

Minor-Universal Graph for Graphs on Surfaces

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Problem

Let \mathcal{F} be a family of finite graphs, and \mathcal{F}_n the family of n -graphs of \mathcal{F} .

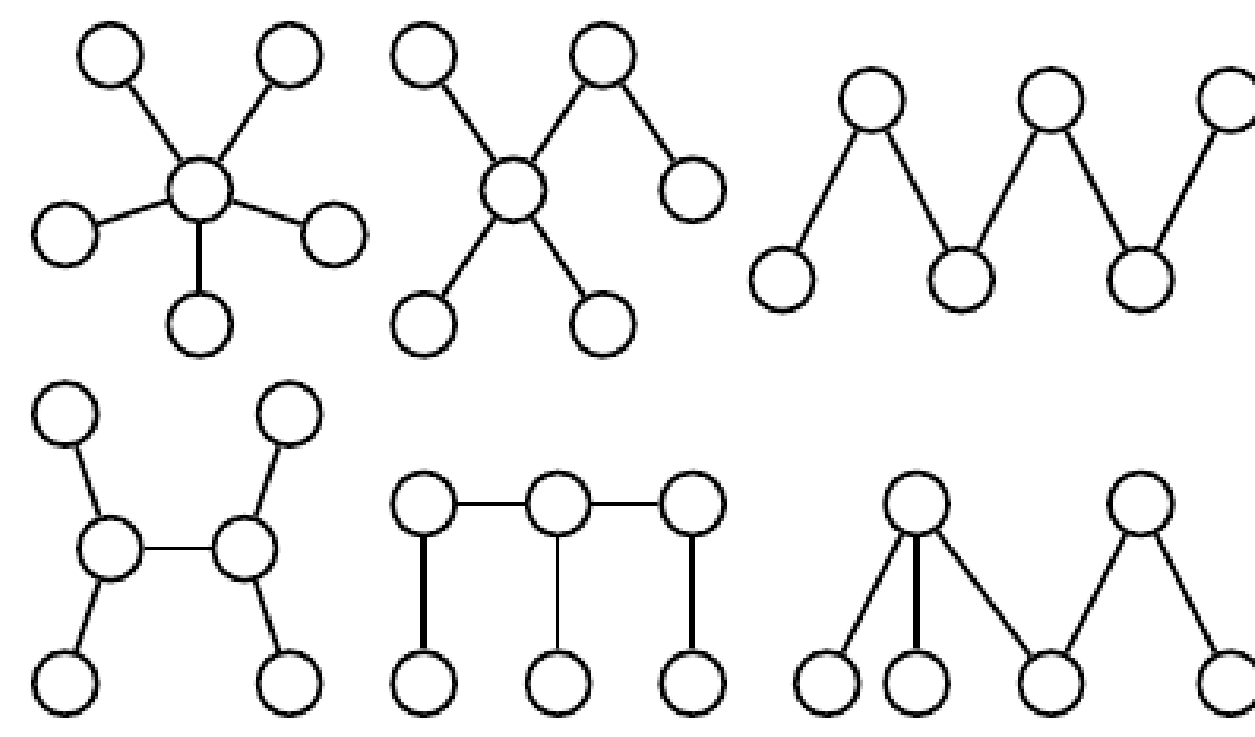
U is **minor-universal** for \mathcal{F}_n if any $G \in \mathcal{F}_n$, G is a minor of U .

Question

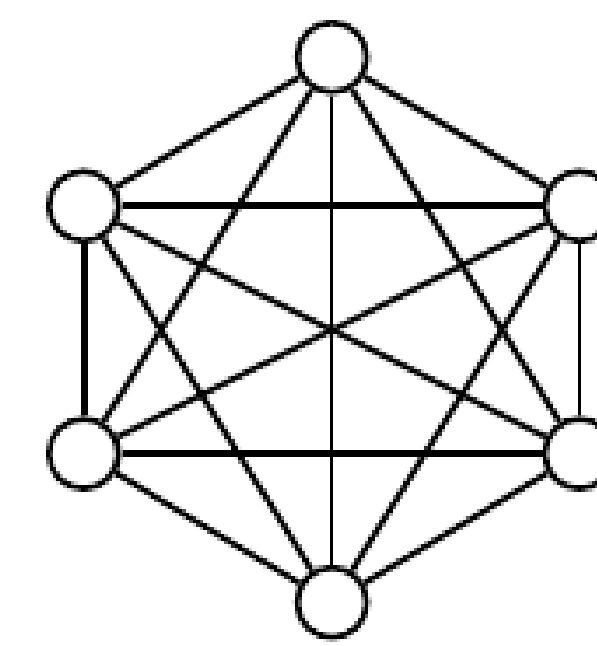
For every n , what is the **smallest number of vertices** of a graph of \mathcal{F} minor-universal for \mathcal{F}_n ?

Example

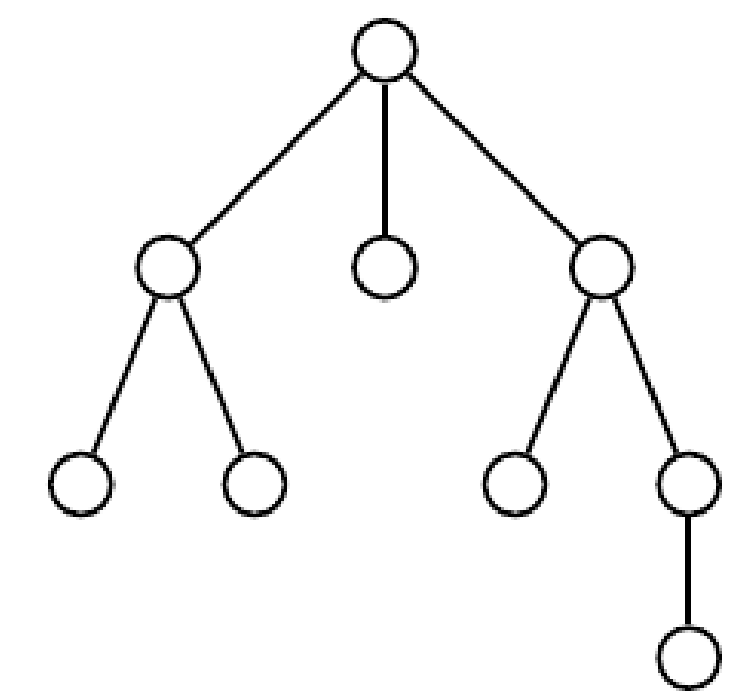
The **6-vertex trees**:



Minor-universal with **no constraint**:



Minor-universal that is a **tree**:



For Trees [GKLPS18]

For every n , a **tree** minor-universal for the n -vertex trees has size $\Omega(n^{1.724\dots})$ and $O(n^{1.895\dots})$.

For Planar graphs [RST94]

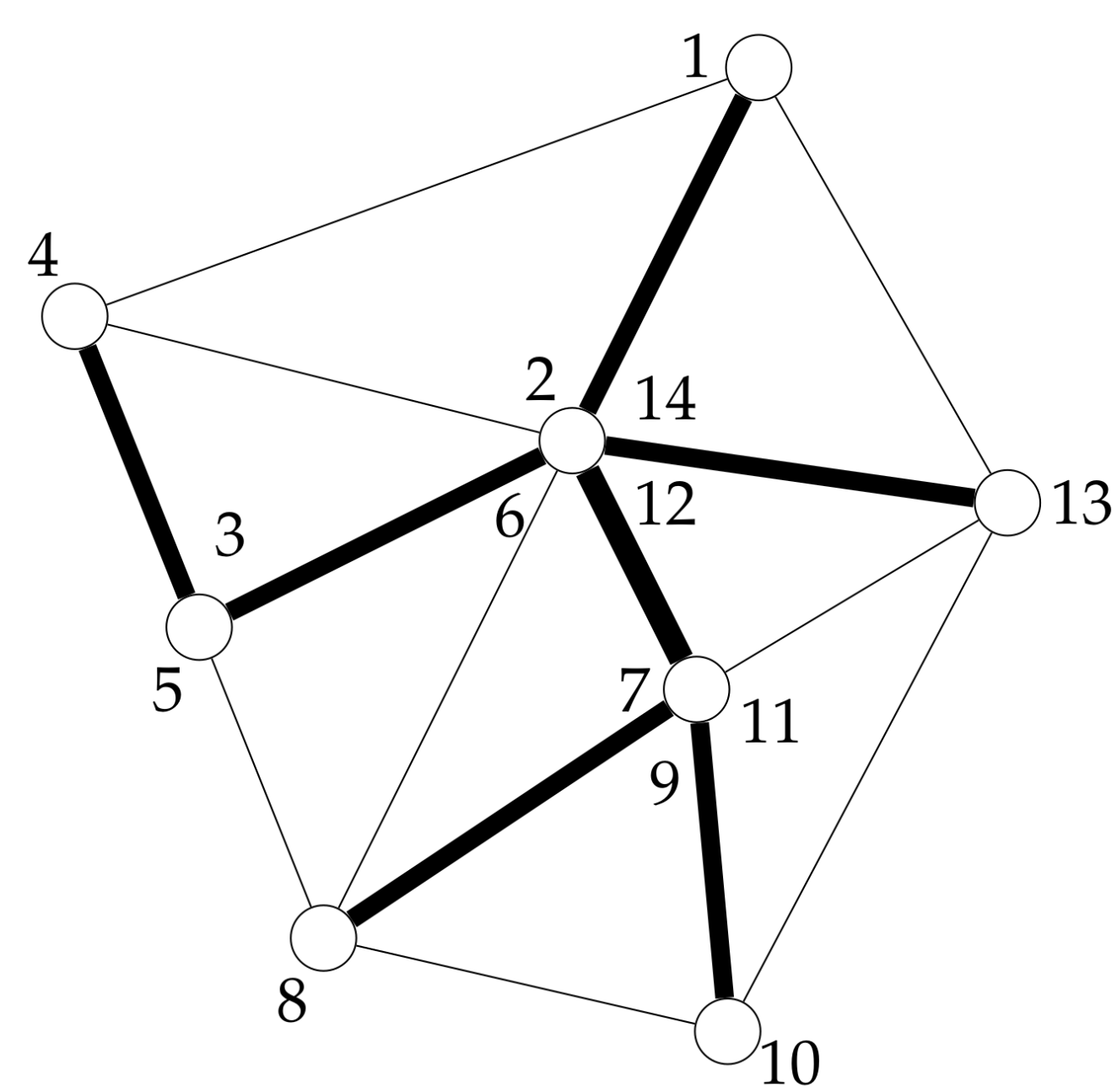
For every n , there is a **planar** graph on $O(n^2)$ vertices that is minor-universal for the **planar** n -vertex graphs.

For graphs on surfaces [Gavoille and H 2023+]

For every n and every surface Σ of Euler genus $g \geq 0$, there is a graph **embedded on** Σ with $O(g^2(n+g)^2)$ vertices minor-universal for the n -vertex graphs **embeddable on** Σ .

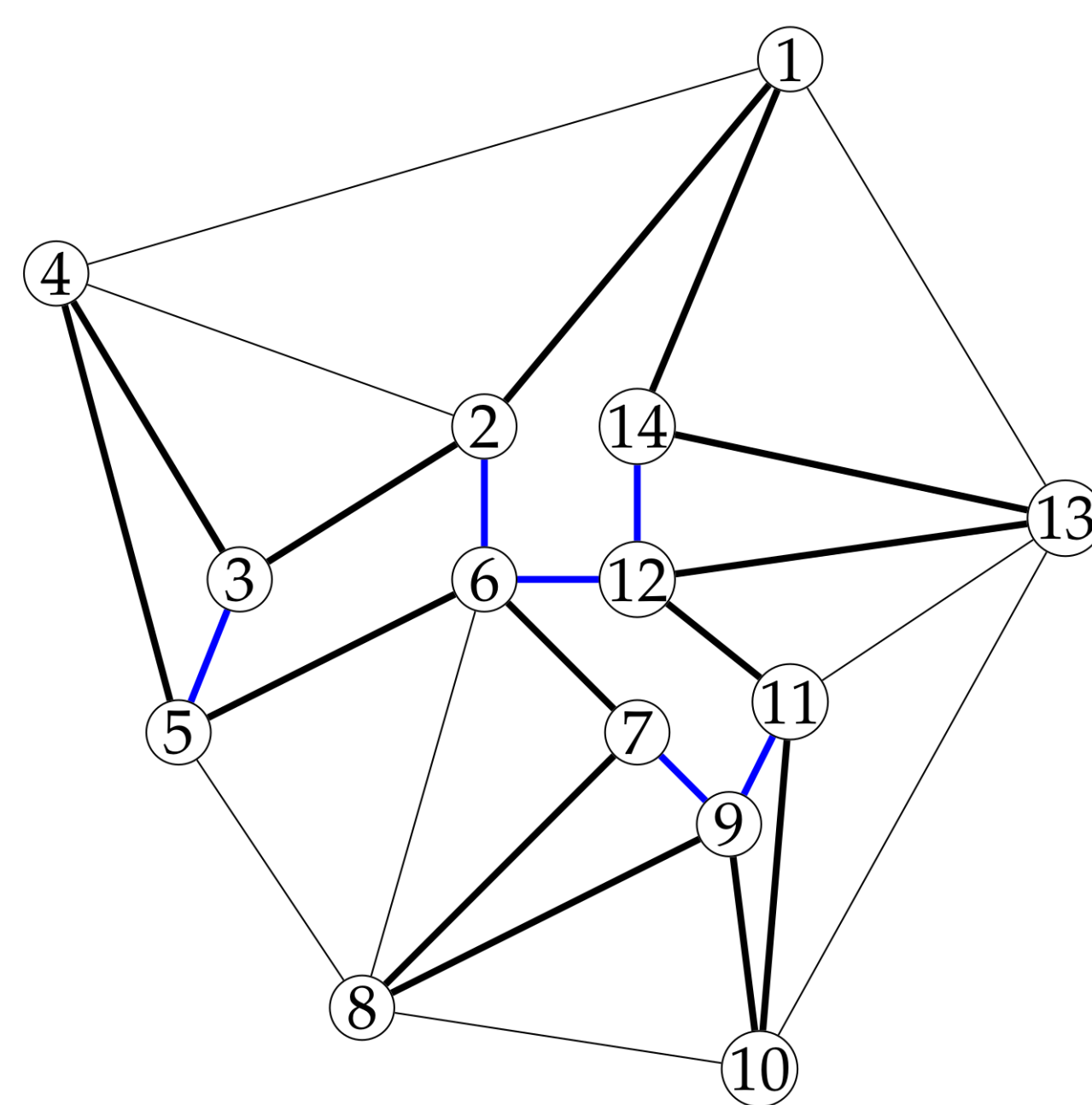
Sketch of the proof for planar graphs

Every planar n -vertex graph is a minor of...



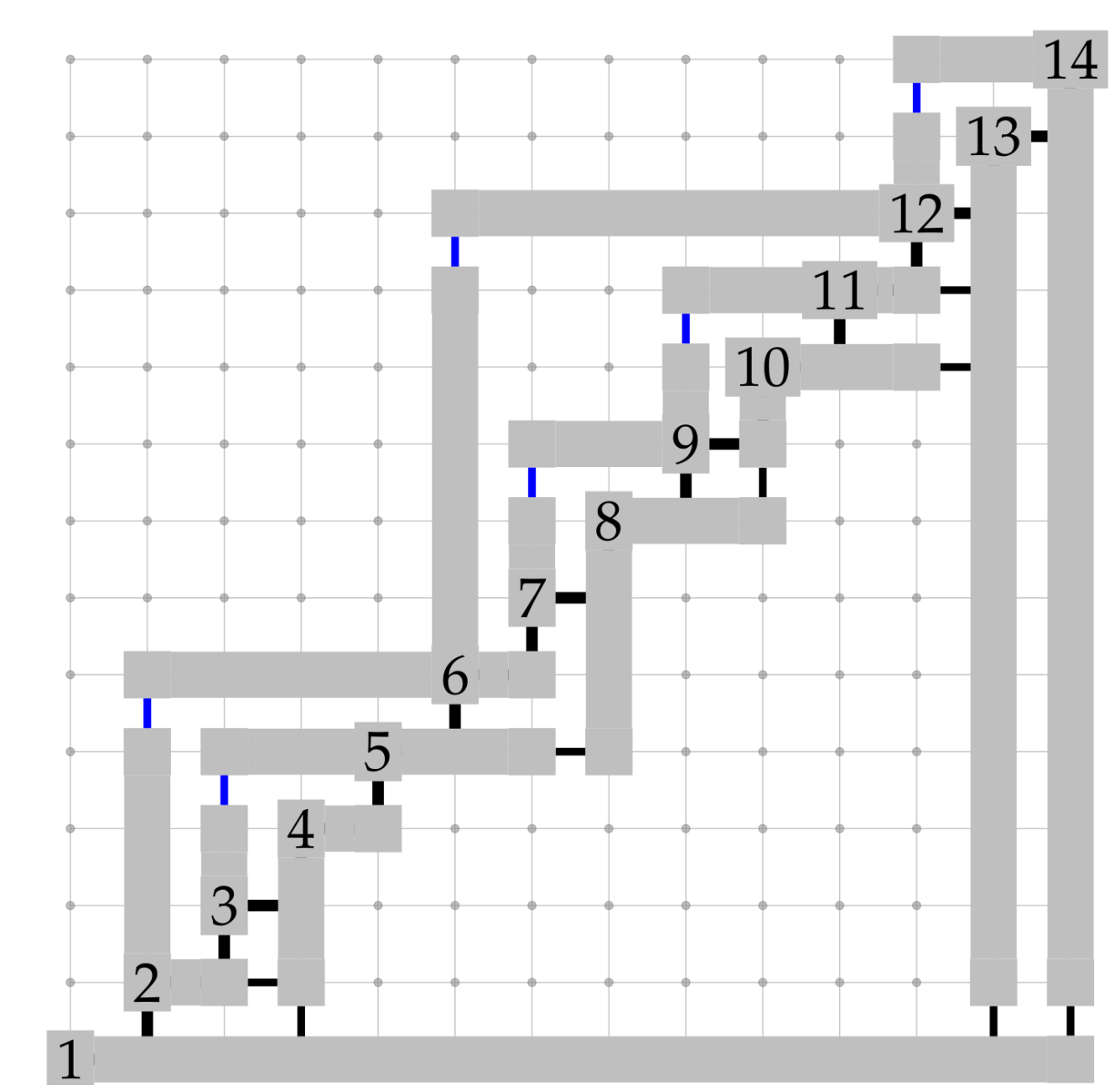
Step 1

... a Hamiltonian planar $2n$ -vertex graph...



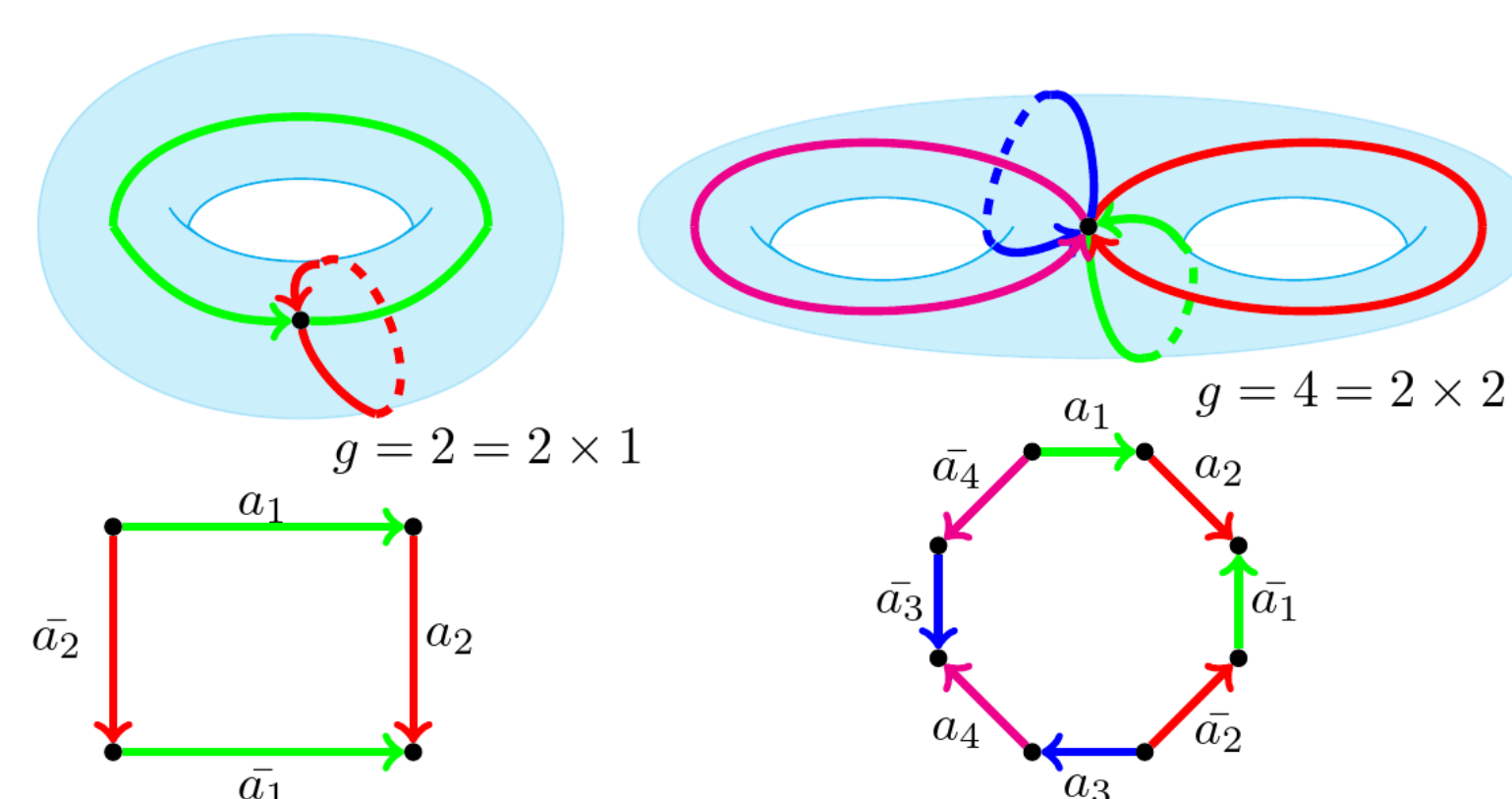
Step 2

... which is a minor of the $2n \times 2n$ -grid.

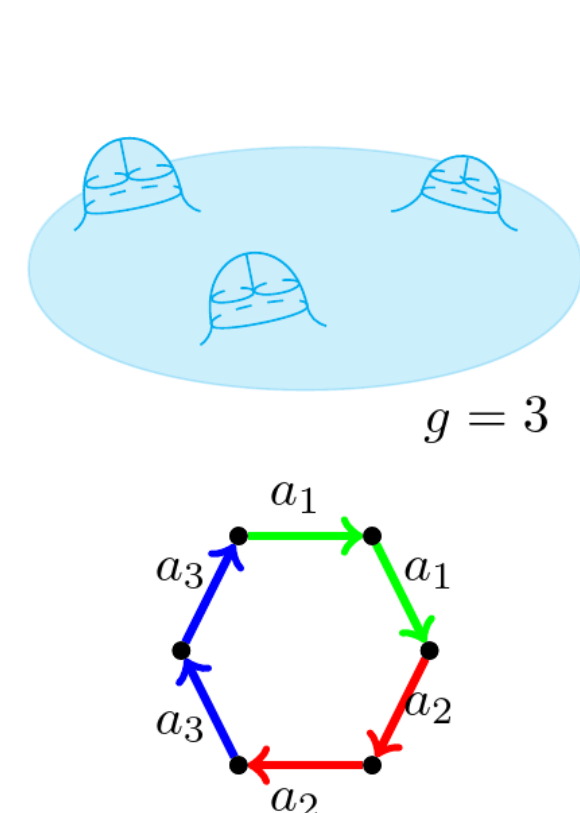


Surfaces & Euler genus

Orientable:

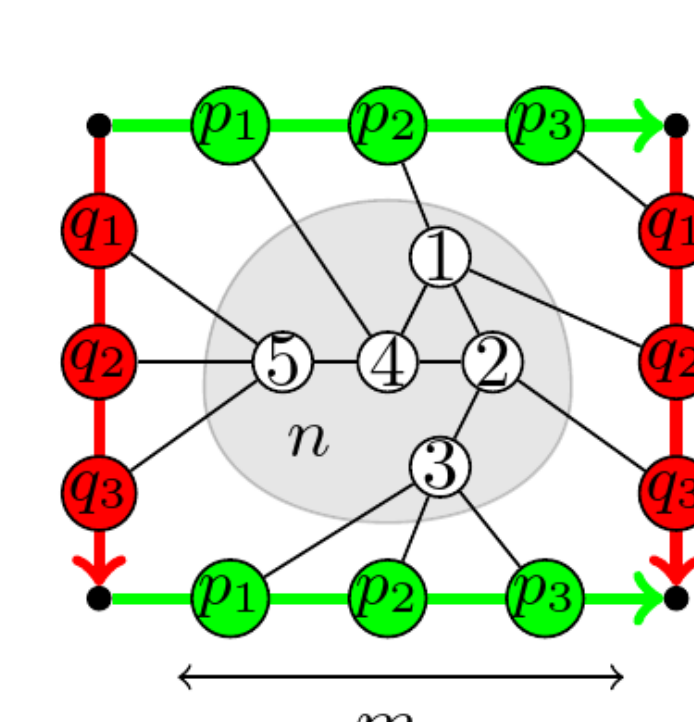
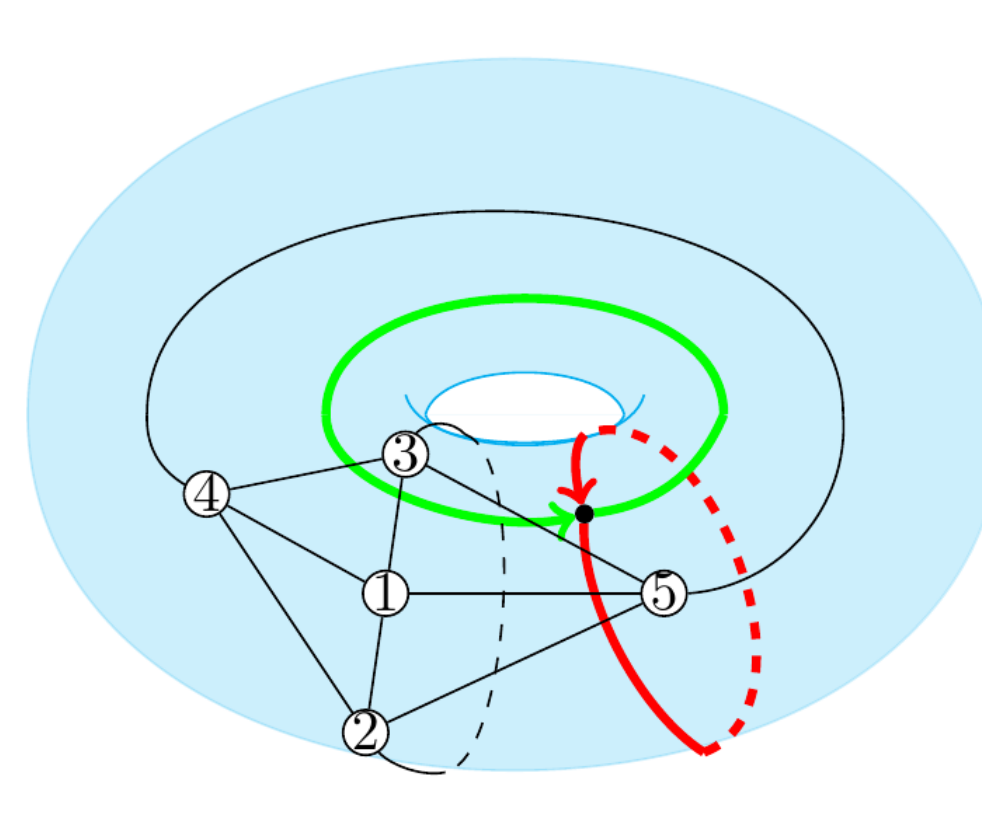


Non-orientable:



A surface is characterized by its **orientability** and **Euler genus**, and can be represented with a **polygonal schema**.

Polygonal embedding

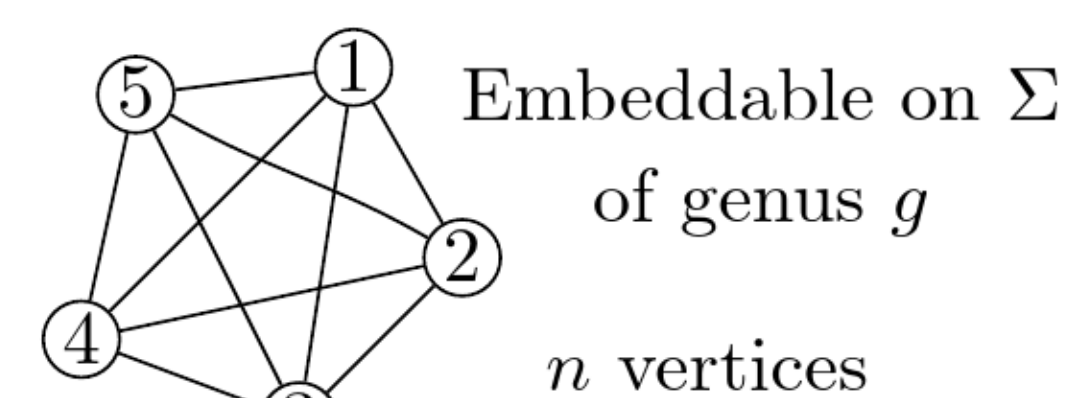


Polygonal embedding $P_\sigma(m, n)$:

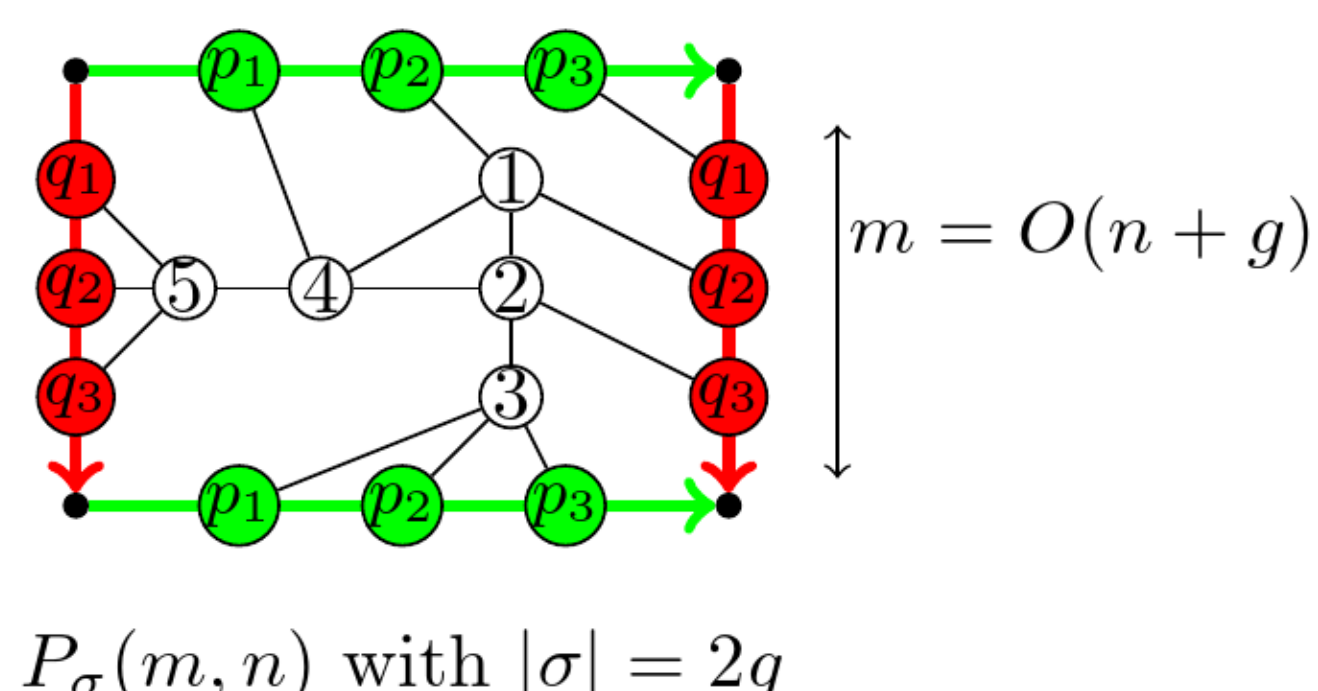
- ▶ $|\sigma|$: signature of the polygonal schema (\rightarrow, \rightarrow);
- ▶ at most n internal vertices (1,2,3,4,5);
- ▶ at most m vertices on each side ($p_1, p_2, p_3, q_1, q_2, q_3$).
 $m = O(n+g)$
[LPVV01, FHdM22]

Sketch of the proof:

Consider polygonal embedding instead of graphs on surfaces:

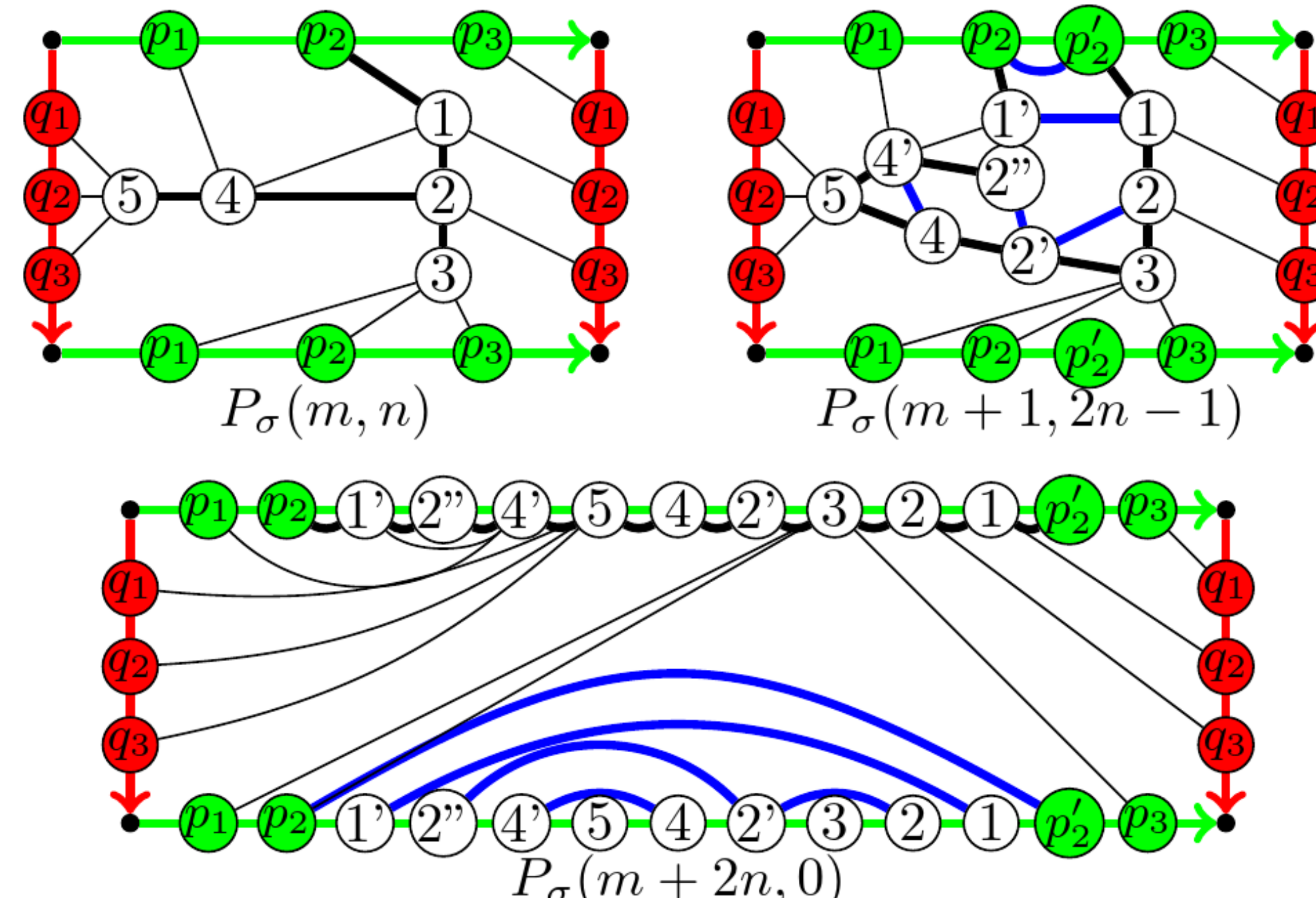


minor of



Step 1: getting an outerplanar-like embedding

Every graph with a **polygonal embedding** $P_\sigma(m, n)$ is minor of a graph with polygonal embedding $P_\sigma(m+2n, 0)$.



Step 2: the minor-universal graph

This graph with **polygonal embedding** $P_\sigma(m', \frac{(|\sigma|m')^2}{2})$ is minor-universal for the graphs with **polygonal embedding** $P_\sigma(m', 0)$.

