Distributed and Non-Distributed Computational Models

Ami Paz
IRIF – CNRS and Paris Diderot University
Message Passing Models

1. Local
2. Congest
3. Clique
Message Passing Models

- A graph $G = (V, E)$ representing the network’s topology
- $n$ unbounded processors, located on the nodes
- Communicating on the edges
- Synchronous network

- Compute / verify graph parameters
The **Local** Model

- **Unbounded** messages
- Solving local tasks:
  - Coloring
  - MST
  - MIS
- Anything* solvable in $O(D)$ rounds
Two Examples

- **Triangle detection**
  - Easy, in one round
  - Send all your neighbors your list of neighbors

- **Computing the diameter** $D$
  - Takes $\Theta(D)$ rounds
Diameter Lower Bound

- Computing $D$ takes $\Omega(D)$ rounds
  - Indistinguishability argument

\[ D = \frac{n}{2} \]

\[ D = n - 1 \]
Diameter Lower Bound

- Computing $D$ takes $\Omega(D)$ rounds
  - Indistinguishability argument

View after $n/2 - 1$ rounds

Cannot distinguish $D = n/2$

View after $n/2 - 1$ rounds

$D = n - 1$
The **CONGEST** Model

- Bounded message size; typically $b = O(\log n)$
- All **LOCAL** lower bounds still hold
- Some **LOCAL** algorithms still work
  - But not all!
**Congest – Typical Lower Bound [HW12]**

- Communication complexity problem
- Inputs encoded by a graph
- Split the graph between Alice and Bob
- CC lower bounds imply message lower bounds

Disjointness on $\Theta(n^2)$ bits.

Diam 2 or 3?
- Diam 2 – disjoint
- Diam 3 – not disjoint

$\tilde{\Omega}(n)$ rounds are needed
**Congest – Another Lower Bound**

$\Omega(\sqrt{n}/b)$ lower bound

Verification: MST, bipartiteness, cycle, connectivity…

Approximation: MST, min cut, shortest s-t path…

Verifying MST is harder than finding one!
So Far:

- **Local model:**
  - Unbounded messages
  - Everything is solvable in $O(D)$ rounds

- **Congest model:**
  - Message = $O(\log n)$ bits
  - Lower bounds of $\tilde{\Omega}(\sqrt{n} + D)$
  - Tight for many problems

- Question: is $\Omega(\sqrt{n})$ due to congestion?
The **CLIQUE** Model

- All-to-all message passing – a clique network
- Diameter of 1
- No distance – only congestion

- MST in $O(\log^* n)$ rounds [GP16]
  - Fast triangle detection, diameter, APSP, …
**CLIQUE – Lower Bound?**

- Diam = 1
- Larger set – more outgoing edges

- No nontrivial lower bound is known
- Simple counting argument [DKO14]
  - many functions need $n - 5 \log n$ rounds
Parallel Systems
Parallel Systems

- $n$ synchronous processors, $k$ inputs to each
- Connected by a communication graph
- Typical graphs:
  - Clique
  - Cycle
  - Torus (Grid)
- Known topology, known identities
- Bounded message size
- Bounded memory
- Bounded computational power
Parallel vs. **Congest**

- **Parallel is more restrictive:**
  - Bounded memory
  - Bounded computational power

- **Different focus:**
  - Specific communication graphs
  - Algebraic questions vs. graph parameters
Circuits
Circuits

- **Algebraic** computation model
- A computation graph (circuit) composed of:
  - Inputs, output, and operation gates
- Represent many algorithms:
  - Matrix multiplication, determinant, permanent
- **Complexity measures**:
  - Depth, number of gates, fan-in, fan-out
Circuits Families

- Arithmetic circuits
- Boolean circuits
- Boolean circuits augmented with:
  - mod $m$ gates
  - Threshold gates
  - ...

![Diagram of a circuit with a top mod 3 gate and several AND gates connected to it.]
Circuits Lower Bounds

- What can be computed in constant depth?
- Counting argument:
  - Many functions cannot be computed using Boolean circuits
  - ... or even using augmented circuits
- But:
  - No explicit function is known
Circuits ⇔ Clique
**CLIQUE** vs. Circuits

- **CLIQUE** can simulate circuits [DKO14]
  - Each node simulates a set of gates in a layer
  - Circuit’s depth = # of rounds
**CLIQUE vs. Circuits**

- **Main idea:**
  - Simulate each layer of the circuit in $O(1)$ rounds
Clique vs. Circuits

- Main idea:
  - Simulate each layer of the circuit in $O(1)$ rounds
Clique vs. Circuits

Main idea:
- Simulate each layer of the circuit in $O(1)$ rounds
Clique vs. Circuits

- Main idea:
  - Simulate each layer of the circuit in $O(1)$ rounds
**Clique vs. Circuits**

- **Clique** can simulate circuits
  - Non-constant rounds lower bound for the **Clique** \(\Rightarrow\)
    Non-constant depth lower bound for circuits

- There is also a reduction in the other direction [DKO14]
  - A circuit can simulate the **Clique**
Parallel ⇔ Clique
Matrix Multiplication

- Base for many algebraic problems
- Thoroughly studied in parallel computing
- Several algorithms:
  - different topologies, input / output partitions

\[ Q \times S = T \]
Matrix Multiplication

This talk:
- The 3D algorithm [ABG+95]
- For $n \times n$ matrices and $n$ processors
- Adaptation of parallel algorithm to the Clique [CHK+16]
Matrix Multiplication

- Parallel 3D algorithm \( \Rightarrow \)
- **CLIQUE** matrix multiplication in \( O(n^{1/3}) \) rounds
  - Implies triangle detection, \( D, \) APSP, …
  - In similar time [CHK+16]

\[
Q = S \cdot T
\]
Fast Matrix Multiplication

- **Standard** matrix multiplication:
  - Compute $n^2$ entries, each need $n$ multiplications
  - Total: $\Theta(n^3)$ time

- There exist **faster algorithms**:
  - Strassen $O(n^{2.807})$ [1969]
  - Coopersmith-Vinograd $O(n^{2.376})$ [1990]
  - ...
  - Le Gall $O(n^{2.373})$ [2014]

- Can be implemented in the **CLIQUE**
  - Distributed matrix multiplication in $O(n^{0.158})$ rounds
Some Results & Conclusion
Triangle Detection in the **Clique**

1. Combinatorial algorithm:
   - $O\left(n^{1/3}\right)$ rounds [DLP12]

2. Reduction from circuits for matrix multiplication:
   - $(n^{\omega-2}) \approx O(n^{0.373})$ rounds, randomized [DKO14]

3. Using a technique from parallel matrix multiplication:
   - $O\left(n^{1-2/\omega}\right) \approx O(n^{0.158})$ rounds [CHK+16]

   - 2,3 Imply similar complexities for:
     - APSP, diameter, girth

---

**Sequential matrix multiplication:** $O(n^\omega)$ operations
Conclusion

- Several models:
  - Message passing
    - *Local, Congest and Clique*
  - Parallel systems
  - Circuits
    - Arithmetic, Boolean, augmented
- Many connections and similarities
- Approach different questions
- Using different techniques

Thank You!