Routing v.s. Spanners

Spanner et routage compact : similarités et différences

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Outline

Spanners

Routing

The Question and the Answer
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Spanners

Routing

The Question and the Answer
What is a Spanner?

A spanner of a graph $G$ is a subgraph spanning $V(G)$.
What is a Spanner?

A **spanner** of a graph $G$ is a subgraph spanning $V(G)$

A spanning tree

![Diagram of a spanner]
What is a Spanner?

A **spanner** of a graph $G$ is a subgraph spanning $V(G)$

- a spanning tree
- a Hamiltonian cycle
- ...
Approximate Distance Spanners

There are two “natural” criteria for a spanner of $G$:

- **size**: its number of edges.
- **stretch**: its maximum distance distortion from $G$. 
Approximate Distance Spanners

There are two “natural” criteria for a spanner of $G$:

- **size**: its number of edges.
- **stretch**: its maximum distance distortion from $G$.

 Goals:

- find a good skeleton of the graph;
- decrease the size of the graph while preserving distances;
- optimize stretch-size tradeoffs.
A complete Euclidian graph on 15 nodes
A (minimum cost) spanner with stretch 1.2
A (minimum cost) spanner with stretch 1.7
A (minimum cost) spanner with stretch 2.0
A (minimum cost) spanner with stretch 3.0
Delaunay triangulation
For specific metrics, geometric spaces ...

- Every 2D Euclidean graph has a stretch-2 spanner of size $3n - 6$ [STOC ’86] - Delaunay triangulations get stretch-2.42 (conjectured stretch-1.57)
- Every 2D Euclidean graph has a stretch-$(1 + \varepsilon)$ spanner of size $O(\varepsilon^{-1}n)$ [Yao- & $\Theta$-Graphs]
- Generalization to doubling dimension and higher dimension spaces, also to hyperbolic spaces ...
For general (connected) graphs ... 

- Every weighted graph has a stretch-$(2k - 1)$ spanner of size \(O(n^{1+1/k})\) [Greedy algorithm (Kruskal)]

- Every unweighted graph has a stretch +2 spanner of size \(O(n^{1+1/2})\) [SICOMP’99], and a stretch +6 spanner of size \(O(n^{1+1/3})\) [SODA’05]
Recently ...

- Every unweighted stretch \((2k - 1)\) spanner must have \(\Omega\left(\frac{1}{k} n^{1+1/k}\right)\) edges in the worst-case [FOCS’06]
- Every unweighted graph has a \(f\)-fault-tolerant stretch-\((2k - 1)\) spanner of size \(k^3 f^{k+3} \cdot \tilde{O}(n^{1+1/k})\) [STOC’09]
  - [a spanner for \(G \setminus F\) for all subgraphs \(F\) with \(|F| \leq f\)\]
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Routing with Compact Tables
Compact Routing Schemes

Two “natural” criteria for designing a routing scheme for $G$:

- **size**: routing table size (and also address/header size)
- **stretch**: maximum route length distortion from distances in $G$. 
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- **size**: routing table size (and also address/header size)
- **stretch**: maximum route length distortion from distances in $G$.

**Goals:**
- find an efficient routing scheme for $G$;
- compress routing table size while preserving near-shortest routes;
- optimize stretch-size tradeoffs.
Routing: Some Results (1/3)

For metrics ...

- Every 2D Euclidean graph has a stretch-$14.0$ labeled routing scheme with $8 \log D$-bit tables [Yao-Graph $Y_7 +$ greedy routing]
- Every 2D Euclidean graph has a stretch-$(1 + \varepsilon)$ labeled routing scheme with $O(\varepsilon^{-1} \log D)$-bit tables [cf. Abraham-Malkhi: PODC’04]
- Extension to 3D and more ...
Routing: Some Results (2/3)

For specific (connected) graphs ...

- Every weighted planar graph has a stretch-$(1 + \varepsilon)$ labeled routing scheme with $\tilde{O}(\varepsilon^{-1})$-bit tables, addresses and headers [Thorup: J. ACM’05]
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- Similar bounds for weighted $H$-minor free graphs [Abraham-G.: PODC’06]
Routing: Some Results (3/3)

For general (connected) graphs ...

- Every weighted graph has a stretch-$(4k - 5)$ labeled routing scheme with $\tilde{O}(n^{1/k})$-bit tables and polylog addresses and headers [Thorup-Zwick: SPAA’01]
- Every weighted graph has a stretch-$O(k)$ name-independent routing scheme with $\tilde{O}(n^{1/k})$-bit tables and polylog headers [Abraham et al.: SPAA’06]
Outline

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The Question and the Answer
### Summary

<table>
<thead>
<tr>
<th>stretch</th>
<th>size</th>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2k - 1$</td>
<td>$O(n^{1+1/k})$</td>
<td>Spanner: Greedy Algorithm</td>
</tr>
<tr>
<td>$2k - 1$</td>
<td>$O(kn^{1+1/k})$</td>
<td>Spanner: Tree Cover</td>
</tr>
<tr>
<td>$4k - 5$</td>
<td>$\tilde{O}(kn^{1/k})$</td>
<td>Routing: Tree Cover ++</td>
</tr>
<tr>
<td>+2</td>
<td>$O(n^{1+1/2})$</td>
<td>Spanner: Tree Cover</td>
</tr>
<tr>
<td>+6</td>
<td>$O(n^{1+1/3})$</td>
<td>Spanner: ad-hoc</td>
</tr>
<tr>
<td>$+f(k)$</td>
<td>$O(n^{1+1/k})$</td>
<td>Open for every $k &gt; 3$</td>
</tr>
</tbody>
</table>

**Fact:** Spanner and Routing problems use similar techniques, and get similar bounds.
Can we make additively stretched spanners routable? [Baswana, Elkin, Pettie, ...]

Is there a routing scheme with sublinear space and additive stretch for all graphs?
Yes or No?

PRO:

Numerology!

Spanner: stretch-3 for size $O(n^{1+1/2})$

Routing: stretch-3 for size $\tilde{O}(n^{1/2})$

Spanner: stretch-$O(k)$ for size $O(n^{1+1/k})$

Routing: stretch-$O(k)$ for size $\tilde{O}(n^{1/k})$

Just a coincidence?

There exist spanners of size $o(n^2)$ with constant additive stretch (ex: size $n^{1+1/2}$ or $n^{1+1/3}$ for stretch + 2 or + 6).

It should exist sublinear compact routing scheme with constant additive stretch!!!
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**CON:** Spanners do not tell us how to route on sparse graphs.

The problem is:
- Spanner: prove $\exists$ a near-shortest path
- Routing: construct a near-shortest path
An Impossibility Result

There are graphs with sparse additive spanners but no additive compact routing schemes

Theorem (2009)

Every routing strategy providing, for each unweighted connected $n$-node graph, a labeled routing scheme with tables and addresses $\leq \mu$ bits, produces, for some graph of less than $2n$ edges, an additive stretch $\Omega(n^{1/3}/\mu^{2/3})$. 

Corollary:
The additive stretch of every universal routing strategy with tables and addresses in $o(\sqrt{n})$ is unbounded.

In particular, the stretch-7 routing scheme of [TZ] ($k=3$) with $\tilde{O}(n^{1/3})$-bit tables must have an additive stretch of $n^{1/9} - o(1)$. 
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THANK YOU!
The Graph Family $\mathcal{F} = \mathcal{F}(p, \delta)$

Graphs of $\mathcal{F}$ are constructed from $p \times p$ boolean matrices. Sets of $p$ nodes: $S = \{s_i\}$, $A = \{a_i\}$, $B = \{b_i\}$, $T = \{t_j\}$.

Connect a path of length $\delta$ between: $s_i \leadsto a_i$, $s_i \leadsto b_i$, and $t_j \leadsto a_i$ if $M[i, j] = 1$, and $t_j \leadsto b_i$ if $M[i, j] = 0$. 

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$