

# Implicit Representation and Distance Labeling of Graphs

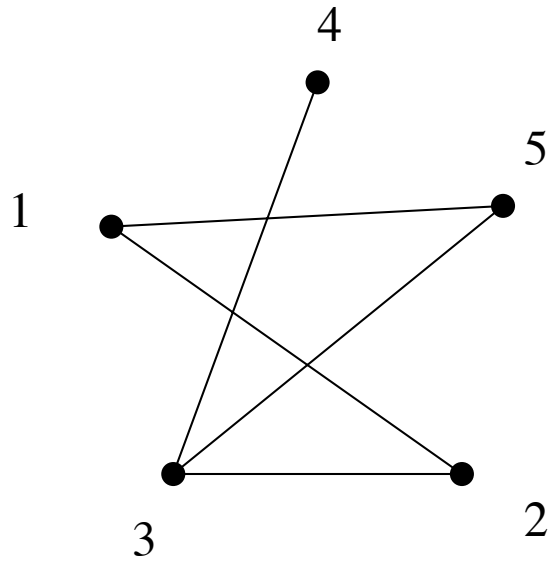
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## Representation of a Graph



	1	2	3	4	5
1		1	0	0	1
2			1	0	0
3				1	1
4					0

1:	2,5
2:	1,3
3:	2,4,5
4:	3
5:	1,3

Adjacency list/matrix

1 node = 1 pointer in the  
data structure

## Implicit Representation / Model of a Graph

To give more **information** to nodes

$$u \rightarrow L(u)$$

such that *adjacency* between  $u$  and  $v$  can be computed from  $L(u)$  and  $L(v)$  without any extra information (no adjacency list or matrix).

**Implicit Representation**: Kannan, Naor, Rudich [KNR88]

Labels of size  $O(\log n)$  bits

## Definition

A pair  $(L, f)$  is an **implicit representation** for a given class of graphs  $\mathcal{F}$  if for every  $G \in \mathcal{F}$ :

1. for every  $u \in V(G)$ ,  $L(G, u) = L_G(u) \in \{1, \dots, 2^\ell\}$

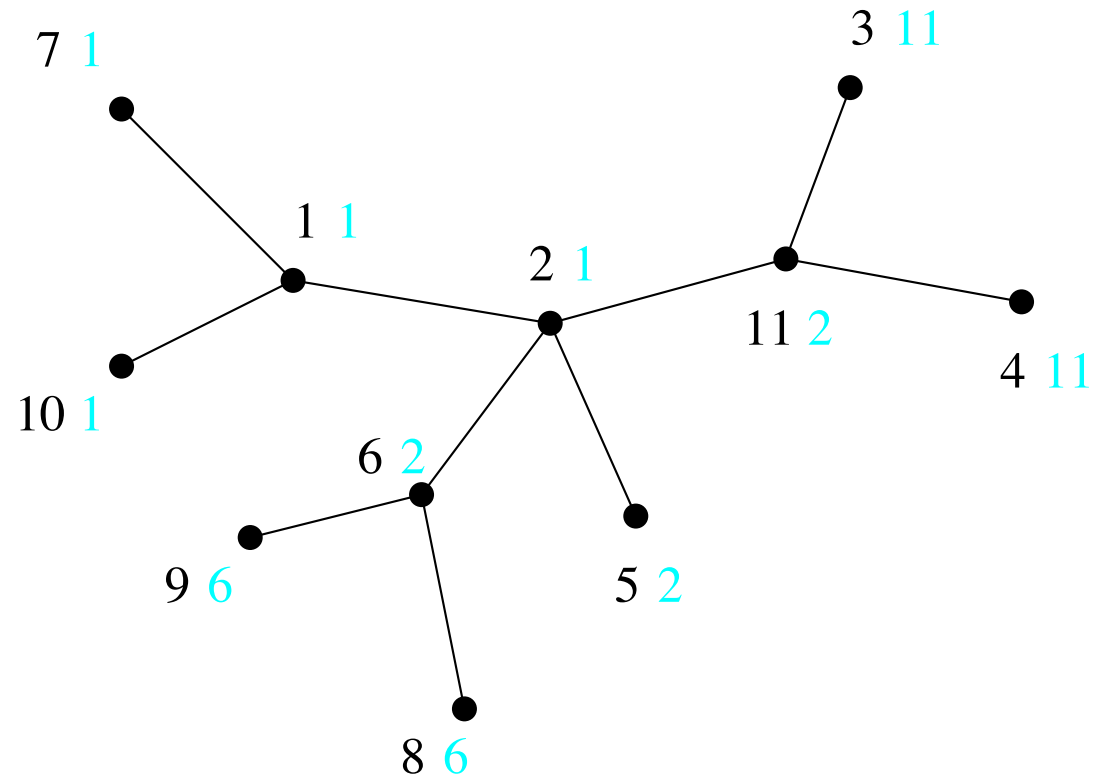
2. for every  $u, v \in V(G)$ ,

$$f(L_G(u), L_G(v)) = \begin{cases} 1 & \text{if } \{u, v\} \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

**Goal:** to minimize  $\ell$

( $\ell_{\min} = \ell(\mathcal{F})$  depends on  $\mathcal{F}$  only).

## Basic example: $\mathcal{F} := \text{trees}$

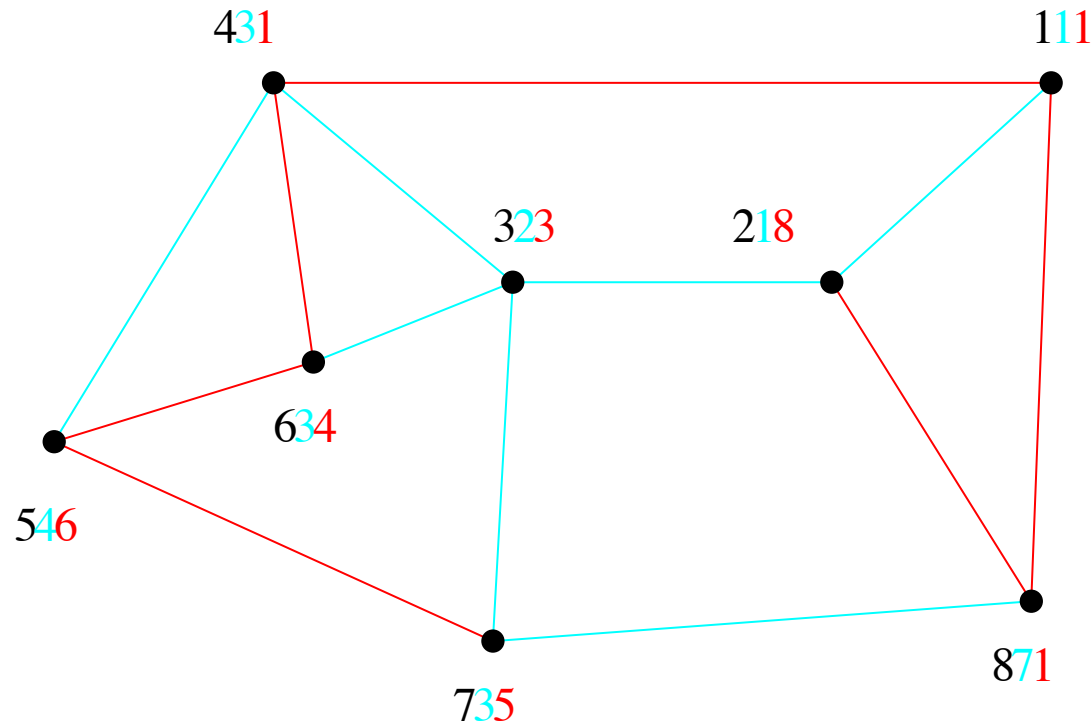


$$f(xy, uv) = 1 \quad \text{iff} \quad y = u \text{ or } v = x \\ L(u) \in \{1, \dots, n^2\}$$

$$\Rightarrow \ell(\text{trees}) \leq 2 \log n$$

## A Generalization: arboricity

$a(G) :=$  minimum number of edge-disjoint forests whose union is  $G$



$$f(x\ yz, u\ vw) = 1 \quad \text{iff} \quad v = x \text{ or } w = x \text{ or } y = u \text{ or } z = u$$

$$\Rightarrow \ell(\text{arboricity } a) \leq (a + 1) \log n \quad \text{[KNR88]}$$

## Applications

$$a(\text{clique } K_n) \leq \lceil n/2 \rceil$$

$$a(\text{max degree } d) \leq \lfloor d/2 \rfloor + 1$$

$$a(\text{genus } \gamma) \leq O(\sqrt{\gamma})$$

$$a(\text{\#edges } m) \leq \left\lceil \frac{1}{2} \sqrt{2m + n} \right\rceil$$

$$a(\text{planar bipartites}) \leq 2$$

$$a(\text{planars}) \leq 3$$

$$\Rightarrow \ell(\text{planars}) \leq 4 \log n$$

## Generalization: Distance Labeling (Peleg WG'99)

Distance Labeling for  $\mathcal{F}$ . Pair  $(L, f)$  such that

1.  $L_G(u) \in \{1, \dots, 2^{\bar{\ell}}\}$
2.  $f(L_G(u), L_G(v)) = \text{dist}_G(u, v)$

$s$ -approximate distance labeling  $s \in \mathbb{R}$ ,  $s \geq 1$ , if:

$$\frac{1}{s} \cdot \text{dist}_G(u, v) \leq f(L_G(u), L_G(v)) \leq s \cdot \text{dist}_G(u, v)$$

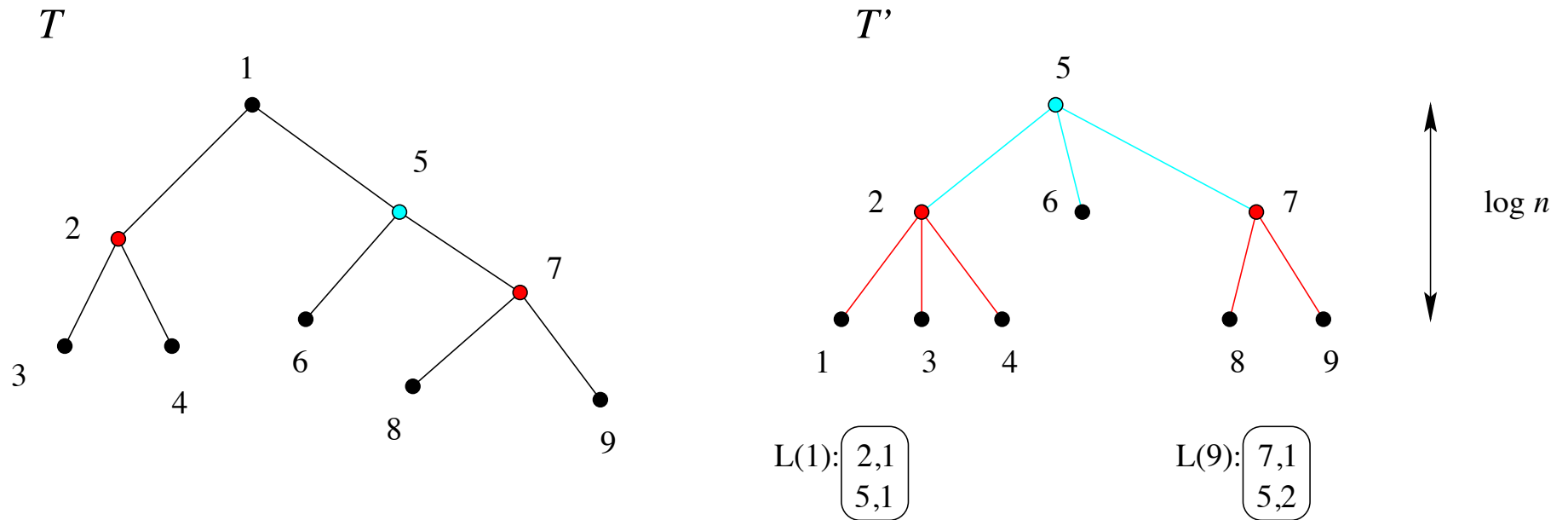
$\bar{\ell}(\text{general graphs}) \leq O(\log^3 n)$  with  $s \leq \sqrt{2 \log n}$

[P99]



## Trees on $n$ nodes

Centroid of  $T :=$  node  $u$  such that all the connected components of  $T - \{u\}$  have less than  $n/2$  nodes.



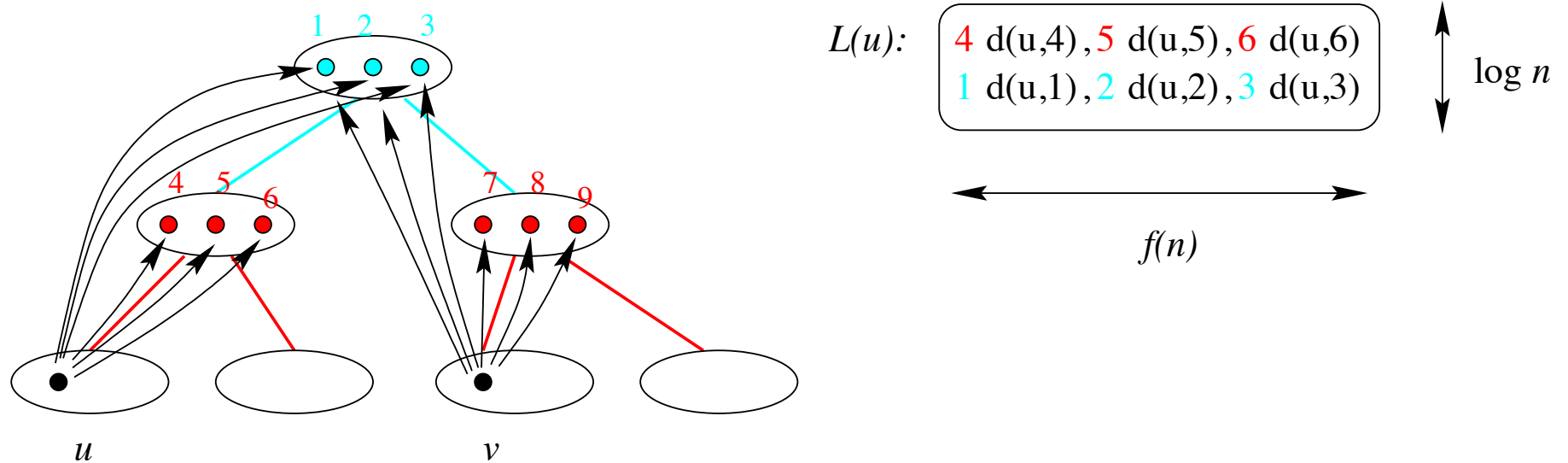
$$\bar{\ell}(\text{trees}) \leq 2 \log^2 n$$

[P99]

$$\bar{\ell}(\text{trees}) \geq \frac{1}{8} \log^2 n - O(\log n)$$

[GPPR99]

## $f(n)$ -separator



$$\bar{\ell}(f(n)\text{-separator}) \leq O(F(n) \log n) \quad [\text{GPPR99}]$$

where  $F(n) := \sum_{i=0}^{\log n} f(n/2^i) \geq \log^2 n$

- $\Rightarrow \bar{\ell}(\text{bounded path-width}) \leq O(\log^2 n)$
- $\Rightarrow \bar{\ell}(\text{bounded tree-width}) \leq O(\log^2 n)$
- $\Rightarrow \bar{\ell}(\text{bounded clique-width}) \leq O(\log^2 n)$
- $\Rightarrow \bar{\ell}(\text{planar}) \leq O(\sqrt{n} \log n)$

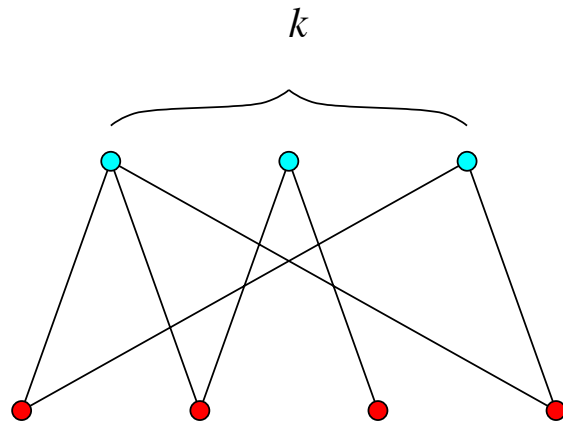
## Lower Bounds?

$$[ \bar{\ell}(\text{trees}) = \Theta(\log^2 n) ]$$

$$\bar{\ell}(\text{planar}) \leq \log^{O(1)} n ?$$

## $k$ -separator: lower bound

$\mathcal{B}_k$  := bipartite  $n$ -node graphs, partition of cardinality  $k \leq n/2$



$$X = \{x_i\}$$

$$S_G = 1101 \ 0110 \ 1001$$

$$Y = \{y_j\}$$

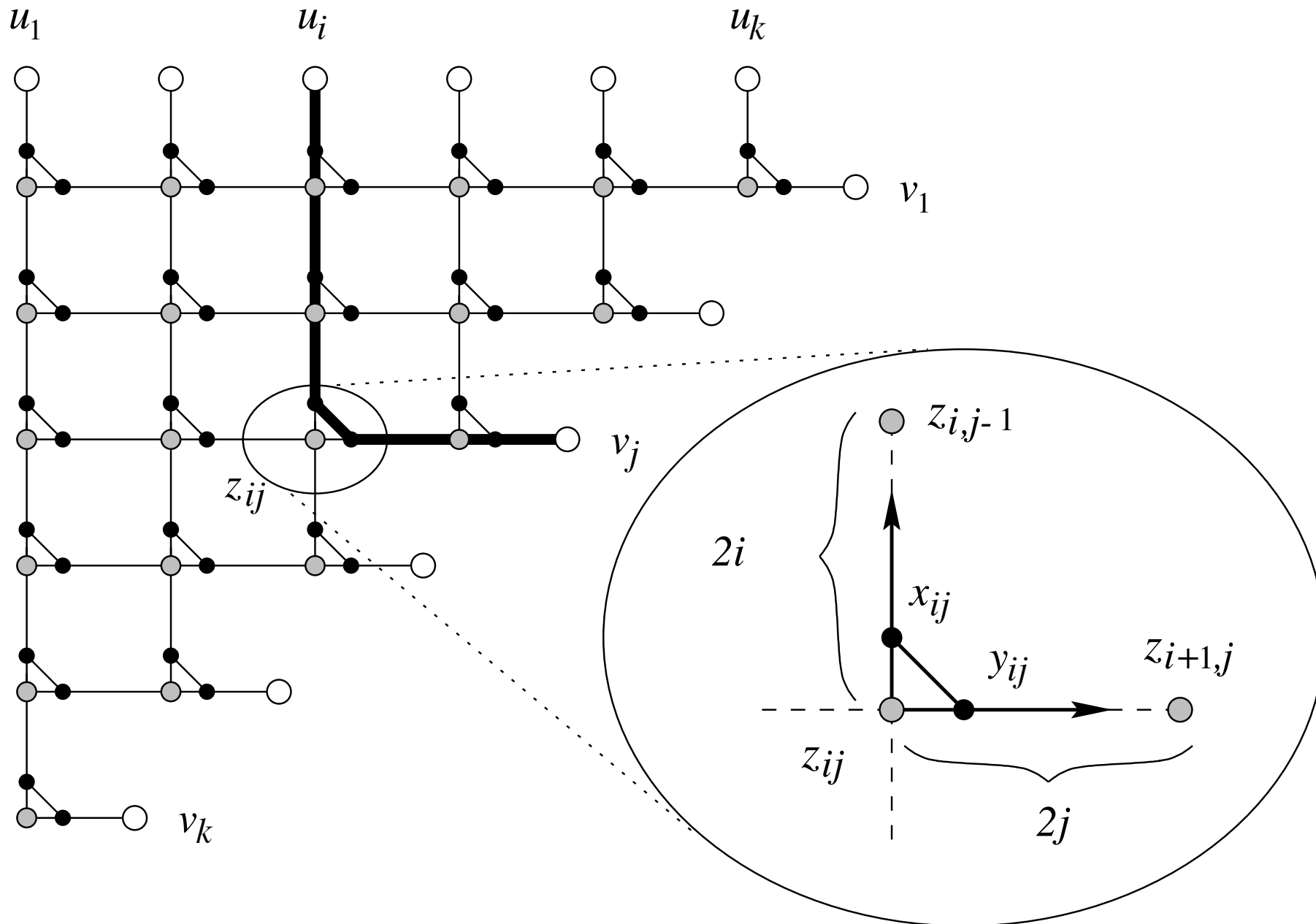
$$L_G := \langle L(x_1), L(x_2), \dots, L(y_1), L(y_2), \dots \rangle, \quad \mathcal{L} := \{L_G \mid G \in \mathcal{B}_k\}$$

$$\exists G_0 \in \mathcal{B}_k, \quad |L_{G_0}| \geq \log(\#S) \geq k(n-k)$$

$$\Rightarrow \exists u, |L_{G_0}(u)| \geq k(n-k)/n \geq k - o(k) \quad \text{for } k \leq n^\epsilon$$

$$\Rightarrow \bar{\ell}(\sqrt{n}\text{-separator}) \geq \sqrt{n} \quad \text{but non-planar!}$$

[GPPR99]



## Planar: lower bound

$\mathcal{P} :=$  set of  $n$ -node planar graphs,  $n \leq k^3$

$$|S_G| = k(k+1)/2, \quad \#S = 2^{k(k+1)/2}$$

Let  $W := \{u_1, \dots, u_k, v_1, \dots, v_k\}$ ,  $|W| = 2k$

$L_G := \langle L_G(u_1), \dots, L_G(u_k), L_G(v_1), \dots, L_G(v_k) \rangle$ ,  $\mathcal{L} := \{L_G \mid G \in \mathcal{P}\}$

$$\exists G_0 \in \mathcal{P}, \quad |L_{G_0}| \geq \log(\#S) \geq \frac{1}{2}k^2$$

$$\exists u \in W, \quad |L_{G_0}(u)| \geq \frac{1}{4}k \geq \frac{1}{4}n^{1/3}$$

QED

## Conclusion: Open Problems

$$\Omega(n^{1/3}) \leq \bar{\ell}(\text{planar}) \leq O(n^{1/2} \log n)$$

$$\Omega(\log n) \leq \bar{\ell}(\text{interval graphs}) \leq O(\log^2 n) \quad [\text{KKP99}]$$

Rem:

$O(n^{1/3} \log n)$ -bit label for planar with multiplicative approx.  $\approx 1.7$

$O(\log n)$ -bit label for interval graphs with additive approx. 1