On the Dilation of Interval Routing

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In this paper we deal with interval routing on n-node networks of diameter D. We show that, for all n and D such that $2 \leq D \leq \Theta(n)$, there exists a network on which every interval routing scheme with less than $\Omega(n/(D\log(n/D)))$ intervals per link has a routing path length at least $\lfloor 3D/2 \rfloor - 1$. It improves the lower bound on the routing path lengths for the range of very large number of intervals. Moreover, we build a network of bounded degree, for all n and D such that $\Theta(\log n) \leq D \leq \Theta(n)$, on which every interval routing scheme with less than $\Omega(n/D^2)$ intervals per link has a routing path length at least $3D/2 - O(\log n)$.

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1. INTRODUCTION

The *dilation* of a routing scheme is the length of the longest routing path. Assuming that the time cost of message delivery is a function of the routing path length or of the number of routers crossed, the dilation is a parameter of the worst-case time complexity. At the same time, fast routers must be easily implemented with a small amount of hardware.

Interval Routing is a routing scheme implementing compact routing tables, and allowing fast routing decisions at each node [12, 17] (see [4] for a recent survey). It consists in labeling nodes by a unique integer taken in $\{1, \ldots, n\}$, n the number of routers, and in assigning to each link at every router a set of intervals of destinations, such that any message can reach its destination from any source. Such a labeling scheme on a network G is called an *interval routing scheme* on G. Each router locally finds the next link to forward a message to its destination by choosing the link that contains the label of the destination in one of its intervals. At each node, the intervals must be pairwise disjoint and cover the set of all the labels, maybe excepted the label of the node itself. The local routing decision time is bounded by $O(\log n)$ whereas the space complexity of the router is at most $O(kd \log n)$ bits, for a router of d links, and if at most k intervals per link are used[2]. In particular, such a routing scheme is efficient, i.e., compact and fast, if the degree of the network and the number of intervals per link are both low[3] relatively to the number of routers of the network.

The interval routing scheme is used in the last generation of C104 routing chips for the INMOS T9000 Transputer design [10], and in the RCube router [18]. Since the number of intervals is limited in each routing chip, we are interested in finding the minimum dilation for interval routing scheme using a fixed number of intervals. The dilation is expressed in term of the diameter of the networks, which is a common lower bound of the dilation for all networks. On the other side every network of diameter D supports an interval routing scheme of dilation $2D$ with one interval per link. Indeed, it is sufficient to route along a minimum spanning tree of the network, which supports an interval routing scheme with only one interval per link [12]. So, the lower bound on dilation ranges between D and $2D$ depending on the number of intervals we allow.

The following table summarizes the best known lower bounds and our contribution about the dilation of interval routing schemes using at most k intervals per link on n-node networks of diameter D. Note that the bounds presented in the table hold for infinitely many values of D, but D is possibly a suitable function of n (cf. Paragraph 2.3 for more details).

Number of intervals	Dilation (lower bound)	Reference
$k = 1$	$2D - 3$	[13]
$k = O(\sqrt{n})$	$3D/2 - 3$	[8]
$k = O(n^{2/3})$	$D + O(1)$	[16]
$k = O(n/\log n)$	$\lfloor 3D/2 \rfloor - 1$	Corollary 3.3

The next section presents the notations and previous works about the dilation. In Section 3, we prove the main theorem. We extend the result to diameters which depend on n. We prove a trade-off of

[2]For each interval $[a_i, b_i]$ one can store its associated output port p_i, so a triple (a_i, b_i, p_i). There are at most kd such triples we can sort with respect to a_i's in a data structure of size $O(kd \log n)$ bits. Actually b_i's can be removed from the data structure, and given a destination label $x \in [a_i, b_i]$, p_i can be found by a binary search in $\log(kd) = O(\log n)$ integer operations.

[3]For instance k and d satisfying $kd/\log d = o(n/\log n)$ may provide a scheme more compact than the standard routing tables which need of $\Theta(n \log d)$ bits per router.

$\Omega(n/(D\log(n/D)))$ intervals required for every interval routing scheme of dilation less than $\lfloor 3D/2 \rfloor - 1$, for every D such that $2 \leq D \leq \Theta(n)$. In Section 4, we prove a bound of $\Omega(n/(D^2 \log d))$ intervals required for every interval routing scheme of dilation less than $3D/2 - O(\log_d n)$ for networks of diameter $D = \Omega(\log_d n)$ and degree bounded by $d \geq 3$. As a result, for every constant $\varepsilon > 0$, there exist networks of maximum degree 3 on which every interval routing scheme of dilation less than $3D/2 - o(D)$ requires $\Omega(n/\log^{2+\varepsilon} n)$ intervals. Finally, we propose some open problems in Section 5.

2. STATEMENT OF THE PROBLEM

2.1. Notations

The model of networks is a undirected connected graph G, each vertex representing a router. The distance between any two vertices x and y is the minimum number of edges of paths connecting x and y, and is denoted $\mathrm{dist}(x, y)$. In all the rest of the paper, n will denote the order of the graph, and D its diameter.

An *interval* means a set of consecutive integers taken in $\{1, \ldots, n\}$, n and 1 being considered as consecutive. For every arc[4] e, all the intervals associated to e form a set of integers, i.e., a set of labels of destinations, denoted by I_e. An interval routing scheme R on a graph G is a pair $(\mathcal{L}, \mathcal{I})$, where $\mathcal{L}$ is the labeling of vertices of G, and $\mathcal{I}$ is the set of all the I_e's.

2.2. A simple example

Let us consider the following example of interval routing on a graph G_0 of 7 vertices depicted on the top of Figure 1. Vertices are labeled by integers from 1 to 7, and intervals are assigned to each arc. If vertex 5 sends a message to vertex 1, the message will successively be forwarded along the arc $(5, 7)$, then along $(7, 1)$, because $1 \in I_{(5,7)} = [7, 2] = \{7, 1, 2\}$, and $1 \in I_{(7,1)} = [1] = \{1\}$. Each set of destinations I_e, e arc of G_0, is composed of at most two intervals of consecutive labels. One can check that every routing path on G_0 is of minimal length. Therefore, the dilation of this routing scheme is the diameter of G_0, here 2.

This interval routing scheme on G_0 qualifies as *shortest path* interval routing scheme because all the routing paths in the graph are of minimal length. A classical problem for interval routing is to compute the minimum number of intervals per link needed to guarantee a shortest path interval routing scheme on a given graph. Such a number depends on the graph only, and is termed *compactness*. The compactness of a graph is at most $n/2$, because any set of destinations and any labeling of these destinations can not contain more than $n/2$ non consecutive integers. In [1] it has been shown that there are families of random graphs in which for every $\varepsilon > 0$ a compactness of $\Omega(n^{1-\varepsilon})$ holds with high probability. Moreover, in [6] the authors have given effective constructions of graphs with compactness at least $n/12$. Recently in [7], authors tighten the bounds, and showed a lower bound of $n/4 - o(n)$, and an upper bound of $n/4 + o(n)$. Therefore, $n/4$ is a tight bound of compactness of n-vertex graphs.

[4]For convenience, each undirected edge of the graph is considered as a pair of two symmetric arcs.

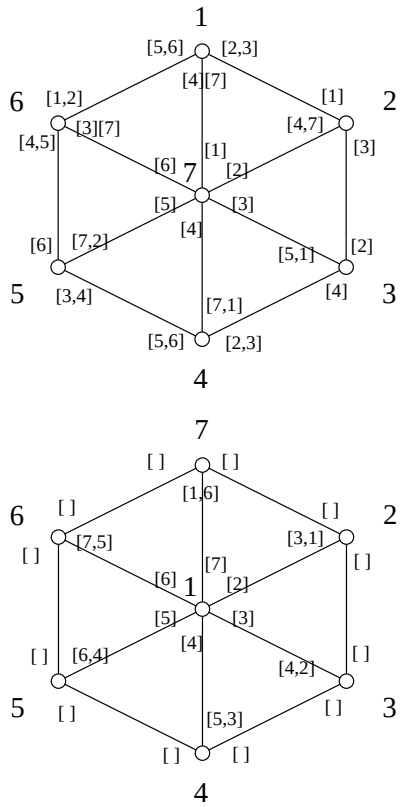

FIGURE 1. Two interval routing schemes of dilation 2 on the same graph G_0. (We denote by [] the *empty interval*, corresponding to an empty set of destinations.)

Actually, G_0 has no shortest path routing scheme with only one interval per link (see [2](page 171) for a proof). Hence the compactness of G_0 is 2. The second picture of Figure 1 shows another interval routing scheme on G_0, but with only one interval per arc. It has also a dilation 2, the diameter. The *dilation* problem is, given a graph G and an integer k, k being less than the compactness of G, to determine an interval routing scheme on G using at most k intervals per link which minimizes the longest routing path. This general question is important in practice whenever a low number of intervals is forced by the hardware of the router, and whenever message delivery time must be as short as possible.

Fundamentally, the compactness problem consists in measuring the compression of the "routing information" whenever paths are of minimum length. Its dual problem, the dilation problem, consists in measuring the efficiency of the routing scheme when the compression rate is limited, which defines the size of the routing in-

formation in each vertex. Both the problems contribute to design some trade-off between time and space used by a router in a communication network.

2.3. Related works

Lower bound for the dilation problem for interval routing was first addressed by Ružička in [11]. He built a *globe-graph* (a subdivision of the bipartite graph $K_{2,q}$) on which every interval routing scheme with one interval has dilation at least $3D/2+1$, for every even D such that $O(1) \leq D \leq \Theta(n)$. This result has been improved by Tse and Lau in [13] for one interval with a lower bound of $2D-3$, for every even D such that $O(1) \leq D \leq \Theta(\sqrt{n})$. More generally they gave a lower bound of $2D - O(D^2/n)$, for every D such that $O(1) \leq D \leq \Theta(n)$. These results are based on an extension of the globe-graph, called the *multi-globe graph*, a subdivision of the bipartite graph $K_{p,q}$.

Considering a larger number of intervals, Tse and Lau proved recently in [15] a lower bound of $D + \Theta(D/\sqrt{k})$ up to $k = O(n/D) \leq O(n^{2/3})$ intervals, but for even $D = \Theta(n^{1/3})$. Still using the multi-globe graph, Kráľovič, Ružička, and Štefankovič proved in [8], a lower bound of $3D/2-3$ up to $O(n/(D+n/D)) \leq O(\sqrt{n})$ intervals, for every even D such that $O(1) \leq D \leq \Theta(n)$. No result was known for a number of intervals larger than $\Theta(n^{2/3})$ [15].

In the other side, in [16], it is proved that, for every network and for every $\alpha \in (0,1)$, $O(n/(\alpha D))$ intervals suffice to guarantee a dilation of at most $\lceil (1+\alpha)D \rceil$. For $\alpha = 2/3$, i.e., dilation at most $\lceil 5D/4 \rceil$, the number of intervals can be reduced to $O(\sqrt{n})$ (cf. [16]). Moreover for $\alpha = 1/2$, i.e., dilation at most $\lceil 3D/2 \rceil$, [8] showed that $O(\sqrt{n \log n})$ intervals suffice. Intuitively, the more intervals used, the lower the dilation.

In this paper we extended the range of possible number of intervals up to $O(n/(D \log(n/D))) \leq O(n/\log n)$, and we prove a dilation of at least $\lfloor 3D/2 \rfloor - 1$, for every D such that $2 \leq D \leq \Theta(n)$. Our result expresses the existence of a gap on the number of intervals required for a dilation close to 3/2 the diameter.

3. CONSTRUCTION OF THE WORST-CASE

The main Theorem of this section is the following:

THEOREM 3.1. For all integers $D \geq 2$, and $k \geq 2$, there exists a graph of diameter D with $n = O(Dk \log k)$ vertices on which every interval routing scheme with less than k intervals per link has dilation at least $\lfloor 3D/2 \rfloor - 1$.

In Theorem 3.1 the number of vertices is expressed as a function of the diameter and of the number of intervals, thus yielding more general results. For instance, Theorem 3.1 shows that for every constant k there exists an n-vertex graph of diameter $D = \Theta(n)$ on which every interval routing scheme using k intervals has a dilation at least $\lfloor 3D/2 \rfloor - 1$.

COROLLARY 3.2. For every integer D such that $2 \leq D \leq \Theta(n)$, there exists an n-vertex graph of diameter D on which every interval routing scheme of dilation less than $\lfloor 3D/2 \rfloor - 1$ requires $\Omega(n/(D \log(n/D)))$ intervals.

Corollary 3.2, and Theorem 3.1 can be improved for the particular case $D = 2$. Indeed, in [3] it is shown that there exists graphs of diameter two on which every interval routing scheme using $n/4 - o(n)$ intervals has dilation at least 3.

COROLLARY 3.3. For every constant integer $D \geq 2$, there exists an n-vertex graph of diameter D on which every interval routing scheme of dilation less than $\lfloor 3D/2 \rfloor - 1$ requires $\Omega(n/\log n)$ intervals.

Note that neither Theorem 3.1, Corollary 3.2, or 3.3, says that every interval routing scheme of dilation at most $\lfloor 3D/2 \rfloor - 1$ requires $\Omega(n/\log n)$ intervals for some arbitrary n-vertex graphs of diameter D.

Sketch of the proof of Theorem 3.1.
Basically, we use a similar technique to establish lower bounds of the compactness (shortest paths), and to prove a lower bound of the dilation with a large number of intervals. Our construction is an adaptation of the graph defined in [6]. For simplification, in this sketch of proof we assume that D is an odd fixed constant ≥ 3. We build a graph which has the two following properties:

1) Some vertices require an interval routing scheme using $k = \Omega(n/\log n)$ intervals on some arcs to route along the shortest paths between vertices at distance $t = (D-1)/2$.
2) Any interval routing scheme which does not route along the shortest paths between these vertices has routing path lengths at least $3t$.

Any interval routing scheme of dilation $< 3t$ on this graph requires at least k intervals, or equivalently, any interval routing scheme that uses at most $k-1$ intervals per link has dilation at least $3t$.

The graph construction.
Our construction is function of a boolean matrix M and of an integer D. It is denoted by $G_{M,D}$. More precisely, for every $p \times q$ boolean matrix M and for every integer $D \geq 2$, we define the graph $G_{M,D}$ as follows: we associate with each row i of M, $i \in \{1, \ldots, p\}$, a vertex v_i in $G_{M,D}$. At each column j of M, $j \in \{1, \ldots, q\}$, we associate a pair of vertices $\{a_j, b_j\}$ which are connected by an edge. We set $t = 1$ if $D = 2$, and $t = \lfloor (D-1)/2 \rfloor$ for every $D \geq 3$. For every $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$, we connect v_i to a_j by a path of length t, i.e., such that $\mathrm{dist}(v_i, a_j) = t$, if and only if $m_{i,j} = 0$. Similarly, we connect v_i to b_j by a path of length t if and only if $m_{i,j} = 1$. See Figure 2 for an example. The construction is slightly different for D even and $D \geq 4$. For every even $D \geq 4$, we subdivide each edge

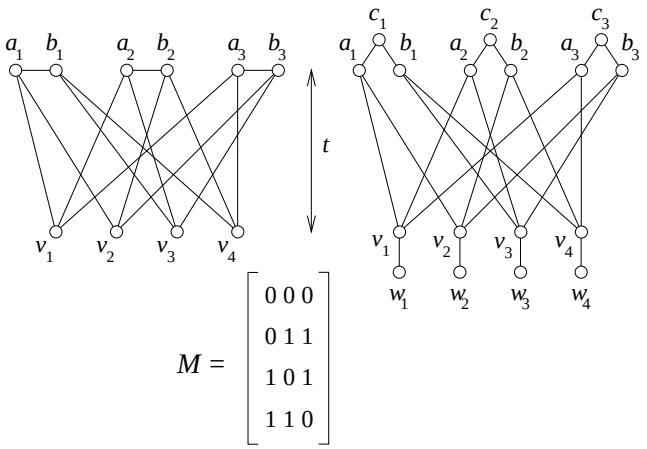

FIGURE 2. The graph $G_{M,D}$: 1) of diameter $D \leq 2t+1$, and 2) of diameter $D \leq 2t+2$.

a_j, b_j with a new vertex c_j, and with the two new edges: $\{a_j, c_j\}$ and $\{c_j, b_j\}$. Also, we add p vertices of degree 1, w_i, for $i \in \{1, \ldots, p\}$. The vertex w_i is connected to v_i.

We define the *compactness* of a boolean matrix M as the smallest integer k such that there exists a matrix obtained by row permutation of M having at most k blocks of consecutive ones per column. The first and the last entry of a column are considered as consecutive. For example, the matrix M described on Figure 2 has its first and its third column composed of one block of consecutive ones, whereas its second column is composed of two blocks (each block being composed of only one 1-entry). The reader can check that the compactness of M is 2, as whatever one permutes the rows of M, there exists at least one column with two blocks of consecutive ones. Therefore, in this example, any interval routing scheme using only one interval on all the arcs of the form (a_j, b_j) cannot optimally reach all the v_i's, and should have a dilation at least $3t$. In fact, every vertex labeling corresponds to a row permutation of M (according to the relative order of the labels of v_i's) and, since an arc (a_j, b_j) belongs to a shortest path to a vertex v_i if and only if $m_{i,j} = 1$, every interval associated to an arc (a_j, b_j) corresponds (according to labels of the v_i's it contains) to at most one block of consecutive one entries in column j of M.

Without loss of generality we can always assume that in M there is no column whose entries are all equal to 0 or all equal to 1, since a column does not influence the compactness of M. The graph obtained from $G_{M,D}$ by contraction of the edges $\{a_j, b_j\}$, $j \in \{1, \ldots, q\}$, is a complete bipartite graph $K_{p,q}$ if $t = 1$, and a multiglobe graph (see [8, 14, 13]) if $t > 1$, i.e., a subdivision of $K_{p,q}$.

In the sequel, columns of boolean matrices are seen as binary strings. log-functions are assumed in base two. The following lemmas are exploited in the proof of Theorem 3.1. We use the notation $f(n) \sim g(n)$ to express the fact that $f(n)$ and $g(n)$ are asymptotically equivalent, i.e., $\lim_{n\to\infty} f(n)/g(n) = 1$.

LEMMA 3.4. Let p, q be two sufficiently large integers. Let $\mathcal{M}$ be the set of $p \times q$ boolean matrices having $\lfloor p/2 \rfloor$ 1-entries per column. Let $\mathcal{M}_1$ be the subset of matrices of $\mathcal{M}$ such that all the rows are pairwise non complemented. Let $\mathcal{M}_2$ be the subset of matrices of $\mathcal{M}$ such that for every pair of columns the $2 \times p$ matrix composed of the pair of columns contains the submatrix[5] $\left[\begin{smallmatrix} 0\,0\,1\,1 \\ 0\,1\,0\,1 \end{smallmatrix}\right]$ up to a column permutation. Then, if $p = o(2^{q/2})$, and $q = o(2^{p/4})$ then $|\mathcal{M}_1 \cap \mathcal{M}_2| \sim |\mathcal{M}|$.

This is a consequence of a result proved in [5]. Hereafter, we set $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$. The matrix M depicted on Figure 2 belongs to $\mathcal{M}_0$. The will see later that the graphs $G_{M,2}$ built from any $M \in \mathcal{M}_0$ have diameter 2 exactly.

LEMMA 3.5. For every sufficiently large integer p, there exist a constant $\alpha \leq 416$ and a $p \times \lfloor \alpha \log p \rfloor$ matrix of $\mathcal{M}_0$ of compactness at least $p/5$.

Proof. We use a counting argument which can be formalized by the Kolmogorov Complexity (see [9] for an introduction). Basically, the Kolmogorov Complexity of an individual object X is the length (in bits) of the smallest program, written in a fixed programming language, which prints X and halts. A simple counting argument allows to say that no program of size less than K can print certain X_0 taken from a set of more than 2^K elements.

Let $\mathcal{M}$ be the set of $p \times q$ boolean matrices with $\lfloor p/2 \rfloor$ 1-entries per columns. Let us begin to show that compactness of some matrices of $\mathcal{M}$ is linear in p for $q = \Theta(\log p)$, then we extend the result to $\mathcal{M}_0$.

For every $M \in \mathcal{M}$, we define $\mathrm{cl}(M)$ the subset of the matrices of $\mathcal{M}$ obtained by row permutation of M. There exists a matrix $M_0 \in \mathcal{M}$ such that all the matrices of $\mathrm{cl}(M_0)$ have a Kolmogorov Complexity at least $C = \log|\mathcal{M}| - \log(p!) - 3\log p$. Indeed, by contradiction let $M_0' \in \mathrm{cl}(M_0)$ be a matrix of Kolmogorov Complexity $C' < C$, for any $M_0 \in \mathcal{M}$. M_0 may be described by a pair (i_0, M_0'), where i_0 is the index of the row permutation of M_0' into M_0. Such an index can be coded by at most $\log(p!) + 2\log p + O(1)$ bits. $2\lceil \log p \rceil$ bits are sufficient to describe p in a self-delimiting way. Hence the Kolmogorov Complexity of M_0 would be at most $C' + \log(p!) + 2\log p + O(1) < \log|\mathcal{M}|$, that is impossible for any matrix $M_0 \in \mathcal{M}$ by the Pigeon Hole Principle.

$|\mathcal{M}| = \binom{p}{\lfloor p/2 \rfloor}^q = \Theta(2^p/\sqrt{p})^q$. By the Stirling's formula $\log|\mathcal{M}| = pq - O(q\log p)$, and $\log(p!) = p\log p - O(p)$. Hence,

$$C = \log|\mathcal{M}| - \log(p!) - 3\log p \quad (1)$$
$$= pq - p\log p - O(p + q\log p)\,. \quad (2)$$

All the matrices of $\mathcal{M}$ have q columns, each one of Kolmogorov Complexity bounded by $p + O(1)$. Therefore,

[5] A is a submatrix of B if A can be obtained from B by removing some columns and rows.

there exists a matrix M_0 such that every matrix obtained by row permutation of M_0 has a column of Kolmogorov Complexity at least

$$C_0 \;=\; \frac{C}{q} - 2\log p \;=\; p - \frac{p\log p}{q} - O\left(\frac{p}{q} + \log p\right). \quad (3)$$

The term $2\log p$ codes the length of the description of such a column in a self-delimiting way. From [9](Theorem 2.6.1, page 162), every binary string of length p bits, and of Kolmogorov Complexity at least $p - \delta(p)$ contains at least

$$\frac{p}{4} - \sqrt{(\delta(p) + c)p\,\frac{3}{2}\ln 2} \quad (4)$$

occurrences of 01-sequences, for any *deficiency* function δ, and some constant c. Since each 01-sequence in a binary string starts necessarily a new block of consecutive ones, we get a lower bound on the number of blocks of consecutive ones for such strings.

By choosing $q = \lfloor \alpha \log p \rfloor$, Equation (3) provided

$$C_0 \;=\; p - \frac{p}{\alpha} - O\left(\frac{p}{\log p}\right) \;>\; p - \delta(p),$$

$$\text{with } \delta(p) \;=\; \frac{p}{\alpha} + o(p).$$

By Equation (4), it follows that M_0 has a compactness at least

$$\frac{p}{4} - \sqrt{(1 + o(1))\; p^2\,\frac{3}{2\alpha}\ln 2} = \quad (5)$$

$$p\left(\frac{1}{4} - \sqrt{\frac{3}{2\alpha}\ln 2}\right) - o(p) \;>\; \frac{p}{5}, \text{ for } \alpha \;=\; 416. \quad (6)$$

Let us show that the result holds also for the compactness of some matrices of $\mathcal{M}_0$. We remark that the lower bound in the form $p/5$ we proved on the compactness of M_0 depends of the cardinality of $\mathcal{M}$ only (see Equation (1), (3), (4) and (6)). From Lemma 3.4, since $p = o(2^{q/2})$ implies $p = o(2^{208\log p}) = o(p^{208})$, and $q = o(2^{p/4})$ because $q = O(\log p)$, we get $|\mathcal{M}_0| \sim |\mathcal{M}|$. Clearly it implies that $\log|\mathcal{M}_0| = \log|\mathcal{M}| + o(1)$, and thus Equations (1), (3), (4), and (6) hold for $\mathcal{M}_0$ as well, that completes the proof. ∎

Remark. The proof of Lemma 3.5 is not constructive. As a result, we can prove only the existence of such a worst-case graph $G_{M,D}$.

Proof of Theorem 3.1.
Let D be an integer ≥ 2, let p be an integer large enough, and let $M \in \mathcal{M}_0$ be a matrix satisfying Lemma 3.5. We consider the graph $G_{M,D}$. Let t be the length of the chains between the vertices $\{a_j, b_j\}$'s and the vertices v_i's. By the construction of $G_{M,D}$, $t = 1$ if $D = 2$, and $t = \lfloor (D-1)/2 \rfloor$ otherwise. Let us remark first that for every integer $D' > D$ and for every graph G of diameter D on which every interval routing scheme using k intervals has a dilation δ, there exists a graph G' of diameter D' on which every interval routing scheme using k intervals has a dilation at least δ (not a function of D'). Indeed, to obtain G' it is sufficient to add a path of length $D' - D$ to one vertex of G which is of eccentricity D. Hence, to prove a lower bound for the dilation of graphs of diameter D it is sufficient to prove a lower bound for dilation in graphs of diameter at most D.

Fact 1. For odd $D \geq 3$, the diameter of $G_{M,D}$ is at most $2t + 1$.

By construction $G_{M,D}$ has no vertex w_i, and no vertex c_j. Let us show that, for every pair of vertices x and x', $\mathrm{dist}(x, x') \leq 2t + 1$. From the particular shape of the matrices of $\mathcal{M}_0$, no column is the string 0^p or the string 1^p. Thus, for $p \geq 2$, all the vertices have degree at least 2. Let us show that there is a cycle of length at most $4t + 2$ that cuts x and x'. x (respectively x') belongs to a chain of length t that directly connects a vertex, namely v_i (respectively $v_{i'}$), to a vertex $\alpha_j \in \{a_j, b_j\}$ (respectively $\alpha_{j'} \in \{a_{j'}, b_{j'}\}$). It follows that x and x' belongs to a cycle $(x, \alpha_j, \overline{\alpha_j}, v_{i'}, x', \alpha_{j'}, \overline{\alpha_{j'}}, v_i, x)$, where $\overline{\alpha_j}$ denotes the complement of α_j in $\{a_j, b_j\}$ (similarly for $\overline{\alpha_{j'}}$). Its length is clearly bounded by $4t + 2$. It follows that $\mathrm{dist}(x, x') \leq (4t + 2)/2 = 2t + 1$. Therefore, for odd $D \geq 3$, $G_{M,D}$ has a diameter bounded by $D \geq 2t + 1$.

Fact 2. For any interval routing scheme $R = (\mathcal{L}, \mathcal{I})$ on $G_{M,D}$, for every arc (a_j, b_j) and for every vertex v_i, if $m_{i,j} = 1$ and $\mathcal{L}(v_i) \notin I_{(a_j,b_j)}$, or if $m_{i,j} = 0$ and $\mathcal{L}(v_i) \in I_{(a_j,b_j)}$, then the dilation of R is at least $3t$.

This results from the fact that there is no path shorter than $2t$ between two any vertices $v_i \neq v_{i'}$. Moreover any wrong decision for routing from a_j to v_i induces a path that cuts a vertex $v_{i'} \neq v_i$ before to reach v_i.

Fact 3. Let k be the compactness of M and $R = (\mathcal{L}, \mathcal{I})$ be any interval routing scheme on $G_{M,D}$. If R uses less than k intervals per arc, then its dilation is at least $3t$.

Assume R is fixed, and uses only $k - 1$ intervals. Let j_0 be a column of M composed of at least k blocks of consecutive ones. Such a column exists because the compactness of M is k. Let us consider the sequence $u = (u_1, \ldots, u_p)$ defined by: for every $i \in \{1, \ldots, p\}$, $u_i = 1$ if $\mathcal{L}(v_i) \in I_{(a_{j_0},b_{j_0})}$, and $u_i = 0$ otherwise. Since the set $I_{(a_{j_0},b_{j_0})}$ is composed of at most $k-1$ intervals, u is composed of at most $k-1$ blocks of consecutive ones. Thus the column j_0 and the sequence u differ in at least one place. Set i_0 the index such that $m_{i_0,j_0} \neq u_{i_0}$. If $u_{i_0} = 1$, then $\mathcal{L}(v_{i_0}) \in I_{(a_{j_0},b_{j_0})}$ and $m_{i_0,j_0} = 0$. If $u_{i_0} = 0$, then $\mathcal{L}(v_{i_0}) \notin I_{(a_{j_0},b_{j_0})}$ and $m_{i_0,j_0} = 1$. We conclude by applying Fact 2.

Fact 4. For every even $D \geq 4$, the diameter of $G_{M,D}$ is at most $2t+2$. Moreover, if R uses less than k intervals per arc, then its dilation is at least $3t + 2$.

Note that for $D = 2$ we do not need Fact 4, because actually the statement of Theorem 3.1 holds for every

graph of diameter 2.

The graph $G_{M,D}$ has in this case the vertices w_i of degree 1 and a path $\{a_j, c_j, b_j\}$ between a_j and b_j. Let x, x' be two non-w_i vertices. Similarly to Fact 1, there exists a cycle of length at most $4t+4$ that cuts x and x'. Hence $\mathrm{dist}(x, x') \leq 2t+2$. Let us show that $\mathrm{dist}(v_i, v_{i'}) \leq 2t$. Indeed, $M \in \mathcal{M}_0$ has the property that the rows i and i' of M are not complemented. Thus there exists j such that $m_{i,j} = m_{i',j}$. It follows that either a_j either b_j is connected by a path of length t to v_i and $v_{i'}$. Hence $\mathrm{dist}(w_i, w_{i'}) \leq 2t+2$. Let us show that $\mathrm{dist}(x, v_i) \leq 2t+1$. Assume that x belongs to a path connecting $\alpha_j \in \{a_j, b_j\}$ to $v_{i'}$. Let $t_x \geq 0$ be the distance between x and $v_{i'}$. There are two paths between x and v_i: one through $v_{i'}$ of length $t_x + 2t$, and another one through α_j of length at most $t - t_x + 2 + t$. Thus $\mathrm{dist}(x, v_i) \leq \min\{t_x + 2t, 2t + 2 - t_x\} = 2t + \min\{t_x, 2 - t_x\} = 2t+1$. It follows that $\mathrm{dist}(x, w_i) \leq 2t+2$, and thus the diameter of $G_{M,D}$ is bounded by $2t+2 = D$ for even $D \geq 4$. The dilation of R, using less than k intervals, is clearly two more longer than the case of odd diameter, because: 1) for a wrong routing path we start with a new edge of the form $\{c_j, a_j\}$ or $\{c_j, b_j\}$, and 2) the unique shortest path from c_j to any w_i cuts v_i.

The order of $G_{M,D}$ is $n \leq 2p + 2q + pq(t-1) + c$, where $q = \Theta(\log p)$ and $c \in \{0, 1, 2\}$ is the number of vertices added to have a diameter D exactly. We have $t = \Theta(D)$, and by Lemma 3.5, the compactness of M, k, satisfies $k = \Theta(p)$. Therefore, for every $D \geq 2$, $n = O(Dk \log k)$. By Fact 3, for odd D, the dilation is $3t = 3\lfloor (D-1)/2 \rfloor = 3D/2 - 3/2$, whereas by Fact 4, for even D, the dilation is $3t+2 = 3\lfloor (D-1)/2 \rfloor + 2 = 3(D/2-1)+2 = 3D/2-1$. Therefore, in all the cases, the dilation is at least $\lfloor 3D/2 \rfloor - 1$, that completes the proof. ∎

Theorem 3.1 allows to establish a trade-off between the order, the diameter of the graph, and the number of intervals required for a dilation $\lfloor 3D/2 \rfloor - 1$. For instance, as already observed the matrix M of Figure 2 has a compactness 2. Then, Fact 3 of Theorem 3.1 shows that the worst-case graph of Figure 2, which is of maximum degree 3 and of diameter $D \sim n/6$, has a dilation $\lfloor 3D/2 \rfloor - 1 \sim n/4$ for 1 interval. However in general, the graph $G_{M,D}$ has a maximum degree $\Theta(p+q)$ which is $\Theta(n/(D \log(n/D)))$, for $D \geq 5$.

4. A BOUNDED DEGREE WORST-CASE

We saw that the degree of our construction is bounded by $\Theta(n/(D \log(n/D)))$, for every $D \geq 5$. We adapt our previous construction to show a lower bound for dilation in networks of degree bounded by d, however with a little degradation of the number of intervals.

THEOREM 4.1. For all integers $d \geq 3$, $k \geq 2$, and $D \geq 3\log_{d-1}(k) + 31$, there exists a graph of diameter D, maximum degree d, and with $n = O(Dk \log k)$ vertices on which every interval routing scheme with less than k intervals per link has dilation at least $3D/2 - 4.5\log_{d-1}(k) - 48$.

Proof. In the following proof we make no attempt in the optimization of the constants. If $d = O(n/(D \log(n/D)))$ one can use directly Theorem 3.1. In particular, for $k = 2$ and $d = 3$ one can use the matrix given in Figure 2 that gives a better construction.

We consider a $p \times q$ boolean matrix $M = (m_{i,j})$ satisfying Lemma 3.5. Let k be its compactness. Hence its compactness is $k = \Theta(p)$, and $q = \Theta(\log p)$. by Lemma 3.5, we have $p \leq 5k$, and $q \leq 416\log(5k)$.

Let T_l be a $(d-1)$-ary tree composed of l leaves and having a minimum number of vertices such that all the leaves are at the same distance from the root. We denote by h_l the height of T_l. Such a tree can be built from a complete $(d-1)$-ary tree of height $h_l - 1$, by adding one more level of exactly l leaves. Vertices which are not along a path from such leaves to the root can be deleted. The height h_l of T_l must satisfy $(d\text{–}1)^{h_l - 1} < l \leq (d\text{–}1)^{h_l}$, i.e., $h_l = \lceil \log_{d-1} l \rceil$, because in a complete k-ary tree there are k^h vertices at distance h from the root. The order of T_l is:

$$|T_l| \;\leq\; \sum_{i=0}^{h_l - 1} (d-1)^i + l \;=\; \frac{(d-1)^{h_l} - 1}{d-2} + l \;<\; 2l,$$

since $d \geq 3$. Hence $|T_l| = \Theta(l)$. Let $\delta = 2(h_p + h_q)$. We have

$$\delta \;\leq\; 2\left(\lceil \log_{d-1}(5k) \rceil + \lceil \log_{d-1}(416\log(5k)) \rceil\right) .$$

Since $k \geq 2$, and $d \geq 3$, we remark that $\delta \leq 3\log_{d-1}(k) + 29$. Note that $D - \delta \geq 2$. So, the graph $G_{M,D-\delta}$ defined in Section 3 can be considered.

Let us consider $G_{M,D-\delta}$ that we modify as follows: for every $i \in \{1, \ldots, p\}$ a copy of T_q is rooted in v_i, and for every $j \in \{1, \ldots, q\}$ a copy of T_p is rooted in a_j. All the paths previously connected to v_i are now connected to a leaf of the tree T_q associated to v_i, in such way that each leaf is connected to a unique path. Similarly, paths connected to a_j are moved to a unique leaf of a_j. The same transformation is applied to all the vertices b_j. See Figure 3.

The graph obtained is denoted by $G_{M,D,d}$ and is of degree bounded by d. By construction, v_i such that $m_{i,j} = 0$ (respectively $m_{i,j} = 1$) are at distance $t + h_p + h_q$ from a_j (respectively from b_j), where t is the common length of the paths between the vertices v_i and a_j in $G_{M,D-\delta}$. Here $t = \lfloor (D - \delta - 1)/2 \rfloor$, thus $t > D/2 - 1.5\log_{d-1}(k) - 16$.

From the remark in the proof of Theorem 3.1, it is sufficient to show that the diameter of $G_{M,D,d}$ is at most D. The graph $G_{M,D-\delta}$ has a diameter at most $D - \delta$. Every shortest path between any two vertices of $G_{M,D,d}$ cuts at most two leaves of the T_p trees and cuts at most two leaves of the T_q trees. Thus the diameter of $G_{M,D,d}$ is bounded by $D - \delta + 2h_p + 2h_q = D$.

Let k be the compactness of M. Similarly to Fact 3 of the proof of Theorem 3.1, any interval routing scheme

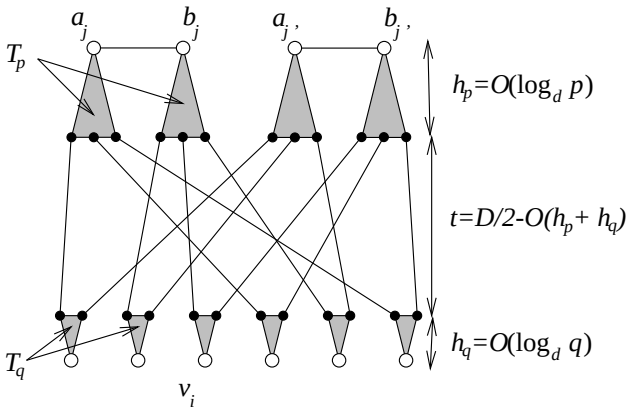

FIGURE 3. Adaptation of the graph $G_{M,D}$ for graphs of maximum degree d.

using less than k intervals routes messages from some a_{j_0} to v_{i_0} along a path of length at least $3t$. Therefore, every interval routing scheme on $G_{M,D,d}$ of dilation less than $3t$ requires k intervals.

The order of $G_{M,D,d}$ is $n = p|T_q| + q|T_p| + (t-1)pq = O(tpq) = O(Dk\log k)$ because $t = O(D)$, $p = \Theta(k)$, and $q = O(\log p)$. The dilation is at least $3t$, i.e., at least $3D/2 - 4.5\log_{d-1}(k) - 48$, that completes the proof. ∎

COROLLARY 4.2. For all integers d and D such that $3 \leq d \leq \Theta(n)$, and $\Theta(\log_d n) \leq D \leq \Theta(n)$, there exists an n-vertex graph of diameter D and maximum degree d on which every interval routing scheme of dilation $3D/2 - O(\log_d n)$ requires $k = \Omega(n/(D^2\log d))$ intervals.

Proof. Because $D = \Omega(\log_d k)$, we can apply Theorem 4.1 that shows that $n = O(Dk\log k)$. Since $D = \Omega(\log_d k)$ implies $\log k = O(D\log d)$, we have $n = O(D^2 k\log d)$. It turns out that $k = \Omega(n/(D^2\log d))$. The dilation is at least $3D/2 - O(\log_d k) \geq 3D/2 - O(\log_d n)$. ∎

COROLLARY 4.3. For every integer D large enough, there exists a graph of diameter D and maximum degree 3 on which every interval routing scheme of dilation $3D/2 - o(D)$ requires $k = \Omega(n/\log^{2+\varepsilon} n)$ intervals, for every constant $\varepsilon > 0$, where n is the number of vertices of the graph.

Proof. Choose n such that $D = \Theta(\log^{1+\varepsilon/2} n)$, and apply Corollary 4.2 with $d = 3$. We get $O(\log_d n) = o(D)$, hence the dilation is $3D/2 - o(D)$, and $k = \Omega(n/\log^{2+\varepsilon} n)$. ∎

We can increase the number of intervals if the degree is unbounded. For instance, for degree at most $\log n$, and choosing n such that $D = \Theta(\log n/\sqrt{\log\log n})$, we can prove by Corollary 4.2 a dilation of $3D/2 - o(D)$ for $k = \Omega(n/\log^2 n)$ intervals. More generally, we can choose $D = \Theta(\log n/\sqrt{\log d})$ and prove a dilation of $3D/2 - O(D/\sqrt{\log d})$, which is $3D/2 - o(D)$ for unbounded d, and with $k = \Omega(n/\log^2 n)$ intervals.

5. CONCLUSION

We essentially shows in this paper that there is no hope do guarantee, for arbitrary network, interval routing schemes with a dilation shorter than $\lfloor 3D/2 \rfloor - 1$ with less than $\Omega(n/(D\log(n/D)))$ intervals per link. So, it is still open to find a tight bound on the smallest number of intervals required for a dilation exactly $\lfloor 3D/2 \rfloor - 1$ or higher. For instance, we do not know whether the upper bound of $O(\sqrt{n\log n})$ intervals for dilation at most $\lceil 3D/2 \rceil$ is tight or not (cf. [8]). It would also interesting to find trade-offs for particular class of graphs, like planar graphs. We propose the two following problems, already mentioned in [4]:

Open question 1: Is there a constant k_0 such that every planar graph of diameter D supports an interval routing scheme of dilation at most $\lceil 3D/2 \rceil$ and using at most k_0 intervals?

Open question 2: Are there constants k_0 and s_0 such that every graph supports an interval routing scheme with at most k_0 intervals and such that the ratio between the routing path length and the distance in the graph between any pair of nodes (the stretch factor) is bounded by s_0?

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