

# Robustness of distances and diameter in an aging network

Timothée Corsini

joint work with A. Casteigts, H. Hocquard and A. Labourel

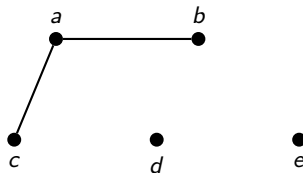
ANR Descartes

November 10th 2021

## Temporal graph

A temporal graph is a graph that changes with time.

→ Can represent interactions : epidemiology, social networks, connected objects...

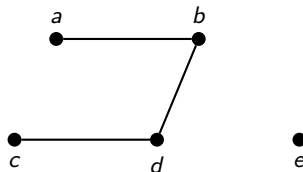


Snapshot  $t = 1$

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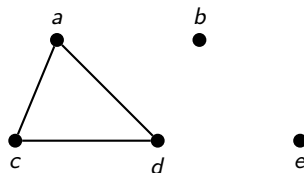


Snapshot  $t = 2$

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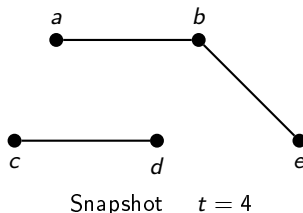


Snapshot  $t = 3$

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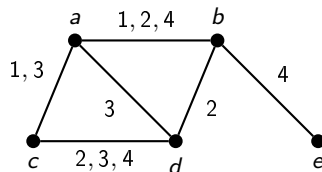
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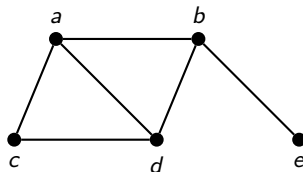
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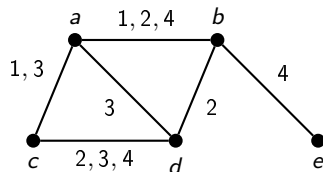


footprint of the temporal graph

## Journey

A journey (or temporal path) is a path whose edges, additionally, use increasing times.

→ the increase in time can either be strict or not

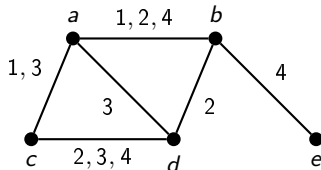




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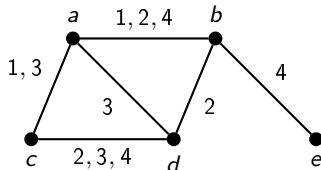


Journey from  $e$  to  $a$  but not a strict journey.

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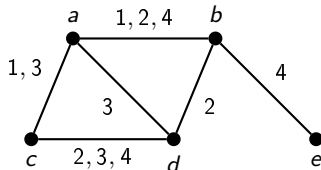


Journey from  $c$  to  $e$  but not from  $e$  to  $c$ .

## Journey

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Not temporally connected.

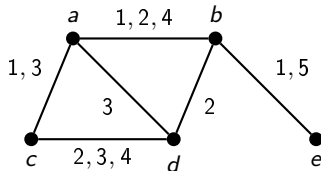
## Temporal connectivity

$G$  is temporally connected if for every ordered pair of vertices  $(u,v)$ , there is a journey from  $u$  to  $v$ . If so, we note  $G \in \mathcal{TC}$ .

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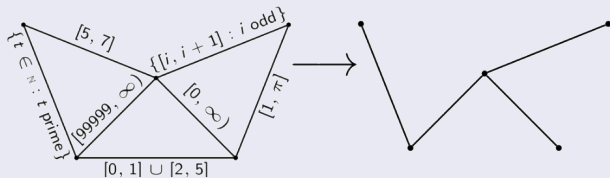


Temporally connected.

## Temporal connectivity

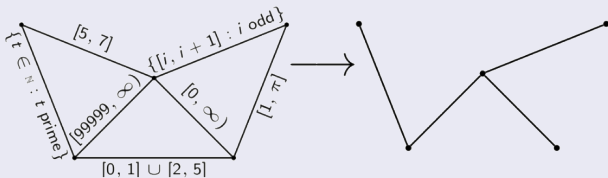
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## Eventual footprint in temporal graphs

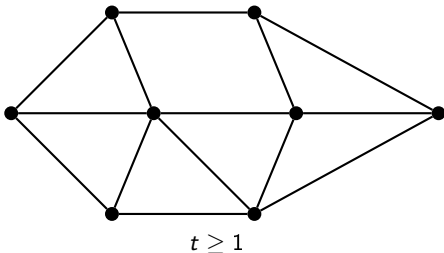


Connected eventual footprint  $\implies G \in \mathcal{TC}^{\mathcal{R}}$ .

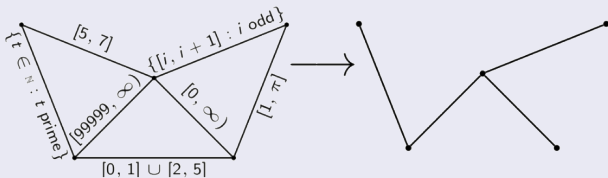
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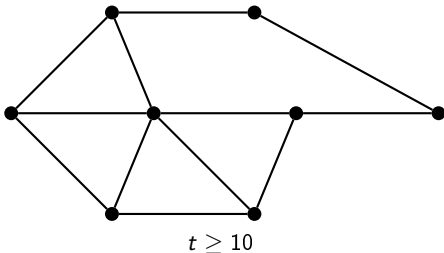
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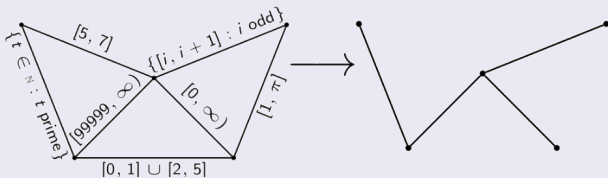
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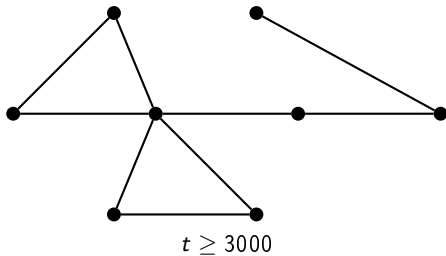
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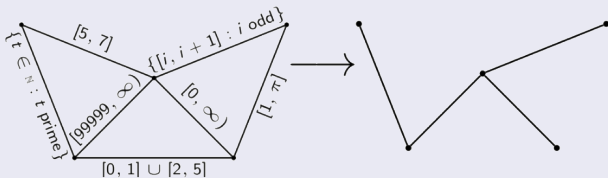


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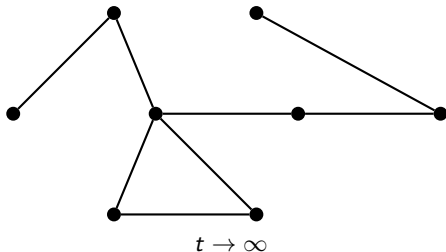




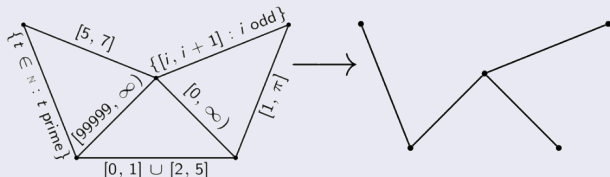
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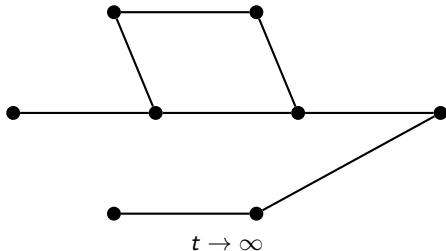
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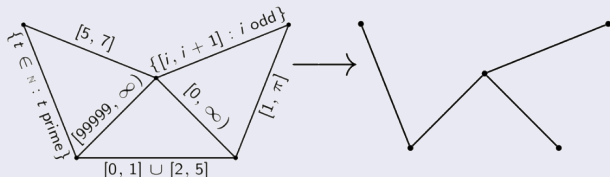


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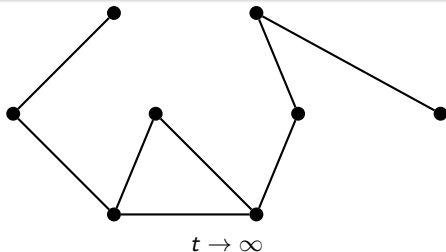


What happens if the eventual footprint is unknown?  
 What kind of properties are preserved?

## Eventual footprint in temporal graphs



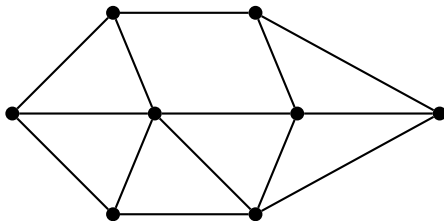
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Robustness [A. Casteigts, S. Dubois, F. Petit, J. M. Robson, 2020]

A property  $P$  is robust in  $G$  if and only if it is satisfied in every connected spanning subgraph of  $G$  (including  $G$  itself).

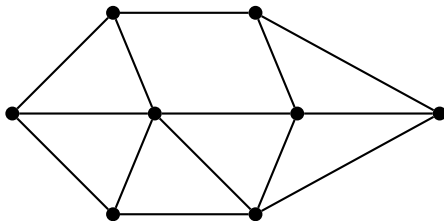


Similar definitions :

- Hereditary : if the property holds in any induced subgraph ;
- Monotone : if the property holds in any subgraph.

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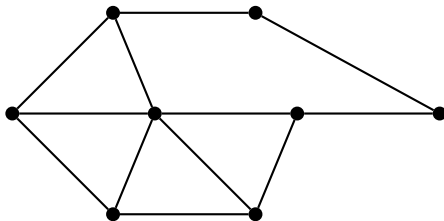


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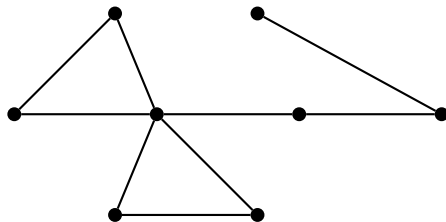


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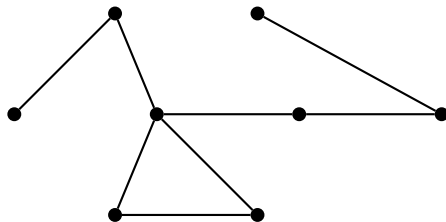


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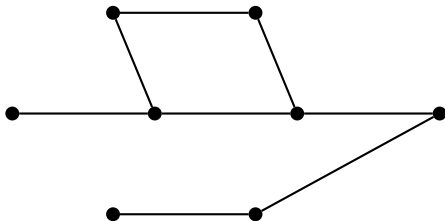
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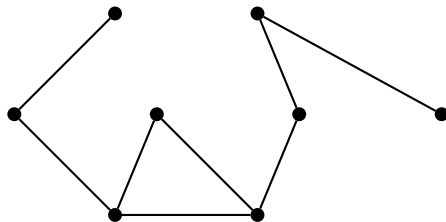


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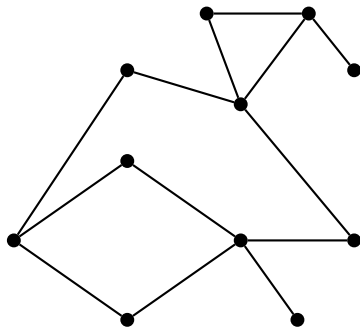
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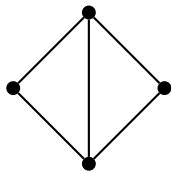
Consider an aging network :



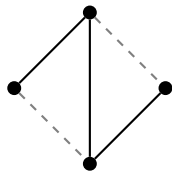
### Problem

Is  $diam(G) = k$  robust ?

# Is the diameter robust ?

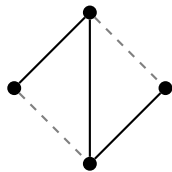


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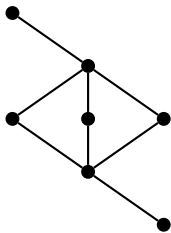


**NO** :  $diam < 3$

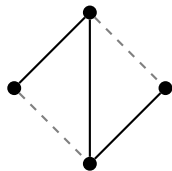
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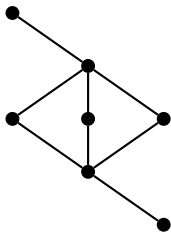
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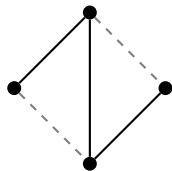


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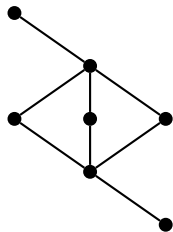


**YES** :  $diam = 4$

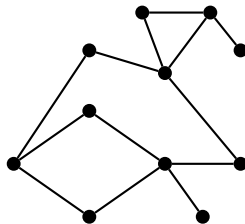
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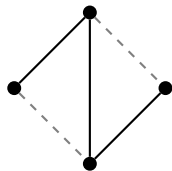


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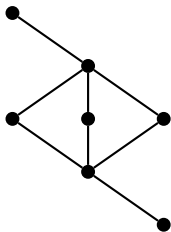




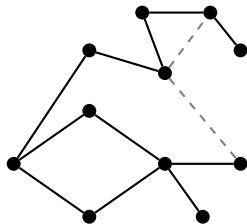
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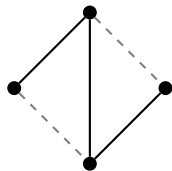


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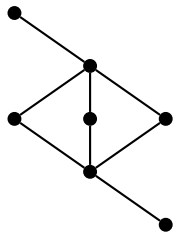


**NO** :  $diam < 8$

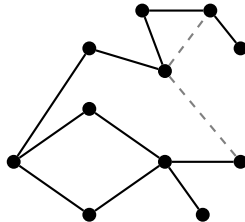
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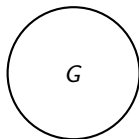


**NO** :  $diam < 8$

$diam(G)$  robust  $\iff$  it is equal to the longest path.

Reminder : deciding if  $G$  has an hamiltonian path is NP-complete.

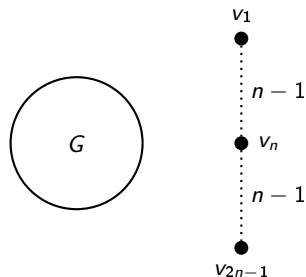
Proof : reducing the hamiltonian path problem to the robustness of the diameter.



Let  $G$  be any graph of order  $n$ .

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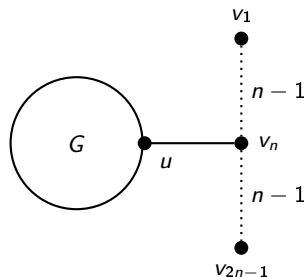
Proof : reducing the hamiltonian path problem to the robustness of the diameter.



A path  $P_{2n-1}$ .

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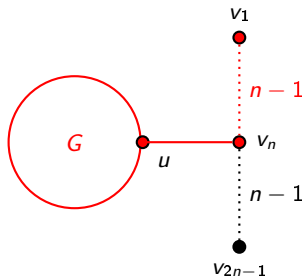


The graph  $H$ .

# Robustness of the diameter is co-NP-complete

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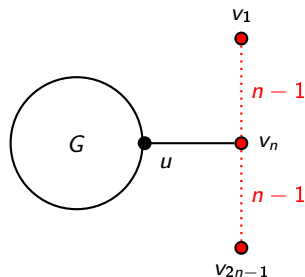


If the diameter is not robust, there is a longer path crossing  $G$ .

# Robustness of the diameter is co-NP-complete

Reminder : deciding if  $G$  has an hamiltonian path is NP-complete.

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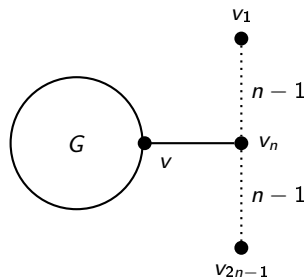


Otherwise, the diameter is robust.

## Robustness of the diameter is co-NP-complete

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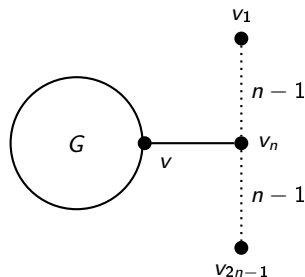
Repeat for each vertex of  $G$ .



## Robustness of the diameter is co-NP-complete

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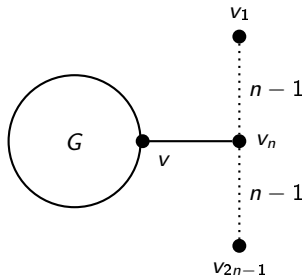


$G$  has an hamiltonian path  $\iff$  one of the graphs  $H$  doesn't have a robust diameter.

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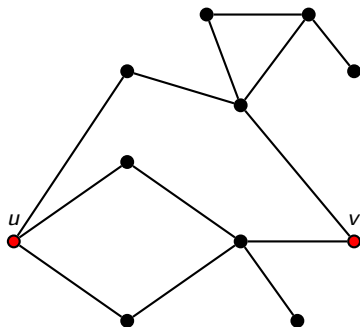
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$G$  has an hamiltonian path  $\iff$  one of the graphs  $H$  doesn't have a robust diameter.

Robust diameter is co-NP-hard.

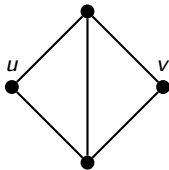
Consider an aging network with two *important* vertices :



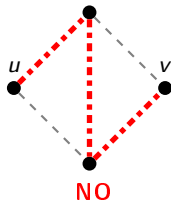
### Problem

Let  $u, v$  be 2 vertices of a connected graph  $G$ ,  $\text{dist}(u, v) = d$ , is  $\text{dist}(u, v) = d$  robust ?

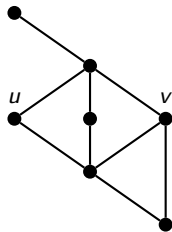
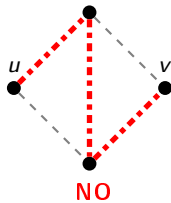
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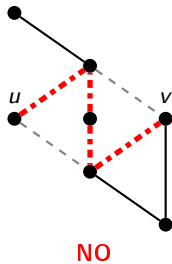
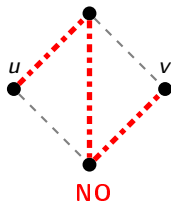
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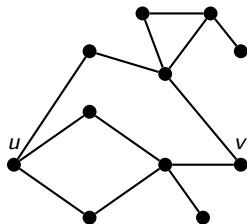
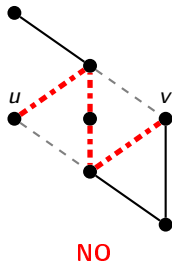
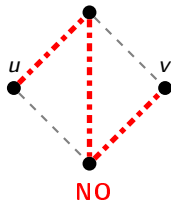
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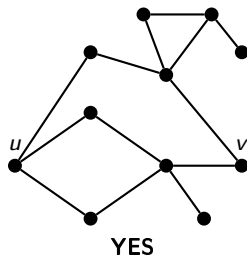
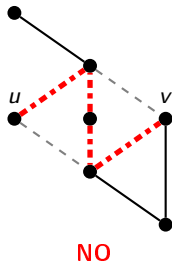
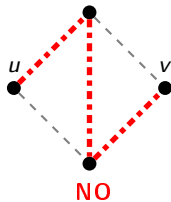


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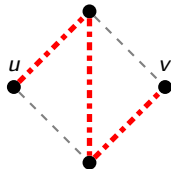




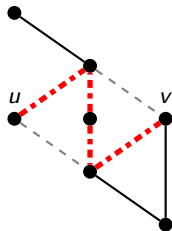
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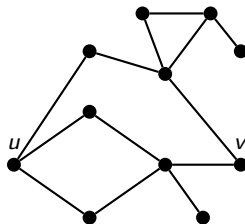
# Is the distance robust?



NO



NO

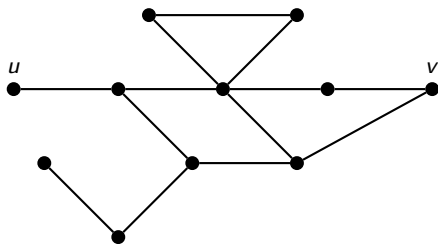


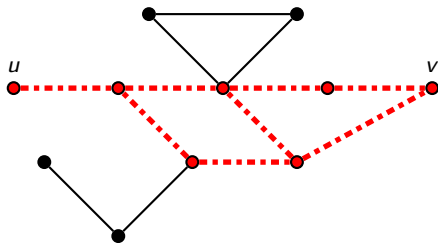
YES

$dist(u, v)$  robust  $\iff$  it is equal to the length of the longest path between  $u$  and  $v$ .

Which graphs have this property?

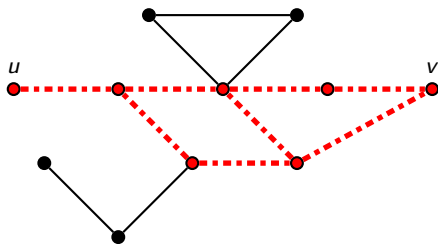
## Distance robustness : remove unnecessary parts of the graph





## Lemma 1

$dist(u, v)$  is robust in  $G$  if and only if it is robust in the graph  $H$  induced by all paths from  $u$  to  $v$ .



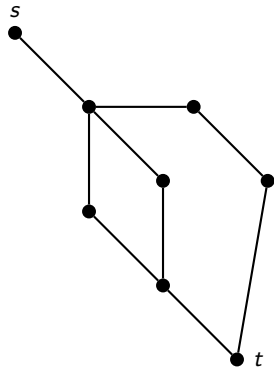
## Lemma 1

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→ what is the nature of  $H$ ?

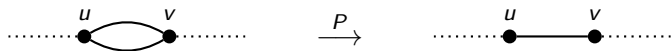
Multiple definitions :

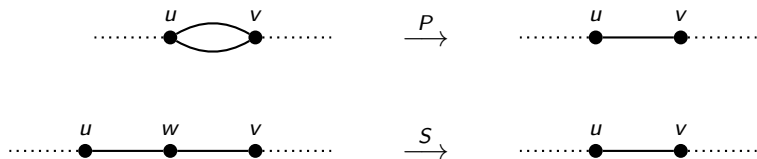
- 1 Composition of series-parallel graphs;
- 2 Subdivision of edges;
- 3 Ear decomposition.



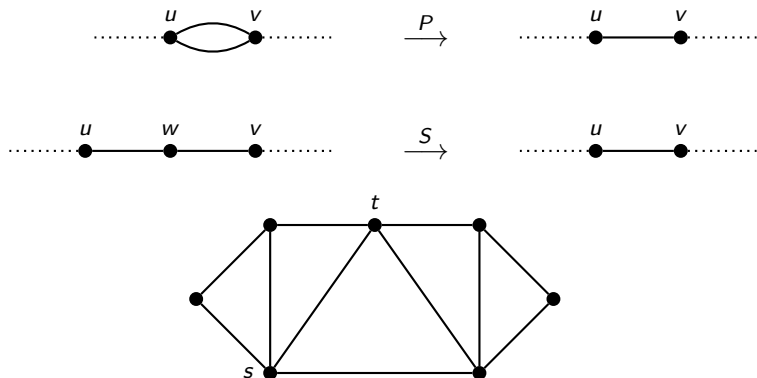
Can be used to model series and parallel electric circuits.

Can be recognised in linear time ! [K. Takamizawa, T. Nishizeki, N. Saito, 1982]

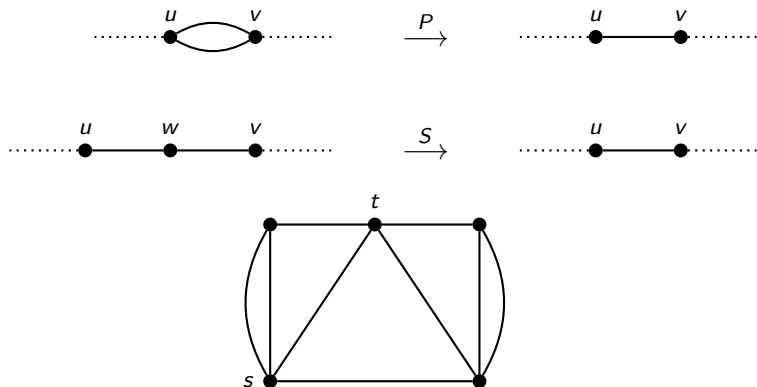




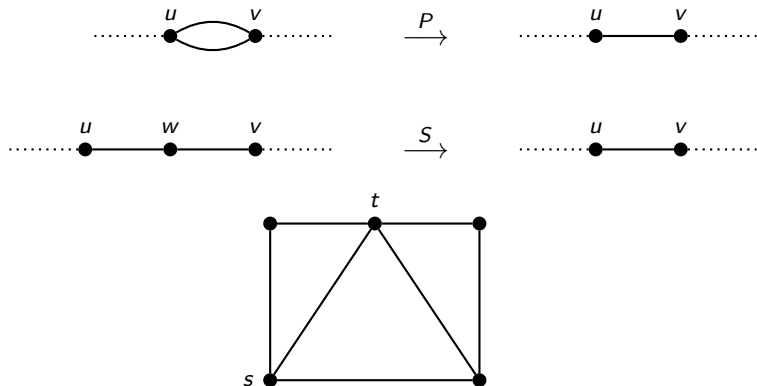




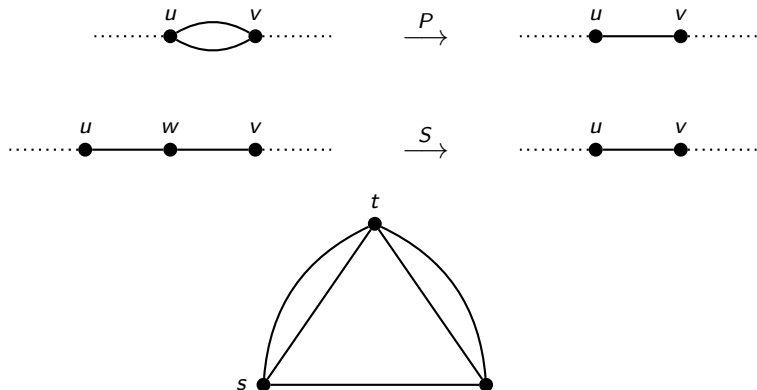
All parallel edges are removed as well as vertices of degree 2.



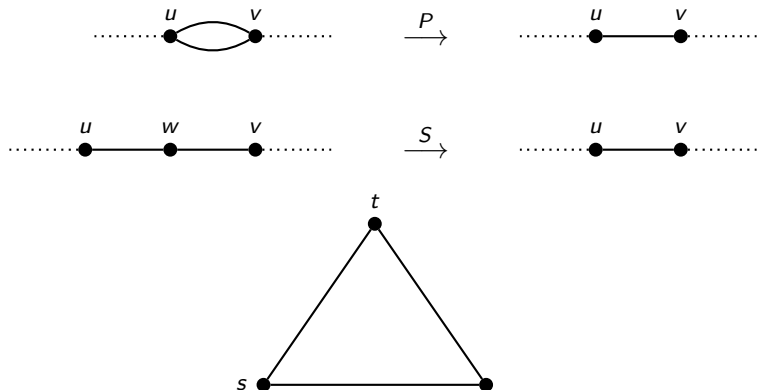
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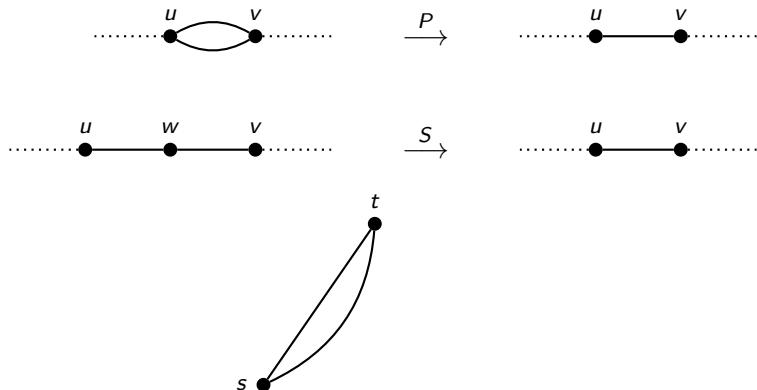
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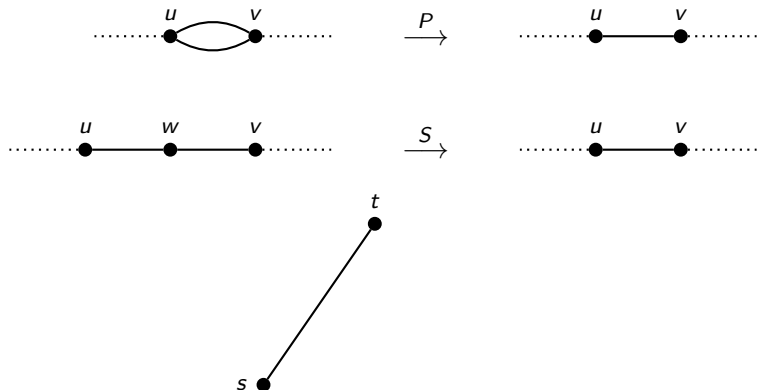
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## Lemma 2 : robustness requires TTSP

Let  $G$  be a multigraph with every vertex  $w$  on a path from  $u$  to  $v$ . If  $\text{dist}(u, v)$  is robust, then  $G$  is a TTSP.



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Suppose that  $G$  is not a TTSP. After the application of  $P$  and  $S$ ,  $G$  should have the following properties :

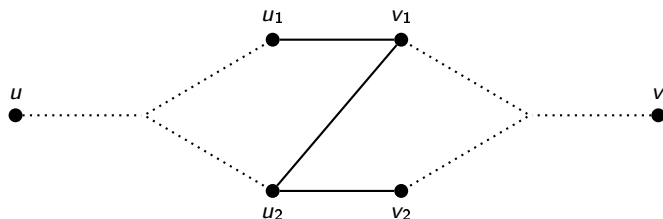
- $G$  is not  $K_2$  ;
- each vertex of  $G$  (except  $u$  and  $v$ ) have a degree  $\geq 3$  ;
- $G$  has more than 3 vertices.

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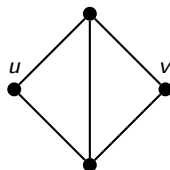


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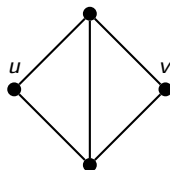
TTSP  $\iff$  forbidden rooted-minor diamond.

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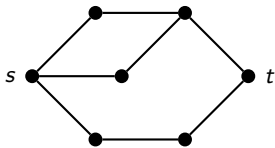
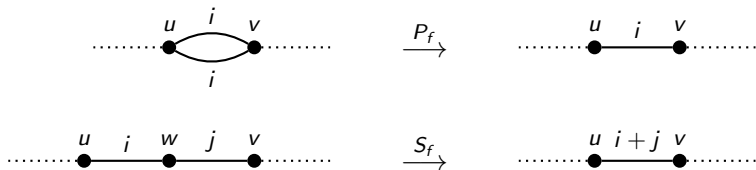
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TTSP  $\iff$  forbidden rooted-minor diamond.

All graphs with robust distance are TTSP, but not all TTSPs have a robust distance.

# TTSP of fixed length

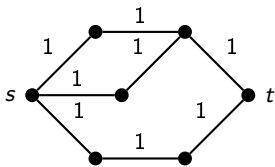
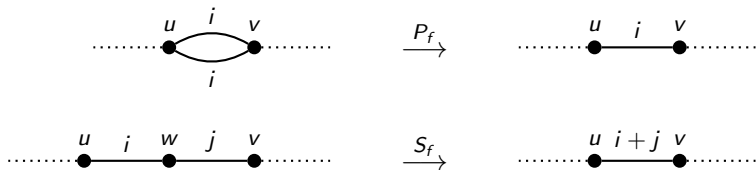


Is  $G$  a TTSP of fixed length?

## TTSP of fixed length

Let  $G$  be a multigraph. By putting a weight of 1 on each edge, if  $G$  can be reduced with the operations  $P_f$  and  $S_f$  to  $K_2$  then it is a TTSP of fixed length between the two remaining vertices.

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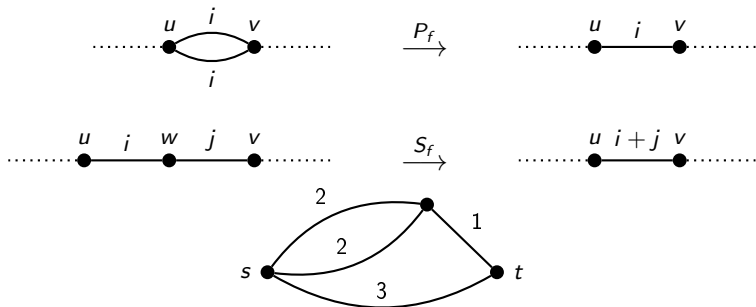


Put a weight of 1 on each edge.

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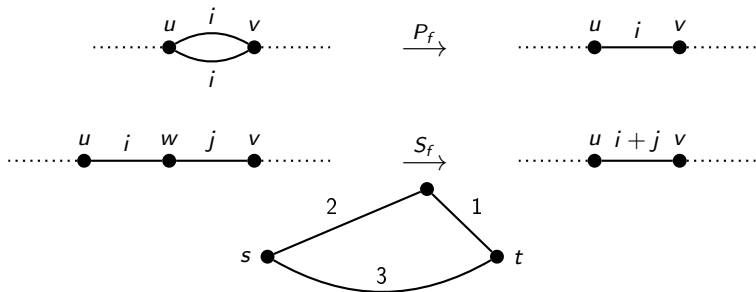


Apply the operations  $P_f$  and  $S_f$  until  $K_2$  is obtained.

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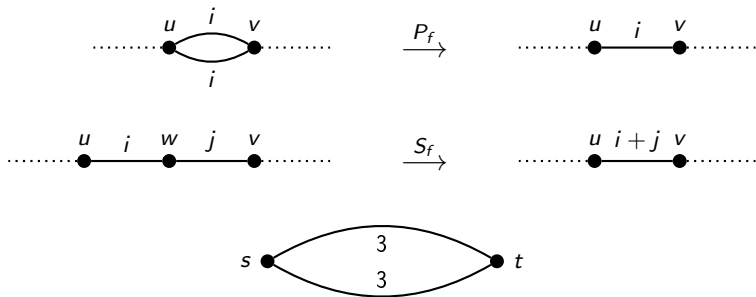
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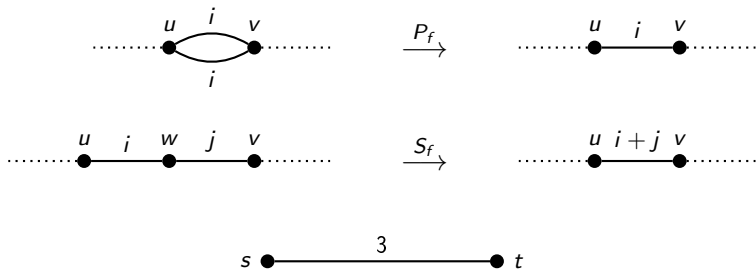


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A weighed  $K_2$  is obtained, thus  $G$  is a TTSP of fixed length.

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### Lemma 3 : TTSP of fixed length and robustness

$G$  is a TTSP (with  $u$  and  $v$ ),  $G$  has a robust distance between  $u$  and  $v$   $\iff$   $G$  is a TTSP of fixed length between  $u$  and  $v$ .

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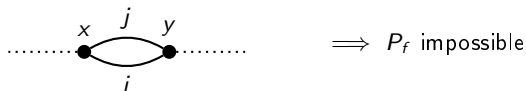
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- $P_f$  cannot be applied on parallel edges  $\iff$  there is a longer path from  $u$  to  $v$ ;
- Thus  $\text{dist}(u, v)$  robust  $\iff$  the TTSP is also a TTSP of fixed length.



What we proved :

- $G$  has a robust distance between  $u$  and  $v \iff H$  induced by the paths from  $u$  to  $v$  has a robust distance ;



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Therefore :

## Theorem

*Let  $G$  be a connected graph,  $u, v \in V(G)$ ,  $\text{dist}(u,v)$  is robust  $\iff$  the graph induced by the paths from  $u$  to  $v$  is a TTSP of fixed length between  $u$  and  $v$ .*

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$\rightarrow$  Distance robustness can be computed in linear time.

- robust diameter  $\implies$  robust distance between the vertices on the periphery ;
- there could be a longer path in the graph !
- all distances are robust  $\iff G$  is a tree.

Results :

- robustness of the diameter : co-NP-complete ;
- robustness of a distance : linear with the use of series-parallel graphs ;
- robustness of the diameter and the distance are mostly unrelated.

Other results :

- robustness of a distance with a multiplicative constant : consider  $dist(u, v) = d$ ,  $c$  a constant value, is the property  $dist(u, v) \leq c \times d$  robust ? Hard for any  $c > 1$ .
- vertices with robust distances form equivalent classes : if  $dist(u, v)$  and  $dist(v, w)$  are robust, then  $dist(u, w)$  is also robust.

Thanks for your attention