On the Dilation of Interval Routing

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Abstract. In this paper we deal with interval routing on n-node networks of diameter D. We show that for every fixed $D \geq 2$, there exists a network on which every interval routing scheme with $O(n/\log n)$ intervals per link has a routing path length at least $\lfloor 3D/2 \rfloor - 1$. It improves the lower bound on the routing path lengths for the range of very large number of intervals. No result was known about the path lengths whenever more than $\Theta(\sqrt{n})$ intervals per link was used. Best-known lower bounds for a small number of intervals are $2D - O(1)$ for 1 interval [11], and $3D/2 - O(1)$ up to $\Theta(\sqrt{n})$ intervals [5]. For $D = 2$, we show a network on which any interval routing scheme using less than $n/4 - o(n)$ intervals has a routing path of length at least 3. Moreover, we build a network of bounded degree on which every interval routing scheme with routing path lengths bounded by $3D/2 - o(D)$ requires $\Omega(n/\log^{2+\varepsilon} n)$ intervals per link, where ε is an arbitrary non-negative constant.

1 Introduction

The *dilation* of a routing scheme is the length of the longest routing path. Assuming that time cost of message delivery is function of the routing path length or of the number of routers crossed, dilation is a parameter of the worse-case time complexity. Concurrently, fast routers must be easily implemented with a small amount of hardware.

Interval Routing is a routing scheme implementing compact routing tables, and allowing fast routing decisions in each node [9, 12]. Interval routing consists in labeling nodes by an unique integer taken in $\{1, \ldots, n\}$, n the number of routers, and in assigning to each link at every router a set of intervals of destinations, such that any message can reach its destination from any source. Such a labeling scheme on a network G is so-called an *interval routing scheme* on G. Each router locally finds the next link to forward a message to its destination by choosing the link that contains the number of the destination in one of its intervals. At each node, the intervals must be pairwise disjoint, and cover the set of all the labels, maybe excepted the label of the node it-self. The local routing decision time is bounded by $O(d \log k \cdot \log n)$ bit operations[1] whereas the space complexity of the router is at most $O(kd \cdot \log n)$ bits, for a router of d links, and if at most k intervals per link are used. In particular, such a routing scheme is efficient, i.e., compact and fast, if the degree of the network and the number

[1] For each link the router can perform a binary search among the k sorted intervals, every operations being on integers of size $O(\log n)$ bits.

of intervals per link are both low[2] relatively to the number of routers of the network.

The interval routing scheme is used in the last generation of C104 routing chips used in the INMOS T9000 Transputer design [7]. Since the number of intervals is limited in each routing chip, we are interested in finding the minimum dilation for interval routing scheme using a fixed number of intervals. The dilation is expressed in term of the diameter of the networks which is a common lower bound of the dilation for all networks.

The following table summarizes the best lower bounds known, and our contribution about the dilation of interval routing schemes using at most k intervals per link on n-node networks of diameter D. Note that for every network, there is an interval routing scheme with one interval and of dilation $2D$, where D is the diameter of the network. Indeed, it is sufficient to route along a minimum spanning tree of the network which supports an interval routing scheme [9] with only 1 interval per link.

Number of intervals	Dilation (lower bound)	Reference
$k = 1$	$2D - 3$	[11]
$2 \leq k \leq \Theta(\sqrt{n})$	$3D/2 - 3$	[5]
$2 \leq k \leq \Omega(n/\log n)$	$\lfloor 3D/2 \rfloor - 1$	Theorem 1

The next section presents the notations, and previous works about the dilation. In Section 3, we prove the main theorem. Also, we extend the result to diameters which depend on n. We prove a trade-off of $\Omega(n/(D\log(n/D)))$ intervals required for every interval routing scheme of dilation less than $\lfloor 3D/2 \rfloor - 1$. In Section 4, we adapt our construction to bounded degree networks, and we improve the multiplicative constant of the result of [4] about the maximum compactness of n-node networks. Moreover the lower bound is proved for a network of diameter 2, and of maximum degree $n/2$.

2 Statement of the Problem

2.1 Notations

The model of networks is a undirected connected graph G, each node representing a router. The distance between any two nodes x and y is the minimum number of edges of paths connecting x and y, and is denoted $\mathrm{dist}(x, y)$. In all the rest of the paper, n will denote the order of the graph, and D its diameter.

An *interval* means a set of consecutive integers taken in $\{1, \ldots, n\}$, n and 1 being considered as consecutive. For every arc[3] e, all the intervals associated to e form a set of integers, i.e., a set of labels of destinations, and is denoted by I_e. An interval routing scheme R on a graph G is denoted by a pair $(\mathcal{L}, \mathcal{I})$, where $\mathcal{L}$ is the labeling of vertices of G, and $\mathcal{I}$ is the set of all the I_e's.

[2] For instance k and d satisfying $kd/\log d = o(n/\log n)$ may provide a scheme more compact than the standard routing tables which need of $\Theta(n \log d)$ bits per router.

[3] For convenience, each edge of the graph is considered as a pair of two symmetric arcs.

2.2 A simple example

Let us consider the following example of interval routing on a graph G_0 of 7 vertices depicted on the left-hand side of Figure 1. Nodes are labeled by integers from 1 to 7, and intervals are assigned to each arc. If the node 5 sends a message to node 1, the message will successively be forwarded along the arc $(5,7)$, then along $(7,1)$, because $1 \in I_{(5,7)} = [7,2] = \{7,1,2\}$, and $1 \in I_{(7,1)} = [1] = \{1\}$. Each set of destinations I_e, e arc of G_0, is composed of at most two intervals of consecutive labels. One can check that every routing path on G_0 is of minimal length. Therefore, the dilation of this routing scheme is the diameter of G_0, here 2.

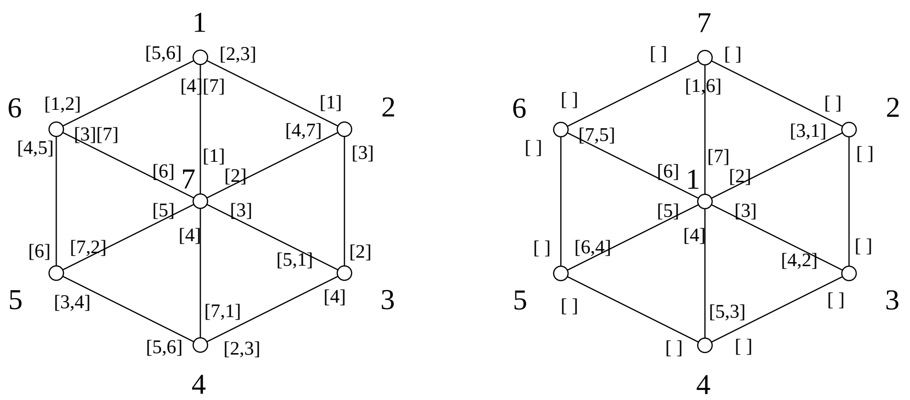

Fig. 1. Two interval routing schemes of dilation 2 on the same graph G_0. (We denote by [] the *empty interval*, corresponding to an empty set of destination.)

This interval routing scheme on G_0 qualifies as *shortest path* interval routing scheme because all the routing paths in the graph are of minimal length. A classical problem for interval routing is to compute the minimum number of intervals needed to guarantee a shortest path interval routing scheme on a given graph. A such number depends on the graph only, and is termed *compactness*. Compactness of a graph is at most $n/2$, because any set of destinations, and any labeling of these destinations, can not contain more than $n/2$ non consecutive integers. In [4], it has been built graphs of compactness at least $n/12$. Therefore $\Theta(n)$ is a tight lower bound of compactness of n-vertex graphs.

Actually, G_0 has no shortest path routing scheme with only one interval per link (see [2, page 793] for a proof). Hence the compactness of G_0 is 2. The right-hand side of Figure 1 shows another interval routing scheme on G_0, but with only one interval per arc. It has also a dilation 2, the diameter. The *dilation* problem is, given a graph G, and an integer k, k being less than the compactness of G, to determine an interval routing scheme on G using at most k intervals per link which minimizes the longest routing path. This general question is important in practice whenever a low number of intervals is forced by the hardware of the router, and whenever message delivery time must be as short as possible.

Fundamentally, the compactness problem consists in measuring the compression of the "routing information" whenever paths are of minimum length. Its dual problem, the dilation problem, consists in measuring the efficiency of the routing scheme when the compression rate is limited, which defines the size of the routing information in each node. Both the problems contribute to design some trade-off between time and space used by a router in a communication network.

2.3 Related works

Lower bound for the dilation problem for interval routing was first addressed by Ružička in [8]. He built a *globe-graph* on which every interval routing scheme with one interval has dilation at least $3D/2-O(1)$. This result has been recently improved by Tse and Lau in [11], for one interval with a lower bound of $2D-3$ (based on an extension of the globe-graph), and generalized in [10] to more than one interval with a lower bound still of $3D/2-O(1)$ up to $\Theta(\log n)$ intervals, and of $5D/4-O(1)$ up to $\Theta(\sqrt{n})$ intervals. Recently, in [5], Královič, Ružička, and Štefankovič improved the lower bound for the range of intervals from $\Theta(\log n)$ up to $\Theta(\sqrt{n})$, with $3D/2-3$, using a *multi-globe graph*. No result was known for a larger number of intervals. Intuitively, the more intervals are used, the lower the dilation. In [5], it is also proved that $O(\sqrt{n \log n})$ intervals suffice to guarantee a dilation of at most $\lceil 3D/2 \rceil$ on every network.

In this paper we extended the range of possible number of intervals up to $\Omega(n/\log n)$, and we prove a dilation of at least $\lfloor 3D/2 \rfloor - 1$. In [1], an independent and similar lower bound has been shown, however for a dilation $3D/2-2$, and for even D. Our result expresses the existence of a gap on the number of intervals required for a dilation close to 3/2 the diameter. Indeed, the routing with a dilation δ might be done with only k intervals, whereas $k' = \omega(k)$ intervals might be required to route with a dilation $\delta - 1$.

3 Construction of the Worst-Case

The main Theorem of this section is the following:

Theorem 1. *For every integer $D \geq 2$, there exists an n-vertex graph of diameter D on which every interval routing scheme of dilation less than $\lfloor 3D/2 \rfloor - 1$ requires $\Omega(n/(D \log(n/D)))$ intervals.*

In Theorem 1, the number of intervals is expressed as a function of the diameter. It turns out more general results. For instance, Theorem 1 shows that for every constant k there exists an n-vertex graph of diameter $D = \Theta(n)$ on which every interval routing scheme using k intervals has a dilation $\lfloor 3D/2 \rfloor - 1$. Note that all the previous results were proved for an arbitrary but fixed diameter. The result of [5] is improved just by applying Theorem 1 with D an arbitrary constant.

Corollary 2. *For every constant integer $D \geq 2$, there exists an n-vertex graph of diameter D on which every interval routing scheme of dilation less than $\lfloor 3D/2 \rfloor - 1$ requires $\Omega(n/\log n)$ intervals.*

Sketch of the proof.
Basically we use a similar technique to establish lower bounds of the compactness, and to prove a lower bound of the dilation with a large number of intervals. Our construction is an adaptation of the graph defined in [4]. For the sketch we assume that D is an odd fixed constant ≥ 3. We build a graph which has the two following properties:

1) Some vertices require an interval routing scheme using $k = \Omega(n/\log n)$ intervals on some arcs to route along the shortest paths between vertices at distance $t = (D-1)/2$.
2) Any interval routing scheme which does not route along the shortest paths between these vertices has routing path lengths at least $3t$.

Any interval routing scheme of dilation $< 3t$ on this graph requires at least k intervals, or equivalently, any interval routing scheme that uses at most $k-1$ intervals per link has dilation at least $3t$.

The graph construction.
Our construction is function of a boolean matrix M, and of an integer D. It is denoted by $G_{M,D}$. More precisely, for every $p \times q$ boolean matrix M and for every integer $D \geq 2$ we define the graph $G_{M,D}$ as follows: we associate with each row i of M, $i \in \{1, \ldots, p\}$, a vertex v_i in $G_{M,D}$. At each column j of M, $j \in \{1, \ldots, q\}$, we associate a pair of vertices $\{a_j, b_j\}$ which are connected by an edge. We set $t = 1$ if $D = 2$, and $t = \lfloor (D-1)/2 \rfloor$ for every $D \geq 3$. For every $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$, we connect v_i to a_j by a path of length t if and only if $m_{i,j} = 0$. Similarly, we connect v_i to b_j by a path of length t if and only if $m_{i,j} = 1$. See Figure 2 for an example. Note that the graph obtained by contraction of the edges $\{a_j, b_j\}$, $j \in \{1, \ldots, q\}$, of the graph $G_{M,D}$ is a complete bipartite graph $K_{p,q}$. The construction is slightly different for D even. For every even $D \geq 4$, we subdivide each edge a_j, b_j with a new vertex c_j, and with the two new edges: $\{a_j, c_j\}$ and $\{c_j, b_j\}$. Also, we add p vertices of degree 1, w_i, for $i \in \{1, \ldots, p\}$. The vertex w_i is connected to v_i.

We define the *compactness* of a boolean matrix M as the smallest integer k such that there exists a matrix obtained by row permutation of M having at most k blocks of consecutive ones per column. The first and the last entry of a column are considered as consecutive. For example, the matrix M described on Figure 2 has its first and its third column composed of one blocks of consecutive ones, whereas its second column is composed of two blocks (each block being composed of only one 1-entry). The reader can check that whatever one permute the rows of M, there exists at least one column with two blocks of consecutive ones. Therefore, the compactness of M is 2. Intuitively, on this example, any interval routing scheme using only one interval on all the arcs of the form (a_j, b_j) cannot optimally reach all the v_i's, and should have a dilation of $3t$.

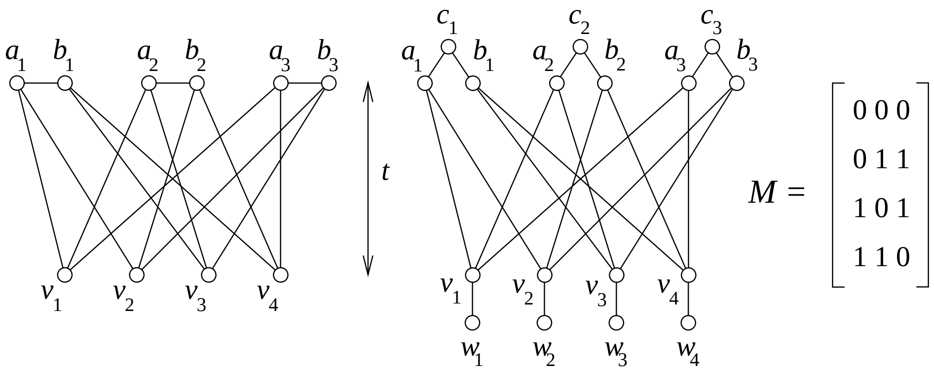

Fig. 2. The graph $G_{M,D}$: 1) of diameter $D \leq 2t+1$, and 2) of diameter $D \leq 2t+2$.

In all the following, columns of boolean matrices are seen as binary strings. log-functions are given for base two. We need of the following lemmas to prove Theorem 1 and Theorem 5:

Lemma 3. *Let p, q be two sufficiently large integers. Let $\mathcal{M}$ be the set of $p \times q$ boolean matrices having $\lfloor p/2 \rfloor$ 1-entries per column. Let $\mathcal{M}_1$ be the subset of matrices of $\mathcal{M}$ such that all the rows are pairwise non complemented. Let $\mathcal{M}_2$ be the subset of matrices of $\mathcal{M}$ such that for every pair of columns the $2 \times p$ matrix composed of the pair of columns contains the submatrix*[4] $\left[\begin{smallmatrix} 0\,0\,1\,1 \\ 0\,1\,0\,1 \end{smallmatrix}\right]$ *up to a column permutation. Then,*

- *if $p = o(2^{q/2})$, then $|\mathcal{M}_1| \sim |\mathcal{M}|$;*
- *if $q = o(2^{p/4})$, then $|\mathcal{M}_2| \sim |\mathcal{M}|$.*

This is a consequence of a result proved in [3]. In all the following of the paper, we set $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$. The matrix M depicted on Figure 2 belongs to $\mathcal{M}_0$. The will see later that the graphs $G_{M,2}$ built from any $M \in \mathcal{M}_0$ have diameter 2 exactly. Futhermore, almost all matrices are in $\mathcal{M}_0$.

Lemma 4. *For every sufficiently large integer p, there exist a constant α, and a $p \times \lfloor \alpha \log p \rfloor$ matrix of $\mathcal{M}_0$ of compactness at least $p/5$.*

Proof. We use a counting argument which can be formalized by the Kolmogorov Complexity (see [6] for an introduction). Let $\mathcal{M}$ be the set of $p \times q$ boolean matrices with $\lfloor p/2 \rfloor$ 1-entries per columns. Let us begin to show that compactness of some matrices of $\mathcal{M}$ is linear in p for $q = \Theta(\log p)$.

For every $M \in \mathcal{M}$, we define $\mathrm{cl}(M)$ the subset of the matrices of $\mathcal{M}$ obtained by row permutation of M. There exists a matrix $M_0 \in \mathcal{M}$ such that all the matrices of $\mathrm{cl}(M_0)$ have a Kolmogorov Complexity at least $C = \log|\mathcal{M}| - \log(p!) -$

[4] A is a submatrix of B if A can be obtained from B by removing some columns and rows in B.

$3 \log p$. Indeed, by contradiction let $M_0' \in \mathrm{cl}(M_0)$ be a matrix of Kolmogorov Complexity $C' < C$, for any $M_0 \in \mathcal{M}$. M_0 may be described by a pair (i_0, M_0'), where i_0 is the indexe of the row permutation of M_0' into M_0. Such an integer can be coded by $\log(p!) + 2\log p + O(1)$ bits. $2\lceil \log p \rceil$ bits are sufficient to describe p, thus the length of any $i_0 \leq p!$, in a self-delimiting way. Hence the Kolmogorov Complexity of M_0 would be at most $C' + \log(p!) + 2\log p + O(1) < \log |\mathcal{M}|$, that is impossible for any matrix $M_0 \in \mathcal{M}$.

$|\mathcal{M}| = \binom{p}{\lfloor p/2 \rfloor}^q = \Theta(2^p/\sqrt{p})^q$. By the Stirling's formula $\log |\mathcal{M}| = pq - O(q \log p)$, and $\log(p!) = p \log p - O(p)$. Hence,

$$C = \log |\mathcal{M}| - \log(p!) - 3\log p = pq - p\log p - O(p + q\log p). \qquad (1)$$

All the matrices of $\mathcal{M}$ have q columns, each one of Kolmogorov Complexity bounded by $p + O(1)$. Therefore there exists a matrix M_0 such that every matrix obtained by row permutation of M_0 has a column of Kolmogorov Complexity at least

$$C_0 = \frac{C}{q} - 2\log p = p - \frac{p \log p}{q} - O\left(\frac{p}{q} + \log p\right). \qquad (2)$$

The term $2 \log p$ codes the length of the description of such a column in a self-delimiting way. From [6, Theorem 2.15, page 131], every binary string of length p bits, and of Kolmogorov Complexity at least $p - \delta(p)$ contains at least

$$\frac{p}{4} - \sqrt{(\delta(p) + c)p\,\frac{3}{2}\ln 2} \qquad (3)$$

occurrences of 01-sequences, for any *deficiency* function δ, and some constant c. Since each 01-sequence in a binary string starts necessarily a new block of consecutive ones, we get a lower bound on the number of blocks of consecutive ones for such strings.

By choosing $q = \lfloor \alpha \log p \rfloor$, Equation 2 provided

$$C_0 = p - \frac{p}{\alpha} - O\left(\frac{p}{\log p}\right) > p - \delta(p), \text{ with } \delta(p) = \frac{p}{\alpha} + o(p).$$

By Equation 3, it follows that M_0 has a compactness at least

$$\frac{p}{4} - \sqrt{(1 + o(1))\ p^2\,\frac{3}{2\alpha}\ln 2} = p\left(\frac{1}{4} - \sqrt{\frac{3}{2\alpha}\ln 2}\right) - o(p) > \frac{p}{5}, \text{ for } \alpha = 416. \qquad (4)$$

Let us show that the result holds also for the compactness of some matrices of $\mathcal{M}_0$. From Lemma 3, because $p = o(2^{q/2}) = o(2^{208 \log p}) = o(p^{208})$, we get that $|\mathcal{M}_1| \sim |\mathcal{M}|$. Similarly, $q = O(\log p) = o(2^{p/4})$, thus $|\mathcal{M}_2| \sim |\mathcal{M}|$, and therefore $|\mathcal{M}_0| \sim |\mathcal{M}|$. It implies that $\log |\mathcal{M}_0| = \log |\mathcal{M}| + o(1)$, and thus Equations 1, 2, 3, and 4 hold for $\mathcal{M}_0$ as well, that completes the proof. □

Proof of Theorem 1.

Let D be an integer ≥ 2, let p be an integer large enough, and let $M \in \mathcal{M}_0$ be a matrix satisfying Lemma 4. We consider the graph $G_{M,D}$. Let t be the length of the chains between the nodes $\{a_j, b_j\}$'s and the nodes v_i's. By the construction of $G_{M,D}$, $t=1$ if $D=2$, and $t=\lfloor(D-1)/2\rfloor$ otherwise. Let us remark first that for every integer $D'>D$, and for every graph G of diameter D on which every interval routing scheme using k intervals has a dilation δ, there exists a graph G' of diameter D' on which every interval routing scheme using k intervals has a dilation at least δ. Indeed, to obtain G' it is sufficient to add a path of length $D'-D$ to any vertex of G which is of eccentricity D. Hence, to prove a lower bound for the dilation of graphs of diameter D it is sufficient to prove a lower bound for dilation in graphs of diameter at most D.

Fact 1. For odd $D \geq 3$, the diameter of $G_{M,D}$ is at most $2t+1$.

By construction $G_{M,D}$ has no vertex w_i, and no vertex c_j. Let us show that, for every pair of vertices x and x', $\text{dist}(x,x') \leq 2t+1$. From the particular shape of the matrices of $\mathcal{M}_0$, no column is the string 0^p or the string 1^p. Thus, for $p \geq 2$, all the nodes have degree at least 2. Let us show that there is a cycle of length at most $4t+2$ that cuts x and x'. x (respectively x') belongs to a chain of length t that directly connects a node, namely v_i (respectively $v_{i'}$), to a node $\alpha_j \in \{a_j, b_j\}$ (respectively $\alpha_{j'} \in \{a_{j'}, b_{j'}\}$). It follows that x and x' belongs to a cycle $(x, \alpha_j, \overline{\alpha_j}, v_{i'}, x', \alpha_{j'}, \overline{\alpha_{j'}}, v_i, x)$, where $\overline{\alpha_j}$ denotes the complement of α_j in $\{a_j, b_j\}$ (similarly for $\overline{\alpha_{j'}}$). Its length is clearly bounded by $4t+2$. It follows that $\text{dist}(x,x') \leq (4t+2)/2 = 2t+1$. Therefore, for $D \geq 3$, $G_{M,D}$ has a diameter bounded by $D \geq 2t+1$.

Fact 2. For any interval routing scheme $R=(\mathcal{L},\mathcal{I})$ on $G_{M,D}$, for every arc (a_j, b_j), and for every vertex v_i, if $m_{i,j}=1$ and $\mathcal{L}(v_i) \notin I_{(a_j,b_j)}$, or if $m_{i,j}=0$ and $\mathcal{L}(v_i) \in I_{(a_j,b_j)}$, then the dilation of R is at least $3t$.

This results from the fact that there is no path shorter than $2t$ between two any vertices $v_i \neq v_{i'}$. Moreover any wrong decision for routing from a_j to v_i cuts a vertex $v_{i'} \neq v_i$ before to reach v_i.

Fact 3. Let k be the compactness of M, and $R=(\mathcal{L},\mathcal{I})$ be any interval routing scheme on $G_{M,D}$. If R uses less than k intervals per arc, then its dilation is at least $3t$.

Assume R is fixed, and uses only $k-1$ intervals. Let j_0 be a column of M composed of at least k blocks of consecutive ones. Such a column exists because the compactness of M is k. Let us consider the sequence $u=(u_1,\ldots,u_p)$ defined by: for every $i \in \{1,\ldots,p\}$, $u_i=1$ if $\mathcal{L}(v_i) \in I_{(a_{j_0},b_{j_0})}$, and $u_i=0$ otherwise. Since $I_{(a_{j_0},b_{j_0})}$ is composed of at most $k-1$ intervals, u is composed of at most $k-1$ blocks of consecutive ones. Thus the column j_0 and the sequence u differ in at least one place. Set i_0 the indexe such that $m_{i_0,j_0} \neq u_{i_0}$. If $u_{i_0}=1$, then $\mathcal{L}(v_{i_0}) \in I_{(a_{j_0},b_{j_0})}$, and $m_{i_0,j_0}=0$. If $u_{i_0}=0$, then $\mathcal{L}(v_{i_0}) \notin I_{(a_{j_0},b_{j_0})}$, and $m_{i_0,j_0}=1$. We conclude by applying Fact 2.

Fact 4. For every even $D \geq 4$, the diameter of $G_{M,D}$ is at most $2t+2$. If R uses less than k intervals per arc, then its dilation is at least $3t+2$.

Note that for $D=2$, we do not need of Fact 4, because actually the statement of Theorem 1 holds for every graph of diameter 2.

The graph $G_{M,D}$ has it this case the vertices w_i of degree 1, and a path $\{a_j, c_j, b_j\}$ between a_j and b_j. Let x, x' be two non-w_i vertices. Similarly to Fact 1, there exists a cycle of length at most $4t+4$ that cuts x and x'. Hence

$\text{dist}(x,x') \leq 2t+2$. Let us show that $\text{dist}(v_i,v_{i'}) \leq 2t$. Indeed, $M \in \mathcal{M}_0$ has the property that the rows i and i' of M are not complemented. Thus there exists j such that $m_{i,j} = m_{i',j}$. It follows that either a_j either b_j is connected by a path of length t to v_i and $v_{i'}$. Hence $\text{dist}(w_i,w_{i'}) \leq 2t+2$. Let us show that $\text{dist}(x,v_i) \leq 2t+1$. Assume that x belongs to a path connecting $\alpha_j \in \{a_j,b_j\}$ to $v_{i'}$. Let $t_x \geq 0$ be the distance between x and $v_{i'}$. There is two paths between x and v_i: one trough $v_{i'}$ of length t_x+2t, and another one trough α_j of length at most $t-t_x+2+t$. Thus $\text{dist}(x,v_i) \leq \min\{t_x+2t, 2t+2-t_x\} = 2t+\min\{t_x, 2-t_x\} = 2t+1$. It follows that $\text{dist}(x,w_i) \leq 2t+2$, and thus the diameter of $G_{M,D}$ is bounded by $2t+2 = D$ for even $D \geq 4$. The dilation of R, using less than k intervals, is clearly two more longer than the case of odd diameter, because: 1) for a wrong routing path we start with a new edge of the form $\{c_j,a_j\}$ or $\{c_j,b_j\}$, and 2) the unique shortest path from c_j to any w_i cuts v_i.

The order of $G_{M,D}$ is $n \leq 2p+2q+pq(t-1)+c$, with $q = \Theta \log p$, and where $c \in \{0,1,2\}$ is the number of nodes added to have a diameter D exactly. For $D \leq 4$, that is $t=1$, $n = O(p+q)$, thus $p = \Omega(n)$. For $D \geq 5$, that is $t \geq 2$, $n = \Theta(Dp\log p)$, and thus $\log p = \Theta(\log(n/D))$. It follows that $p = \Theta(n/(D\log(n/D)))$. Therefore for every $D \geq 2$, $p = \Omega(n/(D\log(n/D)))$. By Lemma 4, the compactness of M is at least $p/5$. By Fact 3, for odd D, the dilation is $3t = 3\lfloor (D-1)/2 \rfloor = 3D/2-3/2$, whereas by Fact 4, for even D, the dilation is $3t+2 = 3\lfloor (D-1)/2 \rfloor + 2 = 3(D/2-1)+2 = 3D/2-1$. Therefore, in all the cases, the dilation is at least $\lfloor 3D/2 \rfloor - 1$, that completes the proof. □

Theorem 1 allows to establish a trade-off between the order, the diameter of the graph, and the number of intervals required for a dilation $\lfloor 3D/2 \rfloor - 1$. For instance, we saw that the matrix M of Figure 2 has a compactness 2. Fact 3 of Theorem 1 shows that the worst-case graph of Figure 2, which is of maximum degree 3 and of diameter $D \sim n/6$, has a dilation $\lfloor 3D/2 \rfloor - 1 \sim n/4$ for 1 interval. However in general, the graph $G_{M,D}$ has a maximum degree $\Theta(p+q)$ which is $\Theta(n/(D\log(n/D)))$, for $D \geq 5$.

4 Other Results

Let us recall that the compactness of a graph G is the smallest integer k such that G supports a shortest path interval routing scheme using at most k intervals per link.

Theorem 5. *For every sufficiently large integer n, there exists an n-vertex graph G of diameter 2, of maximum degree at most $n/2$, and of compactness at least k such that $k > n/4 - O(n^{3/4}\sqrt{\log n})$. Moreover, every interval routing scheme on G using $k-1$ intervals per link has dilation at least 3.*

Proof. (outline) We use the structure of matrices of $\mathcal{M}_0$, and Equation 2 of the proof of Lemma 4 for $q = \lfloor \sqrt{n \log n} \rfloor$. □

Corollary 6. *For every sufficiently large integer n, and for every integer $D \geq 2$, $D = o(n)$, there exists an n-vertex graph of diameter D, of maximum degree at most $n/2$, and of compactness at least $n/4 - o(n)$.*

Proof. It is sufficient to add to any vertex of $G_{M,2}$ a path of length $D-2$ to obtain a new graph G' of diameter D exactly. The number of nodes of G' is $n+D-2 \sim n$, for $D = o(n)$. □

We saw that the degree of our construction is not bounded by a constant. We adapt our previous construction to show a lower bound for dilation in networks of degree bounded by 3, however with a little degradation of the number of intervals. A proof of the next theorem is given in the full version of the paper.

Theorem 7. *For every constant $\varepsilon > 0$, and for every sufficiently large integer D, there exists an n-vertex graph of diameter D, and of maximum degree 3 on which every interval routing scheme of dilation less than $3D/2 - O(D/\log^{\varepsilon} D)$ requires $\Omega(n/\log^{2+\varepsilon} n)$ intervals.*

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