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A Survey on Interval Routing Scheme

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Abstract

We survey in this paper the classic results, and also the most recent results, on the field of Interval Routing Scheme, a well-known strategy to code in a compact way distributed routing algorithms. Theses results are classified in several themes: characterization, compactness and shortest path, dilation and stretch factor, specific class of graphs (interconnection networks, bounded degree, planar, chordal rings, ...), and the others recent extensions proposed in the literature: dead-lock free, congestion, non-deterministic, and distributed problems related to Interval Routing Schemes. For each of theses themes we propose a state of the art and several open problems.

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1 Introduction

Interval Routing Scheme is a way of implenting routing schemes on arbitrary networks. It is based on representing the routing table stored at each node in a compact manner, by grouping the set of destination addresses that use the same output port into intervals of consecutive addresses. A possible way to represent such a scheme is to use a connected undirected labeled graph, providing the underlying topology of the network. The addresses are assigned to the nodes, and the sets of destination addresses are assigned to each endpoint of the edges. The routing is computed in a distributed way with the following algorithm: at each intermediate node x, the routing process ends if the header y corresponds to x, otherwise it is forwarded with the message through the edge labeled by a set I such that $y \in I$.

As originally introduced in [SK82], the scheme required each set of destinations to consist of a single interval. So, the message is forwarded through the edge labeled by an interval $[a,b]$ such that $a \leqslant y \leqslant b$. The main advantage of this scheme is the low memory requirement to store the routing: $O(d)$ integers (the boundaries of the intervals) for each node of degree d. Early works appear also in [vLT83], and the first article has been published in [SK85]. This scheme was subsequently generalized in [vLT87] to more than one interval per edge. The Interval Routing method is implemented in the INMOS C104 router [MT90]. A performance study (by simulation) of the C104 router is given in [HKR91].

A short survey has already been presented in [vLT94]. Since then, a lot of works in the field have been done.

The paper is organized as follows:

- Section 2 introduces the formal definition of Interval Routing, and its different related notions: compactness, dilation, stretch factor, and some basic characterizations.
- Section 3 presents a large collection of results concerning specific classes of networks: interconnection networks, chordal rings, planar, treewidth bounded, and other classes of graphs and graph operators.
- Section 4 presents some extensions of the standard model of Interval Routing: non uniform cost link models, and the hierarchical results, deadlock-free, non-deterministic, and congestion of Interval Routing, and also the recent use of Interval Routing for solving several distributed tasks.

Each paragraph ends with some open questions and conjectures. Several new results are added: the coding of Interval Routing (Paragraph 2.3, Theorem 1 and Theorem 2), a lower bound concerning the treewidth (Paragraph 3.5, Theorem 32), and a lower bound for K_n under the dynamic link-cost model which implies some hierarchical results of Interval Routing classes, (Paragraph 4.1, Theorem 43 and Corollary 1).

2 The Model of Interval Routing

This section present the basic material for definitions and notations about Interval Routing.

2.1 Definition of Interval Routing

The underlying topology is a symmetric digraph $G = (V, E)$, V being the set of routers of the network, and each bidirectional link between the nodes x and y is represented by two opposite arcs of E: (x, y) and (y, x). So, E is the set of arcs. We denote by $\deg(x)$ the number of neighbors of x. For the routing problems, we will also assume that the graph is always finite and connected; moreover it has no loop and no multi-arc. Although the common model of graph is symmetric, i.e., undirected, Interval Routing can be applied to general digraphs. The definition quoted below holds for weighted graphs. The distinction between weighted and unweighted graphs will be relevant later in Section 2.2 when the length of the routing paths induced will be taken in consideration.

Definition 1 *Let* $G = (V, E)$ *be a graph. A pair* $(\mathcal{L}, \mathcal{I})$ *is an* Interval Routing Scheme *on* G *(IRS for short), if the following conditions are satisfied:*

1. $\mathcal{L}$ *is a one-to-one labeling of* V, $\mathcal{L} : V \to \{1, \dots, |V|\}$;
2. $\mathcal{I}$ *is an arc-labeling,* $\mathcal{I} : E \to 2^{\mathcal{L}(V)}$, *such that*[1]*:*
   2a. *for every* $x \in V$, $\{\mathcal{I}(x, y) \mid (x, y) \in E\} \cup \mathcal{L}(x) = \{1, \dots, |V|\}$;
   2b. *for every distinct arcs* (x, y) *and* (x, z) *of* E, $\mathcal{I}(x, y) \cap \mathcal{I}(x, z) = \emptyset$;
3. *for every* $x, y \in V$, $x \neq y$, *there exists a sequence of nodes* $(u_0, \dots, u_t)$ *such that* $u_0 = x$, $u_t = y$, *and for every* $i \in \{1, \dots, t\}$, $\mathcal{L}(y) \in \mathcal{I}(u_{i-1}, u_i)$.

Sometimes the set $\{0, \dots, |V| - 1\}$ is used for the definition of the node labeling $\mathcal{L}$. Emphasize that any ordered set can be used as well.

An IRS on G induces a *routing function* on G, which is a function that returns for every source-destination pair (x, y) a path from x to y defined by the sequence of nodes of Condition 3. Such a path is called *routing path.*

When a pair $(\mathcal{L}, \mathcal{I})$ of labeling satisfies Condition 1 and 2, it is called an Interval Labeling Scheme (ILS for short). An ILS on G is not necessary an IRS on G. In particular, Condition 3 fails if there exists a z, for some edge $\{x, y\}$, such that $z \neq x$, $z \neq y$ and $z \in \mathcal{I}(x, y) \cap \mathcal{I}(y, x)$. In this case we have a bouncing between x and y whenever the destination of the message is z. Note that some label $\mathcal{I}(x, y)$ may be empty, and the link (x, y) not used. An IRS on a graph is a *valid* ILS, that is an ILS satisfying Condition 3. For the routing problem, we are only interested in valid ILS. The validity of every ILS (Condition 3) can be checked in $O(n^2)$ time [vLT83], where n is the number of nodes of the graph.

[1] $2^{\mathcal{L}(V)}$ denotes the power set of $\mathcal{L}(V)$, i.e., the set of all the subsets of $\{1, \dots, |V|\}$.

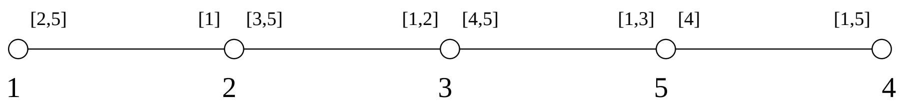

Figure 1: A valid interval labeling scheme for a path.

We can check that Condition 1, 2, and 3 for the example of Figure 1 are satisfied. Of course many different labeling are possible for a same graph.

2.2 Compactness, Linearity and Strictness of Interval Routing

The notion of compactness and linearity are related to the way of coding the labels of the arcs, whereas strictness is related to the construction of the IRS it-self. For every integers n, $a, b \leqslant n$, an *interval* $[a, b]$ with respect to n is the set of consecutive integers between a and b, n and 1 being considered as consecutive. Formally, $[a, b] = \{i \mid a \leqslant i \leqslant b\}$, if $a \leqslant b$, and $[a, b] = \{i \mid b \leqslant i \leqslant n\} \cup \{i \mid 1 \leqslant i \leqslant a\}$ otherwise. Theses two kinds of intervals are respectively called *linear* and *cyclic* intervals. For every set $I \subset \{1, \ldots, n\}$, the *number of intervals* of I is the length of the smallest sequence $(a_1, b_1, \ldots, a_k, b_k)$ such that $I = [a_1, b_1] \cup \ldots \cup [a_k, b_k]$. The number of linear intervals of I is defined similarly, but all intervals must be linear. The number of intervals of the empty set is 0.

Definition 2 *Let $R = (\mathcal{L}, \mathcal{I})$ be an IRS on a graph G. The* compactness *of R is the maximum, over all the arcs (x, y) of G, of the number of intervals of $\mathcal{I}(x, y)$.*

The linear compactness is defined similarly from the number of linear intervals of the arc-labels. Compactness and linear compactness differ by at most 1: every cyclic interval equals two linear intervals. We denote by k-IRS every IRS of compactness k. A k-IRS that has also a linear compactness k is called a k-Linear Interval Routing Scheme (k-LIRS for short). Moreover, an IRS is qualified of *strict*, denoted by SIRS, if every arc (x, y) satisfies $\mathcal{L}(x) \notin \mathcal{I}(x, y)$. The light will be done later on the reason of the distinction of all its variants.

Hence we have for variants for IRS depending on linearity and strictness: k-IRS, k-LIRS, k-SIRS, and k-SLIRS. Section 4.2 gives more details about the hierarchy of the classes induced by theses different IRS.

The IRS depicted in Figure 1 is linear, non strict, and of linear compactness 1. So this is a 1-LIRS.

2.3 Coding of Interval Routing

The routing information of a node is entirely determined by the knowledge of all the labels (its local name and the labels of its incident arcs). Implicitly, the

Interval Routing model corresponds to the possibility of permuting the output ports numbers in advance. We mean that for routing decision into a node x towards a destination y, x is able to determine the output port number such that $\mathcal{L}(y) \in \mathcal{I}(e)$, and to send the message onto this port (i.e., through the edge e) without extra information, excepted, of course, the knowledge of the label $\mathcal{I}(e)$.

We invite the reader to see [BHV96] for a discussion of the impact of the node and/or port relabeling on routing information complexity. In all the following, the function log denotes the logarithm in base 2.

Theorem 1 *Every k-IRS (and its variants) on an n-node graph can be implemented in each node x with $\log\binom{n}{K} + \log\binom{K}{d} + (K-d)\log d + O(\log n)$ bits, which is at most $O(dk\log(n/k))$ bits, where K is the total number of intervals for the node x, and d the number of arcs incident to x that brings a non-empty label.*

Proof. Let x be a node. For each $i \in \{1,\dots,K\}$, we denote by $[a_i, b_i]$ the i-th interval of x (with Condition 2b the intervals do not overlap), and by e_i the output port number on which is assigned this interval. First, we remark that only 2 intervals at most can be non strict and/or cyclic. So, with an overhead of $O(\log n)$ bits only, it is easy to implement any k-IRS, k-SIRS, or k-LIRS, from its strict and linear version.

W.l.o.g. we assume that the a_i's are sorted by increasing order, the intervals $[a_i, b_i]$ are strict and linear, and we know the integers n, d, K, and $\mathcal{L}(x)$ with an overhead of $5\log n$ bits[2]. To code all the labels in x, it suffices to store the sequences $S_1 = (a_1,\dots,a_K)$ and $S_2 = (e_1,\dots,e_K)$. Indeed the b_i's can be computed as follows: if $i < K$, then $b_i = a_{i+1} - 1$ ($= a_{i+1} - 2$ if $\mathcal{L}(x) = a_{i+1} - 1$), otherwise $b_K = n$ ($= n-1$ if $\mathcal{L}(x) = n$). S_1 is a sequence of K distinct integers in the range 1 to n, therefore it can be stored with $\log\binom{n}{K}$ bits [LV93].

Let d be the number of output ports used by the IRS in x, i.e., $d = |\{e_i \mid 1 \leqslant i \leqslant K\}|$. The IRS does not use necessary all the arcs incident to x, so d might be less than the degree of x. The number of ways to obtain S_2 is at most $\binom{K}{d}d^{K-d}$. Indeed, $S_2 = (e_1,\dots,e_K)$ is a sequence composed of exactly d different values e_i taken from $\{1,\dots,d\}$ located on K possible places, and of $K-d$ independent integers of $\{1,\dots,d\}$. So, there is at most $d!\binom{K}{d}$ choices of the d different values of the sequences, and d^{K-d} ways for the $K-d$ other elements; thus a total of $d!\binom{K}{d}d^{K-d}$ different sequences $(e_1,\dots,e_K)$. However the permutation of the output ports numbers can be made in advance. Since d ports are used, at least $d!$ sequences are equivalent up to a permutation of the ports, coding the same IRS. Each different IRS has a sequence S_2 which can be coded with $\log(d!\binom{K}{d}d^{K-d}/d!) = \log\binom{K}{d} + (K-d)\log d$ bits.

Thus the total number of bits used to store all the labels in x is bounded by

[2] $4\log n + O(\log\log n)$ bits suffices to store n, d, K, $\mathcal{L}(x)$ in a self-delimiting way, which is bounded by $5\log n$ for n large enough.

$M = \log\binom{n}{K} + \log\binom{K}{d} + (K-d)\log d + c\log n$, for a suitable constant $c \leqslant 5$. On the other hand, $k \leqslant K \leqslant dk$. Moreover $\binom{n}{K} \leqslant (ne/K)^K \leqslant (ne/k)^{dk}$, and $\log\binom{K}{d} \leqslant K$. Hence

$$\log\binom{n}{K} + \log\binom{K}{d} \leqslant K\left(\log\frac{n}{k} + \log(2e)\right) \leqslant \log(2e)\, dk \log\frac{n}{k}.$$

We note that k is in the range $1 \leqslant k \leqslant n/2$, thus for large n enough, $(n/k)^k \geqslant n$, and therefore $c\log n \leqslant ck\log(n/k) \leqslant c\,dk\log(n/k)$, for every constant $c \geqslant 0$, and for every $d \geqslant 1$. It follows that

$$\log\binom{n}{K} + \log\binom{K}{d} + c\log n \;\leqslant\; (c + \log(2e))\, dk\log\frac{n}{k}.$$

To prove that $M = O(dk\log(n/k))$, it remains to show that

$$(K-d)\log d \;\leqslant\; \alpha\, dk\log\frac{n}{k}, \quad \text{for a suitable constant } \alpha \geqslant 1. \tag{1}$$

Let assume that $n > dk$. In this case:

$$d < \frac{n}{k} \Rightarrow \log d < \log\frac{n}{k} \Rightarrow K\log d < dk\log\frac{n}{k} \Rightarrow (K-d)\log d < dk\log\frac{n}{k}.$$

It remains to show Inequality 1 for $n \leqslant dk$. Let $\beta = dk/n$, $\beta \geqslant 1$. Since $K \leqslant n$, we get $K \leqslant dk/\beta$. To show Inequality 1 it suffices to show:

$$\begin{aligned} K\log d \leqslant \alpha\, dk\log\frac{d}{\beta}, \quad \text{or} \quad & \frac{dk}{\beta}\log d \leqslant \alpha\, dk\log\frac{d}{\beta} \\ \Leftrightarrow \quad & \log d \leqslant \alpha\beta\log\frac{d}{\beta} \\ \Leftrightarrow \quad & d \leqslant \left(\frac{d}{\beta}\right)^{\alpha\beta}. \end{aligned}$$

Let $f(\beta) = (d/\beta)^{\alpha\beta}$. $f(\beta)$ increases if $f'(\beta) \geqslant 0$ for $\beta \geqslant 1$.

$$f'(\beta) = \alpha f(\beta)\left(\ln\left(\frac{d}{\beta}\right) - 1\right).$$

$\alpha f(\beta) > 0$. Thus $f'(\beta) \geqslant 0$ if $\ln(d/\beta) \geqslant 1$, or $d \geqslant \beta e$. So, if $d \geqslant \beta e$ then $f(\beta) \geqslant f(1) = d^\alpha$. Therefore if $d \geqslant \beta e$ then Inequality 1 holds.

So, let us assume $d < \beta e$. If, furthermore, $\beta e \leqslant \gamma\log d$, where $\gamma = 3/\log 3 \approx 1.892$, then it implies that $d < \gamma\log d$, which is impossible for $d \geqslant 1$. So, $\beta e > \gamma\log d$. In this case it implies that:

$$\begin{aligned} n\beta e > \gamma n\log d \quad &\Rightarrow \quad dke > \gamma n\log d, \quad \text{because } dk = n\beta \\ &\Rightarrow \quad dke > \gamma(K-d)\log d, \quad \text{because } n > K-d \\ &\Rightarrow \quad (K-d)\log d < \frac{e}{\gamma}dk\log\frac{n}{k}. \end{aligned}$$

In total,

$$M \;<\; \left(c + \log(2e) + \frac{e}{\gamma}\right) dk\log\frac{n}{k} \;<\; 9\, dk\log\frac{n}{k}.$$

□

Theorem 1 implies that every 1-IRS (and its variants) can be coded with $n + O(\log n)$ bits per node. Moreover, the implementation is quite easy using an n-bits vector coding within the 1's the left-boundary of each interval. (The time complexity to locally compute the routing function is linear in n in this case). We can even say better for trees.

Theorem 2 *Every n-node tree has a 1-SIRS which can be implemented with $O(\sqrt{n})$ bits in each node.*

Proof. Let T be an n-node tree, and r be a node of T chosen as the root of T. For every edge (u, v), the graph obtained by removing (u, v) in T is composed of two connected components. We denote by $T_{(u,v)}$ the component that contains v. We say that $T_{(u,v)}$ is the subtree of T induced by (u, v). For all integers n, k, a k-partition of n is an integer sequence $(n_1, \ldots, n_k)$ such that $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$, and $\sum_{i=1}^{k} n_i = n$.

We label the nodes of T with a particular depth first search scheme. We initialize the labeling process by labeling r with 1. For each node x, let $y_1, \ldots, y_k$ be the children of x ordered such that $|V(T_{(x,y_1)})| \leqslant \ldots \leqslant |V(T_{(x,y_k)})|$. We start to label recursively the subtrees $T_{(x,y_1)}, \ldots, T_{(x,y_k)}$ in this order. If $x \neq r$, we assign the output port number 1 to the edge towards r, i.e., the edge (x, y) with y the father of x, and for each $i \in \{1, \ldots, k\}$ the output port number $i+1$ to the edge (x, y_i). For $x = r$ we assign the output port number i to the edge (x, y_i).

Consider a node $x \neq r$. We set $k = \deg(x) - 1$, $n_0 = |V(T_{(x,y)})|$ for y the father of x, and for each $i \in \{1, \ldots, k\}$, set $n_i = |V(T_{(x,y_i)})|$. We remark that $\sum_{i=0}^{k} n_i = n$, or equivalently $(n_1, \ldots, n_k)$ is a k-partition of $n - n_0$. We store in x:

- the label of x;
- the values n, n_0, and k;
- the k-partition of $n - n_0$: $(n_1, \ldots, n_k)$;
- an $O(1)$-bits size discussion.

For $x = r$ we set $k = \deg(x)$, and $n_0 = 1$, and we store similarly the values n, k, and the k-partition of $n - n_0$ defined by $(n_1, \ldots, n_k)$. To simplify, x and y represent the labels of the node x and y respectively.

The routing scheme is the following: assume the node x must route a message to the destination y. If $x = y$ then the routing process ends. If $y \notin [x+1, x+n-n_0]$, then the message is forwarded to the father of x through the output port 1 (the case never happens if $x = r$). Otherwise, one computes the unique integer $p \geqslant 1$ such that $y \in [x + 1 + \sum_{i=1}^{p-1} n_i, x + \sum_{i=1}^{p} n_i]$. The message is forwarded through the output port $p + 1$, if $x \neq r$, and p for $x = r$. Clearly such a scheme correspond to a 1-SIRS, because the union of all the intervals cover $\{1,\dots,n\}$, and all the intervals are pairwise disjoint. Moreover they never contain x.

The process routes correctly. Indeed, by the construction of the node-labeling if $y \in [x+1+\sum_{i=1}^{p-1} n_i, x+\sum_{i=1}^{p} n_i]$, then y is necessary a node of the subtree $T_{(x,y_p)}$, and the output port assigned to the edge (x, y_p) is $p+1$ if $x \neq r$, and p if $x = r$. And if $y \notin [x+1, x+n-n_0]$, then y is not a descent of x, and hence must be forwarded to its father.

Let us compute the amount of information required. The integers values of n, n_0, k, and the label of x can be stored on $O(\log n)$ bits. Knowing n and k, any k-partition of n, P, can be coded using at most $\lceil \log U_n \rceil$ bits, where U_n is the total number of partitions of n. Indeed, there exists very simple algorithm that, knowing n, enumerates all the partitions of n. So, it suffices to store the index of P in such enumeration. Furthermore, we have the well-known formula due to Harder and Ramanujan in 1918 [Hal86, Equation (4.2.7) page 44]:

$$U_n \sim \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{2n/3}}.$$

Globally $3.71\sqrt{n}$ bits per node suffices to describe the routing algorithm for n large enough, that completes the proof. □

We remark that the distinction between the variants of Interval Routing has no real impact on the coding of IRS in local memory of the nodes, up to an additive term of $O(\log n)$ bits. The interest of theses variants will appear in Theorem 40 of Section 3.7.

Since every edge brings $O(k \log n)$ bits of information to code its labels, the total of routing information for the entire graph $G = (V, E)$ is bounded by $O(|E| k \log n)$ bits.

2.4 Characterizations

Every graph supports an IRS: label arbitrary the nodes and select any routing function with simple paths (without loop) to generate the label of the arcs. Because Interval Routing has been investigated to give compact implementation of routing functions, we are interested to find IRS with small compactness.

Using a depth search scheme for the labeling of nodes, we have the following result:

Theorem 3 ([SK85]) *Every acyclic digraph has a* 1-*SIRS.*

It follows that trees have a 1-SIRS, and thus, by the use of a spanning tree, every graph has a 1-SIRS. This result has been extended:

Theorem 4 ([vLT87]) *Every graph has a* 1*-SIRS such that all the arcs have non empty labels.*

For the latter result, more details can be found in [Tel94, page 136]. A comparison of the labeling of [SK85] and of [vLT87] is presented in [SML94]. However the result does not hold for linear intervals.

Theorem 5 ([FG94a])

- *A graph has a* 1*-LIRS if and only if it is not a lithium graph, i.e., a graph with three bridges connected to a same* 2*-edge-connected component, the bridges corresponding to edges of nodes of degree* 1 *excluded.*
- *A graph has a* 1*-SLIRS if and only if it is not a weak-lithium graph, i.e., a graph with three bridges connected to a same* 2*-edge-connected component.*

An simple and efficient algorithm for the labeling of graphs supporting a 1-LIRS and 1-SLIRS is presented in [EMZ97b]. Note that for all its characterizations nothing is said about the length of the routing paths.

2.5 Efficiency, Dilation and Stretch Factor

Whereas compactness, linearity, and strictness are qualitative parameters of the labeling, we can also define several other criteria to measure the quality of the routing induced by the IRS. Here the graphs considered have uniform weights on all the links.

Definition 3 *Let R be an IRS on a graph G. R is a* shortest path *IRS if the routing paths induced by R on G are only composed of shortest paths in G.*

Definition 4 *The* compactness *of a graph G, denoted by* $\mathrm{IRS}(G)$*, is the smallest integer k such that G supports a shortest path k-IRS.*

The compactness of $G = (V, E)$ is called sometimes the IRS number of G. As we saw previously the space complexity for the storage of a routing scheme in x is $O(dk \log(n/k))$ bits or $O(|E| k \log n)$ bits in total. Therefore the compactness, k, is an important for the design of compact routing schemes.

We can define similarly the linear compactness ($\mathrm{LIRS}(G)$) and the strict or strict-linear compactness ($\mathrm{SIRS}(G)$ and $\mathrm{SLIRS}(G)$) of graphs. However, up to an additive constant theses variants are equivalent.

Definition 5 *Let R be an IRS on a graph G. The* dilation *of R, denoted by* $\mathrm{dilation}(R)$*, is the length of the longest routing path induced by R. The k-*dilation *of G, denoted by k-*dilation(G)*, is the minimum, over all the k-IRS R on G, of the dilation of R.*

By taking a labeling like a depth first scheme along a minimum spanning tree, the result of [SK85] implies that every graph G of diameter D satisfies k-dilation$(G) \leqslant 2D$ (more precisely, twice the radius of G). Moreover, by combining Theorem 2, we can conclude that every graph of diameter D has a 1-SIRS of dilation at most $2D$ which can be implemented with $O(\sqrt{n})$ bits in each router.

Although, using this scheme, the routing path lengths remains always linear in the diameter, this length might be very large for nodes that are relatively close.

Definition 6 *Let R be an IRS on a graph G. The* stretch factor *of R, denoted by* stretch(R)*, is the smallest real s such that for every pair of nodes (x, y) the length of the routing path induced by R from x to y is at most s times longer than the distance in G between x and y. The k*-stretch *of G, denoted by k*-stretch(G)*, is the minimum, over all the k-IRS R on G, of the stretch factor of R.*

Shortest path IRS are IRS of stretch factor 1. The next result shows that the behavior between 1-IRS and 1-LIRS is different for the dilation.

Theorem 6 ([EMZ96]) *For every fixed D, there exists a graph G of diameter at least D such that every 1-LIRS on G has a dilation at least $D^2/18$. Moreover G is planar and of maximum degree 4.*

Open question 1
- Is the lower bound $\Omega(D^2)$ tight?

Theorem 7 ([KRŠ97a]) *For every n-node graph G of diameter D, there exists a $k \leqslant \left\lceil \sqrt{n \ln n} \right\rceil + 1$ such that k*-dilation$(G) \leqslant \lceil 3D/2 \rceil$.

Theorem 8 ([Gav97]) *For every $D \geqslant 2$, there exists an n-node graph G such that k*-dilation$(G) \geqslant \lfloor 3D/2 \rfloor - 1$*, for every $k \leqslant O(n/(D \log(n/D)))$.*

It should be noticed that the dilation is quite sensitive to the additive term around 3/2 the diameter.

Open question 2
- What is the smallest integer k (as function of n and D) such that k-dilation$(G) \leqslant \lceil 3D/2 \rceil$, for every graph G?

For bounded degree we have:

Theorem 9 ([FN96]) *For every $D \geqslant 4 \log n$ there exists a bounded degree graph G such that k*-dilation$(G) \geqslant 3D/2 - 2\log(n/D)$*, for every $k \leqslant 0.05\, n/(D \log(n/D))$.*

Here the proof that the IRS proposed by [SK85] based on a spanning tree is close to the optimality.

Theorem 10 ([TL97a]) *For every even D, there is an n-node graph G of diameter D, $n = \Theta(D^2)$ and of girth $2D$, such that* $1\text{-dilation}(G) \geqslant 2D - 3$. *It implies that* $1\text{-stretch}(G) = \Omega(\sqrt{n})$.

Definition 7 *Let R be an IRS on a graph G, and $k \geqslant 0$ be an integer. R is a* k-regional *IRS on G if for every node x of G the induced routing paths by R from x to any node y at distance at most k of x are shortest paths.*

Theorem 11 ([vLT87])

- *Every n-node graph G has a 1-regional 2-SIRS, hence* $2\text{-stretch}(G) \leqslant n/2$.
- *Every Hamiltonian n-node graph H has a 1-regional 1-SIRS, hence* $1\text{-stretch}(H) \leqslant n/2$.

Open question 3
- Is there a n-node graph G, for every n large enough, such that $1\text{-stretch}(G) = \Omega(n/\log n)$? $\Omega(n)$?
- What are the general bounds for k-stretch(G), $k > 1$?
- Do all graphs support a 1-regional 1-IRS?

We will see that there is no hope to extended Theorem 11 for 2-regional IRS. Indeed in Paragraph 3.1, Theorem 15 states that shortest path IRS in graphs of diameter two can required $n/4$ intervals in the worst-case. The 2-regional linear IRS satisfy the following property:

Theorem 12 ([ZTRP94]) *If G supports a 2-regional 1-SLIRS then every edge of G is either a bridge, or belongs to a cycle of length 3 or 4.*

The parameters dilation and stretch factor do not necessary reflect the quality of a routing scheme. The density of long routing paths may be small. Although few papers of the field have considered the stretch factor, we believe that for a better analysis of the real networks it would be relevant to consider the average dilation and the average stretch factor.

In the next definition $\rho(x,y)$ denotes the routing path length induced by the routing scheme R from x to y, and $s(x,y)$ the ratio between $\rho(x,y)$ and the distance between x and y.

Definition 8 *Let R be an IRS on an n-node graph G. The* average dilation *(respectively* average stretch factor*) of R, is the real*

$$\frac{1}{n(n-1)} \sum_{x \neq y} \rho(x,y) \quad \textit{respectively,} \qquad \frac{1}{n(n-1)} \sum_{x \neq y} s(x,y).$$

The average k-dilation(G) *(respectively average k*-stretch(G)*) is the minimum, overall the shortest path k-IRS R on G, of the average dilation of R (respectively stretch factor).*

The average stretch factor has been studied for Interval Routing in 2D-grids in [SML94].

Open question 4
- What are the bounds on average dilation and stretch factor of k-IRS for arbitrary graphs?

2.6 Complexity of Characterization

Characterization for IRS with at most 1 interval per arc (linear or cyclic) are clearly polynomial whenever there is no constraint about the induced routing path lengths, see Paragraph 2.4. The problem becomes much harder when, for example, the shortest paths are required. Because the labeling can be optimized over all the possible node and arc labeling (including all the possible choice of routing paths), it appears that determine whether or not a graph has a shortest path 1-IRS is difficult in practice. The best-known NP-completeness results are the following:

Theorem 13

- **[EMZ97a]** *For every s, $1 \leqslant s < 3/2$, the problem of determining whether a graph supports a 1-SLIRS (or its variants) of stretch factor s is NP-complete.*
- **[Fla97b]** *For every s, $1 \leqslant s < 3/2$, the problem of determining whether a graph supports a 2-SLIRS (or its variants) of stretch factor s is NP-complete.*

Thus it is NP-complete for the case of weighted graphs, and shortest path IRS ($s = 1$). The result concerning the stretch factor was not mentioned in the original paper of [Fla97b]. However, since their construction is a graph of diameter two, is quite easy to see that G has a shortest path 1-SLIRS (or its variant) if and only if G has a 1-SLIRS of stretch factor $s < 3/2$.

In [FGS95] it is also mentioned that the problem of determining the minimum K such that a given weighted graph supports a shortest path IRS with a total number of K intervals, for the entire graph, is NP-hard.

Open question 5
- Is it NP-complete for every fixed $k \geqslant 3$?
- Is it NP-complete for $k = 1$ and bounded degree graphs? planar graphs?
- What is the best approximation polynomial time algorithm for the problem of

characterization of the class of graphs supporting a 1-SLIRS (and its variants)?
- Does the following problem remain NP-hard: find the smallest k such that k-stretch$(G) \leqslant s$, for some constant $s \geqslant 3/2$?
- NP-completeness for k-dilation of IRS.

In Paragraph 4.1 we will see that the problem becomes polynomial under some assumptions of non uniform link costs (dynamic link-costs).

The general problem of shortest path k-IRS is difficult to solve, however shortest path k-IRS have been found out for many large classes of graphs.

3 Specific Class of Graphs

In this section we refer the results known about the compactness of graphs, that is the smallest integer k that provides a shortest path k-IRS. We start with some general results. Recall that the compactness of G is denoted by $\mathrm{IRS}(G)$, $\mathrm{LIRS}(G)$ (respectively $\mathrm{SIRS}(G)$ and $\mathrm{SLIRS}(G)$) denotes the linear compactness (respectively strict and strict-linear compactness). See Paragraph 2.5.

For convenience, for every class of graphs $\mathcal{G}$, we denote by $\mathrm{IRS}(\mathcal{G})$ the value defined by:

$$\mathrm{IRS}(\mathcal{G}) = \max_{G \in \mathcal{G}} \mathrm{IRS}(G)$$

that is a worst-case complexity of the compactness. In the following n always denotes the number of nodes.

3.1 Extremal Compactness

Because Interval Routing is devoted to implement routing in general networks, it is natural to wonder what is the compactness of an n-node graph.

The compactness of an n-node graph cannot exceed $n/2$, because a necessary condition to have k intervals for a set I is to have $|I| \leqslant n-k$, and clearly the number of intervals of I is at most $|I|$. The next result proposes a little improvement. The three next results use randomization.

Theorem 14 ([GP97]) *Every n-node graph G, $n \geqslant 1$, satisfies*

$$\mathrm{IRS}(G) < \frac{n}{4} + \frac{1}{4}\sqrt{2n \ln (3n^2)}.$$

Theorem 14 is actually a consequence of a deeper result: for every family $\mathcal{F}$ of at most $e^{n/2}/n$ subsets of $\{1, \ldots, n\}$ there exists a permutation π of $\{1, \ldots, n\}$ such that all the subsets of $\mathcal{F}$ are composed of at most $n/4 + O(\sqrt{n \ln n})$ intervals under π. It can be clearly apply on IRS models where Condition 2a and/or 2b are relaxed, $\mathcal{F}$ representing the set of arcs, each subsets representing an arc-label, and the permutation π representing the node labeling.

Due to the next lower bound, $n/4$ is a tight bound.

Theorem 15 ([Gav97]) *For every n large enough, there exists a n-node graph G such that* $\mathrm{IRS}(G) > n/4 - 1.72\, n^{2/3} \ln^{1/3} n$. *Moreover G is of diameter* 2, *and for every $k <$* $\mathrm{IRS}(G)$, *k*-dilation$(G) \geqslant 3$. *Therefore* k-stretch$(G) \geqslant 3/2$.

To a certain extent Interval Routing cannot efficiently used for the *regional routing problem* posted by Peleg in [SRS97], that consists in optimal routing up to distance k (see also Definition 7 in Paragraph 2.5).

Note there exists a constructive proof for a worst-case compactness of at least $n/12$ [GG95] (historically the first $\Theta(n)$-lower bound). A third proof, using randomization, for a $\Theta(n)$-lower bound is given in [KK96]. They also show that $\Theta(n)$ intervals might be required simultaneously on $\Theta(n)$ edges.

Against intuition, the next result shows that, in general, the compactness does not depend on the number of edges of the maximum degree of the graph. It may be quite high even for sparse graphs.

Theorem 16 ([GP96a]) *For every n large enough, there exists a cubic graph G with at most n nodes such that* $\mathrm{IRS}(G) = \Theta(n)$. *Moreover this number can be required on $\Theta(n)$ edges.*

As a corollary, there exists some graph with $(1+\varepsilon)n$ edges, for every $0 < \varepsilon \leqslant 1/2$, and of compactness $\Theta(\varepsilon n)$. Indeed, it suffices to choose a graph satisfying Theorem 16 with $2\varepsilon n$ nodes, and to connect $(1-2\varepsilon)n$ nodes of degree 1 to some nodes. Clearly such a new graph has n nodes, $3\varepsilon n + (1-2\varepsilon)n = (1+\varepsilon)n$ edges, and compactness $\Theta(\varepsilon n)$.

A way to prove Theorem 16 is to use the result of [GP96b] about the *memory requirement* of a graph, which defines the smallest amount of memory needed to code any shortest path routing function (see [FG97] for an introduction to Universal Routing Schemes). In [GP96b] it has been built n-node graphs of degree bounded by d, $3 \leqslant d \leqslant \varepsilon n$ for $\varepsilon < 1$, of local memory requirement $\Theta(n \log d)$. This result, combined with Theorem 1, implies Theorem 16 (for $d = 3$ with a slightly modification to have a 3-regular graph).

We have also the following isolated results to quickly check whether a graph has compactness 1:

Theorem 17 ([GG95])

- *Let G be an n-node graph with d nodes of degree $n-1$. Let m be the number of connected components of at least two nodes in the complement of G. If $d \geqslant (n-m)/2$ then* $\mathrm{SLIRS}(G) = 1$. *Therefore, every graph G obtained from K_n by removing c edges, for $c \leqslant (n+1)/4$, verifies* $\mathrm{SLIRS}(G) = 1$.

- *Every graph G on $n < 7$ nodes and $m < 8$ edges verifies* $\mathrm{SIRS}(G) = 1$*. Moreover there exists a counter-example of* 7 *nodes and* 8 *edges with compactness* 2 *(a cycle on* 5 *nodes with a path of length* 2 *connecting two non-adjacent nodes).*

3.2 Interconnection Networks

We list in this paragraph the results about the graphs used for multi-processor based architectures.

Theorem 18 ([SK85])

- SIRS(trees) = 1.
- SIRS(rings) = 1.

Theorem 19 ([vLT87])

- SLIRS(paths) = 1.
- SLIRS(2D-grids) = 1.
- SIRS(2D-grids with column-wrap-around) = 1.
- SLIRS(complete graphs) = 1.
- SIRS(complete bipartite graphs) = 1.

Theorem 20 ([BvLT91])

- SLIRS(d-dimensional grids) = 1.
- SLIRS(hypercubes) = 1.
- SLIRS(d-dimensional tori) = 1*, if each dimension is of length at most* 4*.*

Applying the results of the product of graphs [KKR93, FG94b] it is quite easy to show:

Theorem 21 ([KKR93, FG94b])

- SLIRS(generalized hypercubes) = 1.
- SIRS(d-dimensional tori) = 1*, if the second largest dimension is of length at most* 4*, otherwise* SLIRS(d-dimensional tori) = 2.

Theorem 22 ([KRŠ97a])

- IRS(Shuffle-Exchange) = $\Omega(n^{1/2-\epsilon})$*, for every* $\epsilon > 0$*.*
- IRS(Connected-Cube-Cycle) = $\Omega(\sqrt{n/\log n})$.
- IRS(Butterfly) = $\Omega(\sqrt{n/\log n})$.

- IRS(Star graph) $= \Omega(n(\log\log n/\log n)^5)$.

Open question 6
- Are the bounds of Theorem 22 tight?

There are some experimental results in [SML94] about the stretch factor and the average stretch factor of 2D-grids and Tori with different several general labeling algorithms.

Theorem 23 ([SML94]) *For every* $d \geqslant 3$, 1-stretch(d-dimensional tori) $\leqslant (d-2)/2$.

3.3 Chordal Rings

A chordal ring is an augmented ring. Formally it is defined by the pair (n, L) where n is the number of nodes of the ring, and L is the set of integers or chords, $L \subset \{2, \ldots, n-2\}$. Each chord $l \in L$ connects every pair of nodes of the ring that are at distance l in the ring. We denote by $\mathcal{C}(n; l_1, \ldots, l_t)$ the chordal ring defined by $(n, \{l_1, \ldots, l_t\})$. t is called the dimension. Emphasize the degree of chordal rings is $2t$ in general, excepted whenever there is some chord of length $n/2$.

Theorem 24 ([NO97])

- *For every* $k < n/2$, SIRS$(\mathcal{C}(n;k)) \leqslant 2\sqrt{n}$.
- *For every* $k \leqslant \lceil\sqrt{2n-1}\,\rceil$, 1-dilation$(\mathcal{C}(n;k)) \leqslant \lceil 3D/2 \rceil$, *where* D *is the diameter of* $\mathcal{C}(n;k)$.
- *For every* $i \geqslant 2$, 1-stretch$(\mathcal{C}(2i^2+2i+1; 2i+1)) \in]1, 2]$.

According to the third result the smallest chordal ring known to be of compactness greater than 1 is $\mathcal{C}(13, 5)$, since 1-stretch$(\mathcal{C}(13, 5)) > 1$.

A "natural" labeling of the nodes is the cyclic labeling: consecutive labeling of nodes around the ring. This labeling is not always the best possible one. For instance $\mathcal{C}(14; 4)$ has no shortest path 1-IRS with cyclic labeling whereas the following one shows that SIRS$(\mathcal{C}(14; 4)) = 1$: the nodes are labeled respectively with $(1, 2, 3, 4, 5, 6, 9, 8, 7, 10, 11, 12, 13, 14)$, and the arc-labeling is $\mathcal{I}(1, 2) = [2, 3]$, $\mathcal{I}(1, 5) = [4, 8]$, $\mathcal{I}(1, 11) = [9, 12]$, and $\mathcal{I}(1, 14) = [13, 14]$ (the other intervals are obtained by shifting). The next result gives a characterization of graphs supporting a 1-IRS according a cyclic labeling of the nodes.

Theorem 25 ([KL95])

- *A chordal ring* $\mathcal{C}(n; k)$ *has a shortest path* 1-*IRS with cyclic labeling if and only if* $n \bmod k = 0$, $n \bmod n-k = 0$, *or* $n = sk+1$, *or* $n = sk-1$ *and* s *is even, or* $n = sk+2$ *and* s *is even.*

- *For every $k \leqslant 3$,* $\mathrm{SIRS}(\mathcal{C}(n;k)) = 1$.

[KL95] also showed that $\mathcal{C}(29,3)$ has no shortest path 1-IRS with cyclic labeling whereas it is of compactness 1 ([KL95] gave the labeling). Other results about directed chordal rings are mentioned. [NO97] showed that there exists some chordal rings, $\mathcal{C}(2i^2+2i+1; 2i+1)$, where every shortest path IRS using a cyclic labeling requires $\sqrt{n/2}$ intervals, and also that $\mathcal{C}(35;20)$ has no 1-IRS of stretch factor 2 with a cyclic labeling.

Open question 7
- Is $\mathrm{SIRS}(\mathcal{C}(n,4)) = 1$ for every n?
- Is there some chordal ring of compactness $\Omega(\sqrt{n})$?
- Characterize chordal rings of compactness k with cyclic labeling.

For two chords we have:

Theorem 26 ([FGS94]) *Let i, j be two integers, and $C = \mathcal{C}(n;i,j)$.*

- *if* $n \bmod i = 0$, *and* $n \bmod j = 0$, *then* $\mathrm{IRS}(C) = 1$.
- *if* $j \bmod i = 0$, $n \bmod j = r \neq 0$, *and* $n/j \geqslant \max\{j-r-1, r\}$, *then* $\mathrm{IRS}(C) = O(\min\{n, j^2\}/i)$.
- *if* $n \bmod j = 0$, *and* $j = i+1$, *then* $1\text{-stretch}(C) \leqslant 3/2$.
- *if* $j = i+1$, *or* $2j < i$, $1\text{-stretch}(C) \leqslant (i+2)/4$.
- *if* $2i > j$, *then* $1\text{-stretch}(C) \leqslant (j-i+2)/2$.

And for higher dimensions,

Theorem 27 ([FGS94])

- *For every $k \geqslant 1$,* $\mathrm{IRS}(\mathcal{C}(n;1,\ldots,k)) = 1$.
- *For every $r, p \geqslant 2$, and for every subset $\{l_1,\ldots,l_k\} \subset \{1,\ldots,p-1\}$,* $\mathrm{IRS}(\mathcal{C}(r^p; r^{l_1},\ldots,r^{l_k})) = 1$.

In [FGS94] is presented several results concerning IRS without the disjointness property of the labels assigned to the arcs, i.e., with Condition 2b of Definition 1 relaxed. See Paragraph 4.4 for complement.

Open question 8
- Is there a general upper bound for the compactness of chordal rings of dimensions d of the form: $f(d)n^{1-1/d}$, for some function f independent of n?

3.4 Planar Graphs

Let us recall that a *plane* graph is a planar graph embedded in a plane without crossing edge. An *outerplanar* graph is a plane graph whose all its nodes are lying on one faces (this includes trees).

Theorem 28 ([FJ88]) *Every plane graph G satisfies* $\mathrm{SIRS}(G) \leqslant 3p/2$, *where p is the smallest number of disjoint faces that cover all the nodes. Therefore, every outerplanar graph G verifies* $\mathrm{SIRS}(G) = 1$.

Unfortunately the number of disjoint faces, p, can attain $\Theta(n)$. In [FJ88] the result is generalized for graph of genus g with an upper bound of $3p/2+g$. Other results, but slightly weaker, are presented in the same article. None bound has been proved yet to be tight.

Theorem 29 ([GP96a]) *For every integer n large enough there exists:*

- *an n-node planar graph of compactness at least $\sqrt{n}/2.6$;*
- *a cubic planar graph of at most n nodes of compactness at least $\sqrt{n}/5.6$;*
- *an n-node planar graph on which every shortest path IRS has a total number of intervals required for the entire graph at least $\Omega(n^{3/2})$, i.e., $\Omega(\sqrt{n})$ intervals on a constant fraction of all the edges;*
- *an n-node triangulated*[3] *plane graph of bounded degree of compactness $\Omega(\sqrt{n})$.*

The following conjecture already appears as open question in [Gav96].

Conjecture 1 *Every n-node planar graph has compactness $O(\sqrt{n})$.*

Theorem 30 ([TL97b]) *For every even integer $D \geqslant 2$ there exists an n-node planar graph G of diameter D such that:*

- $k\text{-dilation}(G) \geqslant (2k+1)/(2k)D - 1$, *for every* $k \leqslant (n/24)^{1/3}$;
- $k\text{-dilation}(G) \geqslant (4k+1)/(4k)D - 1$, *for every* $k \leqslant (n/38)^{1/2}$.

The latter result uses a *globe-graph* (a subdivision of a complete bipartite graph $K_{2,t}$) which is a serie-parallel graph [Duf65], and also a graph of treewidth two [Bod92]. Thus the general $\Omega(\sqrt{n})$-lower bound for planar graphs holds also for the subclass of graphs of treewidth two (see Paragraph 3.5 for results about treewidth).

The best known lower bound of compactness, regarding the multiplicative constant, for serie-parallel graphs is:

[3] A plane graph where all the faces are triangles.

Theorem 31 ([KRŠ97b])

- *For every integer* $t \geqslant 1$*, there is a serie-parallel graph* G *of* $n = t(t+1)+2$ *nodes such that* $\mathrm{IRS}(G) \geqslant t/4 > \sqrt{n}/4 - 1/2$ *(a globe-graph with* t *paths of length* $t+2$*).*
- *There are bounded degree serie-parallel graphs of compactness* $\Omega(\sqrt{n})$ *(a* $\sqrt{n} \times \sqrt{n}$ *grid where all internal and horizontal edges have been removed).*

Historically, the first $\Omega(\sqrt{n})$-lower bound for a serie-parallel graph has been implicitly proved by [TL95]. The previous bound (using the same graph) was of $\Omega(n^{1/3})$ in [KKR93].

Open question 9
- Is some constants k and c such that every planar graph G of diameter D satisfies k-dilation$(G) \leqslant 3D/2 + c$? or k-stretch$(G) \leqslant c$?

Actually [KRŠ97b] proposed a stronger version of Question 9: For every $\varepsilon > 0$, is there a constant k such that every planar graph G of diameter D satisfies k-dilation$(G) \leqslant (1+\varepsilon)D$?

3.5 Treewidth Bounded Graphs

The next result confirms the lower bound of Theorem 31, and gives a generalization. Note that graphs of treewidth 1 are trees, and therefore have compactness 1 by Theorem 3.

Theorem 32 *For every* k*,* $2 \leqslant k \leqslant (n/3)^{1/3}$*, there exists an* n*-node graph* G *of treewidth* k *such that* $\mathrm{IRS}(G) = \Omega(\sqrt{kn})$ *(a subdivision of the graph* $K_{k,r}$ *with* $r = \Theta(\sqrt{kn})$*).*

Proof. Let k, r, s be three integers such that $r \geqslant k \geqslant 2$, and $l \geqslant 1$. Consider the graph G obtained by subdividing all the edges in $2l+1$ nodes of the bipartite graph $K_{k,r}$. G has $n = k + r + (2l+1)kr$ nodes and a treewidth $\min\{k, r\} = k$, see [Klo94, Lemma 2.2.1, page 18]. We consider a node x initially belonging to the k-partition of $K_{k,r}$, and a node y initially belonging to the r-partition of $K_{k,r}$. Let W be the set of nodes composed of all the nodes at an odd distance of x. We build a set A of arcs as follows: first A is composed of all the incident arcs of x. Then, we add to A all the incident arcs of y. Finally, we add to A all the arcs of the directed path going from x to y and whose the tail is at an even distance from x.

The nodes of W and the arcs of A constitute a *matrix of constraints* of G, i.e., the shortest path from the tail of any arc $(a, b) \in A$ to any node of $w \in W$ is unique (see [FvLMS95, GG95] for a formal description of this technique): we put "1" in such a boolean matrix if the route from a to w must use the arc (a, b); "0" otherwise. The dimension of the matrix is $|W|\times|A|$. We consider the submatrix M obtained by removing the rows corresponding to nodes of W belonging to the subdivision of the induced subgraph $K_{1,r}$ rooted in y. Therefore M has $|W|-(s+1)r=(k-1)r(s-1)$ rows and $|A|=k+r+s$ columns.

Every two rows of M differ by at least one place (emphasize this is false if we take entirely W as set of nodes). Therefore there exists some arc $(a,b)\in A$ such that the set $\mathcal{I}(a,b)$ is composed of at least $I=(|W|-(s+1)r)/(2|A|)$ intervals, whatever the shortest path IRS $(\mathcal{L},\mathcal{I})$. Indeed, the total number of intervals for all the arcs of A is at least $(|W|-(s+1)r)/2$, the number of 01-sequences for the columns of the matrix under row permutation (each such sequence begins a new interval).

Let $r=s$. This is possible because in this case $n=2kr^2+(k+1)r+k$; so it suffices to choose any integer $r\geqslant\sqrt{n/(3k)}$ in order to have at most an n-node graph, and also $r\geqslant k$ for the desired range for k: $2\leqslant k\leqslant (n/3)^{1/3}$.

$$I \;=\; \frac{|W|-(s+1)r}{2|A|} \;=\; \frac{(k-1)r(s+1)}{2(k+r+s)} \;>\; \frac{(k-1)r^2}{2(k+2r)}.$$

Since $r\geqslant k$, $k+2r\leqslant 3r$, and thus $I>(k-1)r/6$. On the other hand $n=2kr^2+r(k+1)+k<8(k-1)r^2$, for every $r\geqslant k\geqslant 2$. Thus $r>\sqrt{n/(8(k-1))}$. We get finally that

$$I \;>\; \frac{k-1}{6}\sqrt{\frac{n}{8(k-1)}} \;=\; \frac{1}{6\sqrt{8}}\sqrt{(k-1)n},$$

which is $\Omega(\sqrt{kn})$ for every $k\geqslant 2$, that completes the proof. □

Theorem 33 ([NN96])

- $\mathrm{SIRS}(2\text{-trees})=3$ *(thus there exists some 2-trees of compactness 3).*
- $\mathrm{SIRS}(k\text{-trees})\leqslant 2^{k+1}$.

Clearly the first result does not hold for partial 2-trees (a 2-tree obtained by deleting some edges), because it is well-known that partial 2-trees have treewidth at most 2, and compactness $\Omega(\sqrt{n})$ by Theorem 32. The second result is particularly interesting since, in general, k is a constant in comparison with n, the number of nodes.

In [FL96], there is another study about 1-IRS on directed serie-parallel graphs (special kind of digraphs). They built a shortest path 1-IRS for the directed case, and gave also a valid 1-ILS for the undirected case, however without explicit bound on the routing path length for the undirected case.

Open question 10
- Is the second bound of Theorem 33 tight?
- Is there some function f such that $\mathrm{IRS}(\mathcal{T}_k)\leqslant f(k)\sqrt{n}$, where $\mathcal{T}_k$ is the class of graphs of treewidth k?

Note that even for the class of graphs having $O(n)$ edges the function $f(k)$ must be at least $\sqrt{k}$ since there exists some graphs with $1.18n$ edges (actually almost every n-node graph with $m \geqslant 1.18n$ edges) and treewidth $\Theta(n)$ [Klo94, Theorem 5.3.2, page 58], and also bounded degree graphs of compactness $\Theta(n)$ (Theorem 16).

Conjecture 2 *Every n-node graph of constant treewidth has compactness $o(n)$.*

Several works have been done about the trade-off between the dilation and the compactness for the class of *multi-globe* graphs. A multi-globe graphs is a subdivision of the graph $K_{a,b}$ such a way all the edges of $K_{a,b}$ are subdivided in the same number of nodes. This class provides a large set a counter-examples (actually used in Theorem 10, and 32) and includes the globe-graphs (used in Theorem 30, and 31).

For convenience, for every class of graphs $\mathcal{G}$, we will denote by k-dilation$(\mathcal{G}) = \max_{G \in \mathcal{G}} k$-dilation$(G)$, and similarly for k-stretch$(\mathcal{G})$.

Theorem 34

- **[TL97a]** 1-dilation(multi-globe) $\geqslant 2D - 3$.
- **[KRŠ97b]** 2-dilation(multi-globe) $\leqslant 1.25D$.
- **[KRŠ97b]** k-dilation(multi-globe) $\leqslant 1.25D - 1$ *implies* $k = \Omega(\sqrt{n})$.
- **[KRŠ97b]** 1-dilation(globe) $\leqslant 1.5D$.
- **[Ruž88]** k-dilation(globe) $\leqslant 1.5D - 1$ *implies* $k > 1$.
- **[TL95]** $O(1/\varepsilon)$-dilation(globe) $\leqslant (1+\varepsilon)D$, *for every* $\varepsilon > 0$.
- **[FGK97]** 1-stretch(globe) $\in [3 - O(1/D), 3]$.

Note that, in most of the papers cited in Theorem 34, the diameter D of globe or multi-globe graphs is even.

3.6 Other Classes of Graphs

$K_{n_1,\ldots,n_r}$ denotes the complete r-partite graph of sets of nodes $V = V_1 \cup \ldots \cup V_r$, $n_i = |V_i|$. The edges are such that for every $i \neq j$, the graph induced by V_i and V_j is a complete bipartite graph, K_{n_i,n_j}. $K_{1,\ldots,1} = K_n$.

Theorem 35 ([KKR93]) *For every $2 \leqslant n_1 \leqslant \ldots, \leqslant n_r$,* $\mathrm{LIRS}(K_{n_1,\ldots,n_r}) = 1$.

It has been slightly improved later for bipartite graphs (improving also a result of Theorem 19):

Theorem 36 ([NS96])

- *For every* $1 \leqslant n_1 \leqslant n_2$, $\mathrm{SIRS}(K_{n_1,n_2}) = 1$.
- *For every* $2 \leqslant n_1 \leqslant n_2$, $\mathrm{SLIRS}(K_{n_1,n_2}) = 1$.

The next class of graphs includes complete graphs, paths and rings. See [Gol80] for an introduction of these classes of graphs.

Theorem 37 ([FG94b])

- SIRS(unit arc-circular graphs) $= 1$, *and* IRS(arc-circular graphs) > 1.
- SLIRS(unit interval graphs) $= 1$, *and* LIRS(interval graphs) > 1.

The latter result has been improved (this class includes trees):

Theorem 38 ([NS96]) SIRS(interval graphs) $= 1$.

The class $\mathcal{G}_{n,p}$ denotes the classic model of n-node random graphs, where p represents the probability to have an edge between any two nodes.

Theorem 39 ([FvLMS95])

- *Let* $p = n^{-1+1/s}$ *for an integer* $s > 0$ *such that there exists* ε *that verifies* $(\ln^{1+\varepsilon} n)/n < p < n^{-1/2-\varepsilon}$. *Then, given a graph* $G \in \mathcal{G}_{n,p}$, $\mathrm{IRS}(G) = \Omega(n^{1-6/\ln(np)-\ln(np)/\ln n})$ *with probability* $1 - o(1)$.
- *Given a graph* $G \in \mathcal{G}_{n,p}$, *for some* $p = n^{-1+1/\Theta(\sqrt{\log n})}$, $\mathrm{IRS}(G) = \Omega(n^{1-1/\Theta(\sqrt{\log n})})$ *with probability* $1 - o(1)$.

By an exhaustive computation of all the possible labeling, we found that the Petersen graph, P, satisfies $\mathrm{SLIRS}(P) = \mathrm{LIRS}(P) = \mathrm{SIRS}(P) = \mathrm{IRS}(P) = 3$. $P = (V, E)$ is an intersection graph[4] defined by $V = \{X \subset \{1, \ldots, 5\} \mid |X| = 2\}$.

Finally, there are several works on Interval Routing with experimental approaches and simulation results: [Bra93, Jug93, HKR91, LW96, Sch97].

3.7 Graph Operators

There are several results about some graph operators: Cartesian product, composition, and join of graphs.

The *Cartesian product* of $G_1 = (V_1, E_1)$ with $G_2 = (V_2, E_2)$, denoted by $G_1 \times G_2$, has the node set $V_1 \times V_2$, and the edge set $\{(u, v) \mid u = (u_1, u_2), v = (v_1, v_2)$, and $[(u_1 = v_1$ and $(u_2, v_2) \in E_2)$ or $(u_2 = v_2$ and $(u_1, v_1) \in E_1)]\}$.

The first point (the linear case) of Theorem 40 is due to [KKR93], the others due to [FG94b]. The lower bounds of this theorem are obtained by application of Theorem 48 in Paragraph 4.1.

[4] I.e., $E = \{(A, B) \in V^2 \mid A \neq B$ and $A \cap B \neq \emptyset\}$.

Theorem 40 ([KKR93, FG94b])

- $\mathrm{SLIRS}(G \times H) = \max\{\mathrm{SLIRS}(G), \mathrm{SLIRS}(H)\}$;
- $\max\{\mathrm{LIRS}(G), \mathrm{LIRS}(H)\} \leqslant \mathrm{LIRS}(G \times H) \leqslant \max\{\mathrm{LIRS}(G), \mathrm{SIRS}(H)\}$;
- $\max\{\mathrm{SIRS}(G), \mathrm{SIRS}(H)\} \leqslant \mathrm{SIRS}(G \times H) \leqslant \max\{\mathrm{SLIRS}(G), \mathrm{SIRS}(H)\}$;
- $\max\{\mathrm{IRS}(G), \mathrm{IRS}(H)\} \leqslant \mathrm{IRS}(G \times H) \leqslant \max\{\mathrm{LIRS}(G), \mathrm{SIRS}(H)\}$.

We can see one motivation to distinguish the strictness and the linearity of IRS. To apply Theorem 40, one graph must support a strict IRS and the other a linear IRS. Note that, in general, $\mathrm{IRS}(G \times H) \neq \max\{\mathrm{IRS}(G), \mathrm{IRS}(H)\}$. The 5×5-torus is a counterexample. In [FG94b], other results are mentioned about the k-dilation in Cartesian product.

The *composition* of $G_1 = (V_1, E_1)$ with $G_2 = (V_2, E_2)$, denoted by $G_1[G_2]$, has the node set $V_1 \times V_2$, and the edge set $\{(u,v) \mid u = (u_1, u_2), v = (v_1, v_2)$, and either $(u_1, v_1) \in E_1$ or $(u_1 = v_1$ and $(u_2, v_2) \in E_2)\}$. The previous result of [KKR93] has been improved with:

Theorem 41 ([NS96])

- $\mathrm{SLIRS}(G[H]) \leqslant \mathrm{SLIRS}(G) + 1$.
- *If H is a unit interval graph, then* $\mathrm{SIRS}(G[H]) \leqslant \mathrm{SIRS}(G)$.

The *join* of $G_1 = (V_1, E_1)$ with $G_2 = (V_2, E_2)$, denoted by $G_1 + G_2$, has the node set $V_1 \cup V_2$, and the edge set $E_1 \cup E_2 \cup \{(u,v) \mid u \in V_1, v \in V_2\}$.

Theorem 42 ([KKR93]) *Suppose G_1 and G_2 are graphs with n_1 and n_2 nodes, minimum degrees δ_1 and δ_2, and maximum degrees Δ_1 and Δ_2 respectively. Then,*

$$\mathrm{LIRS}(G_1 + G_2) \leqslant 1 + \max\{\lceil f_1/n_2 \rceil, \lceil f_2/n_1 \rceil\}$$

where $f_i = \min\{n_i - \delta_i - 1, \Delta_i + 1\}$, $i \in \{1, 2\}$.

4 Variations on the Interval Routing Model

4.1 Fixed and Dynamic Link-Cost Models

In [FJ88] is presented the first results dealing with weighted graphs and dynamic link-cost model. Their nice characterization for compactness 1 have been extended in [BvLT91] for the linear case, and finally [BLTT95] proved that linear algorithm exists for the characterization of graphs of compactness k, for every fixed $k \geqslant 1$, in the setting of dynamic link-cost model.

In this paragraph we consider weighted graphs, that is a pair (G, w) where $G = (V, E)$ is a graph, and w a non negative function that assigns a "weight" for each arc $(x, y) \in E$. This weight can take into account the communication costs along the link between x and its neighbor y. The *cost of a path* in a weighted graph is the sum of the weights of the arcs that compose the path. A *minimum path* is a path of minimum cost between its two end-points.

Definition 9 *A graph G supports a* minimum fixed cost *IRS if for every weighted function w there is an IRS on G such that all the routing paths are minimum paths. The class of graphs having a minimum fixed cost k-IRS is denoted by k-$\mathcal{IRS}_f$.*

Let us denote by k-$\mathcal{IRS}$ the class of graphs that support a shortest path k-IRS. Clearly k-$\mathcal{IRS}_f \subset k$-$\mathcal{IRS}$ because the uniform weight function is a particular case of the model k-$\mathcal{IRS}_f$.

Another model assumes that the addresses of the nodes are fixed once in advance (independently of the link-costs), and then the weights can change. This model takes into account the situation where the link-costs evolute over time, and where the computation of the node-labeling through the network is impossible, whereas the edge-labeling remains locally adjustable accordingly the minimality of the routing path lengths.

Definition 10 *A graph G supports a* minimum dynamic cost *IRS if there exists a node-labeling of G, $\mathcal{L}$, such that for every weighted function w there an IRS on G with respect to $\mathcal{L}$ such that all the routing paths are minimum paths. The class of graphs having a minimum dynamic cost k-IRS is denoted by k-$\mathcal{IRS}_d$.*

Similarly, k-$\mathcal{IRS}_d \subset k$-$\mathcal{IRS}_f$ because for $G \in k$-$\mathcal{IRS}_d$, the node-labeling $\mathcal{L}$ gives a labeling solution for every weighted function w. Hence k-$\mathcal{IRS}_d \subset k$-$\mathcal{IRS}_f \subset k$-$\mathcal{IRS}$, and all the lower bounds proved in the uniform link-cost model hold also for the two other models. We extend the previous notations to the linear and/to strict IRS in a similar way.

The dynamic link-cost model assumes implicitly that addresses of the nodes cannot be permuted in order to optimize the number of intervals of the edge-labeling with the minimality of the routing path lengths. It turns out that Theorem 14 (compactness at most $n/4 + o(n)$ for every graph) holds for the fixed link-cost model, but not for the dynamic link-cost model.

For convenience, a statement which holds independently for the all the classes k-$\mathcal{IRS}$, k-$\mathcal{IRS}_f$, and k-$\mathcal{IRS}_d$ is termed with k-$\mathcal{IRS}_\star$. By definition, we set $K_1 \in 0$-$\mathcal{SLIRS}_\star$.

Theorem 43

- *For every n-node graph G, $n \geqslant 1$,*
  - $G \in \lceil n/2 \rceil$-$\mathcal{SLIRS}_\star$;

- $G \in \lfloor n/2 \rfloor$-$\mathcal{SIRS}_\star$;
- $G \in \lfloor n/2 \rfloor$-$\mathcal{LIRS}_\star$;
- $G \in (\lceil n/2 \rceil - 1)$-$\mathcal{IRS}_\star$.

- *For every* n, $n \geqslant 1$,
  - $K_n \in k$-$\mathcal{SLIRS}_d$ *implies* $k \geqslant \lceil n/2 \rceil$;
  - $K_n \in k$-$\mathcal{SIRS}_d$ *implies* $k \geqslant \lfloor n/2 \rfloor$;
  - $K_n \in k$-$\mathcal{LIRS}_d$ *implies* $k \geqslant \lfloor n/2 \rfloor$;
  - $K_n \in k$-$\mathcal{IRS}_d$ *implies* $k \geqslant \lceil n/2 \rceil - 1$.

Proof. Let us start to prove the first point. Let $R = (\mathcal{L}, \mathcal{I})$ be an IRS on G. Consider $I = \mathcal{I}(x,y) \subset \{1,\dots,n\}$. Let k be the number of intervals of I. Let $I' = \{1,\dots,n\} - I$ be the complement of I, and k' its number of intervals. Clearly, $k + k' \leqslant n$. Moreover, for the linear cases $k \leqslant k' + 1$, and $k = k'$ for cyclic intervals. It implies that

$$2k \leqslant k + k' + 1 \leqslant n + 1 \quad \Rightarrow \quad k \leqslant \left\lceil \frac{n+1}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor,$$

for the linear cases, and $2k = k + k' \leqslant n$ which implies $k \leqslant \lfloor n/2 \rfloor$ for the cyclic cases. Remains to prove the non strict cases.

Assume R is non strict and k is minimum. To show that $G \in \lfloor n/2 \rfloor$-$\mathcal{LIRS}_\star$, it suffices to show that for n odd the previous bound decreases by 1. So assume $n = 2i + 1$, and that I is composed of $\lceil n/2 \rceil = i + 1$ intervals. Note that I' is necessary composed of at least i intervals. Moreover all the intervals that composed I and I' are single, i.e., of the form $[a]$. Indeed if 1 interval (at least) is composed of 2 integers (or more) then the number of elements in $I \cup I'$ would be $(i+1) + i + 1 = 2i + 2 = n + 1$ which is impossible. If $\mathcal{L}(x) \in I$ then $\mathcal{L}(x)$ appears as a single interval, so its removing will win 1 interval. I contains 1 and n, because I and I' are composed of single intervals only, and $|I| > |I'|$. If $\mathcal{L}(x) \notin I$ then its insertion will win 1 intervals because $\mathcal{L}(x) - 1$ and $\mathcal{L}(x) + 1$ are both single intervals and both belong to I (we can assume $n \geqslant 3$ because for $n \leqslant 2$ the result is trivial).

To prove $G \in (\lceil n/2 \rceil - 1)$-$\mathcal{IRS}_\star$ it suffices to prove it for n even. So assume $n = 2i$, and I composed of $\lfloor n/2 \rfloor = i$ intervals. Similarly I and I' are composed of single intervals only, and thus the insertion or deleting of $\mathcal{L}(x)$ in I will decreases by 1 the number of intervals of I.

Let us show the second point. Consider the node labeled 1 and 2. First consider the case of strict intervals ($\mathcal{SLIRS}_d$ or $\mathcal{SIRS}_d$). Choose w such that $w(2,1) = 1$, $w(1, 2i+1) = 1$ for i integer such that $3 \leqslant 2i + 1 \leqslant n$, and $w(x,y) = 3$ otherwise. The shortest paths from 2 to i travels 1 if i is odd (cost $\leqslant 2$, and 3 otherwise). Thus $\mathcal{I}(2,1) = \{1, 3, \dots, 2i+1, \dots\}$. It is composed of $|\mathcal{I}(2,1)| = \lceil n/2 \rceil$ linear intervals (there is no consecutive integers in $\mathcal{I}(2,1)$), and $\lfloor n/2 \rfloor$ cyclic intervals (in the odd case 1 and n can be merged).

For the non strict intervals ($\mathcal{LIRS}_d$ or $\mathcal{IRS}_d$), choose w such that $w(2,1) = 1$, $w(1, 2i) = 1$ for i integer such that $4 \leqslant 2i \leqslant n$, and $w(x,y) = 3$ otherwise. The shortest paths from 2 to i travels 1 if i is even ($i \neq 2$). Hence

$\mathcal{I}(2,1) \subset \{1,4,\ldots,2i,\ldots\}$. $\mathcal{L}(2)$ can belong to $\mathcal{I}(2,1)$ or not, it does not change the number of intervals of $\mathcal{I}(2,1)$. It is composed of $|\mathcal{I}(2,1)| = \lfloor n/2 \rfloor$ linear intervals, and $\lfloor (n-1)/2 \rfloor = \lceil n/2 \rceil - 1$ cyclic intervals. This completes the proof. □

The inherent definition of the link-cost model imposes that a graph can have very different compactness depending on the model (uniform, fixed or dynamic link-cost). Indeed we have seen in Theorem 19 that $K_n \in 1\text{-}\mathcal{SLIRS}$. In the opposite to the class $k\text{-}\mathcal{IRS}$ the fixed and dynamic link-cost models have known characterizations.

The definition of outerplanar graphs is recalled in Paragraph 3.4.

Theorem 44 ([FJ88]) *$G \in 1\text{-}\mathcal{SIRS}_d$ if and only if G is outerplanar.*

Theorem 45 ([BvLT91]) *$G \in 1\text{-}\mathcal{IRS}_d$ if and only if every 2-vertex-connected component of G is outerplanar or K_4.*

A *G-star* is a graph obtained from G by adding zero or more nodes of degree one to the nodes of G. A *centipede* is a P-star, P being a path, where some edges of P are replaced by a K_3 (the edge of the path is identified to an edge of K_3). A Y-graph is a tree on 7 nodes with one node of degree 3, and three nodes of degree 2. Finally, H is a *subgraph of minimum paths* of a weighted graph G if H is a subgraph of G, and if for all $x,y \in V(H)$, all the minimum paths between x and y in G are wholly contained in H.

Theorem 46 ([BvLT91])

- *$G \in 1\text{-}\mathcal{LIRS}_f$ if and only if G is a centipede, a K_3-star or a K_4-star that does not contain a Y-graph as subgraph of minimum paths.*
- *$G \in 1\text{-}\mathcal{LIRS}_d$ if and only if G is a centipede.*

According to [vLT94] we have:

Theorem 47 ([BvLT94]) *$G \in 1\text{-}\mathcal{SLIRS}_d$ if and only if G is a path.*

The following result proposes a tool to show that a given graph has not compactness k. It suffices to show that one of its subgraph of minimum paths is not.

Theorem 48 ([FG94b]) *A given graph belongs to $k\text{-}\mathcal{IRS}_\star$ if all its subgraph of minimum paths belongs to $k\text{-}\mathcal{IRS}_\star$ (idem for the linear and/or strict variants).*

This is also a sufficient condition since G is a subgraph of minimum paths of itself. In the original paper, the result has been mentioned only for the uniform link-cost model. The proof holds for all the models, and can be easily adapted for a version using stretch factor and dilation.

Because IRS (with cyclic intervals) are invariant under shifting of the labels modulo n, it is easy to compose graph by union sharing one cut-vertex. The sufficient condition of the next result is an application of Theorem 48.

Theorem 49 ([GG95]) *For every $k \geqslant 1$, $G \in k\text{-}\mathcal{IRS}_\star$ (respectively $k\text{-}\mathcal{SIRS}_\star$) if and only if all its 2-vertex-connected components are in $k\text{-}\mathcal{IRS}_\star$ (respectively $k\text{-}\mathcal{SIRS}_\star$).*

Here a version for the linear case, and for $k = 1$:

Theorem 50 ([NS96]) *Let G be a graph, let $G_1, \dots, G_k$ be its 2-vertex-connected components, and let $x_{i,j}$ be the cut-vertex between the component G_i and G_j, $1 \leqslant i < j \leqslant k$. $G \in 1\text{-}\mathcal{SLIRS}_\star$ if and only if:*

- *each $G_i \in 1\text{-}\mathcal{SLIRS}_\star$;*
- *the components $G_1, \dots, G_k$ form a path;*
- *$x_{1,2}$ has label $|V(G_1)|$ in the labeling of G_1;*
- *$x_{k-1,k}$ has label 1 in the labeling of G_k;*
- *there is a labeling of each of the graphs G_i, $1 < i < k$, in which $x_{i-1,i}$, has label 1 and $x_{i,i+1}$ has label $|V(G_i)|$.*

The next result establishes an interesting bridge between minor taking theory and minimum dynamic cost IRS.

Theorem 51 ([BLTT95]) *For each $k \geqslant 1$,*

- *the class $k\text{-}\mathcal{IRS}_d$ (and its variants) is closed under minor-taking in the domain of connected graphs, and hence has a linear time recognition algorithm (but the algorithm is unknown);*
- *$G \in k\text{-}\mathcal{IRS}_d$ implies that G has a treewidth at most $4k$ (idem for its variants);*
- *$G \in k\text{-}\mathcal{IRS}_d$ and G is planar implies that G has a treewidth at most $2k+3$ (idem for its variants).*

The two last results of Theorem 51 implies that the class $k\text{-}\mathcal{IRS}_d$ covers a small part of all the graphs since almost all graphs (even sparse) have a treewidth $\Theta(n)$, see [Klo94, Theorem 5.3.2, page 58]. Note that the converse is false because Theorem 32 shows that there are graphs of treewidth bounded by k which have unbounded compactness for the uniform link-cost model, hence for all the models.

Open question 11
- For $G \in k\text{-}\mathcal{IRS}_d$, what is the complexity of finding the node-labeling $\mathcal{L}$ of G?

4.2 Hierarchy of the IRS Classes

The strictness of IRS can be obtained by just removing the label $\mathcal{L}(x)$ from $\mathcal{I}(x,y)$, and therefore by increasing by one at most its compactness (or linearity compactness). Because a cyclic interval is the union of two linear intervals, we have the trivial collection of inclusions:

Theorem 52 (Folklore) *For every $k \geqslant 1$,*

- k-$\mathcal{SIRS}_\star \subset k$-$\mathcal{IRS}_\star \subset (k+1)$-$\mathcal{SIRS}_\star$.
- k-$\mathcal{SLIRS}_\star \subset k$-$\mathcal{LIRS}_\star \subset (k+1)$-$\mathcal{SLIRS}_\star$.
- k-$\mathcal{LIRS}_\star \subset k$-$\mathcal{IRS}_\star \subset (k+1)$-$\mathcal{LIRS}_\star$.
- k-$\mathcal{SLIRS}_\star \subset k$-$\mathcal{SIRS}_\star \subset (k+1)$-$\mathcal{SLIRS}_\star$.

Open question 12
- Is 1-$\mathcal{SIRS}$ = 1-$\mathcal{IRS}$?
- Same question for k-$\mathcal{SIRS}$, $k > 1$?

In the following, $X \subsetneq Y$ means that $X \subset Y$ and $X \neq Y$. Because in [FJ88] and in [BvLT91] it is shown $K_{2,2k+1} \in (k+1)$-$\mathcal{LIRS}_d$ but not in k-$\mathcal{IRS}_d$, we have:

Theorem 53 ([FJ88]) *For every $k \geqslant 1$, k-$\mathcal{IRS}_d \subsetneq (k+1)$-$\mathcal{IRS}_d$.*

Theorem 54 ([BvLT91]) *For every $k \geqslant 1$,*

- k-$\mathcal{LIRS}_d \subsetneq (k+1)$-$\mathcal{LIRS}_d$.
- 1-$\mathcal{LIRS}_d \subsetneq$ 1-$\mathcal{IRS}_d$.

Theorem 53, and 54 can be improved by Theorem 43.

Corollary 1 *For every $k \geqslant 1$,*

- k-$\mathcal{SLIRS}_d \subsetneq k$-$\mathcal{LIRS}_d \subsetneq k$-$\mathcal{IRS}_d$.
- k-$\mathcal{SLIRS}_d \subsetneq k$-$\mathcal{SIRS}_d \subsetneq k$-$\mathcal{IRS}_d$.

Proof. From Theorem 43, for every $k \geqslant 1$, $K_{2k+1} \in \lfloor (2k+1)/2 \rfloor$-$\mathcal{LIRS}_d = k$-$\mathcal{LIRS}_d$, and also $K_{2k+1} \in k$-$\mathcal{SIRS}_d$. From the second point of Theorem 43, $K_{2k+1} \in t$-$\mathcal{SLIRS}_d$ implies that $t \geqslant \lceil (2k+1)/2 \rceil = k+1$. Hence $K_{2k+1} \notin k$-$\mathcal{SLIRS}_d$. It follows that k-$\mathcal{SLIRS}_d \neq k$-$\mathcal{LIRS}_d$ and k-$\mathcal{SLIRS}_d \neq k$-$\mathcal{SIRS}_d$. Similarly $K_{2k+2} \in k$-$\mathcal{IRS}_d$, but $K_{2k+2} \notin k$-$\mathcal{LIRS}_d$, and $K_{2k+2} \notin k$-$\mathcal{SIRS}_d$. $\square$

4.3 Deadlock-Free Interval Routing Schemes

The study of the trade-off between the length of the routing paths and compactness, and more generally the memory requirement, is complex and is the heart of many active research [PU88, FJ90, FG97]. There are still many interesting problems related to Interval Routing that have not been solved yet.

Another interesting aspect related to the routing problem is the traffic and congestion (or load). Moreover problems related to deadlock often arise. A *deadlock* refers to a situation in which a set of messages is blocked forever because each packet in the set holds some resources (links or processors) that are also needed by another packet. Deadlock-free routing is relevant in the framework of *worm-hole* protocols [DS87]. The first studies dealing with deadlock-free Interval Routing have been done in [VDLP94], [SML94], and in [ZBG94]. An IRS on G is *deadlock-free* if its induced routing function on G is deadlock-free (see [DS87]).

We can distinguish two approaches:

- study of k-IRS that induces deadlock-free routing function;
- study of k-IRS with some control mechanisms, in each router, for buffer management: a set of buffers is assigned to each router (eventually to each link) to prevent deadlock situations. E.g., a buffer on each *virtual channel* can be used (see [DS87]).

[VDLP94] showed results on deadlock-free IRS for 2D-grids with faulty regions. For the second approach, [ZBG94] unify the notion of virtual channel and interval assigned to links. For each edge e labeled by the intervals $[a_1, b_1] \cup \ldots \cup [a_k, b_k]$, and for each interval $[a_i, b_i]$, $2 \leqslant i \leqslant k$, it is associated a virtual channel (using the definitions given by [DS87]). In [ZBG94] it is showed that shortest path deadlock-free SLIRS are close under Cartesian product.

Conjecture 3 ([ZTRP94]) *Every graph that supports a shortest path 1-SLIRS supports also a deadlock-free shortest path 1-SLIRS.*

However this approach imposes to use $k-1$ virtual channels if k intervals per edge are used, whereas the number of buffers could be lower than this bound. In [Fla97a] it is built IRS deadlock-free for some classic topologies: trees, rings, grids, complete graphs and chordal rings. Some trade-offs are derived between the compactness of the IRS and the number of buffers in order to guarantee deadlock-free IRS for Cartesian product based graphs: hypercube, grids, and tori. The number of buffers used by the router is taken into account. The notion of buffered k-IRS deadlock-free is introduced. It is based on the notion of *acyclic orientation covering*. An acyclic orientation of a graph if just an orientation of its edges. Given an IRS R on a graph G, the number s, the number of buffers used in the router can be defined as the size of the smallest set of acyclic orientations of G that covers all the routing paths induced by R. In other words, there exists a set of s acyclic orientations of G such that all the arcs of every routing path of R is contained in at least one acyclic orientation of G. Note that $s \geqslant 2$.

In the following an s-buffered deadlock-free k-IRS is denoted by $\langle k, s\rangle$-IRS. Here we give only a part of the results of [Fla97a]:

Theorem 55 ([Fla97a])

- *There exist $\langle 1,2\rangle$-IRS for trees and complete graphs, $\langle 1,3\rangle$-IRS for rings, $\langle 1,2\rangle$-LIRS for grids, $\langle 2,5\rangle$-LIRS for tori.*
- *There exists a shortest path $\langle k, D+1\rangle$-IRS for graphs of diameter D and of compactness k.*
- *There exists a shortest path $\langle 1, \lceil d/2\rceil+1\rangle$-LIRS for d-dimensional grids, and a shortest path $\langle 2, 2d+1\rangle$-LIRS for d-dimensional tori.*

Finally, we would like to mention that [Sch97] proposed deadlock-free IRS with experimental results for augmented grids and augmented tori.

4.4 Non-Deterministic Interval Routing Schemes

The study of *non-deterministic* IRS has been suggested in [vLT86]. It consists in relaxing Condition 2b in Definition 1 in Paragraph 2.1 (or with some static rules), a destination may belong to several arc-labels. In this case, the routing process can choose at random, or following some static rules, a link to forward the message. This allows more flexible routing for dynamic traffic of the network. Such IRSs are called *multipath* IRS. The multipath IRS has been studied in [FGS94] for chordal rings.

An IRS $R = (\mathcal{L},\mathcal{I})$ is called *α-adaptive* if R is a multipath IRS, and if for each node x each destination appears in at least $\min\{\alpha, \deg(x)\}$ different sets $\mathcal{I}(e)$, e incident to x. R is a *full-adaptive* IRS if it is a Δ-adaptive IRS, where Δ is the degree of the graph. An *overall* IRS is a shortest path full-adaptive IRS if it can be used to represent all the shortest paths information. Multipath k-IRS corresponds to the model of *multi-dimensional* IRS of dimension 1. This method is a generalization of Interval Routing, and it has been recently proposed by [FGNT95]. Some results can be found in [RŠ97] (see also Theorem 57 in Paragraph 4.5). Several labeling schemes based also on Interval Routing are proposed in [GV96]. These schemes, called Topological and Hyper-Rectangle Routing Scheme, are compared with Interval Routing for several classes of graphs.

In [Zem97], it has been proposed a particular case of 2-adaptive IRS called *deflecting* IRS. Every destination must appear in exactly two different sets $\mathcal{I}(e)$. The two choices achieves two routing paths: a shortest path and a deflecting path. Surprisingly, this potentiality decreases the compactness of the routing schemes in many cases.

The lower bounds of the compactness of overall IRS, and more generally of shortest path α-adaptive IRS for arbitrary α, are the same than the ones expressed in the theorems 15, 16, 29, and 32 because the proofs are based on the uniqueness of the shortest paths. Whereas they apply for the multi-dimensional model (dimension 1), they do not apply for the 2-adaptive IRS with deflecting path developed by [Zem97] (the deflecting path is not necessary a shortest path).

4.5 Congestion of Interval Routing

The *edge-congestion* of a routing function R is the maximum, taken over all edges, of the number of routing paths using the same edge. Similarly, the *arc-congestion* of R is the maximum number of routing paths using the same arc. The *edge-forwarding index* (respectively *arc-forwarding index*) of G, denoted by $\pi(G)$ (respectively $\vec{\pi}(G)$), is the minimum, over all the routing functions R on G, of the edge-congestion (respectively arc-congestion) of R. Clearly, for every G, $\pi(G) \leqslant 2\vec{\pi}(G)$. See [HMS89] for an introduction. The edge/arc-congestion of an IRS is the edge/arc-congestion of its induced routing function. A first study has been done recently in [CDSF97] for specific topologies like trees, d-dimensional grids, chordal rings.

Theorem 56 ([CDSF97])

- *For each fixed $k \geqslant 1$, the problem of determining whether a graph supports a k-IRS (or its variants) of arc-congestion less than C is NP-complete.*
- *For every $n \times n$-tori T_n there exists an 1-IRS of arc-congestion at most $3n^3/20 + o(n^3) \approx 0.15n^3$. Note that $\vec{\pi}(T_n) \approx 0.125n^3$.*

Congestion of Interval Routing is treated also in [RŠ97] for arbitrary graphs.

Theorem 57 ([RŠ97])

- *For every n-node graph G of maximum degree Δ and for every integer s, $1 \leqslant s \leqslant n$, there exists a multipath k-IRS on G such that its edge-congestion is at most $\pi(G) + n\Delta s$ and $k \leqslant 2 + \lceil n/(2s) \rceil$.*
- *For every n-node planar graph G of bounded degree there exists a multipath k-IRS on G such that its edge-congestion is $O(\pi(G))$ and $k = O(\sqrt{n})$.*

4.6 Interval Routing for Distributed Problems

The general question "how much a labeling can help in the solution of some distributed problems" was posed by [FMS97b], and then further studied by [FMS97a] in the framework of *Sense of Direction*.

Recently, David Peleg proposed in [SRS97] the general question: *"In many contexts and many areas, once a good representation is constructed, it is often* *useful for more than one application. Yet in the particular case of Interval Routing, IRS representations seem to be used* only *for routing. Given that we have invested all this time and efforts in constructing them, it should be nice if we could use them (once constructed) to solve other problems more efficiently, in addition to routing."*

Originally the investigation of Interval Routing came from distributed computing in asynchronous networks. [vLT87] proved that minimum spanning tree, leader-election (when the node labeling is not restricted to $\{1,\ldots,n\}$), and other related distributed problems can be solved in a ring labeled with a 1-IRS by exchanging $O(n)$ messages only, and $O(|E|+n)$ messages for arbitrary graphs, whereas in the general case $O(n\log n)$ messages for the ring, and $O(|E|+n\log n)$ messages for arbitrary graph are required [GHS83].

More precisely, David Peleg posted in [SRS97] the question: *"is it possible to broadcast by exchanging a total of $O(n)$ messages for any graph supporting a shortest path 1-IRS? (adding $O(\log n)$ extra bits to the message)"*

The general problem is still open but there are some partial results in the case of overall k-IRS (see Paragraph 4.4 for the definition of overall IRS). In particular [EPTZ97] shows that:

Theorem 58 ([EPTZ97]) *For every overall* 1*-IRS on a graph G, and for every integer x, the distance in G between the nodes labeled x and $x+1$ is at most* 3.

It implies that the algorithm which consists in sending first the message to destination 1, then, after reception, to destination 2, and so on until the node labeled n, turns out a total of $O(n)$ messages. It is interesting to remark that if G has a full-adaptive 1-IRS of stretch factor s, then by a simulation algorithm, G has a $O(sn)$ messages broadcast algorithm.

Other algorithms have been proposed in [dlTN97] leading also to a linear time solution. Other partial results are known. In particular we have the property:

Theorem 59 ([GM97]) *For every shortest path* 1*-SLIRS on a graph G, and for all integers x and y such $x<y$, then the distance in G between the nodes labeled x and y is at most $y-1$.*

It follows that nodes 1 and 2, and by symmetry the nodes $n-1$ and n, are neighbors.

To conclude, we think that Interval Routing is certainly a sufficiently simple model for the theoretical study of the structural and implicit information of distributed networks. It is quite easy to predict that this model will be extended and/or exploited in many fashions. This is what seems to indicate the paper of [dlTNP97].

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