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Lower Bounds for Interval Routing on Bounded Degree Networks

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Abstract

Interval Routing was introduced to reduce the size of the routing tables: a router finds the direction to forward a message by determining the interval that contains the destination address of the message, each interval being associated to one particular direction. In this paper, we give lower bounds for the minimum number of intervals per edge needed to achieve shortest path routing for the class of 3-regular networks. We prove a tight lower bound of $\Theta(n)$ intervals for a 3-regular network of order at most n. Moreover, for the particular case of 3-regular planar networks, we establish a lower bound of $\Omega(\sqrt{n})$ intervals. This lower bound is also proved for 4-regular and 5-regular planar graphs, and bounded degree planar graphs whose all the faces are triangles.

Keywords: compact routing in distributed networks, shortest path interval routing, regular graphs, planar graphs.

1 Introduction

Table routing is a standard solution to the shortest path routing problem for arbitrary networks. At each router in the network is stored a table listing for each possible destination the output port that should be used to send a message along a shortest path. This solution guarantees shortest paths but requires $O(n \log d)$ bits of memory per node, for an n-router network of maximum degree d. The *compact routing problem* is to implement routing schemes that uses a minimum memory size on each router. The interval routing scheme was introduced by Santoro and Khatib in [19] to improve the space requirements of routing tables.

In [16], we showed that it is not possible to find a "compact" routing scheme for all networks, *i.e.*, universal routing schemes that produce in each router compact data structures for the routing information. We showed that a constant fraction of the routers of the network need to store $\Omega(n \log d)$ bits of information independantly from the routing scheme used.

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This implies that the compact routing problem could be tackled with two dual approaches:

- find the best routing scheme for a given (small) class of networks,
- find all networks requiring small memory requirements for a given routing scheme.

The second approach consists in graph characterization. For the interval routing scheme, this approach has been investigated in [1, 6, 9]. The first approach consists in finding lower and upper bounds for given classes of graphs. The state of the art in the compact routing problem is presented in [8] for the general case, and in [12, page 88] for graphs of maximum degree d. In this paper we study the case of bounded degree networks for the shortest path[1] interval routing scheme.

The model of network is a symmetric digraph (drawn as an undirected graph), each arc represents a link. Graphs are finite, connected, loop-less and without multi-arc. In order to sharpen drawings, we depict bidirectional links with a single edge between two neighbor nodes. The basic idea of the interval routing is the following: (1) each nodes is labeled with an unique integer taken from the set $\{1,\ldots,n\}$, where n is the number of nodes of the network; (2) to each arc e is assigned a set I_e containing all the labels of the destinations using the arc e to receive messages routed along shortest paths. Moreover, the sets assigned to arcs emanating from the same node must be pairwise disjoint, and their union must cover $\{1,\ldots,n\}$. The interval routing coding strategy consists in representing every destination set I_e by an union of disjoint intervals. An "interval" is a subset of consecutive integers taken from $\{1,\ldots,n\}$, where n and 1 are considered consecutive. The routing is performed as follows: upon reception of a message M in a node labeled x, the label of the destination, y, attached to the header of M is first compared with x, to check if the message has arrived at its destination. If not, then M and its header are forwarded trough the unique arc e incident to x such that $y \in I_e$.

The goal of the interval routing is to find a labeling of nodes and an assignment of destination sets that minimizes the number of intervals on each arc. If, for a graph G, this number of intervals is at most k per arc then the memory space of a router of G of maximum degree d is at most $O(dk \log n)$ bits. Indeed, it is sufficient to store the end-points of each interval. The *compactness* of a graph G, denoted by $\textsc{Irs}(G)$, is defined as the smallest integer k such that each destination set of G is composed of at most k intervals. Since the interval routing was introduced in order to reduce the memory space of routers, compactness is an important parameter to consider.

The general study on the compactness of a graph of order n has been investigated by many authors, see in particular [18] and [5]. Unfortunately, Gavoille and Guévremont in [14], build graphs having a compactness of at least $n/12$ intervals, proving uneventful that interval routing scheme is not compact for all graphs of order n. Study of compactness for smallest classes of graphs, like bounded degree graphs, was first proposed in [13]. Their lower bound has been improved in [17], where Kranakis and Krizanc proved that some bounded degree graphs have a compactness of $\Omega(n/\log n)$ intervals.

In [16] we have proved a tight lower bound of the space requirement for any *universal* shortest path routing scheme. We showed that there exist graphs where the total space required to store the routing information of all the nodes is $\Omega(n^2 \log d)$ bits, where d is the maximum degree of the graphs. In particular if d is constant this implies a lower bound of $\Omega(n)$ bits per router for the local routing information. Therefore, we can derive a trivial lower bound of $\Omega(n/\log n)$ (that is not a new result [17]) for the compactness of a bounded degree graph, since a router using interval routing

[1]It means that the routing induces only shortest paths.

scheme on a graph of compactness k and of bounded degree cannot store more than $O(k \log n)$ bits of information. In this paper, we show that this lower bound is not tight by proving that the maximum compactness of a 3-regular graph of order at most n is $\Theta(n)$. Moreover, this number of intervals can be required on $\Omega(n)$ edges, that is asymptotically the total number of edges of the graph. This tight lower bound shows that the worst case for interval routing does not depend on the maximum degree of the graph, whereas the memory requirements of universal routing schemes do [16]. We also show a lower bound of $\Omega(\sqrt{n})$ for 3-regular graphs that are planar, and also show that the lower bound still holds for bounded degree planar graphs whose all the faces are triangles. Note that to prove lower bounds on the compactness of 3-regular graphs is stronger than to prove lower bounds for larger classes of graphs, like bounded degree graphs or even unbounded degree graphs.

In the next section, we introduce definitions and tools to prove the lower bounds of the paper. In Section 3, we prove a tight $\Theta(n)$-lower bound of compactness of 3-regular graphs. In Section 4, we give a worst-case construction establishing a $\Omega(\sqrt{n})$-lower bound for 3-regular planar graphs. In the same section, we show that the lower bound is still in $\Omega(\sqrt{n})$ for 4-regular and 5-regular planar graphs, and also for bounded degree planar graphs with only triangles.

2 Statement of the Problem

2.1 A simple example

Let us consider the following example of interval routing on a graph G_0 of 7 vertices depicted on Figure 1. Nodes are labeled by integers from 1 to 7, and intervals are assigned to each arc. If the node 5 sends a message to node 1, the message will successively be forwarded along the arc $(5,7)$ then along $(7,1)$, because $1 \in I_{(5,7)} = [7,2] = \{7,1,2\}$ and $1 \in I_{(7,1)} = [1] = \{1\}$. Each destination set I_e, e arc of G_0, is composed of at most two intervals of consecutive labels. Therefore, $\textsc{Irs}(G_0) \leq 2$. However, another vertex-labeling and another choice of shortest paths might decrease this number. For example, there exists two shortest paths from 5 to 1, but one would force to add one more interval on the arc $(5,6)$: $I_{(5,6)} = [6] \cup [1]$. We will see in the next paragraph a simple proof showing that the compactness of G_0 cannot be 1, and thus $\textsc{Irs}(G_0) = 2$.

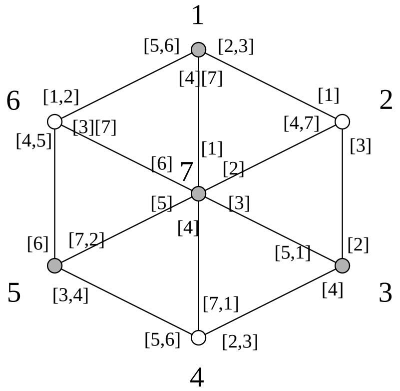


Figure 1: An example of interval routing with at most 2 intervals per link.

It should be borne in mind that the problem of finding the compactness of a graph is NP-complete [4]. More precisely, Flammini in [3], showed that to determine if a graph has a compactness 2 is already NP-complete. Note that for every graph G, $\textsc{Irs}(G) \leq n/2$, since any destination set I_e cannot contain more than $n/2$ non consecutive integers.

2.2 Matrices of constraints

The key of our proof is to use some graphs built from a boolean matrix. This technic, already used in [14] and in [5], is related to the *matrices of constraints*. Informally, for every triple of vertices (u, v, w) of a graph G, where u and v are adjacent and $u \neq w$, three cases may occur for shortest paths from u to w:

1. Every shortest path from u to w uses the arc (u, v).
2. Every shortest path from u to w does not use the arc (u, v).
3. There are shortest paths from u to w that use arc (u, v) and there are shortest paths from u to w that do not use the arc (u, v), depending on the choice of the shortest path routing function.

For the first two cases, the routing decision does not depend on the shortest path routing function. So, we say that arc (u, v) forms a *constraint* for the vertex w[2]. A *matrix of constraints* of a graph G is a $p \times q$ boolean matrix $M = (m_{i,j})$ whose rows are labeled with some vertices of G $\{v_1, \dots, v_p\}$, and whose columns are labeled with some arcs $\{e_1, \dots, e_q\}$ of G such that:

$$\begin{cases} m_{i,j} = 1 \text{ if and only if every shortest path from the tail of } e_j \text{ to } v_i \text{ uses the arc } e_j. \\ m_{i,j} = 0 \text{ if and only if no shortest path from the tail of } e_j \text{ to } v_i \text{ uses the arc } e_j. \end{cases}$$

For example, the matrix M_0 of Equation 1 is a matrix of constraints of the graph draw on Figure 1: $I_{(6,7)}$ must contain 7 and 3, but 5 and 1; $I_{(2,7)}$ must contain 7 and 5, but 3 and 1; $I_{(4,7)}$ must contain 7 and 1, but 3 and 5. Note that a matrix of constraints of a given graph is not unique in general.

$$\begin{array}{c} \\ 7 \\ 3 \\ 5 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} (6,7) & (2,7) & (4,7) \end{array} \\ \boxed{\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}} \end{array} = M_0 \qquad (1)$$

The main interest of matrices of constraints is for proving lower bound on the compactness for particular graphs. Assume that M is a matrix of constraints for a graph G. It is not difficult to see that the maximum number of blocks of consecutive ones in a column of M, minimized over all row permutations of M, is a lower bound on the compactness of G. This fact was first remarked in [5], then developed and generalized in [14]. A natural way of defining the *compactness of a boolean matrix* M, denoted by $\mathrm{I}(M)$, is as the maximum (taken over all the columns of M) of the number

[2] Note that (u, v) is not a constraint for the vertex u itself, since it may depends of the model of interval routing (the *strict* interval routing model assumes that u cannot belong to $I_{(u,v)}$). Indeed, without loss of generality, we assume in order to prove lower bounds that $I_{(u,v)}$ may contain the label of u. The correctness of the routing scheme is preserved since the router first compares the destination with its own label before to check the intervals. This freedom on the model could decrease the compactness. We invite the reader to see [7] for more information about strict and non strict models.

of blocks of consecutive ones in a column of M, minimized over all row permutations of M. For every matrix of constraints M of G, we have of course $\text{IRS}(G) \geq \text{I}(M)$. For instance, $\text{I}(M_0) = 2$, because whatever the row permutation of M_0 there always exists a column that needs two blocks of consecutive ones. Therefore, $\text{IRS}(G_0) \geq 2 = \text{I}(M_0)$.

3 A Tight Lower Bound for 3-Regular Graphs

In this section, we will build a 3-regular graph of order at most n having a matrix of constraints of compactness $\Theta(n)$. The construction is based on the worst-case graph given by Gavoille and Pérennès in [16].

3.1 Graph construction

For every integer $p \geq 2$, we define H_p by induction as follows (see Figure 2 for H_2). In the graph H_p, we distinguish a set of arcs $\{e_1, \ldots, e_p\}$ and a set of 2^p vertices B_p (drawn in gray on Figure 2). Each vertex of B_p is denoted by an unique binary word of p bits. We denote by A_p the set of tails of the arcs e_i, for $i \in \{1, \ldots, p\}$. H_{p+1} is composed of: a copy of H_p; two complete binary trees of 2^{p-1} leaves; 2^p intermediary vertices forming a set B'_p; 2^{p+1} news vertices forming the set B_{p+1} (that is also the set of all binary words of length $p+1$ bits). The roots of the two trees are connected by an edge, and one of its arcs is e_{p+1}. Vertices of B_p and vertices of B'_p are connected by a matching. Each vertex $x \in B'_p$ is connected to the vertices $x0$ and $x1$ of B_{p+1} (x being a binary word of p bits). Moreover, every vertex $w \in B_{p+1}$ is connected to a leaf of the first tree if $w = x0$, and to a leaf of the second tree otherwise, and every leaf of the two trees has a maximum of two neighbors in B_{p+1}.

The graph H_p has an important property: for any vertex $a_i \in A_p$, $i \in \{1, \ldots, p\}$, and for any vertex $w \in B_p$, every shortest path from a_i to w has to use the arc e_i if and only if the i-th bit of w is 1. This fact can be trivially shown by induction on p. Clearly, this property implies that H_p has a particular matrix of constraints that is the $2^p \times p$ boolean matrix whose i-th row is the integer $i-1$ written in binary. Indeed, each arc of $\{e_1, \ldots, e_p\}$ forms a constraint for all vertices of B_p. For instance, in the graph H_3 drawn on Figure 2, the arcs e_1, e_2 and e_3 form constraints for all the vertices of B_3. The arc e_1 represents the first column[3] [00001111], e_2 the second column [00110011], and e_3 the third column [01010101] of such a matrix of constraints for H_3.

The idea is to build a new graph by connecting two copies of H_p by a perfect matching between vertices of the sets B_p. The perfect matching is defined as a permutation π of the set $\{1, \ldots, 2^p\}$. This new graph is denoted by $G_{p,\pi}$. An example of the $G_{p,\pi}$ graph is depicted[4] on Figure 3 for a particular permutation π.

For every integer p and every permutation π of $\{1, \ldots, 2^p\}$, we denote by $M_{p,\pi}$ the $(2^p) \times (2p)$ boolean matrix defined as follows: the i-th row of $M_{p,\pi}$ is composed of $2p$ bits, the first p are the binary writing of $i-1$ and the last p bits are the binary writing of $\pi(i)-1$.

[3] The column [00001111] must be seen has a vertical vector.

[4] Recall that graphs are symmetric digraphs and depicted as undirected. However, some arcs are marked on Figure 2 just to distinguish in the discussion the tail and the head.

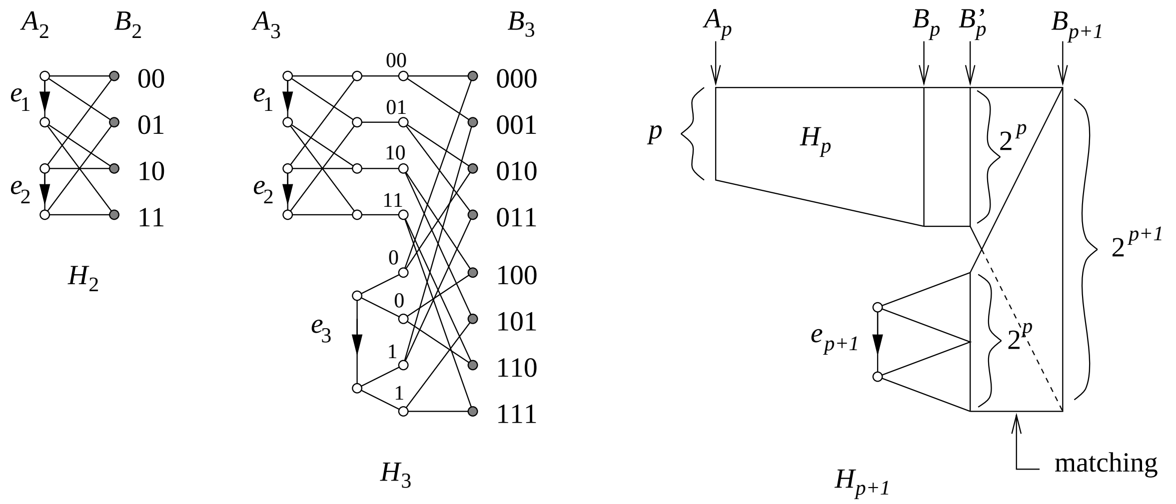


Figure 2: The recursive construction of the graph H_p.

Example: $\pi_3 = (7, 5, 1, 2, 8, 4, 3, 6)$.

$$M_{3,\pi_3} = \begin{bmatrix} 000 & 110 \\ 001 & 100 \\ 010 & 000 \\ 011 & 001 \\ 100 & 111 \\ 101 & 011 \\ 110 & 010 \\ 111 & 101 \end{bmatrix} \tag{2}$$

Lemma 1 *Let $p \geq 2$ and π be a permutation of $\{1, \ldots, 2^p\}$. The graph $G_{p,\pi}$ satisfies:*

1. *$G_{p,\pi}$ is 3-regular and of order $5{\cdot}2^{p+1} - 4p - 16$*
2. *$G_{p,\pi}$ possesses $M_{p,\pi}$ as matrix of constraints*

Proof. By construction, $G_{p,\pi}$ is 3-regular, and its order is $5{\cdot}2^{p+1} - 4p - 16$. Let B_1 be the set B_p of the first copy of H_p (grayed vertices on Figure 3), and A'_p and B'_p the sets of distinguished vertices of the second copy of H_p. $M_{p,\pi}$ is a matrix of constraints of $G_{p,\pi}$. Indeed, from the basic property of graphs H_p there exists a unique shortest path between the vertices of A'_p and the vertices of B'_p, and thus an unique shortest path between the vertices of A'_p and the vertices of the set B_1 (which uses the perfect matching between the two copies of the graph H_p). □

Remark. The $G_{p,\pi}$ graph has only $3n/2$ edges for an order n. However, $G_{p,\pi}$ cannot be planar in general because the matching π between the two copies of the graph H_p.

Now, let us show that this graph requires a large compactness in general.

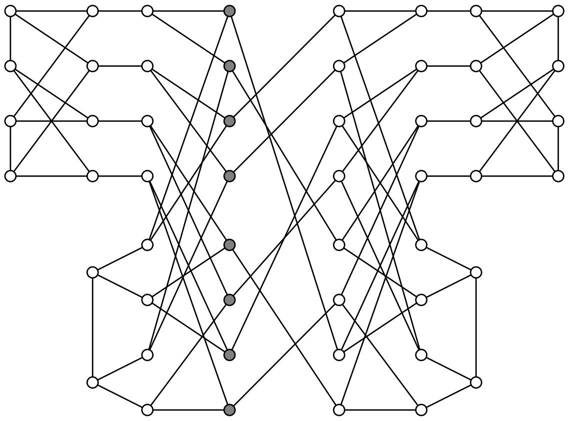


Figure 3: An example of the $G_{p,\pi}$ construction for $p = 3$ and $\pi = \pi_3$ (see definition of π_3 on page 6).

3.2 A $\Theta(n)$-lower bound for compactness

We will first compute the compactness of matrices $M_{p,\pi}$. The matrix M_{3,π_3} of Equation 2 has 4 blocks of consecutive ones in its third column. However, it does not implies that $\mathrm{I}(M_{3,\pi_3}) = 4$, because rows of the matrix can be permuted to decrease this number, corresponding for the graph G_{3,π_3} (on Figure 3) to a possible efficient vertex-labeling. For instance, exchanging the 6-th and 7-th row of M_{3,π_3} turns out $\mathrm{I}(M_{3,\pi_3}) \leq 3$. However, we have the following lower bound for an arbitrary $M_{p,\pi}$ matrix:

Lemma 2 *For every $p \geq 1$, there exists a permutation π such that $\mathrm{I}(M_{p,\pi}) \geq k$ with*

$$\left(\sum_{i=0}^{k} \binom{2^p}{2i}\right)^{2p} \geq (2^p)! \tag{3}$$

Proof. Let p be an integer and let $\mathcal{A}$ be the set of matrices $M_{p,\pi}$ for all permutations π of $\{1, \ldots, 2^p\}$. Let $k = \max\{\mathrm{I}(M) \mid M \in \mathcal{A}\}$, that is the maximum compactness of matrices of $\mathcal{A}$. Let us prove that k satisfies Equation 3. Let f be a function that returns, for every matrix $M \in \mathcal{A}$, a matrix $M' = f(M)$ that is obtained from M by rows permutation such that every column of M' has at most k blocks of consecutive ones. By definition of k, such a function f exists. We denote by $\mathcal{B}$ the set of $(2^p)\times(2p)$ boolean matrices for which all the columns are composed of at most k blocks of consecutive ones. By definition $f(\mathcal{A}) \subset \mathcal{B}$, that implies $|f(\mathcal{A})| \leq |\mathcal{B}|$. Clearly, f is a bijective function because for every matrix $M' \in f(\mathcal{A})$ we can permute the rows of M' in order to obtained the matrix $f^{-1}(M')$, where the first p bits of the rows is the sequence of binary words 0...00, 0...01, 0...10, 0...11, and so on, until 1...11. Therefore, $|f(\mathcal{A})| = |\mathcal{A}| = (2^p)! \leq |\mathcal{B}|$. On the other hand, there exists $\binom{2^p}{2i}$ ways to have exactly i blocks of consecutive ones in a binary column of length 2^p, and thus at most $\sum_{i=0}^{k} \binom{2^p}{2i}$ ways to have at most k blocks of consecutive ones in a column. So, there is at most $(\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$ ways to have a matrix having at most k blocks of consecutive ones on each of its $2p$ columns. In sequel, $|\mathcal{B}| \leq (\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$, and finally k must satisfy $(2^p)! \leq (\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$. $\square$

Corollary 1 *For every $p \geq 3$, there exists a permutation π such that $\mathrm{I}(M_{p,\pi}) = \Omega(2^p)$.*

In a sense this result is optimal because for every p and for every π, $\mathrm{I}(M_{p,\pi}) \leq 2^p/2$, since a matrix of 2^p rows cannot require more than $2^p/2$ intervals per column.

Proof. Applying Lemma 2, for every $p \geq 3$ there exists a permutation π such that Equation 3 is satisfied. Let $k = \mathrm{I}(M_{p,\pi})$. If $2k \geq 2^p/2$ then $\mathrm{I}(M_{p,\pi}) = \Omega(2^p)$. Otherwise, $\sum_{i=0}^{k} \binom{2^p}{2i} \leq (k+1)\binom{2^p}{2k}$, since in this case $\binom{2^p}{2k}$ is the maximum term of the sum. Moreover, since $k < 2^{p-2}$, we get $(\sum_{i=0}^{k} \binom{2^p}{2i})^{2p} \leq 2^{2p(p-2)}\binom{2^p}{2k}^{2p}$. For any two integers n and p, $(n/e)^n \leq n!$ and $\binom{n}{p} \leq (ne/p)^p$ (where $e = 2.718281828...$). Therefore, by Equation 3, k must satisfy:

$$\left(\frac{2^p}{e}\right)^{2^p} \leq 2^{2p(p-2)}\left(\frac{2^p e}{2k}\right)^{4kp} \quad \Rightarrow \quad 2^p \log_2\left(\frac{2^p}{e}\right) \leq 2p(p-2) + 4kp\log_2\left(\frac{2^p e}{2k}\right)$$

Let us define α by $\alpha = 2^p/(2k)$, and let us show that α is upper-bounded by a constant. The latter equation implies:

$$2^p(p - \log_2 e) \leq 2p(p-2) + \frac{2p \cdot 2^p}{\alpha}\log_2(\alpha e) \quad \Rightarrow \quad l(p) = \frac{1}{2} - \frac{\log_2 e}{2p} - \frac{p-2}{2^p} \leq \frac{\log_2(\alpha e)}{\alpha}$$

The left term of the latter inequality, $l(p)$, is an increasing function as p (from $p \geq 3$). In particular for every $p \geq 3$, this implies that α must satisfy $0.134 < l(3) \leq l(p) \leq (\log_2(\alpha e))/\alpha$, implying $\alpha < 54$. Therefore, for every $p \geq 3$, $k = 2^p/(2\alpha) > 2^p/108 = \Omega(2^p)$. □

Using the Stirling approximation of $n!$, we can tighten the upper bound on the constant α and we can show that for some permutation π and for sufficiently large integer p, $\mathrm{I}(M_{p,\pi}) \geq 2^p/19$. However, we think that, with a constructive proof of Lemma 2 and of Corollary 1, this lower bound can still be improved.

Note that matrices requiring a large compactness are not so particular. Indeed, a simple counting argument shows that "almost all" the matrices $M_{p,\pi}$ require $\Omega(2^p)$ intervals (for example a permutation chosen randomly among the $(2^p)!$ ones). Moreover, for such matrices of large compactness, one can check that $\Omega(p)$ columns require $\Omega(2^p)$ blocks of consecutive ones.

We can now derive the main Theorem of this section:

Theorem 1 (Gavoille, Pérennès) *There exists a 3-regular graph G of order at most n such that $\mathrm{IRS}(G) = \Theta(n)$.*

Proof. $\mathrm{IRS}(G) \leq n/2$. Applying Lemma 1 and Corollary 1, there exists a 3-regular graph $G_{p,\pi}$ such that $\mathrm{IRS}(G_{p,\pi}) \geq \mathrm{I}(M_{p,\pi}) = \Omega(2^p)$. The order of $G_{p,\pi}$ is $5 \cdot 2^{p+1} - 4p - 16$. In particular, for every integer n large enough there exists p such that $5 \cdot 2^{p+1} \leq n < 5 \cdot 2^{p+2}$, $G_{p,\pi}$ is of order at most n and of compactness at least $\mathrm{I}(M_{p,\pi}) = \Omega(n)$. □

3.3 Large compactness required by $\Theta(n)$ edges

Even though the local memory requirement seems to be more important in practice for the concrete design of routing chips, the number of routers requiring a large compactness can give a theoretical information about the "routing complexity" of the network. We show that not only the compactness of a 3-regular graph can be very large, but also it can be required by many edges of the graph.

We have already seen that $\Omega(p)$ columns of some matrices $M_{p,\pi}$ require each $\Omega(2^p)$ intervals. Actually, it means that $\Omega(p) = \Omega(\log n)$ arcs of the graph $G_{p,\pi}$ require $\Omega(n)$ intervals. Using the idea devellopped in [16], we can proved that the lower bound of $\Omega(n)$ intervals can be required on $\Omega(n)$ arcs, for another graph construction. In the following we say that a vertex u is a *constrained vertex* for a subset of vertices B if there exists an arc (u, v) such that for every vertex $w \in B$, (u, v) forms a constraint for w (see also Paragraph 2.2 for definition). Figure 2 page 6 shows that each vertex of A_3 is a constrained vertex for the set B_3.

Lemma 3 *Let G be a 3-regular graph of order n and A be a set of p constrained vertices for a subset of vertices B of G. Each constrained vertex $x \in A$ can be "split" in $\Theta(n/p)$ other vertices forming a new graph G' such that every new split vertex has the same constraints as x for the set B. Moreover, G' is 3-regular and is of order $\Theta(n)$.*

The proof of this Lemma can easily be derived from [16, Lemma 6]. An illustration of this Lemma is shown on Figure 4.

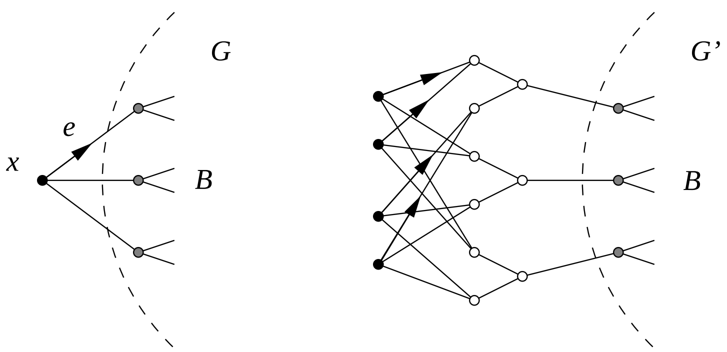


Figure 4: The "splitting" transformation shows that a large compactness can be required by $\Omega(n)$ edges.

Theorem 2 (Gavoille, Pérennès) *There exists a 3-regular graph of order at most n having a compactness of $\Omega(n)$, required by $\Theta(n)$ edges.*

Proof. Let G be the graph of Theorem 1 of order at most n and having $p = \Omega(\log n)$ arcs requiring $\Omega(n)$ intervals each one. Therefore, by the graph transformation of Lemma 3, there exists a 3-regular graph of order at most $n' = \Theta(n)$ and of compactness $\Omega(n) = \Omega(n')$. This number is independently required by $\Theta(n/p)$ groups of p arcs, thus required by a total of $\Theta(n) = \Theta(n')$ arcs. □

In [13], it has been introduced the parameter $\text{IRS}(n, d)$ denoting the maximum of the compactness over all graphs of order n and of maximum degree d. The following theorem generalizes the results given in [14] for bounded and for unbounded graphs:

Theorem 3 *Let ε be a real constant such that $0 < \varepsilon < 1$. For every sufficiently large integer n*

and for every integer d such that $3 \le d \le \varepsilon n$,

$$\mathrm{IRS}(n,d) = \Theta(n).$$

Note that $\mathrm{IRS}(n,2) = 1$.

Proof. From Theorem 2, it is not difficult to build a graph of order exactly n and of maximum degree d, for every n and every $d \le \varepsilon n$. We start from the 3-regular graph built in Theorem 2 with order at most $\lfloor (1-\varepsilon)n \rfloor = \Theta(n)$ and we connect at any vertex the root of a $K_{1,d}$ graph (the complete bipartite graph). Clearly, this graph has a maximum degree d. Since $d \le \varepsilon n$, the order of this construction is at most $\lfloor (1-\varepsilon)n \rfloor + \varepsilon n \le n$. Eventually, we can add a path to complete the order of the graph to obtain exactly n vertices. The compactness of this graph is greater or equal than the compactness of the initial graph, because the initial graph is a *subgraph of shortest paths*. Recall that H is a subgraph of shortest paths of G if H is a subgraph of G and if all the shortest paths between any two vertices of H in G are wholly contained in H (see [7, Theorem 2]). Therefore, applying Theorem 2, its compactness is $\Omega((1-\varepsilon)n) = \Omega(n)$, since ε is a constant. □

Note that the compactness of graphs with high degree is necessary low. Indeed, for every graph G of minimum degree δ, $\mathrm{IRS}(G) \le n - \delta$, from [18, Theorem 1]. It implies in particular that graphs whose minimum degree is $n - O(1)$ have a constant compactness.

4 Lower Bound for Planar Graphs

4.1 Compactness of 3-regular planar graphs

After the study of graphs of bounded degree, we investigate another class of sparse graphs that includes a large range of potential topologies for real networks: the class of planar graphs. In this section we give a lower bound for the maximum compactness of a planar graph of order n. General study of compact routing for planar graphs is principally due to Frederickson and Janardan [9, 10, 11]. They showed in [9] that every outer-planar graph is of compactness 1, that is, a planar graph whose all its vertices lie on a face. The best known lower bound is due to Tse and Lau in [20]. They built a planar graph of maximum degree $\Omega(\sqrt{n})$, and of compactness at least $\sqrt{n}/8.5$ intervals[5] on at least one edge, improving the previous $\Omega(\sqrt[3]{n})$-lower bound of Kranakis, Krizanc and Ravi [18]. We improve this lower bound for a 3-regular planar graph with the following theorem:

Theorem 4 *For every integer n large enough, there exists a 3-regular planar graph G of order at most n such that*

$$\mathrm{IRS}(G) > \frac{\sqrt{n}}{5.6}$$

Moreover, we will see that $\Omega(\sqrt{n})$ intervals can be required on $\Omega(\sqrt{n})$ edges. Our proof is constructive and uses matrices of constraints. For every integer $m \ge 2$, we give the construction of a 3-regular planar graph denoted by P_m. This graph will be used for several worst-case constructions

[5] The exact value is $\sqrt{n/72}$.

of Section 4.

Description of the $P_{\rm m}$ graph.

Let $m \geq 2$ be an integer. Roughly speaking, the P_m graph is composed of a $m \times m$ grid with two trees: one connected to a vertical border, and the other connected to the horizontal border of the grid. In order to have a 3-regular graph, each vertex is removed by small 3-regular structures. Each one is composed of smaller structures namely A, B, C, D, and E as depicted on the right side of Figure 5. They are arranged by their free links. The left side of Figure 5 represents the general topology of the P_m graph with the arrangement of the structures $A, \ldots, E$. The structures $A, \ldots, E$ can be swiveled round in the plane to join their free links. Figure 6 shows an example of the graph P_m for $m = 2$.

Some vertices and some arcs of P_m are distinguished: we say that u is a *gray-vertex* if u is one of the vertices of the structures A, B, or of C drawn in gray on Figure 5. Moreover, we say that e is a *vertical-arc* (respectively *horizontal-arc*) if e is an arc distinguished on the structure D connected on the vertical border (respectively horizontal border) of P_m. Let us recall the every two neighbors are connected by an edge that is composed of two symmetric arcs. The structure D is an arrangement of k blocks of length 3 (two triangles sharing an edge) and must be of minimum length such that $L \geq 2m - 3$ and $L = 3k - 1$. Thus the length of D is such that $2m - 3 \leq L \leq 2m - 1$. The number of blocks for a length L is $k = \lceil (L + 1)/3 \rceil$. Therefore, for every $m \geq 2$, the total number of vertices of D is at most $8k = 8\lceil (L + 1)/3 \rceil \leq 8\lceil 2m/3 \rceil < 16m/3 + 8$.

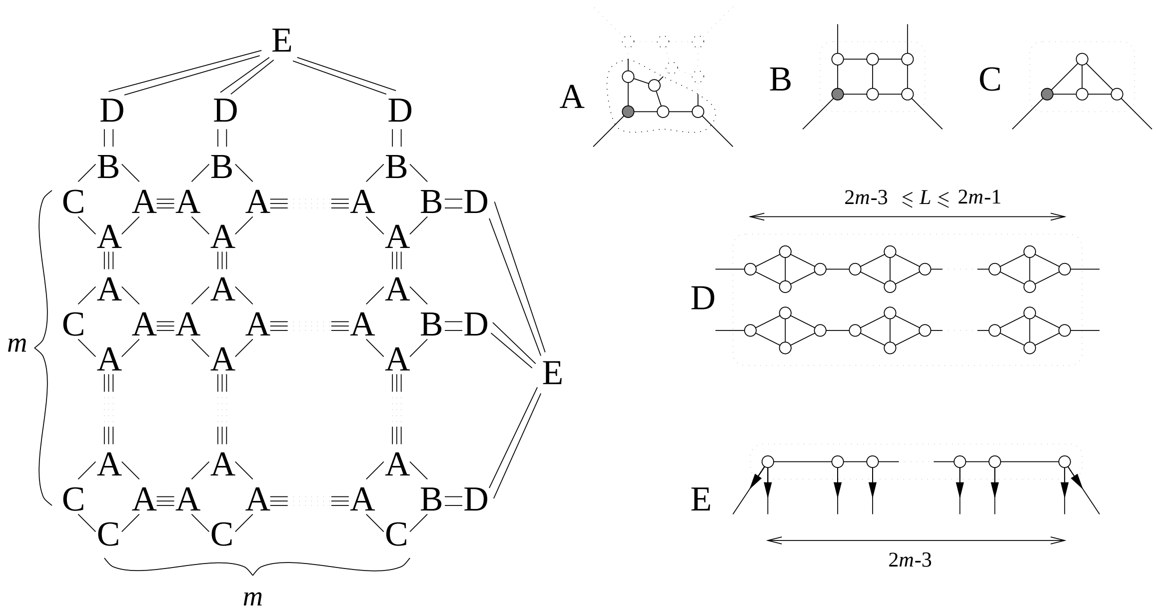
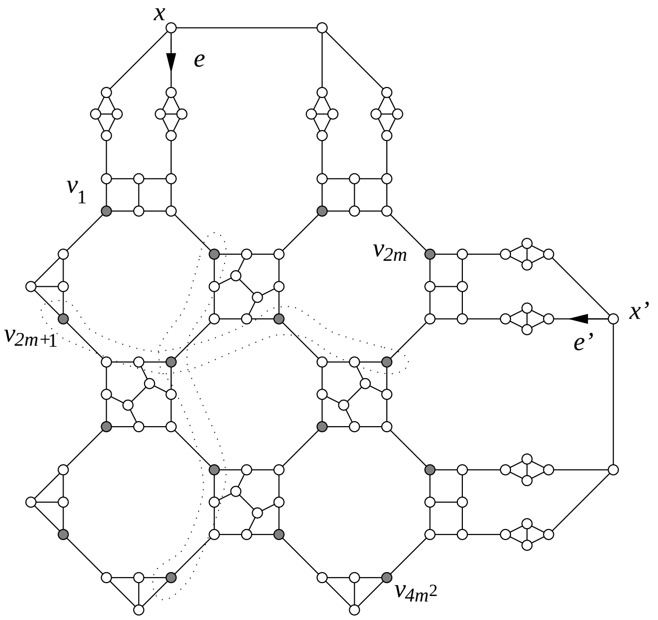


Figure 5: Construction of the P_m graph.

Lemma 4 *For every integer $m \geq 2$,*

- *P_m is 3-regular and planar*
- *P_m is of order at most $\dfrac{92}{3}m^2 - 10m + 56$*

- $\textsc{Irs}(P_m) \geq m$



$$
M_m = \begin{array}{c}
\begin{array}{cccccccc} & e' & & & & e & & \\ & \downarrow & & & & \downarrow & & \end{array} \\
\left[\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
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0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
\begin{array}{l} \\ \\ v_1 \\ \vdots \\ \\ v_{2m} \\ v_{2m+1} \\ \vdots \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ v_{4m^2} \end{array}
$$

Figure 6: The 3-regular planar graph P_m with compactness at least m of order $\Theta(m^2)$, here for $m = 2$.

Proof. Let $m \geq 2$. By construction, P_m is planar and 3-regular. We denote by A, B, $\ldots$, E the order of basic structures of P_m on Figure 5. Note that $A = 5$, $B = 6$, $C = 4$, $D < 16m/3 + 8$, and $E = 2m - 2$. Referring to the left side of Figure 5, the order of P_m is:

$$n = 2E + 2m(D + B + C) + 4(m-2)^2 A + 6(m-2)A + 8A < \frac{92}{3}m^2 - 10m + 56.$$

To prove the third point, we will show that P_m possesses a matrix of constraints, denoted M_m, such that $\mathrm{I}(M_m) \geq m$. M_m is composed of: (1) $4m^2$ gray-vertices $v_1, \ldots, v_{4m^2}$ that we assume indexed first by columns then by rows; (2) horizontal-arcs (the first $2m$ columns) and vertical-arcs (the last $2m$ ones). Figure 6 shows an example of the graph P_m and its matrix of constraints M_m. M_m is a matrix of constraints of P_m because one can easily check that P_m satisfied the following property: for every gray-vertex y and every vertical-arc e of tail x, a shortest path from x to y in P_m uses the arc e if and only if x and y belong to the same "column". "Column" means the same subset of gray-vertices delimited by a dotted line depicted on Figure 6 (the situation is symmetric for vertical-arcs).

First assume that x and y are not on the same column. Let us call P' any shortest path from x to y that uses an arc e' different of e but incident to e. If there exists a shortest path P from x to y using e it has to cross two structures A. It implies that the length of P is at least one more than the length of P' (for P one must pay 2 for structures AA, whereas for P' one pays only 1 for the arc e').

Now, if x and y are located on the same column, a path using the horizontal copy of the structure E cannot be shorter, because of length of the structures D. Indeed, suppose that x is the tail of the most left vertical-arc. x is connected to two vertical-arcs, e_1 and e_2 (assume that e_2 is the left most arc). Finally, let L denote the length of the structure D. The length of any path from x to the gray-vertex v_{4m^2-1} traveling e_1 has of length at most $L+6m+1$, whereas the length of any path using e_2 and the horizontal copy of the structure E is of length at least $3L+2m+8$. It implies that the length of D, *viz* L, must be such that $L+6m+1 < 3L+2m+8$, implying $L > 2m-7/3$, and thus $L \geq 2m-3$. Therefore M_m is a matrix of constraints of P_m.

Let us show that $\mathrm{I}(M_m) \geq m$. Consider each row of M_m as a binary word of length $4m$ bits. Each row is unique and has exactly 2 bits set to 1. It follows that any two rows of M_m differ in at least 2 places. The result of [5, Lemma 5] states that compactness of any $p{\times}q$ boolean matrix M such that any two rows differ of at least r places satisfies $\mathrm{I}(M) \geq rp/(2q)$. Therefore, we can give a trivial lower bound on the compactness of M_m that is $\mathrm{I}(M_m) \geq 2{\cdot}4m^2/(2{\cdot}4m) = m$, that ends the proof. $\square$

It is easy to find a labeling of P_m to state that $\mathrm{IRS}(P_m) = m + O(1)$ ($2m+O(1)$ is trivial, $m+O(1)$ can use a Z-like labeling of gray-vertices). Moreover, the total number of blocks of consecutive ones on $4m$ columns of M_m is $4m^2$. Since there is at most $m+O(1)$ intervals on every arcs, it follows that P_m possesses $\Omega(m)$ arcs with $\Omega(m)$ intervals each.

Proof of Theorem 4.

Let $m = \lfloor x_0 \rfloor$, where x_0 is the positive solution of the equation $92x^2/3 - 10x + 56 = n$ ($x_0 = (\sqrt{276n - 5127}+5)/92$). For every n large enough, $m \geq 2$ and $92m^2/3 - 10m + 56 \leq n$. Applying Lemma 4, the graph P_m has at most n vertices and verifies $\mathrm{IRS}(P_m) \geq m$. For every n large enough this yields $\mathrm{IRS}(P_m) > x_0 - 1 > \sqrt{n}/5.6$, since $(\sqrt{276}/92)^{-1} \approx 5.537$. $\square$

If we relax the constraint on the degree, we can improve a little the multiplicative constant to find a better lower bound for the compactness of a planar graph. However, this new lower bound is still in $\Omega(\sqrt{n})$.

Theorem 5 *There exists a planar graph G of order at most n and of maximum degree $\Theta(\sqrt{n})$ such that*

$$\mathrm{IRS}(G) > \frac{\sqrt{n}}{2.6}$$

Proof. Consider, for every $m \geq 1$, the graph P'_m composed of a $(2m){\times}(2m)$ grid and of two isomorphic trees T whose leaves are respectively connected to an horizontal and to a vertical border of the grid. T is a complete bipartite graph $K_{1,2m}$ where the root is the vertex of degree $2m$ and such that each edge is subdivided in $L-1$ vertices, for a given integer $L \geq 1$. Therefore the distance between the root of T and its leaves is L. Now, let us consider the $4m^2$ vertices of the grid and the $4m$ arcs whose the tail is the root of the horizontal and the vertical tree. One can check that the boolean matrix M_m (defined in the proof of Lemma 4) is a matrix of constraints of P'_m if L satisfies $3L > 2m$. Therefore, for $L = \lfloor 2m/3 \rfloor + 1$, P'_m is of order $n = 4m^2 + 4m\lfloor 2m/3 \rfloor + 2 = 20m^2/3 + O(m)$, and has M_m as matrix of constraints. Since $\mathrm{IRS}(P'_m) \geq \mathrm{I}(M_m) \geq m$, it implies for every integer n large enough that $\mathrm{IRS}(P'_m) > \sqrt{n}/2.6$ ($\sqrt{20/3} \approx 2.581$). The maximum degree of P'_m is $2m = \Theta(\sqrt{n})$. $\square$

4.2 Large compactness required by many edges

Like in Paragraph 3.3, we will see that the compactness of a planar graph of order n can be required by $\Theta(n)$ edges, that is also asymptotically the total number of edges of the graph.

Proposition 1 *There exists a planar graph of order at most n and of maximum degree $\Theta(\sqrt{n})$ having a compactness $\Omega(\sqrt{n})$, required by $\Theta(n)$ edges.*

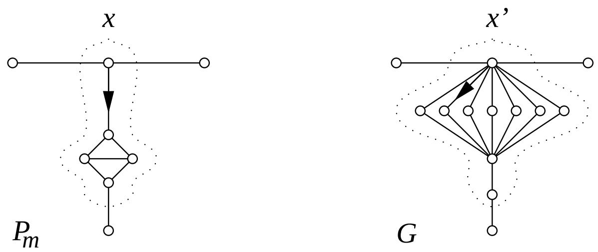

Figure 7: The "planar-splitting" transformation shows that a large compactness can be required by $\Theta(n)$ edges.

Proof. Let $m = \lfloor c\sqrt{n} \rfloor + 2$, for some suitable constant c. By applying Lemma 4, the graph P_m is 3-regular, planar, and has $\Theta(m^2)$ vertices. We already have seen that P_m requires $\Omega(m)$ intervals on $\Omega(m)$ arcs (the distinguished vertical and horizontal-arcs). From P_m we build a new graph G, by transforming each vertical-arc (and horizontal-arc) of tail x by a $K_{2,m}$ as it is shown on Figure 7. Notice that, since $m \geq 2$, the structure D of P_m is composed of at least $\lceil (L+1)/3 \rceil \geq \lceil (2m-3)/3 \rceil \geq 1$ blocks of length 3 (see the description of the graph P_m on page 11). G is planar, is of maximum degree $2m+2$ (due to the vertex x'). The total number of vertices added is $O(m^2)$, thus G has $\Theta(m^2)$ vertices, that is less than n for a suitable constant c. The constraints from x' to any gray-vertex of G remain those of the vertex x in P_m, since the distances are preserved. Therefore, m independent groups of $\Omega(m)$ arcs require each $\Omega(m)$ intervals, so a total of $\Omega(m^2) = \Omega(n)$ arcs requiring each $\Omega(m) = \Omega(\sqrt{n})$ intervals. □

Proposition 1 shows that the $\Omega(\sqrt{n})$-lower bound can be required by $\Theta(n)$ edges, unfortunately for an unbounded degree planar graph. The following theorem generalizes this result. It allows to derive a tradeoff between the compactness, the number of edges requiring such a compactness and the maximum degree.

Theorem 6 *For every integer $d \geq 3$ and for every integer k such that $d-1 \leq k \leq \sqrt{n \log_d n}$, there exists a planar graph of order at most n and of degree at most d having a compactness $\Omega(m)$, required by $\Omega(k \cdot m)$ edges, where $m = k/\log_d k$.*

Proposition 1 is inferred from an application of Theorem 6 with $k = d = \Theta(\sqrt{n})$.

Proof. The main idea is similar to the one of Proposition 1. We split the vertical- and horizontal-arcs of the graph P_m, preserving the constraints of shortest path for the gray-vertices. However, we use a structure of degree bounded by d instead of a graph $K_{2,m}$. Let h be the smallest integer such that $k \leq (d-1)^h$. $h = \lceil \log_{d-1} k \rceil = \Theta(\log_d k)$. Let $m' = \lfloor c \cdot m \rfloor + 2$, for $m = k/\log_d k$ and for a suitable constant c. Let us consider the graph $P_{m'}$, the graph described on page 11. We apply two graph transformations to $P_{m'}$ to prove that the resulting graph, G', has $M_{m'}$ (defined in the proof of Lemma 4) as matrix of constraints for k groups of $\Omega(m)$ independent arcs.

The first transformation produces the graph G and is depicted in the middle part of Figure 8. Each edge of $P_{m'}$ is subdivided in $h+2$ new edges.

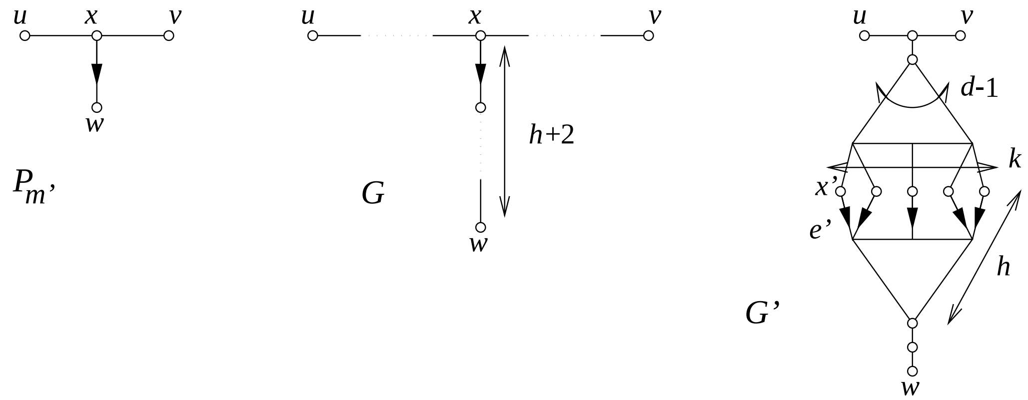

Figure 8: A bounded degree splitting transformation.

The second transformation, depicted on the right side of Figure 8, replaces each structure u, v, w containing a constraint e in G by another structure of maximum degree bounded by d by splitting e in k independent new constraints. This transformation forms the graph G'. Moreover, the distance between x and u, v or w in G is the same than the distance between any vertex x' and u, v or w in G'. The structure is composed of two copies of a complete $(d-1)$-ary tree whose some leaves are removed from the last level in order to have exactly k leaves. These two trees are merged by their k leaves (see the right side on Figure 8). The split constraints are the arcs distinguished on Figure 8. The tail of these arcs are the leaves of the two merged trees. One vertex is added between w and the root of the second tree in order to keep at distance $h+2$ each leaf x' to the vertices u, v or w. G' is planar and of maximum degree bounded by d.

Since $m' \geq 2$, from Lemma 4, $P_{m'}$ has $M_{m'} = (m_{y,e})$ as matrix of constraints, y gray-vertices and e constraints of $P_{m'}$. Let e be an arc forming a constraint for a gray-vertex y of $P_{m'}$. It is easy to show that $m_{y,e} = 1$ implies $m_{y,e'} = 1$, and $m_{y,e} = 0$ implies $m_{y,e'} = 0$, for every new constraints e' of G'. Indeed, assume that every shortest path from x to y in $P_{m'}$ uses e and, by contradiction, assume that a shortest path from x' to y in G' does not use the arc e'. Applying the reversal transformation (the path from x to w in G is obtained from the simple path from x' to w using e' in G' that is necessary of length $h+2$), it would imply that, in G, a shortest path from x and y would not use e, and thus, in $P_{m'}$, there would exist a shortest path from x to y that would not use e: a contradiction. One can use the same kind of argument to prove that $m_{y,e} = 0$ implies $m_{y,e'} = 0$. Therefore, $M_{m'}$ is a matrix of constraints for the graph G', implying $\textsc{Irs}(G') \geq m'$. Moreover, this compactness is required by k independent groups of $\Omega(m')$ arcs. Hence G' has a compactness at least $m' = \Omega(m)$, required by $\Omega(k \cdot m)$ arcs.

Now let us prove that the order of G is bounded by n. The number of vertices of G is $N \leq c_1 \cdot m'^2 + c_2 \cdot m'^2 \cdot h + c_3 \cdot m' \cdot k$, for some constants c_1, c_2 and c_3. Indeed, the number of vertices of $P_{m'}$ is at most $c_1 \cdot m'^2$, its number of edges is linear in its number of vertices, and in G each structure u, v, w ($4m'$ in total) is replaced by a structure based on trees of at most $3k$ vertices. Note that

$m'^2 = O(m' \cdot k)$ and $m'^2 \cdot h = O(m' \cdot k)$. Hence, $N \leq c' \cdot m' \cdot k = c' \cdot c \cdot k^2 / \log_d k$, for some constant c'. For all d and k such that $2 \leq d-1 \leq k \leq \sqrt{n \log_d n}$, $k^2/\log_d k$ is maximum for k maximum. Thus,

$$N \leq c' \cdot c \cdot \frac{n \log_d n}{\frac{1}{2} \log_d n + \frac{1}{2} \log_d \log_d n} < 2c' \cdot c \cdot n.$$

Therefore, for a suitable constant c and for every integer n large enough, G' is planar, of order at most n, of degree at most d, and of compactness $\Omega(m)$, required by $\Omega(k \cdot m)$ edges. $\square$

Combined with Theorem 6, the following proposition will allow to state a similar result for 3-regular planar graphs:

Proposition 2 *Every graph G of order n and of maximum degree 3 can be transformed in a 3-regular graph G' of order at most $6n$ such that* $\text{IRS}(G') \geq \text{IRS}(G)$. *Moreover, G' is planar if G is.*

In particular, by plugging $d = 3$ and $k = \sqrt{n}$ in Theorem 6, it turns out that there exists a 3-regular planar graph having a compactness $\Omega(\sqrt{n/\log n})$, required by $\Theta(n)$ edges.

Proof. For every vertex of degree 1 or 2 of G, we add the planar structure drawn on Figure 9 to transform it in a 3-regular graph G'. For each vertex, the number of added vertices is bounded by 5. Thus, G' is 3-regular and is obtained by adding at most $5n$ vertices. Moreover, G' is planar if G is. All the shortest paths in G' between any two vertices of G are clearly wholly contained in G. Thus G is a subgraph of shortest paths of G'. Therefore, by application of [7, Theorem 2], the compactness of G' is greater or equal than the compactness of G. $\square$

Figure 9: A 3-regular planar graph transformation.

4.3 Compactness of 4-regular and 5-regular planar graphs

In this paragraph we show that the $\Omega(\sqrt{n})$-lower bound for planar graphs does not only hold for 3-regular planar graphs, but also for all the d-regular planar graphs, $d \geq 3$. Note that no d-regular planar graph exists for $d \geq 6$.

Theorem 7

- *There exists a 4-regular planar graph of order at most n having a compactness $\Omega(\sqrt{n})$.*
- *There exists a 5-regular planar graph of order at most n having a compactness $\Omega(\sqrt{n})$.*

The two points actually result of the two following propositions.

Proposition 3 *Every 3-regular graph G of order n having a perfect matching can be transformed in a 4-regular graph G' of order at most $4n$ such that* $\text{IRS}(G') \geq \text{IRS}(G)$. *Moreover, G' is planar if G is.*

Proof. The basic idea is similar to the one used in Proposition 2. We transform G in a 4-regular graph G' by using the edges of a perfect matching C of G. Let us recall that a *perfect matching* of a graph G is a subset of edges of G such that every vertex, but one if G has an odd order, belongs to a unique edge of the perfect matching. The transformation is planar and G is obtained by adding a total of $3n$ vertices. To each edge $e \in C$, we connect the 4-regular structure of 6 vertices depicted on the left side of Figure 10. The both vertices of e are now of degree 4. C has $n/2$ edges (the order of a 3-regular graph is necessary even), thus the order of G' is $n + 6(n/2) = 4n$. The paths from any two vertices of G', originally in G, and using the added vertices is necessary longer. Therefore, G is a subgraph of shortest paths of G', and thus $\text{IRS}(G') \geq \text{IRS}(G)$. □

Figure 10: A 4-regular and 5-regular planar graph transformation from a 3-regular graph (drawn in dotted line).

Proposition 4 *Every 3-regular graph G of order n can be transformed in a 5-regular graph G' of order at most $13n$ such that* $\text{IRS}(G') \geq \text{IRS}(G)$. *Moreover, G' is planar if G is.*

Proof. We use the same argument than for Proposition 2, with the 5-regular and planar transformation of 12 vertices depicted on the right side of Figure 10. □

Proof of Theorem 7.

The first point results from Proposition 3 and the fact that the graph P_m, for $m = \Theta(\sqrt{n})$, has a perfect matching. Indeed, let us consider the basic structures A, B, C, D, E forming P_m, drawn on Figure 5, page 11. Each one of the structures A, B, C, D has a trivial perfect matching which does not contain any edge used for the connections between the structures. Moreover, the structure E has a perfect matching for an odd length, that is the case since its length is $2m - 3$. Therefore, P_m has a perfect matching since the connection edges are not used. The second point, for 5-regular graphs, is straightforward. □

4.4 Compactness of planar graphs with only triangles

In this paragraph, we will consider compactness of planar graphs whose all the faces[6] are triangles (including the outer face). Such graphs are so-called *planar graphs with only triangles*. The following result shows that the same $\Omega(\sqrt{n})$-lower bound holds even for this subclass of planar graphs:

Theorem 8 *There exists a planar graph with only triangles of order at most n and a degree bounded by 9 having a compactness $\Omega(\sqrt{n})$.*

The main idea is to "fill up" all the faces of the plane graph P_m, for $m = \Theta(\sqrt{n})$, that are not a triangle by structures containing only triangles such that the constraints between vertical-arc, horizontal-arc and gray-vertices of P_m are preserved. We use the two following lemmas:

Lemma 5 *For every $l \geq 4$, there exists a planar graph G of order $O(l^2)$ with a plane embedding such that all the faces of G, but the outer face, are triangles, and such that the cycle forming the outer face of G, C, is of length l and is a subgraph of shortest paths. Moreover, G has a degree at most 6 and the degree of the vertices of C is at most 4.*

Proof. Let $D = \lfloor l/2 \rfloor$ and $t = \lceil D/2 \rceil + 1$. G is composed of t set-in levels (labeled from 1 to t), each level being a cycle of length l. A Z-like path is used to connect two consecutive levels of G, see the right side of Figure 11. The last level, the most inside one, is filled with triangles using a similar Z-like path, see the left side of Figure 11). G is planar and of order $t{\cdot}l = O(l^2)$. Each vertex of level i, for $i \in \{1, \ldots, t-1\}$, is connected to two vertices of its level and to at most two vertices of level $i+1$. Therefore, the vertices of level 1, forming the subgraph C and lying the vertices of the outer face of G, have a degree bounded by 4, whereas the other vertices have a degree bounded by 6. To prove that C is a subgraph of shortest paths of G, let us consider x and y any two vertices of C. Let P be a shortest path from x to y. We denote by $d_P(u,v)$, for every vertices u, v of P, the length of the path from u to v in P. For $i \in \{1, \ldots, t\}$, let x_i (respectively y_i) be the vertex of level i that appears the first time (respectively the last time) in P. Let k be the maximum level of a vertex of P. We shall prove that $k = 1$ and thus P is wholly contained in the first level.

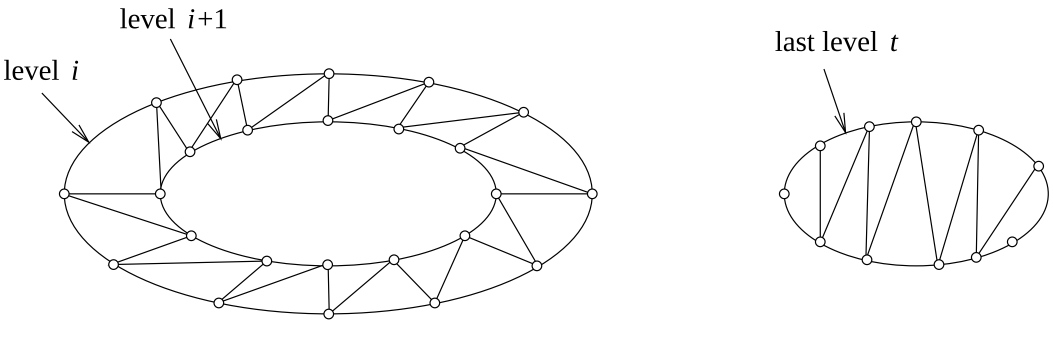

Figure 11: Transformation of planar graphs into planar graphs with only triangles.

We have the following claim: due to the Z-like path connecting two consecutive levels, a shortest path from any two vertices u, v of level i and wholly contained in level i is necessary

[6] The faces of a planar graph G are related to a given plane embedding of G. However, if a plane embedding of G contains only faces that are triangles, then all plane embedding of G contain only triangles.

strictly shorter than any path from u to v only using vertices of level i and $i+1$, for every $i \in \{1, \ldots, t-2\}$ (the claim is false for consecutive levels $t-1$ and t). Let us assume $k \geq 2$. P is a simple path and moreover $x_k \neq y_k$, otherwise P would be not a shortest path. Indeed, in this case x_k has two neighbors u, v of level $k-1$ that belong to P. Thus, by applying the previous claim between u and v on level $k-1$, there exists a path shorter than P using u and v without x_k. In particular, this implies $d_P(x_k, y_k) \geq 1$ and, since P is a simple path, $d_P(x,y) \geq d_P(x_1, x_k) + d_P(x_k, y_k) + d_P(y_k, y_1) \geq 2(k-1)+1$. In the other hand, $d_P(x,y) \leq D$, the diameter of C, implying $2(k-1)+1 \leq D$ and $k \leq D/2 + 1/2 < t$. Since $k-1 \leq t-2$, we can apply the claim for every path from x_i to y_i for all $i \in \{1, \ldots, k-1\}$, to show that $k=1$. Therefore, P is wholly contained in level 1 and C is a subgraph of shortest paths of G. □

Lemma 6 *For every $l \geq 4$, there exists a planar graph G of order $O(l)$ with a plane embedding such that the outer face of G covers l vertices, all the faces of G, but the outer face, are triangles, and such that there exists two adjacent vertices x and y of the cycle forming the outer face of G, C, satisfying: for every vertex z of C, all the shortest paths from x to z and from y to z are wholly contained in C. Moreover, G has a degree at most 6 and the degree of the vertices of C is at most 4.*

Proof. G is composed of 2 set-in levels of length l and of a path of $\lceil l/2 \rceil - 2$ vertices. C forms the first level and the outer face of G. A Z-like path connects the two set-in levels like it is shown on the example of Figure 12, with $l = 11$. Let u, v, w be the neighbors of x and y on the second level, such that v is adjacent to x and y. u and w are connected and a Z-like path connects the second level and the path (see Figure 12). G is planar and is of order $2l + \lceil l/2 \rceil - 2 = O(l)$. Moreover, the vertices of C are bounded by 4 and the other vertices bounded by 6.

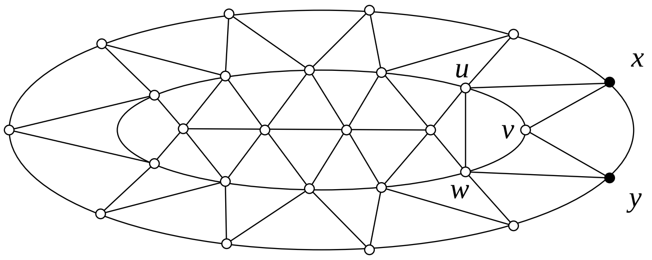

Figure 12: The second transformation of faces into planar graphs with only triangles.

Let z be any vertex of C, and let P be a shortest path from x to z. Let us prove that P is wholly contained in C. If P does not contain any vertex of the middle-path, then by applying the claim of Lemma 5, P is wholly contained in C (it cannot be a shortest path and use only vertices of the first and the second level). However, if P uses a vertex of the middle-path, P is at least 1 more longer than a path using only vertices of C. □

Proof of Theorem 8.

Let us consider the 3-regular planar graph P_m of $\Theta(m^2)$ vertices, for a suitable $m = \Theta(\sqrt{n})$. We will transform P_m in a graph G of order at most n whose all the faces are triangles, and such that G has the matrix M_m as matrix of constraints. P_m has $f = \Theta(m^2)$ faces, including $4m-1$ faces of length $\Theta(m)$, all the other faces being of length bounded by 12 (see Figure 6 on page 12).

The faces of length $\Theta(m)$ are the faces due to the structures D and the outer face of P_m. Clearly, we can apply Lemma 5 to all the constant length faces of P_m and to the outer face of P_m (of length $\Theta(m)$) to obtained a graph H still of order $\Theta(m^2)$ and having P_m has subgraph of shortest paths. H is planar and has a degree bounded by 9, because a vertex u of P_m may belong to 3 faces having each one two connections from u to an inside level.

G is composed of H on which we apply Lemma 6 to all its faces that are not triangles. There are $4m-2$, all of length $\Theta(m)$. G is planar, of degree bounded by 9, and of order $\Theta(m^2)$, that is, for a suitable m, at most n. Let us prove that M_m is a matrix of constraints of G. Let C be any face of P_m, excepted the outer face, having (at most) two constrained vertices, x and y, for the set of gray-vertices. By construction of P_m, x and y are neighbors, C is of length $\Theta(m)$, and in G, C is fill with triangles by applying Lemma 6. Let z be a gray-vertex. All the shortest paths from x to z or from y to z are contained in C. Otherwise there would exists a vertex z' of C such that a shortest path from x to z' (or from y to z') uses not only vertices of C: a contradiction. Therefore, all the shortest paths from constrained vertices and gray-vertices are wholly contained in P_m. This implies that M_m is a matrix of constraints of G, and therefore $\textsc{Irs}(G) \geq m = \Theta(\sqrt{n})$. $\square$

5 Conclusion

We proved a tight lower bound of the compactness for 3-regular networks, showing that compactness could not depend on the maximum degree of the graph in general. We gave also a new lower bound of $\Omega(\sqrt{n})$ intervals for 3-regular (4-regular and 5-regular as well) planar graphs and bounded degree planar graphs with only triangles. In [9], Frederickson and Janardan showed that every planar graph G composed of p *planes* satisfied $\textsc{Irs}(G) \leq 3p/2$. The number of planes of G is defined as being the smallest number of faces needed to cover all the nodes, taken over all possible plane embedding of G. For example, the 1-plane graphs are precisely the outer-planar graphs. Unfortunately the number of planes of a graph of order n can raise up to $\Theta(n)$. It would be interesting to know if a better lower bound can be found for planar graphs (not only for bounded degree planar graphs). Note that for the series-parallel graphs [2], which is a subclass of planar graphs including the outer-planar graphs, Gavoille and Luccio [15] have proved that compactness can rise up to $\Omega(\sqrt{n})$, even for bounded degree series-parallel graphs.

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