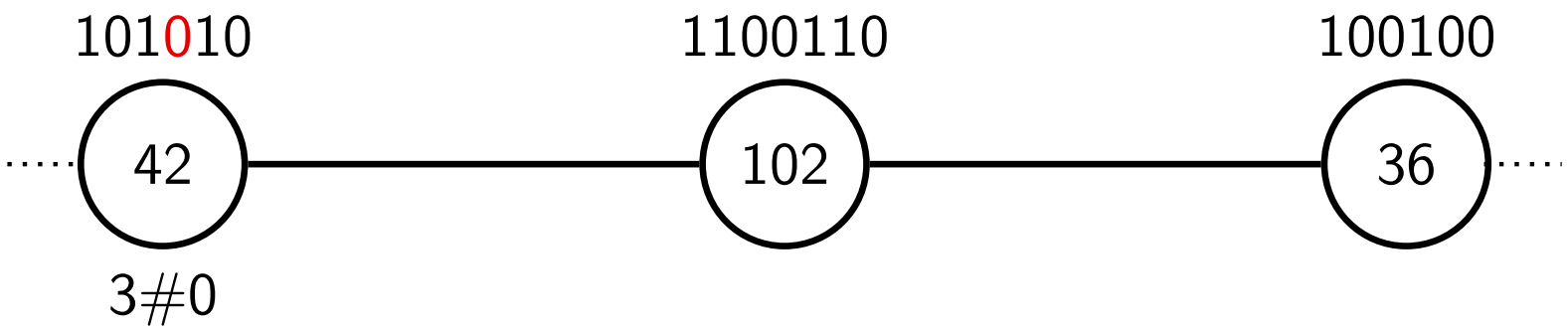

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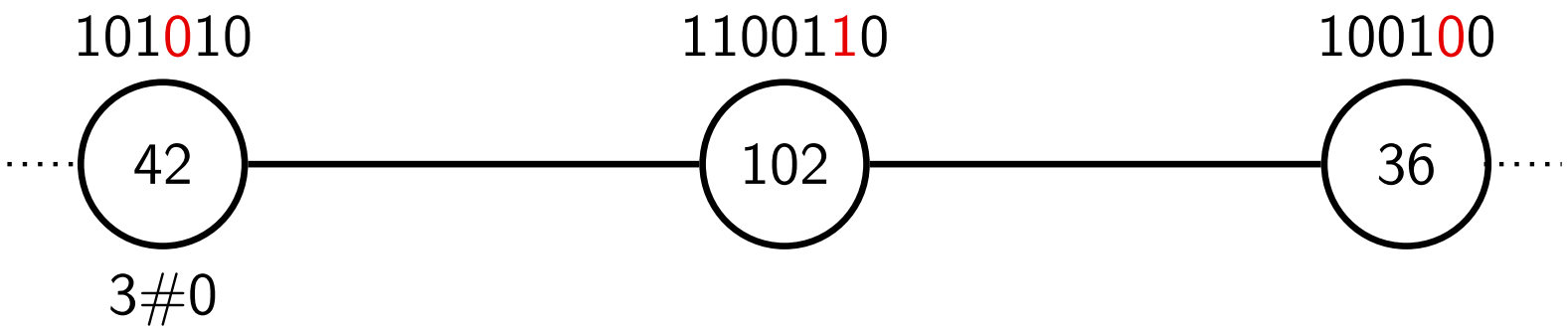

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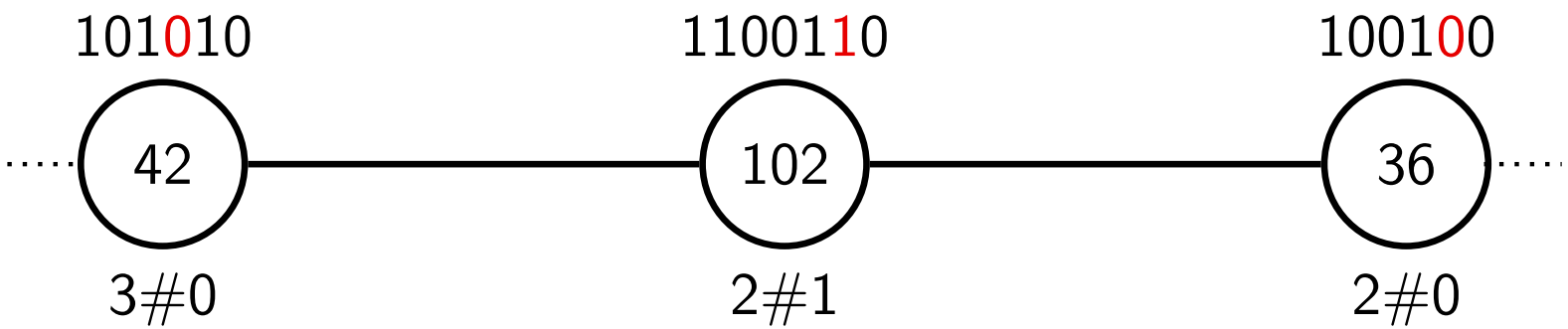

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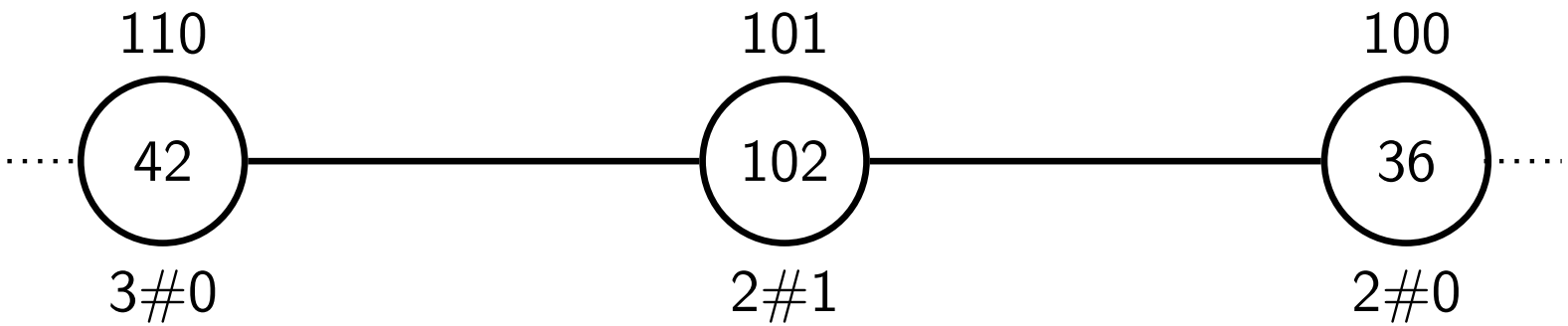

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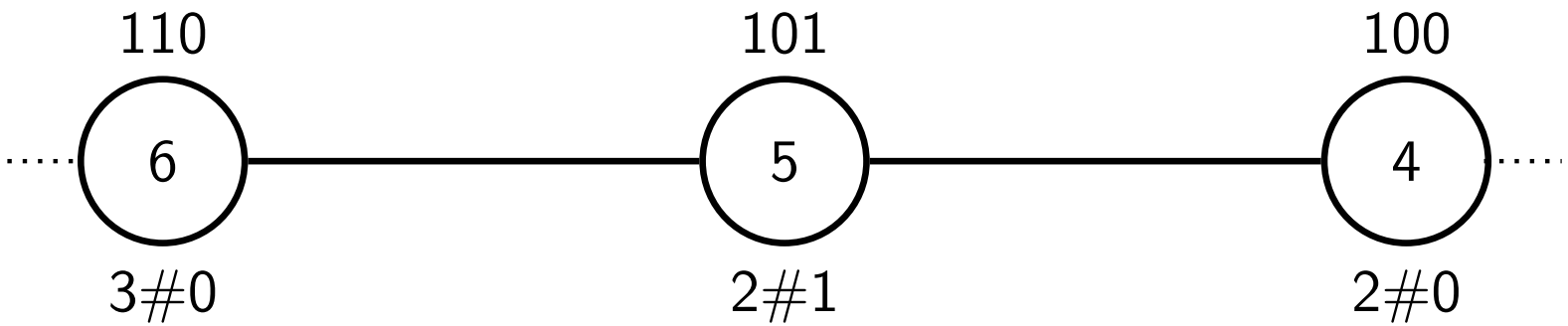

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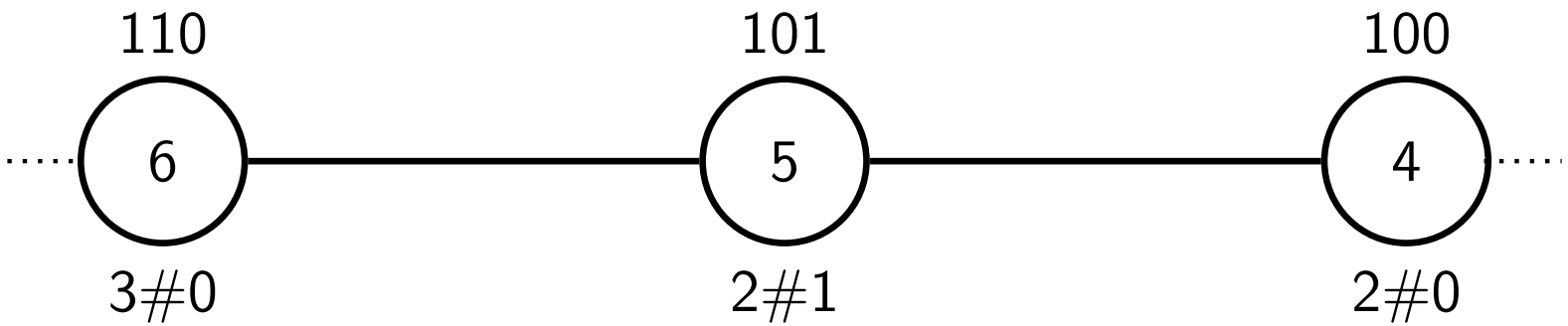

n colors $\Rightarrow \log n$ bits $\Rightarrow 2 \log n$ new colors $\Rightarrow \log \log n + 1$ bits

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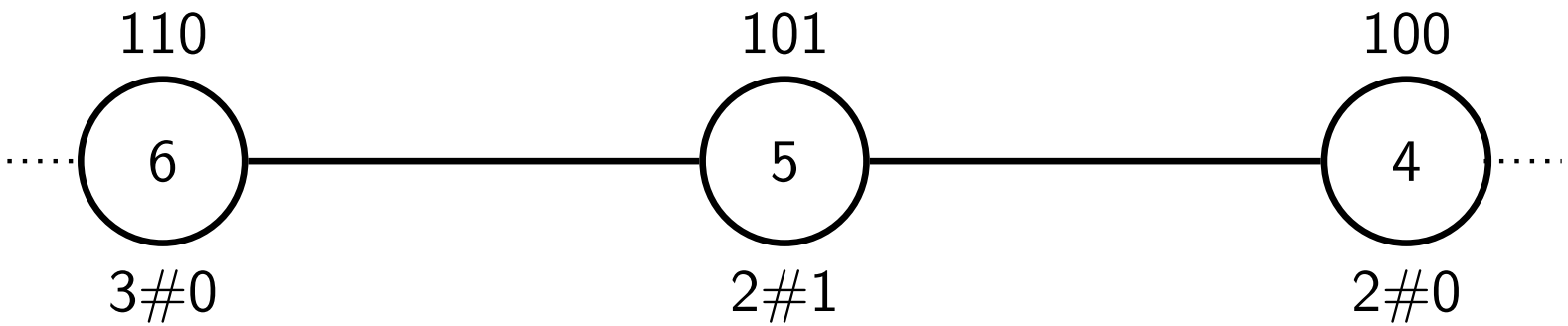

n colors $\Rightarrow \log n$ bits $\Rightarrow 2 \log n$ new colors $\Rightarrow \log \log n + 1$ bits

After $\log^* n$ iterations, $O(1)$ bits.

Coloration Lower Bound

Linial (1992)

An algorithm which colors the n-cycle with three colors requires time at least $\frac{1}{2}(\log^* n - 3)$. The same bound holds also for randomized algorithms.

Speed up Algorithm

A : algorithm that k-colors nodes in T rounds.

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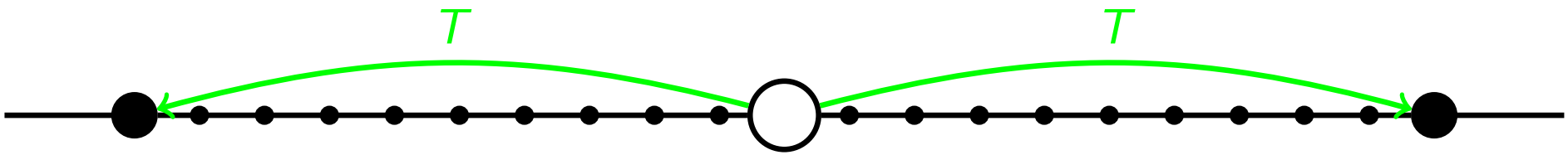

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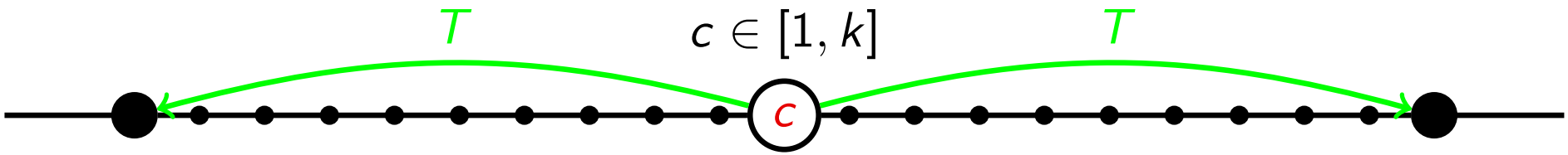

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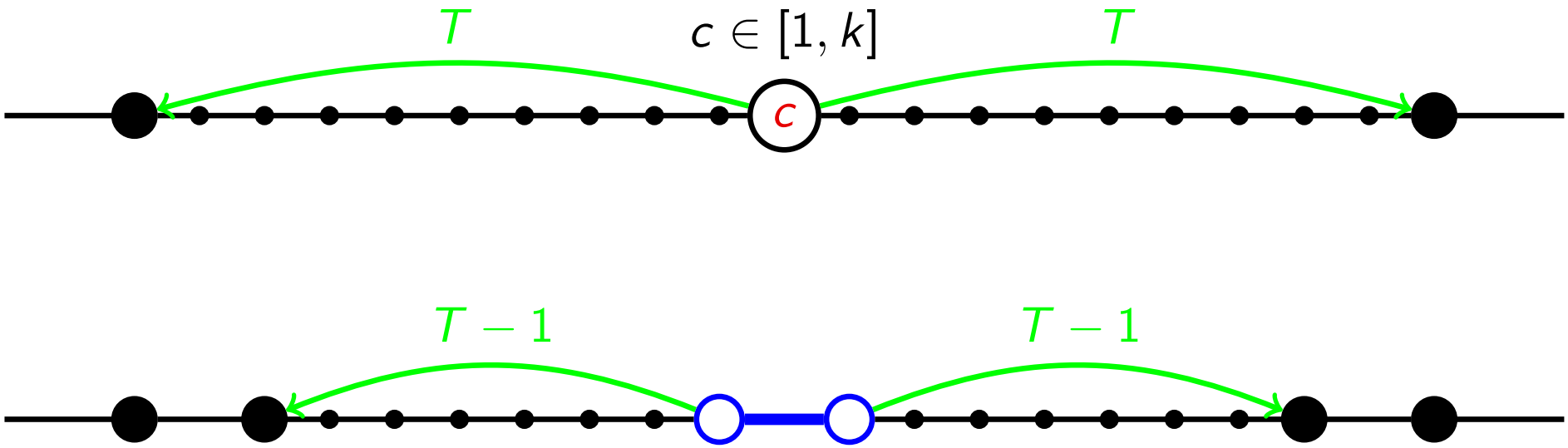

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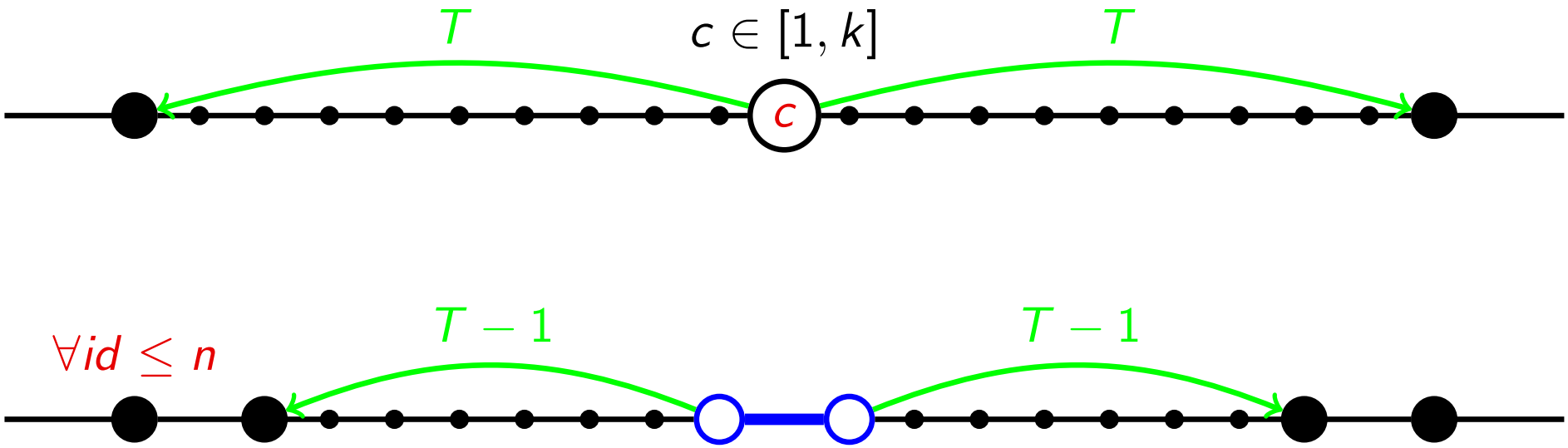

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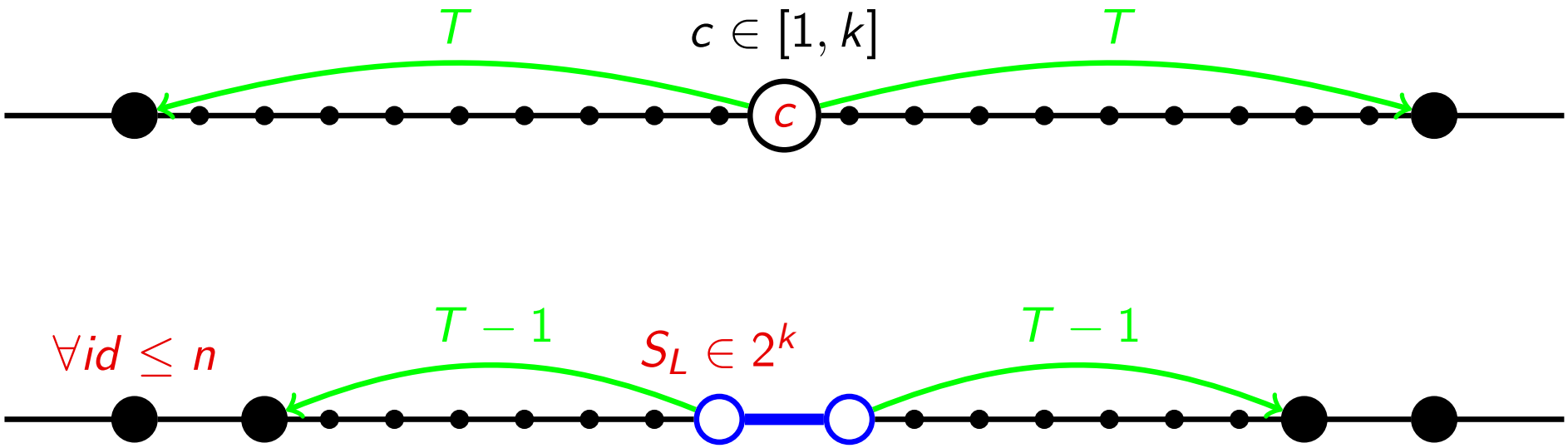

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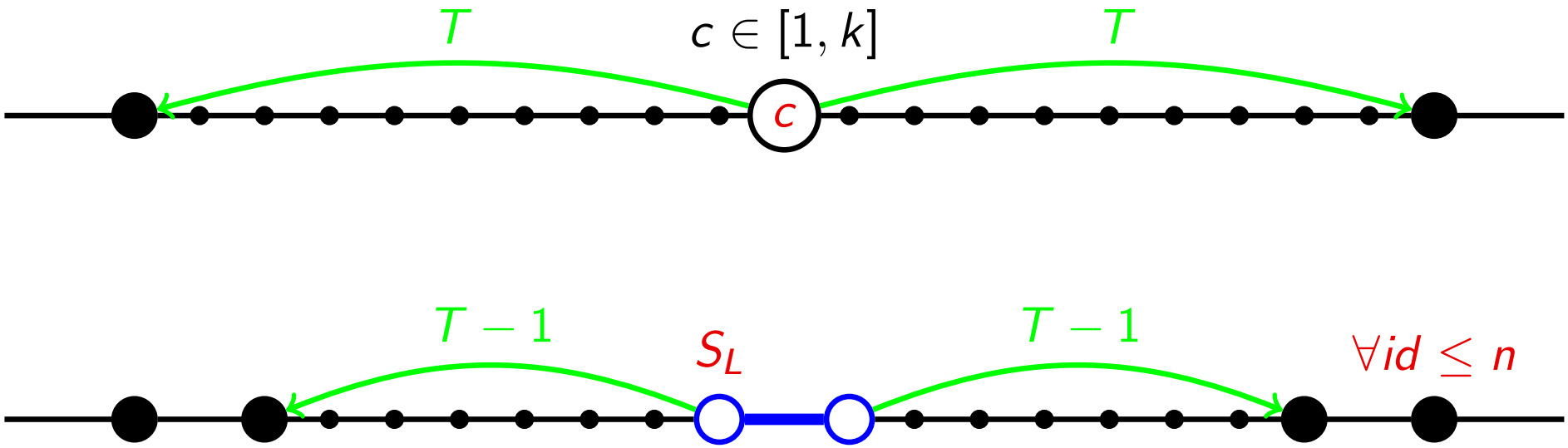

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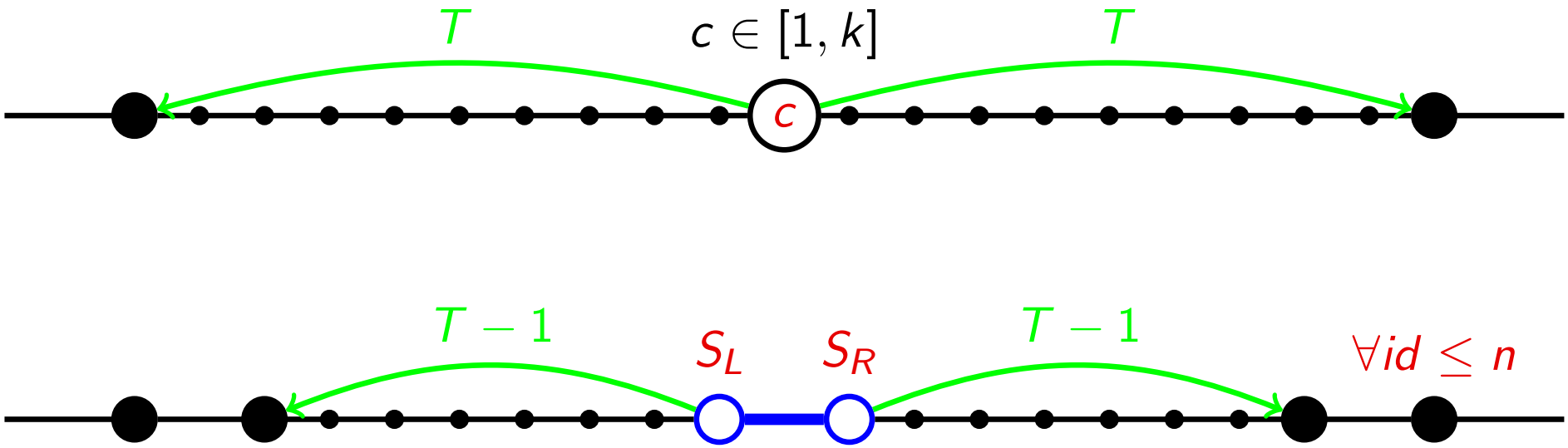

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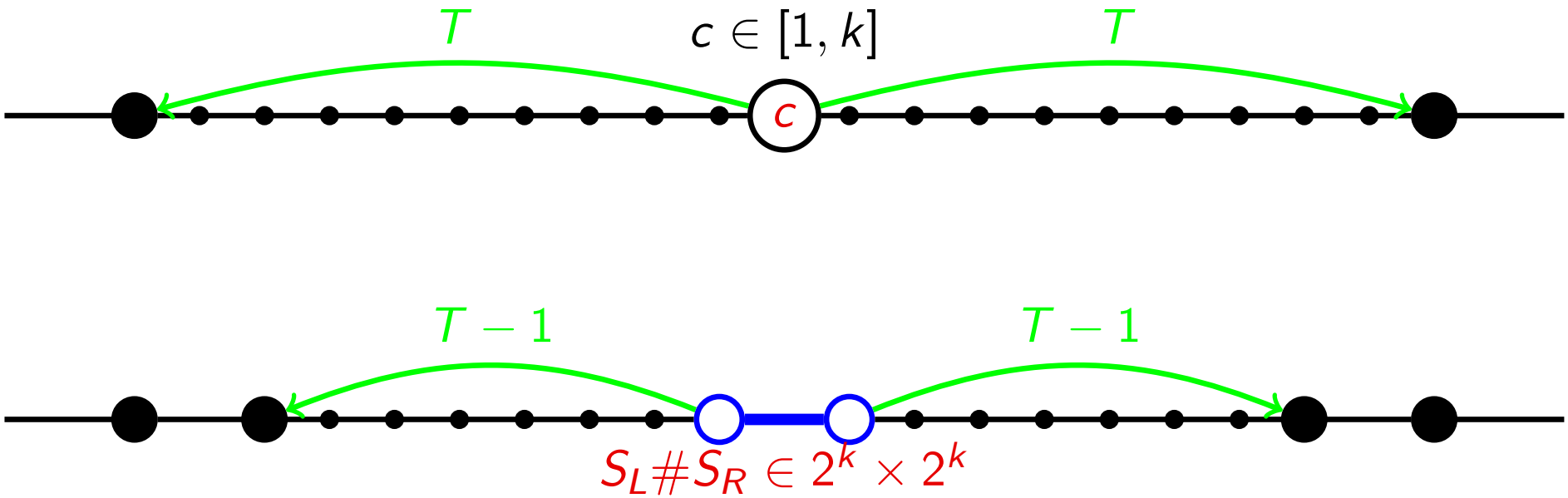

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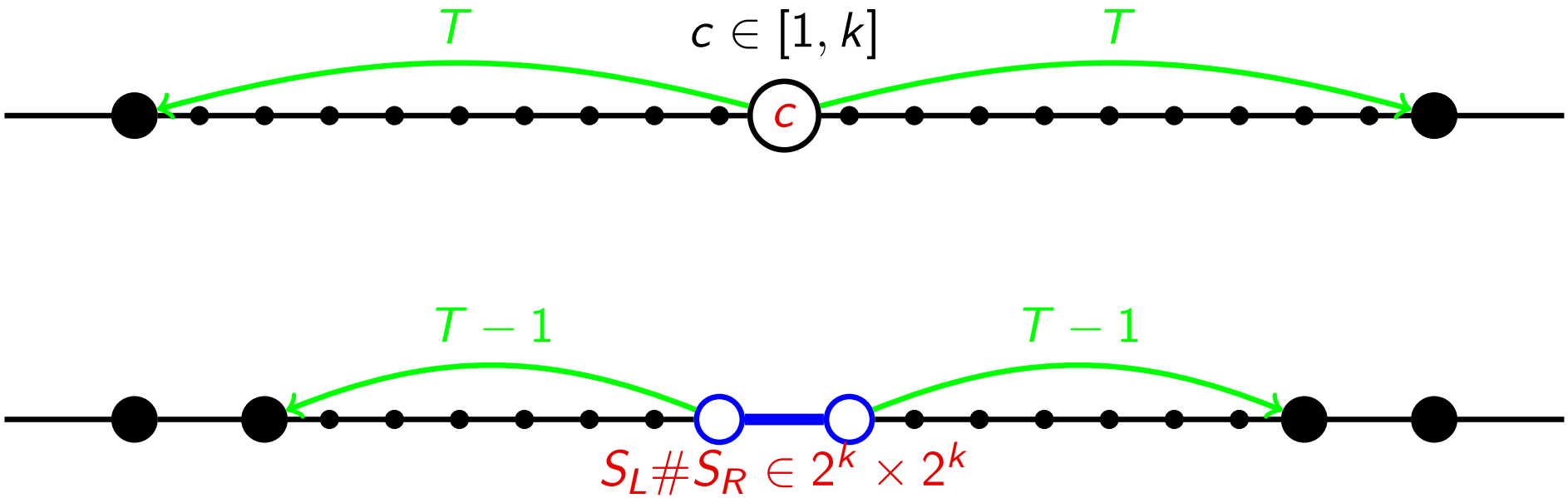

$$S_L \cap S_R = \emptyset$$

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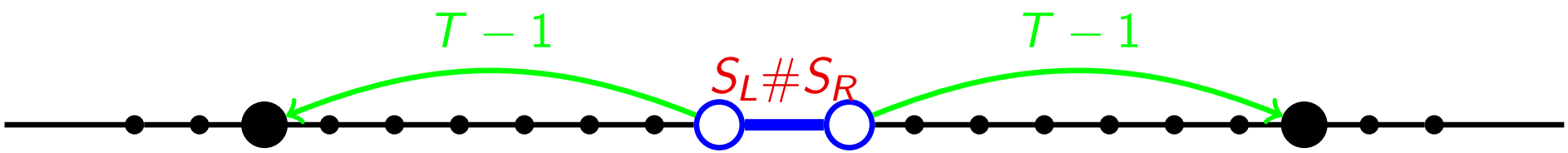

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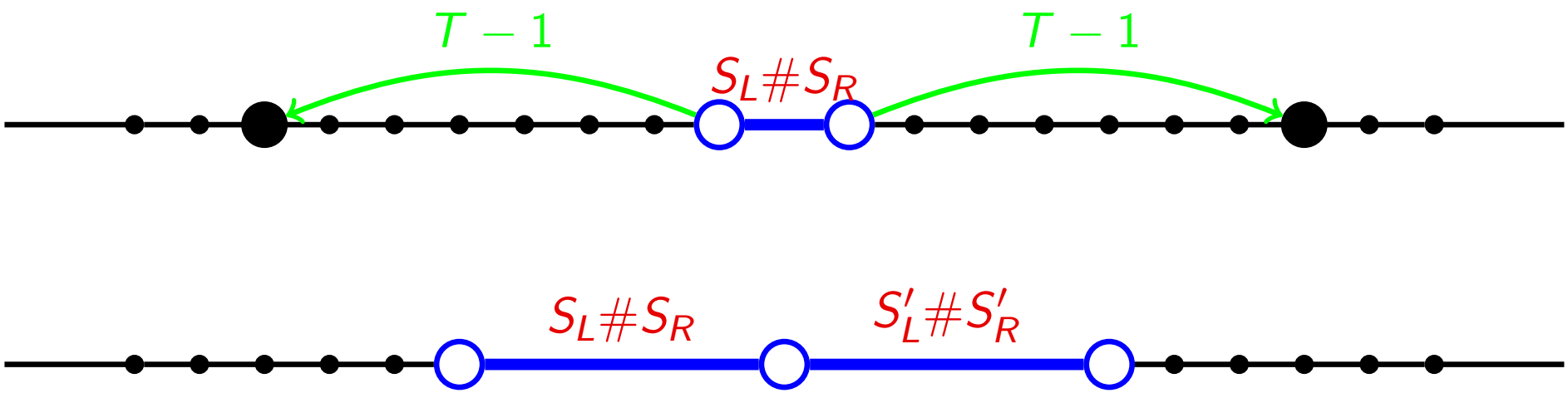

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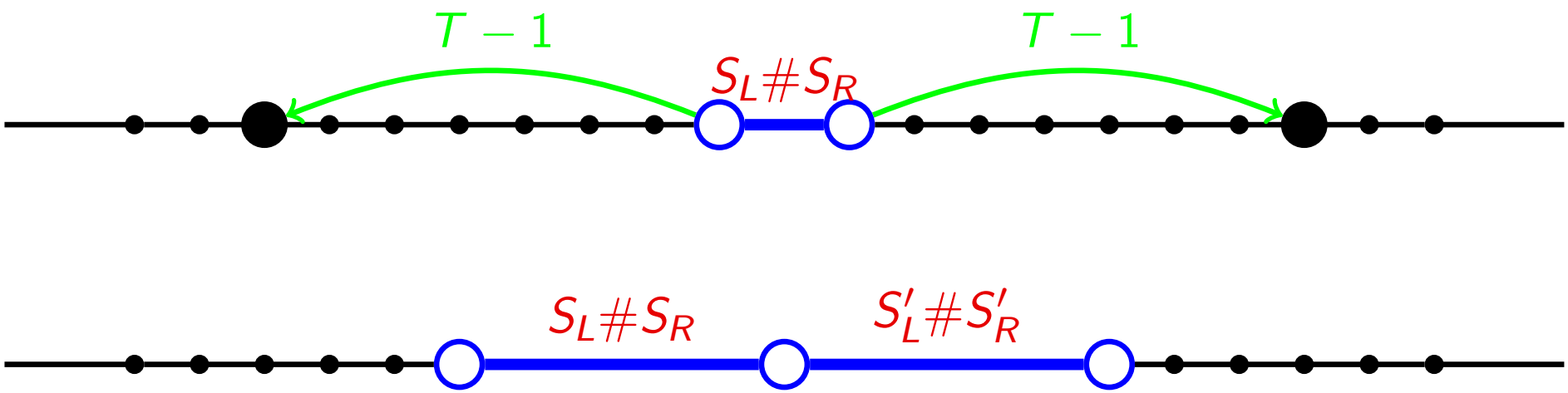

$$S_L \cap S_R = \emptyset \ \& \ S'_L \cap S'_R = \emptyset$$

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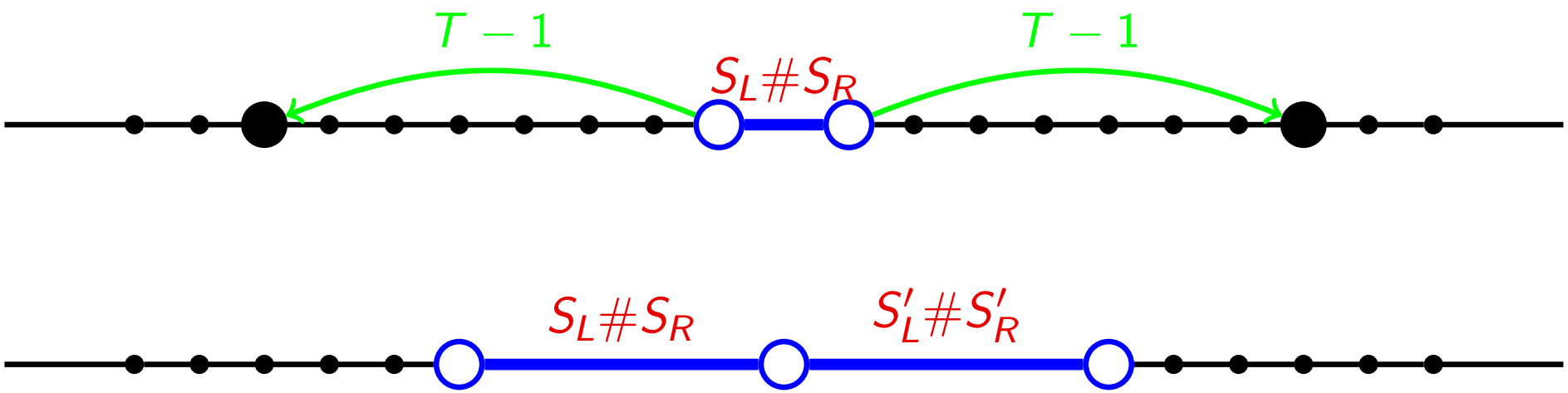

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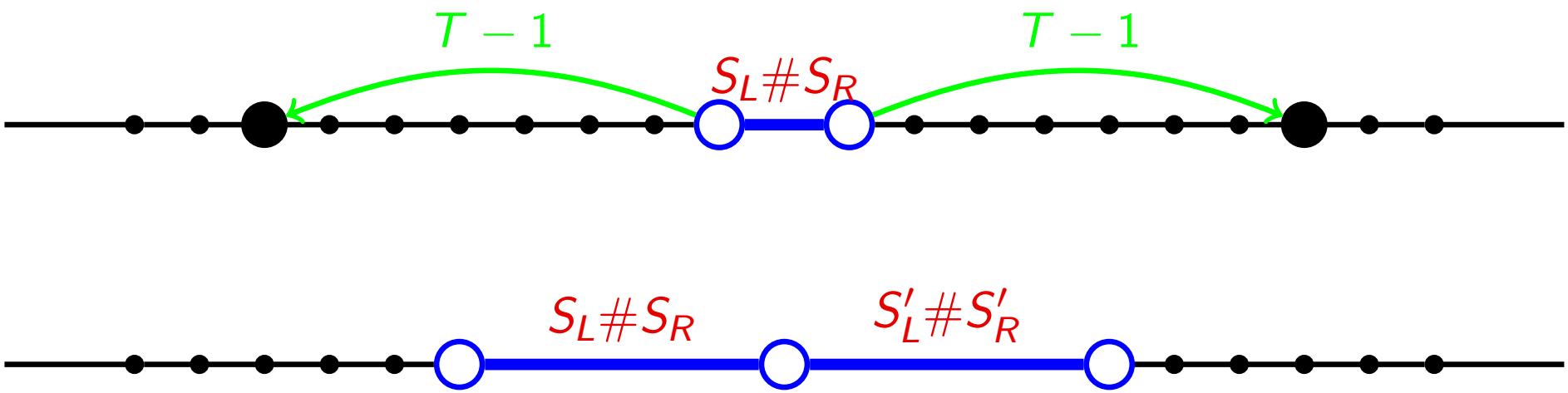

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$$S_L \# S_R \neq S'_L \# S'_R$$

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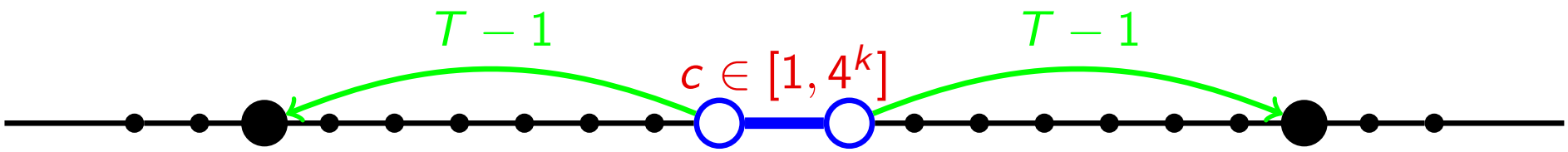
A_1 : algorithm that 4^k-colors edges in $T - 1$ rounds.



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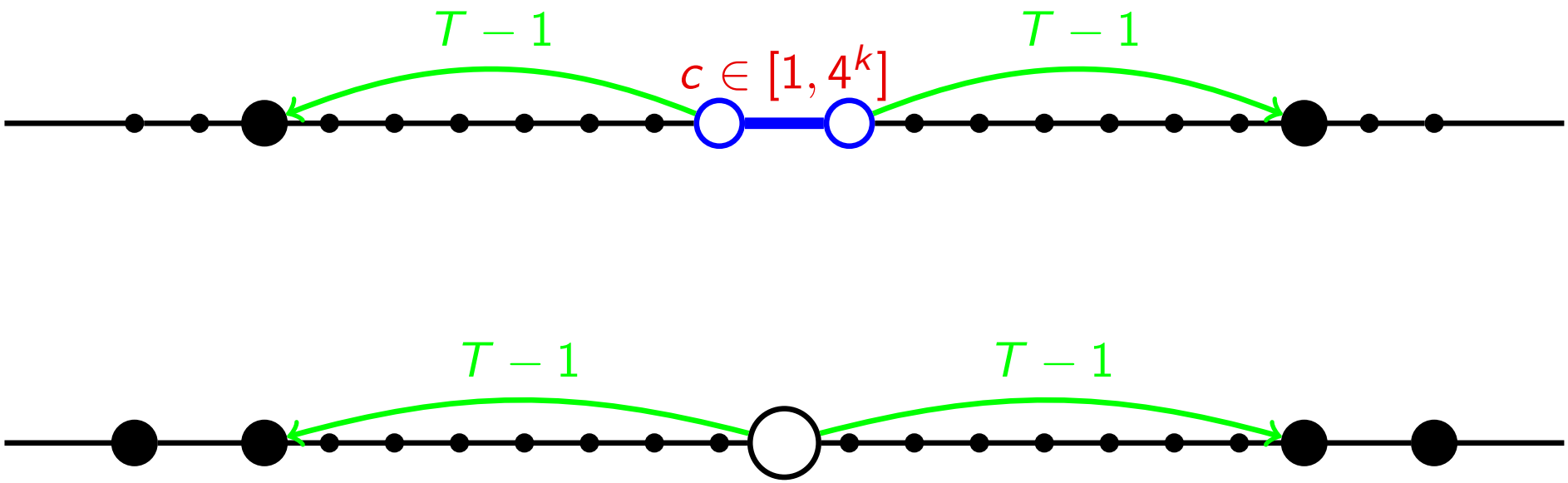

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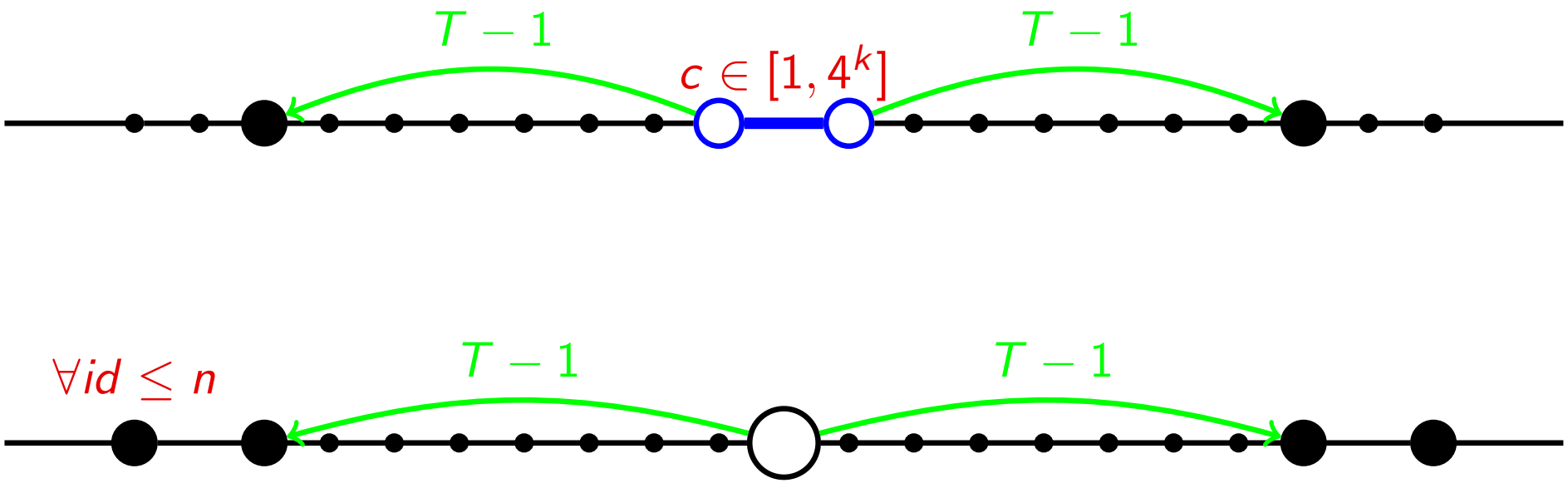

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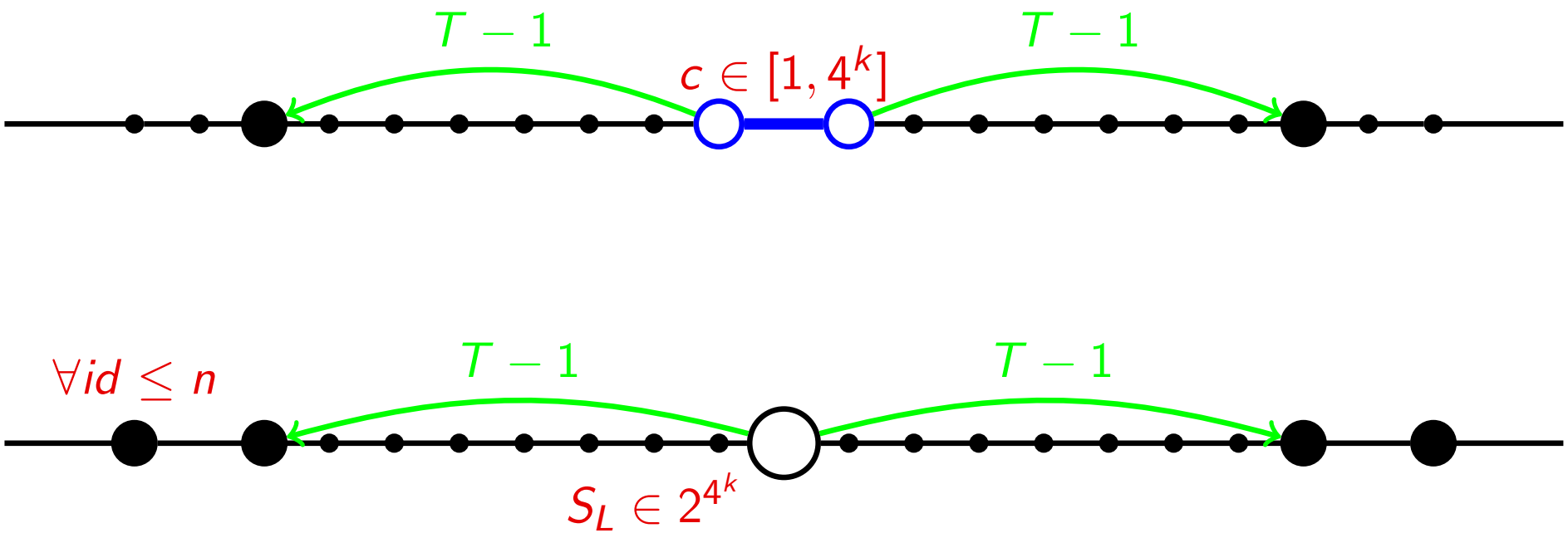

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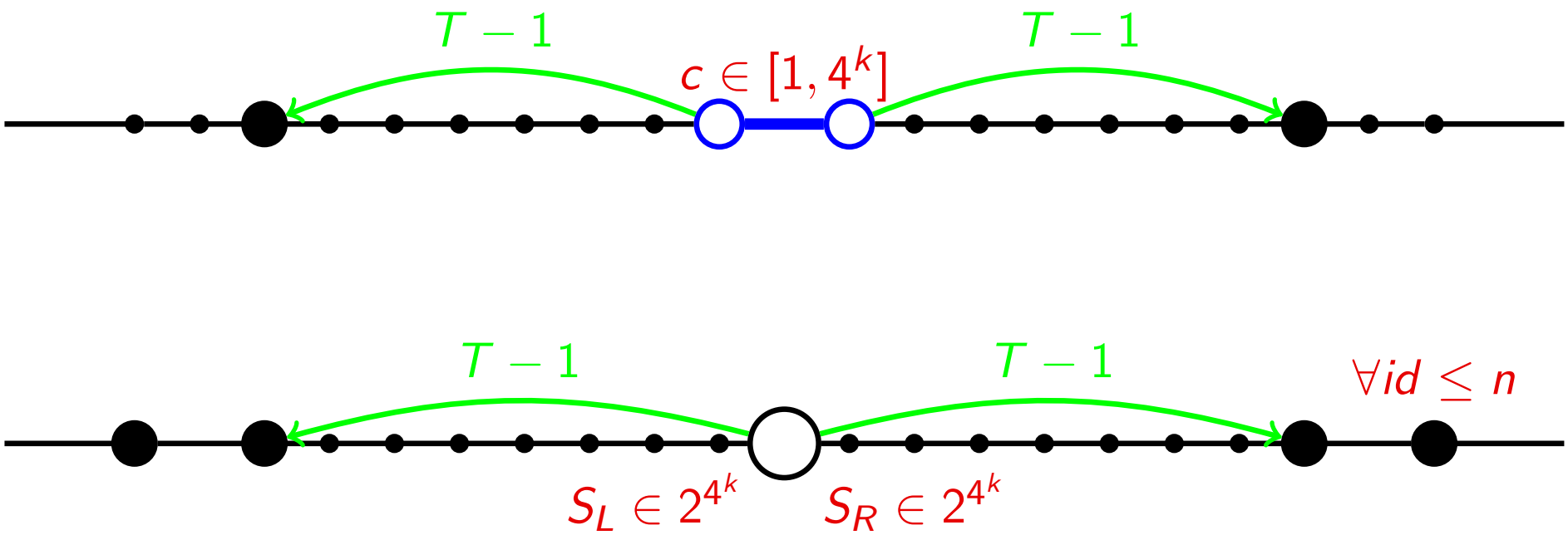

Speed up Algorithm

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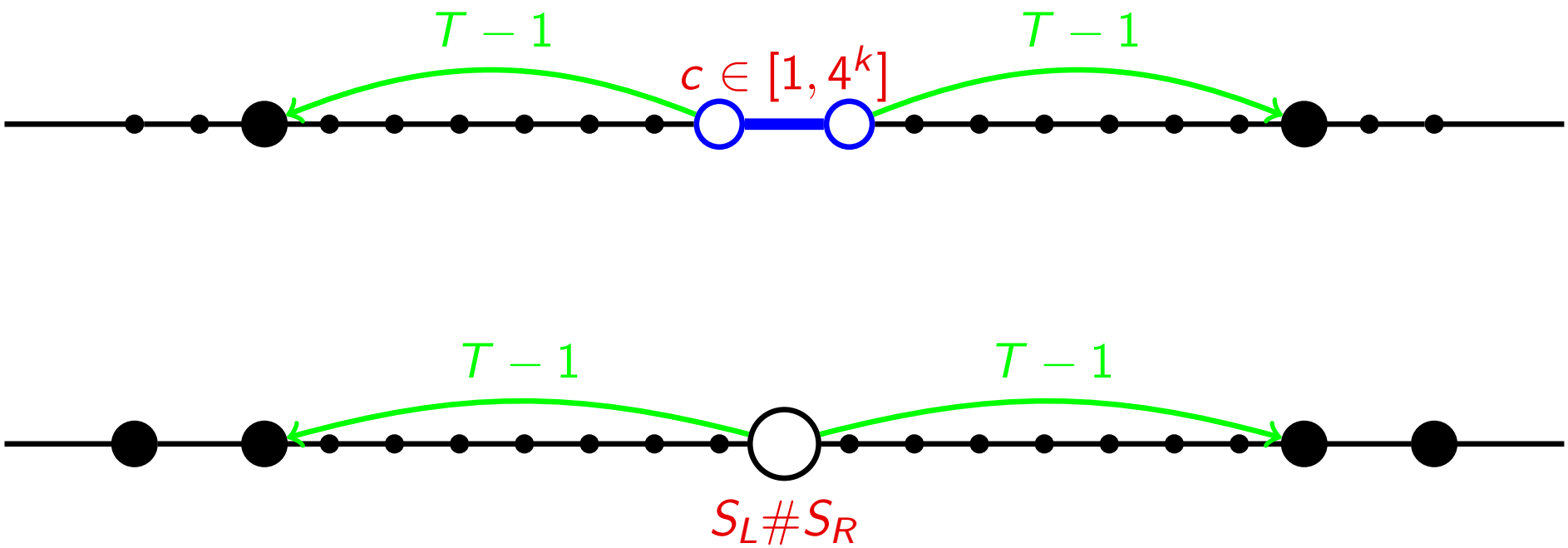

Speed up Algorithm

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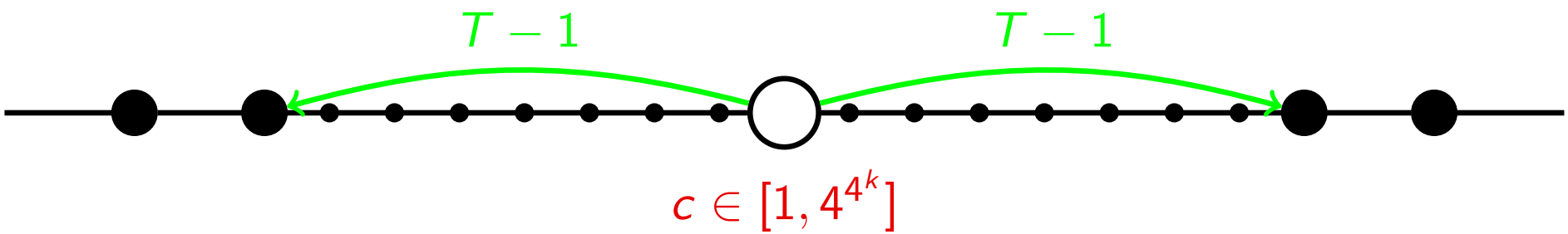

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$\Rightarrow$ any 3-coloring algorithm needs at least $\frac{log^* n}{2}$ rounds.

Independent Set at some Distance

- From 3−coloring to Maximal Independent Set.

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  Nodes at distance 2 or 3.

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Time complexity : $O(s \log^* n)$.

Transition Automata

Node : Sequence of outputs.
Edge : Connecting to an admitable next output.

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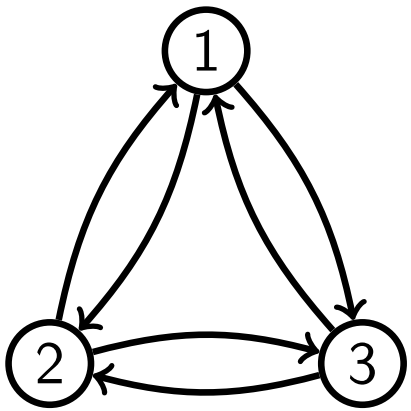

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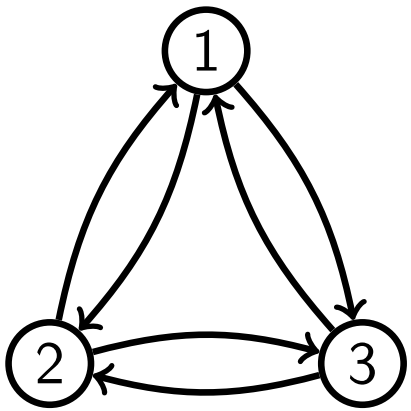

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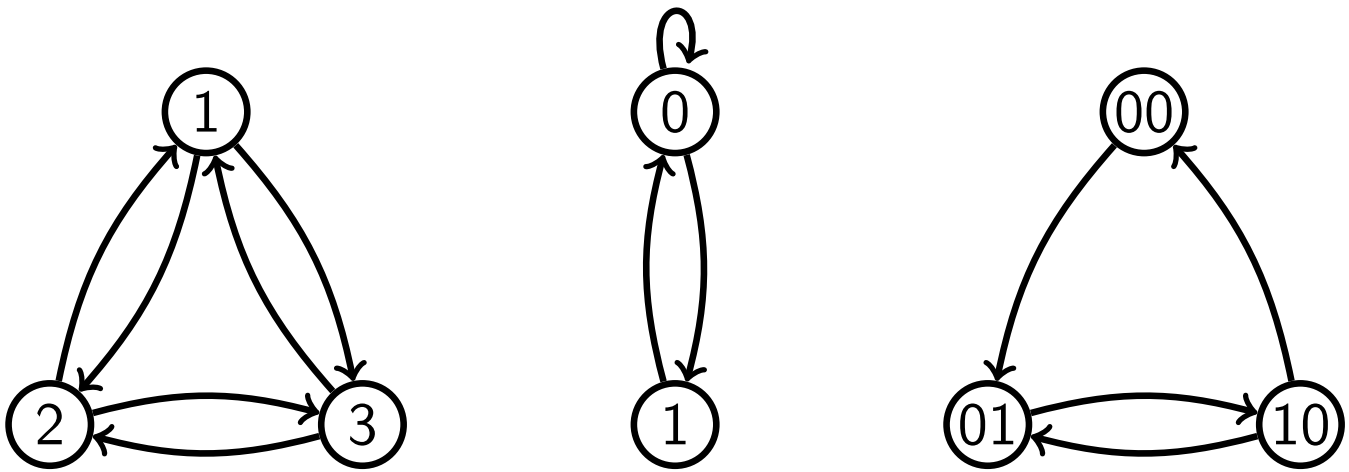

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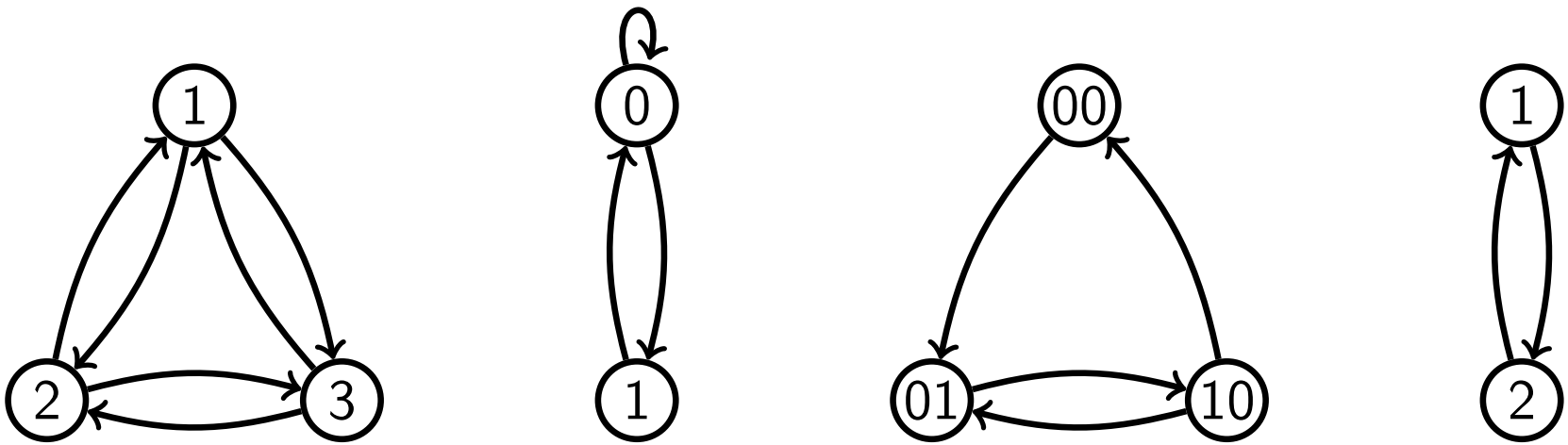

Complexity Separation on Paths

Naor, Sotckmeyer (1995)

If the input graph is an unlabeled path or cycle, the time complexity is decidable.
The different time complexities are $O(1)$, $\Theta(\log^* n)$ and $\Omega(n)$.

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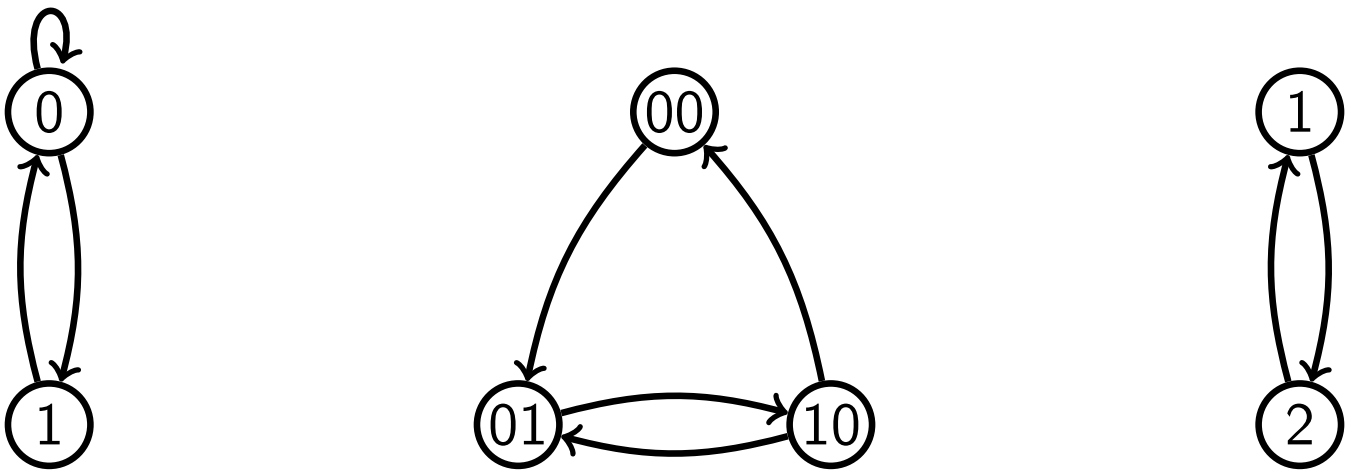
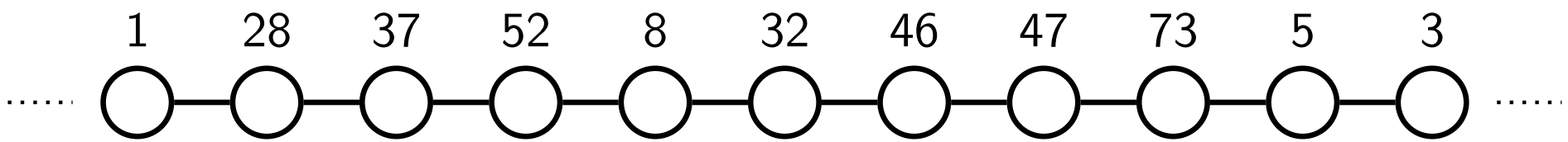


Problem on Paths with Inputs



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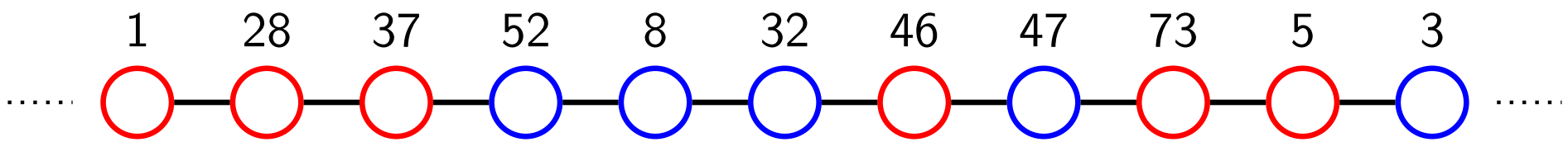
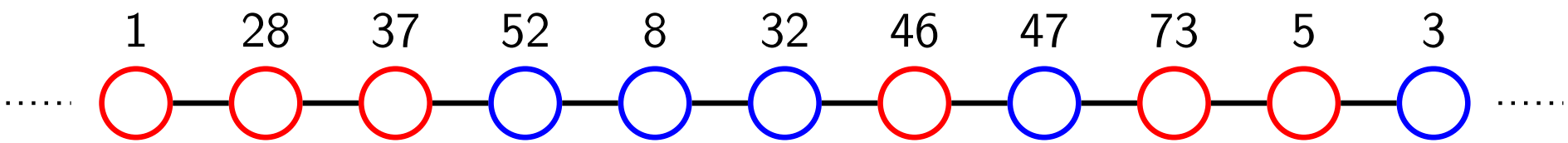
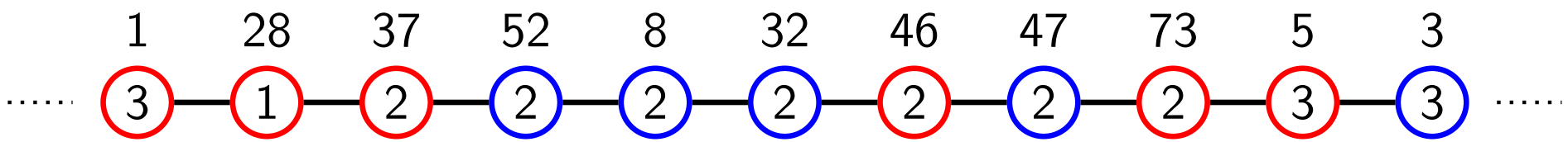


Problem on Paths with Inputs



- 3-color the red nodes.
- Carry the color through the blue nodes.

Problem on Paths with Inputs



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Complexity Separation on Paths with Inputs

Balliu, Brandt, Chang, Olivetti, Rabie, Suomela (2018)

For any LCL problem on cycle graphs, its complexity is either $\Omega(n)$ or $O(\log^* n)$. Moreover, there is an algorithm that decides whether the problem has complexity $\Omega(n)$ or $O(\log^* n)$ on cycle graphs; for the case the complexity is $O(\log^* n)$, the algorithm outputs a description of an $O(\log^* n)$-round deterministic LOCAL algorithm that solves it.

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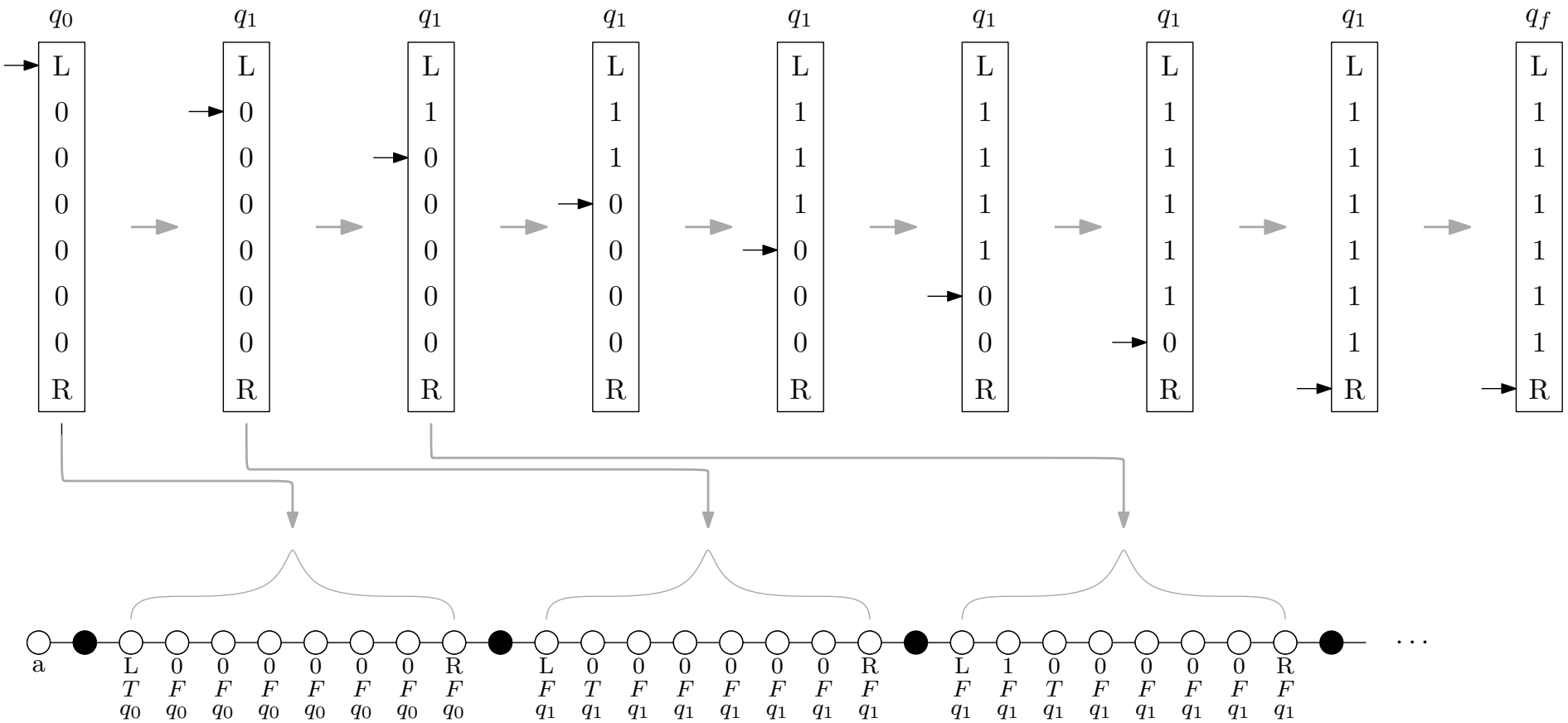
Complexity Separation on Paths with Inputs

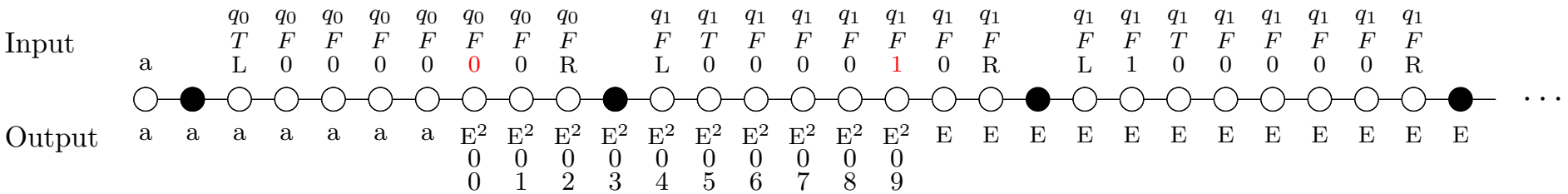
Balliu, Brandt, Chang, Olivetti, Rabie, Suomela (2018)

It is PSPACE-hard to distinguish whether a given LCL problem with input labels can be solved in $O(1)$ time or needs $\Omega(n)$ time on globally oriented path graphs.

Turing Machine Encoding



Error Detection



Complexity Separation on Trees

Balliu, Brandt, Chang, Olivetti, Rabie, Suomela (2018)

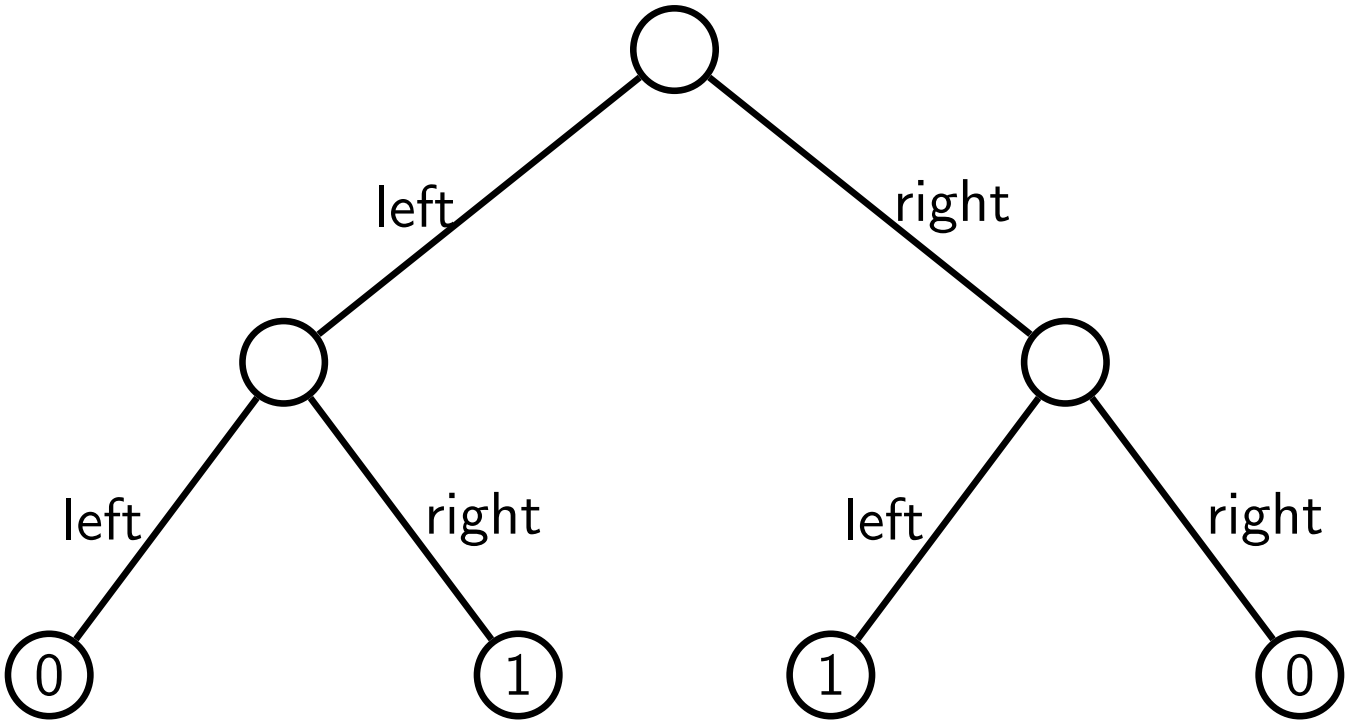
It is PSPACE-hard to distinguish whether a given LCL problem without input labels can be solved in $O(1)$ time or needs $\Omega(n)$ time on trees with degree $\Delta = 3$.

Encoding a Number in a Tree

Encoding $6{=}0110_2$

Encoding a Number in a Tree

Encoding $6{=}0110_2$



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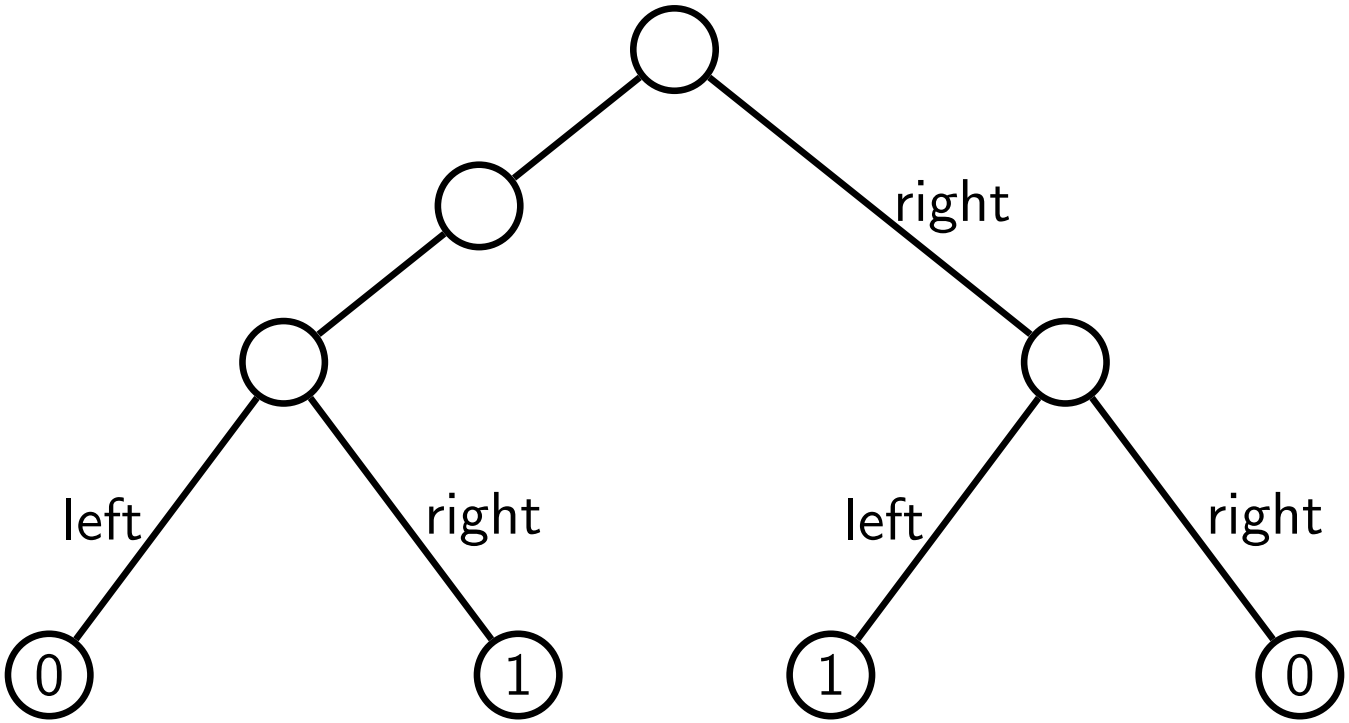

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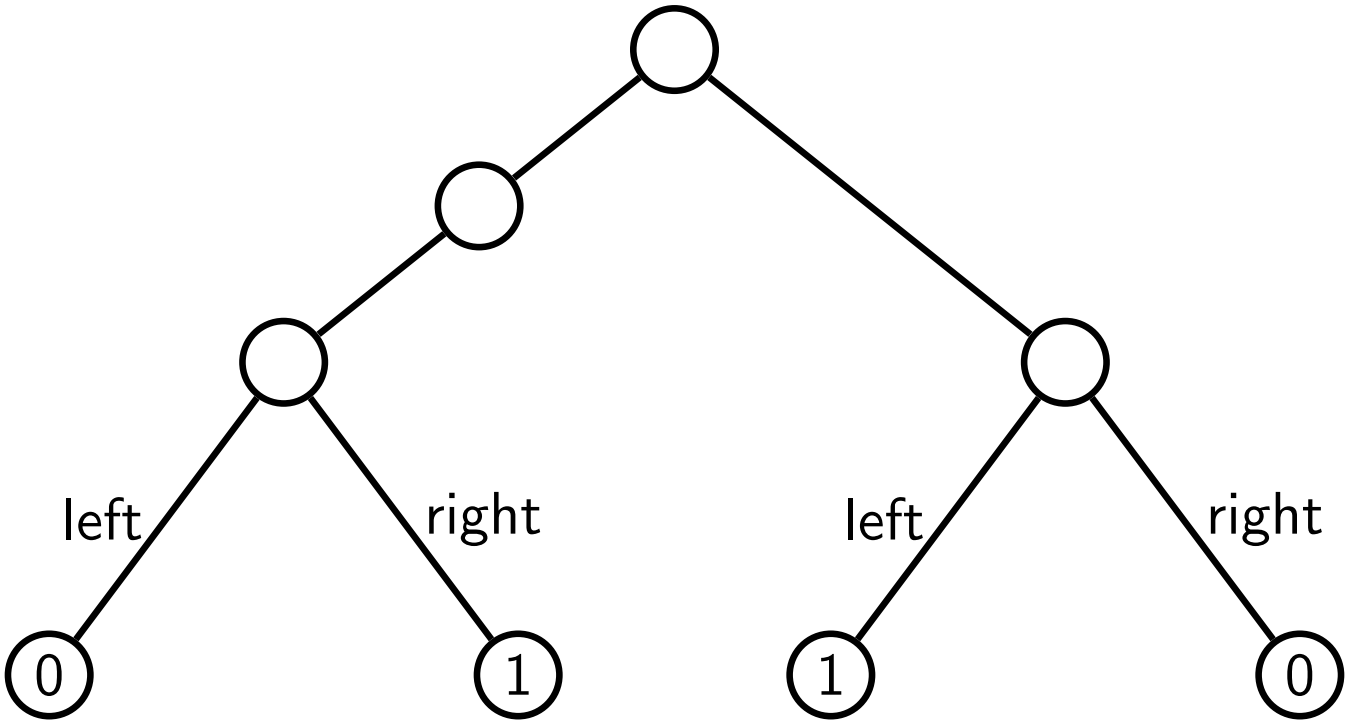

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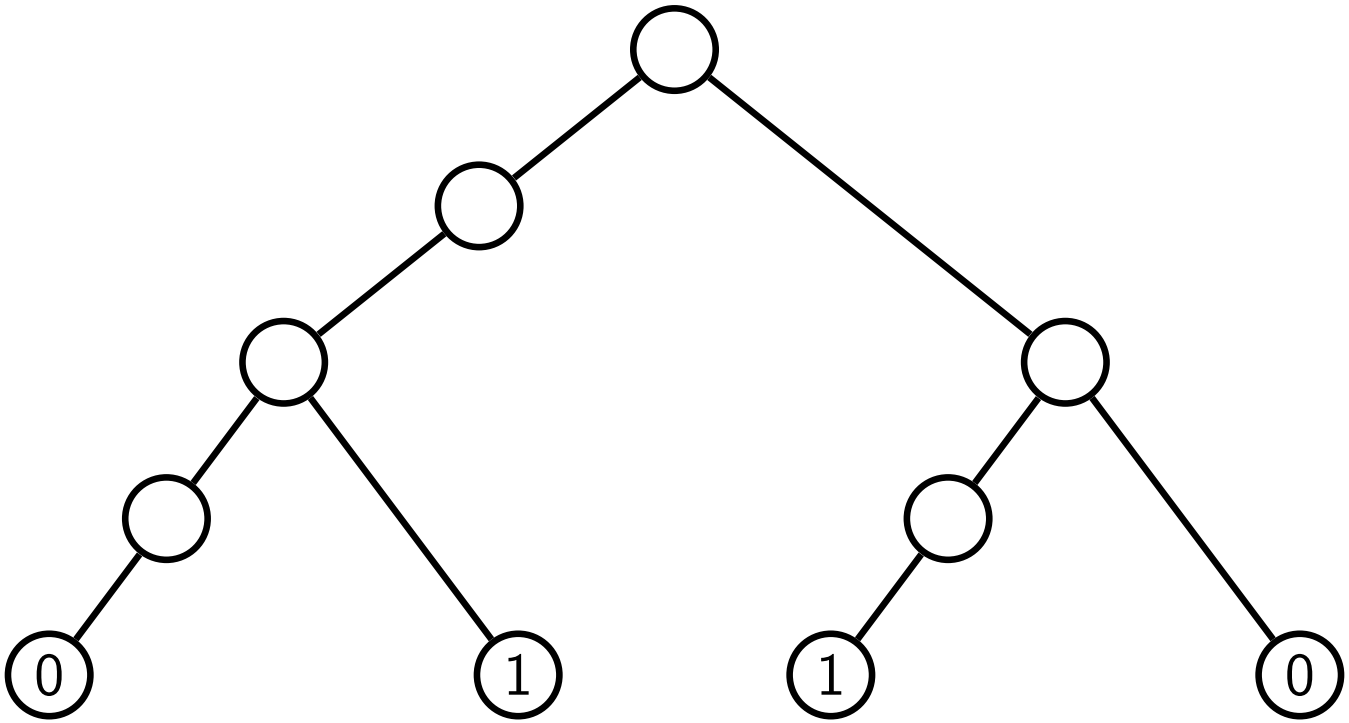

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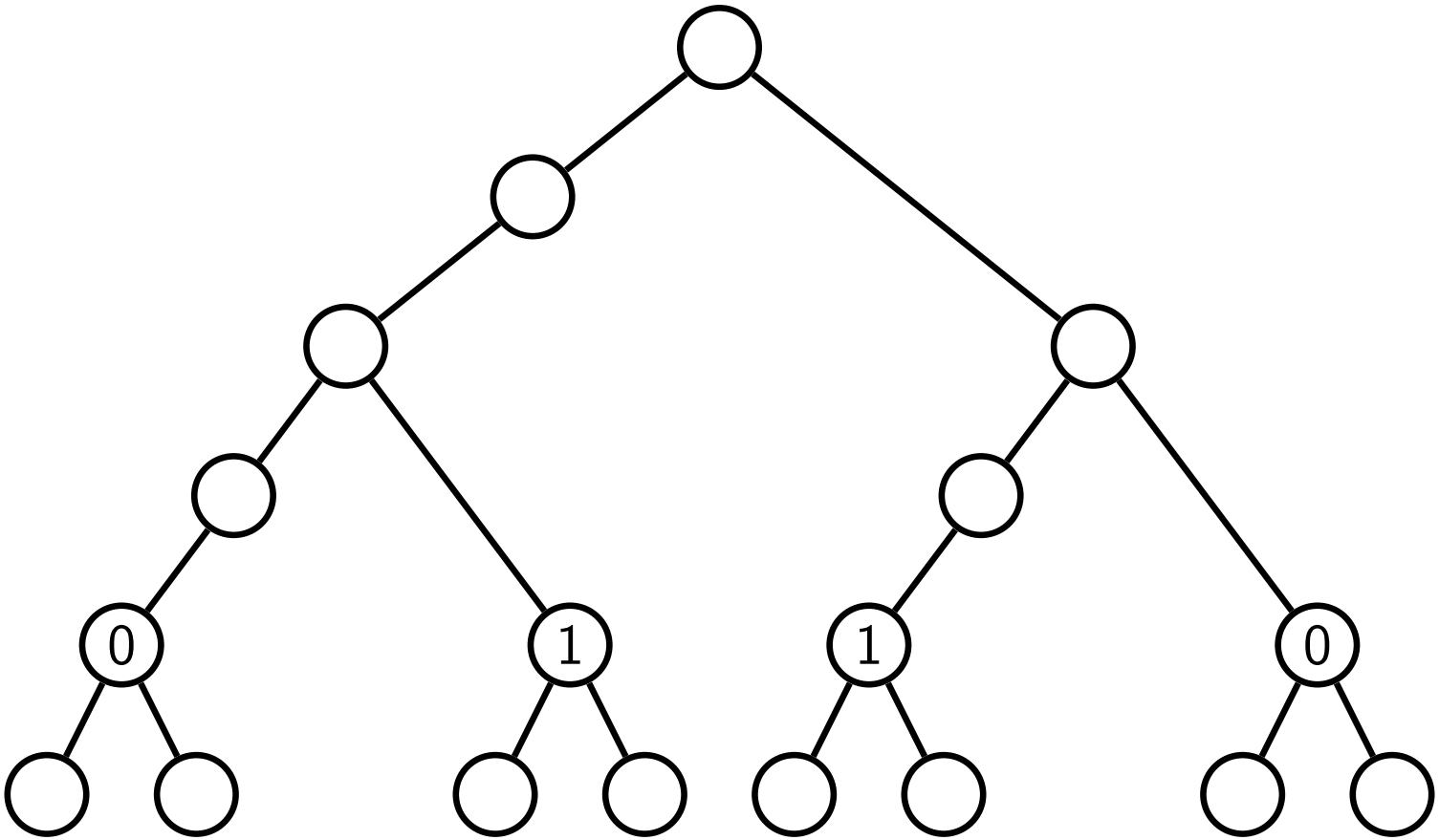

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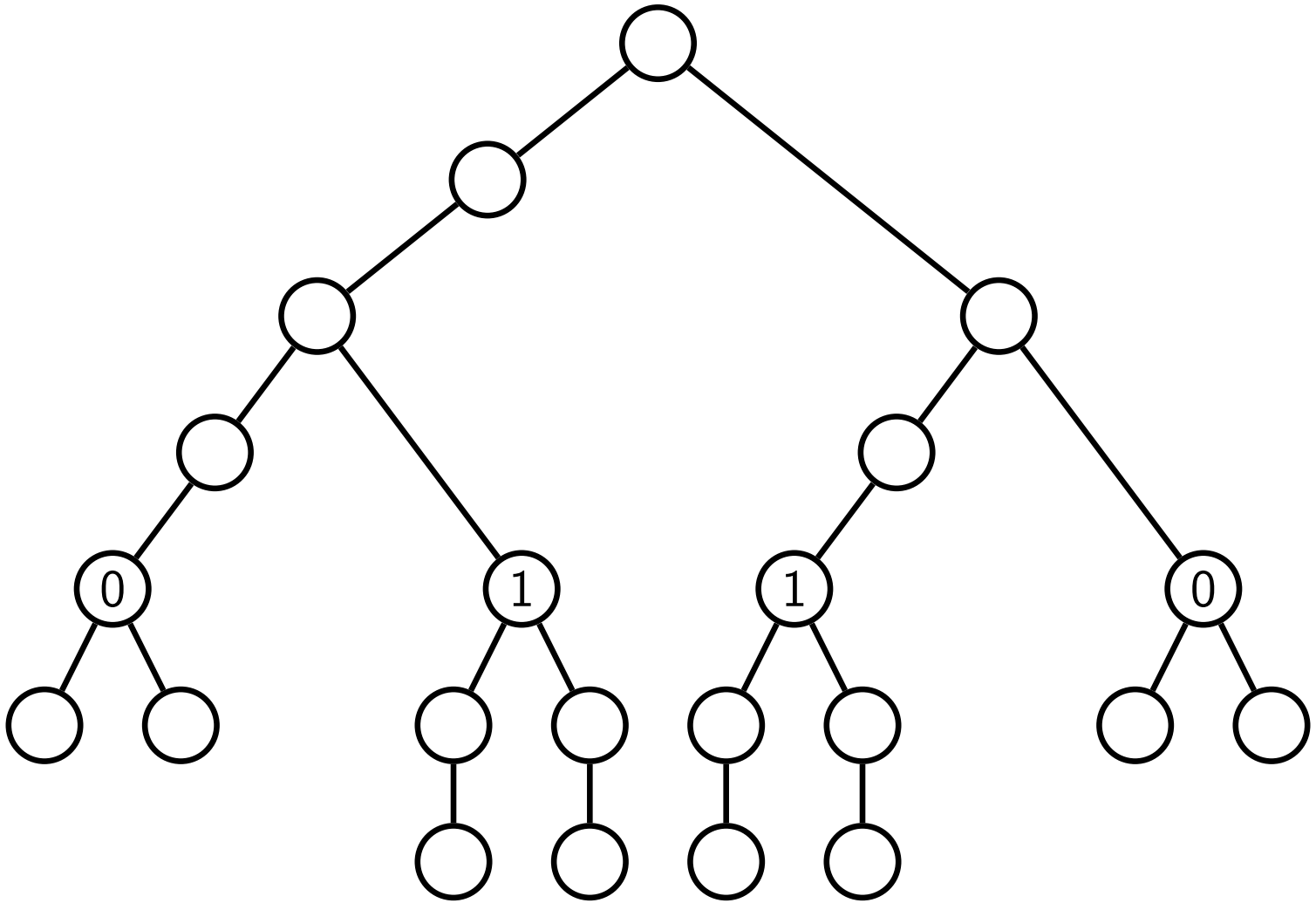

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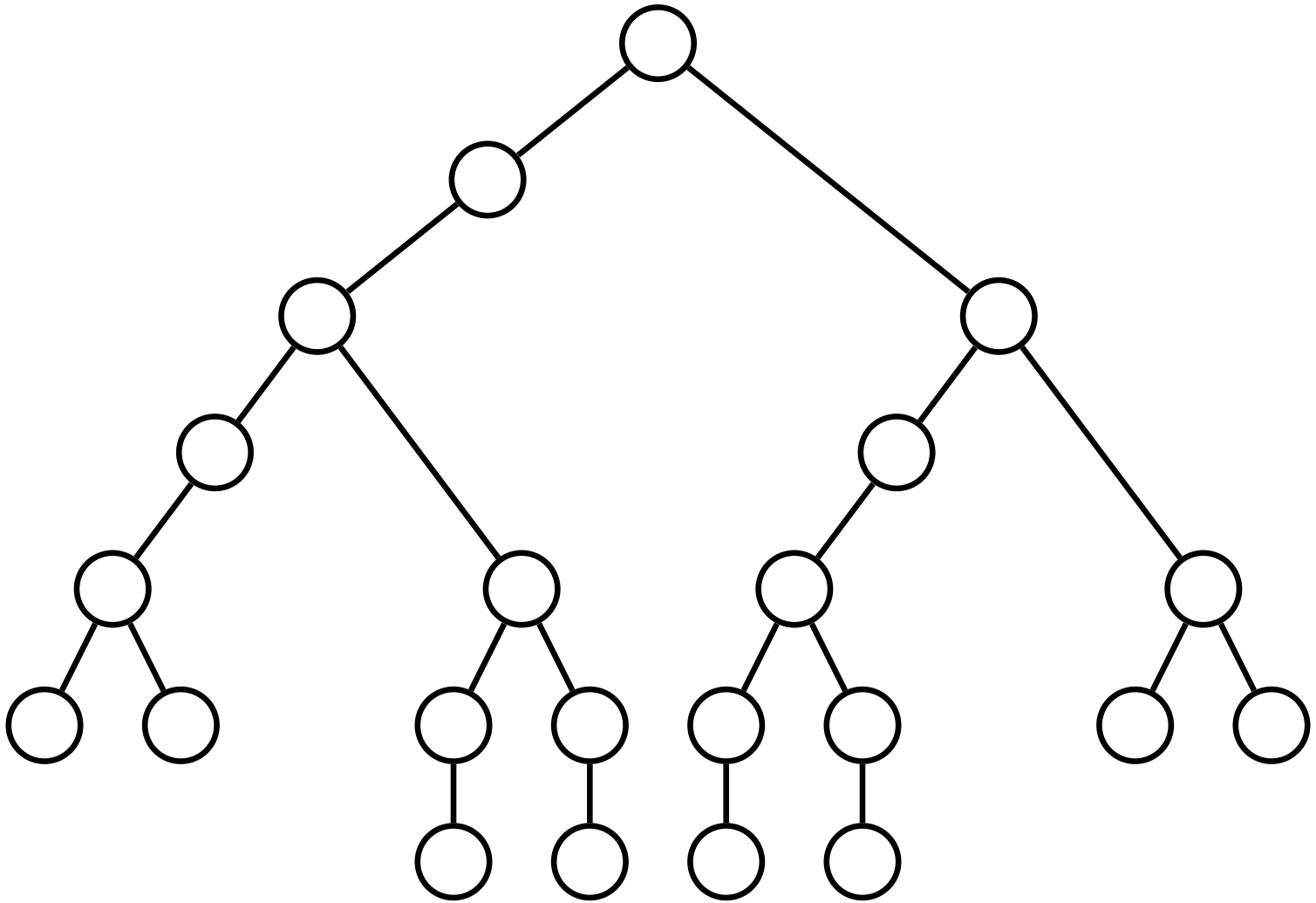
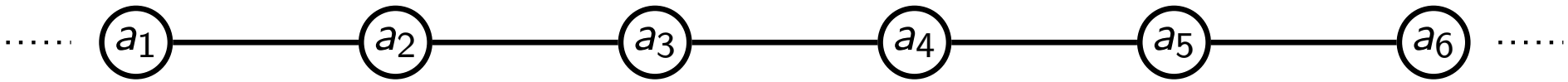


Encoding the Input of the Path



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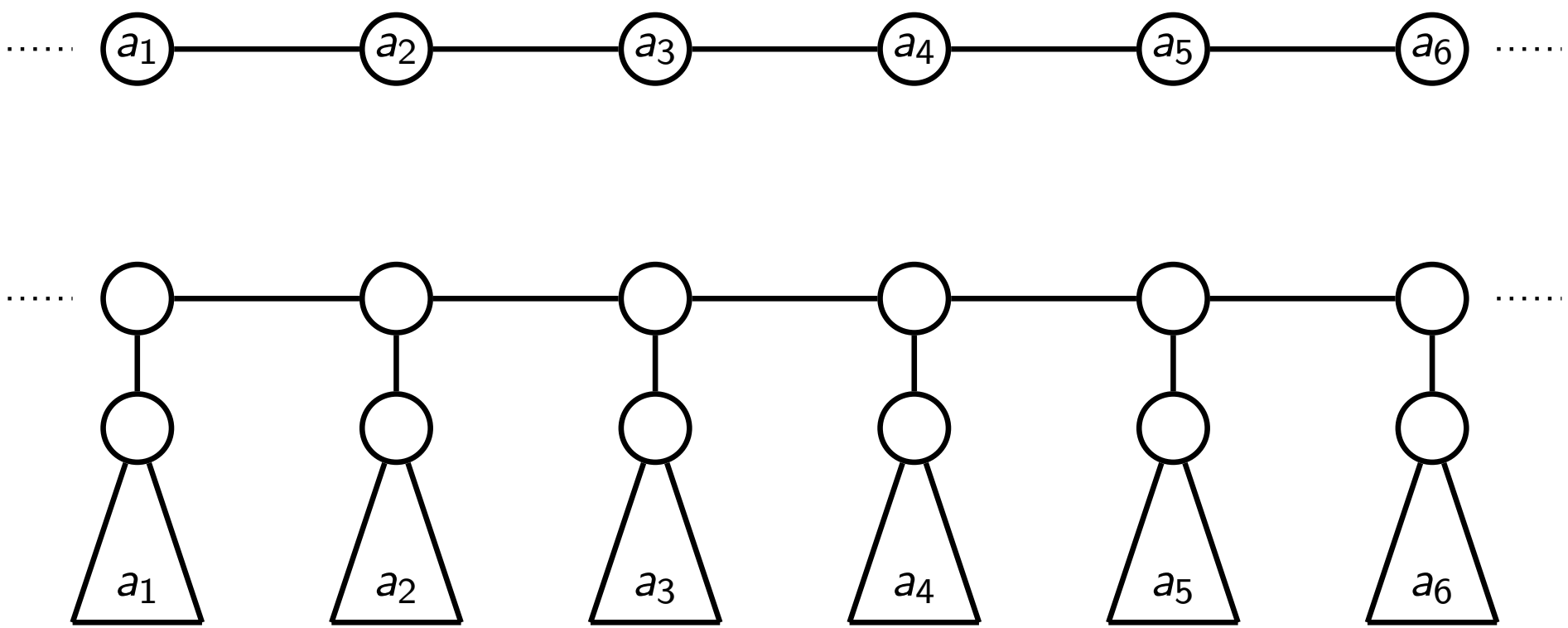

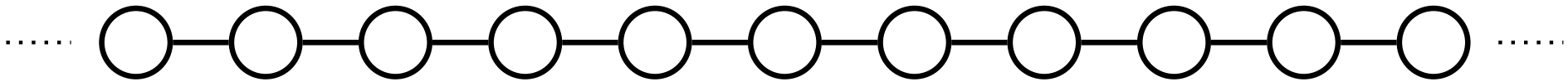

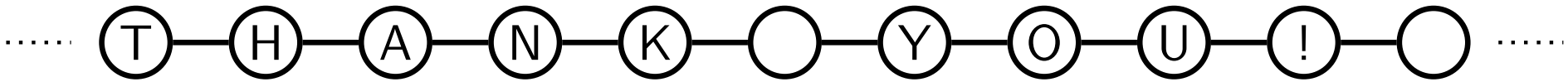
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