On The Colouring Problem In The Physical Local Model

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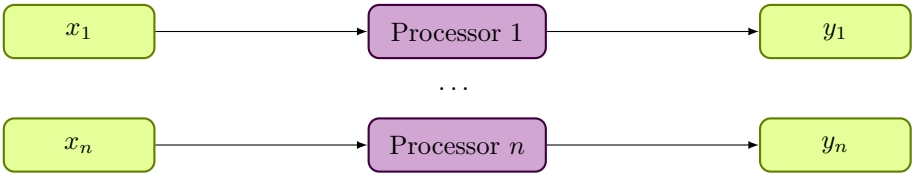
Centralised protocol

Distributed protocol



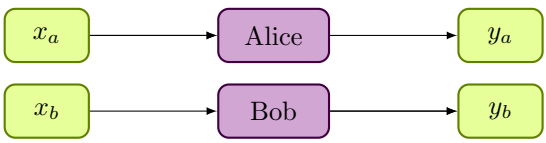
A distributed protocol may use :

- **no randomness**: $\mathbb{P}(y_i^*|x_i^*) = 1$, $\mathbb{P}(y_i^*|x_i) = 0$ for all $x_i \neq x_i^*$.
- **local randomness**: $\mathbb{P}(y_1, ..., y_n|x_1, ..., x_n) = \prod_{i=1}^{n}(y_i, |x_i, \lambda_i)$.
- **shared randomness**: $\mathbb{P}(y_1, ..., y_n|x_1, ..., x_n) = \sum_\lambda \mathbb{P}(\lambda) \prod_{i=1}^{n} \mathbb{P}(y_i|x_i, \lambda)$.
- **quantum entanglement**

[**Bell 64**] : Existence of correlation arising from quantum mechanics that cannot be modelled by a "local hidden variable theory", i.e.

"**shared randomness** $<$ **quantum entanglement**"

CHSH game



Winning condition :

$$y_a \oplus y_b = x_a \wedge x_b$$

Probability of winning :

- Using shared randomness : at most 0.75.
- Using a quantum "Bell state" : $\cos^2(\pi/8) \approx 0.86$.

Correlations arising from the quantum solution are **non-signaling**, i.e. the output of Alice doesn't give any information on the input of Bob and vice-versa.
Mathematically :

$$\sum_{y_b} \mathbb{P}(y_a, y_b | x_a, x_b) = \sum_{y_b} \mathbb{P}(y_a, y_b | x_a, x'_b) = \mathbb{P}(y_a | x_a)$$

and

$$\sum_{y_a} \mathbb{P}(y_a, y_b | x_a, x_b) = \sum_{y_a} \mathbb{P}(y_a, y_b | x'_a, x_b) = \mathbb{P}(y_b | x_b)$$

We have

$$\textbf{Classical} \subsetneq \textbf{Quantum} \subsetneq \textbf{Non-Signaling}$$

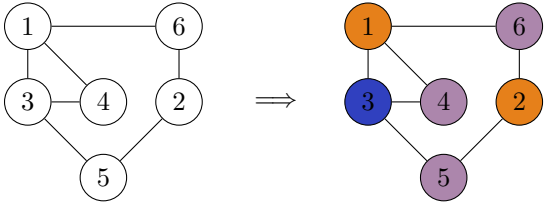
$\Rightarrow$ Not Non-Signaling implies not Quantum

$\Rightarrow$ [**Arfaoui 14**] showed that for 2 players with binary input and ouput and output condition $\neq y_a \oplus y_b$ the best non-signaling probability distribution is classical.

Suppose we have a graph $G = (V, E)$ modelling a communication network.

LOCAL model

- Every node has a (unique) **identifier**.
- One **round** of communication: send & receive information to neighbours & do computation.
- k rounds of communication $\Leftrightarrow$ exchange with neighbours at distance $\leq k$ and do computation
-"Infinite" local computing power and bandwith.



Distributed Colouring Problem in the LOCAL model

How many rounds of communication are necessary and sufficient for q-colouring a graph ?

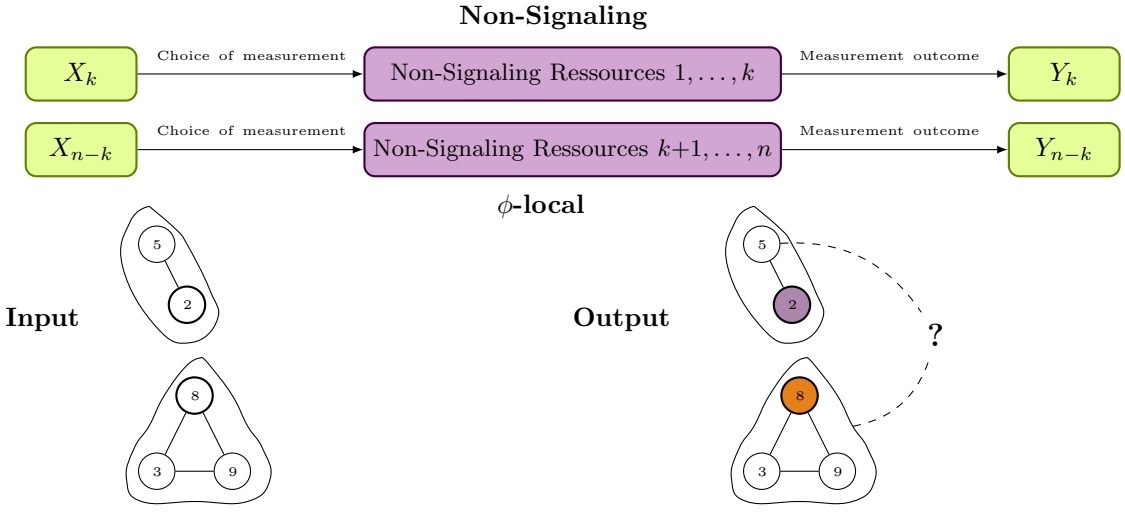
E.g. $q = \Delta + 1$ and graph=cycle or path.

[Cole & Vishkin 86] : $\log^*(n)$ rounds of communcation are sufficient.

[Linial 92] : $\log^*(n)$ rounds of communication are necessary.

Physical Locality

[Gavoille, Kosowki & Markiewicz 09] : Non-Signaling + LOCAL = **ϕ-local**



Physical Locality
An important property

[Barrett, Noah Linden, Massar, Pironio, Sandu, Popescu & Roberts 05] : if the non-signaling property is satisfied for a coalition of $n-1$ players, then it is satisfied for any sub-coalition of $k < n-1$ players.

$$\begin{aligned}\sum_{y_2,y_3} \mathbb{P}(y_1,y_2,y_3|x_1,x_2,x_3) &= \sum_{y_2}\sum_{y_3} \mathbb{P}(y_1,y_2,y_3|x_1,x_2,x_3') \\ &= \sum_{y_3}\sum_{y_2} \mathbb{P}(y_1,y_2,y_3|x_1,x_2',x_3') \\ &= \sum_{y_2,y_3} \mathbb{P}(y_1,y_2,y_3|x_1,x_2',x_3')\end{aligned}$$

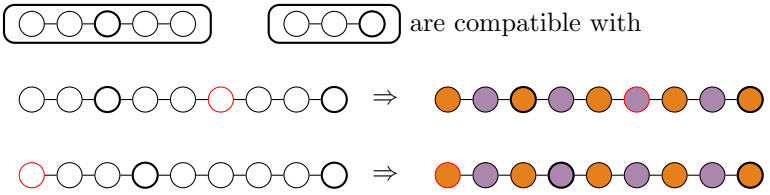
If a coalition of $m \leq n-1$ players such that there is more than one global output corresponding to their inputs satisfies ϕ-local, so does any sub-coalition of $k < m$ players.

$\Rightarrow$ **To check for non-signaling/ϕ-local**, need only look at biggest coalitions.

$\Rightarrow$ **To show non-signaling/ϕ-local isn't satisfied**, small coalitions are sufficient.

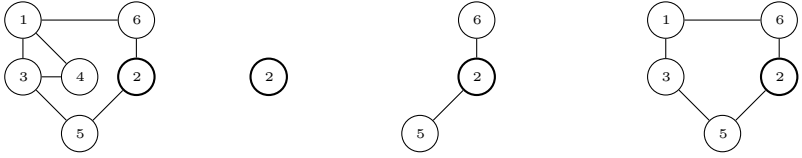
Physical Locality
An example

Example : 2-colouring in an undirected n-path is ϕ-local($\lfloor \frac{n}{3} \rfloor$).
E.g. $n = 9$, $\lfloor \frac{n}{3} \rfloor = 3$



Physical Locality
A formal definition

We define the k-neighbourhood $\mathcal{N}_k(v)$ of a vertex v as the set of vertices at distance less than or equal to k.



$N_0(2)$ $N_1(2)$ $N_2(2)$

Definition

Let $G = (E, V)$ be a directed or undirected graph and let $(X_v)_{v \in V}$ be a stochastic process. $(X_v)_{v \in V}$ is said to be ϕ-local(k) if, for every $m \leq |V|$ and sets of vertices $S = \{s_1, ..., s_m\}$ and $T = \{t_1, ..., t_m\}$ such that the graphs induced by $S \cup \mathcal{N}_k(S)$ and $T \cup \mathcal{N}_k(T)$ are isomorphic, we have

$$\mathbb{P}(X_S) = \mathbb{P}(X_T)$$

Let $(X_n)_{n\in\mathbb{Z}}$ be a stochastic process on $\mathbb{Z}$.

(1) "Radius" of the information necessary to compute the value of a X_n?

r-block factor

$(X_n)_{n\in\mathbb{Z}}$ is r-block factor of an iid process $(Y_n)_{n\in\mathbb{Z}}$ if $X_n = f(Y_n, ..., Y_{n+r-1})$ for every n.

$\Rightarrow$ Distributed computability in LOCAL model.

(2) "Radius" beyond which no information on the value of X_n escapes?

k-dependence

$(X_n)_{n\in\mathbb{Z}}$ is k-dependent if the distributions $(X_{\leq n})$ and if $(X_{\geq m})$ are independent for every n, m with $|m - n| > k$.

Question : is k-dependence the same as ϕ-local(ℓ) for some ℓ ?

It is easy to verify that: k-dependent $\Rightarrow$ directed ϕ-local(k), undirected ϕ-local($\lfloor k/2 \rfloor$).
[**Holroyd & Liggett 15**], [**Holroyd & Liggett 16**] proved the following

k-dependent colouring of $\mathbb{Z}$

There exists a 1-dependent q-colouring process for every $q \geq 4$ and a 2-dependent 3-colouring process but no 1-dependent 3-colouring process.

Directed ϕ-local(1) 4-colouring and undirected ϕ-local(1) or directed ϕ-local(2) 3-colouring of the infinite path is possible.

1-dependent colouring of other graphs

For the following graphs G, the least possible number q of colours such that G admits a 1-dependent colouring is:
- $G = \mathbb{Z}^2$: 9
- $G = \mathbb{Z}^3$: 12
- $G =$ infinite Δ-regular tree, $\Delta \geq 2$: $\frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}}$

Question

is k-dependence the same as ϕ-local(ℓ) for some ℓ ?

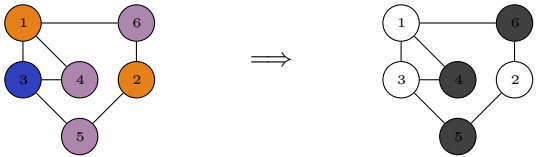
We studied this question by looking at the colouring problem on the path graph.

Our results:

- In general, no.
- In the case of the colouring problem: not exactly, but beyond a certain value of n, ϕ-local(1) 3-colouring of a directed n-path is not possible.

Our results

Idea : given a graph $G = (E, V)$ along with a q-colouring, choose a colour q^* and replace all its occurrences by 1 and put 0 everywhere else. This induces a binary process $(J_v)_{v \in V}$ which is k-dependent if the original colouring is.



$p^* = \sup\{p, \exists \text{ 1-dependent binary process s.t. for every } v \in V \ \mathbb{P}(J_v = 1) = p\}$, then $q \geq \frac{1}{p^*}$.

Theorem [Holroyd & Liggett 2016]

Let $G = (V, E)$ be a graph, p^* as above and $p \in [0, 1]$. We have $p \leq p^*$ iff $Z_A(-p) \geq 0$ for every finite $A \subset V$, where Z_A is the independence polynomial of the induced subgraph of A.

Thus on the n-path, $p^* < 1/3$ as soon as $n = 5$ and $\lim_{n \to \infty} p^* = \frac{1}{4}$.

Our results

Define $p_1 = \mathbb{P}(1)$, $p_2 = \mathbb{P}(1*1)$, $p_i = \mathbb{P}((1*)^i)$.

Catalan numbers : $c_n = \frac{1}{n+1}\binom{2n}{n}$, in particular $\frac{c_n}{c_{n+1}} = \frac{n+2}{2(2n+1)}$

For the directed path graph of length $n = 2\ell$ or $n = 2\ell + 1$ in the ϕ-local(1) model

- $p_1 \leq \frac{c_\ell}{c_{\ell+1}}$
- $p_2 \leq \frac{c_{\ell-1}}{c_\ell} p_1 \leq \frac{c_{\ell-1}}{c_{\ell+1}}$
- $p_i \leq \frac{c_{\ell-i+1}}{c_{\ell-i}} p_{i-1} \leq \frac{c_{\ell-i+1}}{c_{\ell+1}}$

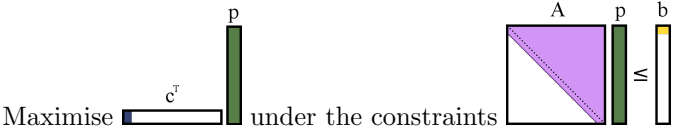
Thus $p_1 < \frac{1}{3}$ as soon as $\ell = 5$, i.e. $n = 10$.

Let $p_i^* = \sup(p_i)$, then $\lim_{n\to\infty} p_1^* = \frac{1}{4}$ and $\lim_{n\to\infty} p_i^* = \frac{1}{4^i} = (p_1^*)^i$.

Our results
Method

The probabilities of each pattern is a linear function of the p_i.

Goal : maximise p_1 such that every probability is between 0 and 1.



Duality theorem : objective function has same value for optimal solutions of dual and primal problems.

We get $p_1 \leq \frac{c_\ell}{c_{\ell+1}} \Rightarrow$ remove first line of A, rearrange and solve again for p_2, etc.

Summary

- ϕ-local model useful for finding if there might be a quantum-classical difference in number of rounds in distributed graph algorithms.
- k-dependence from probability theory is consistent with ϕ-local but stronger.
- $\log^*(n)$ classical rounds necessary and sufficient for solving the distributed graph colouring problem.
- There is a 1-dependent q-colouring probability distribution for $q \geq 4$ and a 2-dependent 3-colouring probability distribution on the path graph.
- Can barely do better in ϕ-local.
- Not easy to study for other families of graphs.

Thank You

For Your Attention