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The Compactness of Interval Routing

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Abstract

The *compactness* of a graph measures the space complexity of its shortest path routing tables. Each outgoing edge of a node x is assigned a (pairwise disjoint) set of addresses, such that the unique outgoing edge containing the address of a node y is the first edge of a shortest path from x to y. The complexity measure used in the context of interval routing is the minimum number of intervals of consecutive addresses needed to represent each such set, minimized over all possible choices of addresses and all choices of shortest paths. This paper establishes asymptotically tight bounds of $n/4$ on the compactness of a n-node graph. More specifically, it is shown that every n-node graph has compactness at most $n/4+o(n)$, and conversely, there exists an n-vertex graph whose compactness is $n/4-o(n)$. Both bounds[1] improve upon known results.

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[1]A preliminary version of the lower bound has been partially published in the proceedings of MFCS '97.

1 Introduction

An *interval routing scheme* is a way of implenting routing schemes on arbitrary networks. It is based on representing the routing table stored at each node in a compact manner, by grouping the set of destination addresses that use the same output port into intervals of consecutive addresses. A possible way of representing such a scheme is to use a connected undirected labeled graph, providing the underlying topology of the network. The addresses are assigned to the nodes, and the sets of destination addresses are assigned to each endpoint of the edges. As originally introduced in [17], the scheme required each set of destinations to consist of a single interval. This scheme was subsequently generalized in [18] to allow more than one interval per edge.

Formally, consider an undirected n-node graph $G = (V, E)$. Since G is undirected, each edge $\{u, v\} \in E$ between u and v can be viewed as two arcs, i.e., two ordered pairs, (u, v) and (v, u). The graph G is said to support an *interval routing scheme* (*IRS* for short) if there exists a labeling $\mathcal{L}$ of V, which labels every node by a unique integer taken from $\{1, \ldots, n\}$, and a labeling $\mathcal{I}$ of the outgoing edges, which labels every exit endpoint of each arc of E by a subset of $\{1, \ldots, n\}$, such that between any pair of nodes $x \neq y$ there exists a path $x = u_0, u_1, \ldots, u_t = y$ satisfying that $\mathcal{L}(y) \in \mathcal{I}(u_i, u_{i+1})$ for every $i \in \{0, \ldots, t-1\}$. The resulting routing scheme, denoted $\mathcal{R} = (\mathcal{L}, \mathcal{I})$, is called a *$k$-interval routing scheme* (*k-IRS* for short) if for every arc (u, v), the collection of labels $\mathcal{I}(u, v)$ assigned to it is composed of at most k intervals of consecutive integers (1 and n being considered as consecutive).

The standard definition of k-IRS assumes a *single* routing path between any two nodes. It therefore forces any two incident arcs $e \neq e'$ to have disjoint labels, i.e., $\mathcal{I}(e) \cap \mathcal{I}(e') = \emptyset$. Here we assume that a given destination may belong to many labels of different arcs incident to a same node. This freedom allows us to implement some adaptive routing schemes, and code for example the full shortest path information, as does the boolean routing scheme [4]. Our upper and lower bounds apply also to the recent extension of interval routing known as *multi-dimensional interval routing* [3].

To measure the space efficiency of a given IRS, we use the compactness measure, defined as follows. The *compactness* of a graph G, denoted by IRS(G), is the smallest integer k such that G supports a k-IRS of single shortest paths, that is, a k-IRS that provides only one shortest path between any pair of nodes.

If the degree of every node in G is bounded by d, then a k-IRS for G is required to store at most $O(dk \log n)$ bits of information per node (as each set $\mathcal{I}(e)$ can be coded using $2k \log n$ bits[2]), and $O(km \log n)$ bits in total, where m

[2] A more accurate coding allows to use only $O(dk \log(n/k))$ bits per node, cf. [7].

is the total number of edges of the graph. The compactness of a graph is an important parameter for the general study of the compact routing, whose goal is to design distributed routing algorithms with space-efficient data structures for each router.

Figure 1 shows an example of a 2-IRS on a graph G. For instance, arc $(7, 1)$ is assigned two intervals: $\mathcal{I}(7, 1) = \{1, 2, 5\}$. Whereas it is quite easy to verify that this labeling is a single shortest path for G, it is more difficult to check whether G has compactness 1. Actually, in [9] it is shown that $\mathrm{IRS}(G) = 2$. Recently, it was proven in [1] that for general graphs, the problem of deciding whether $\mathrm{IRS}(G) = 1$ is NP-complete.

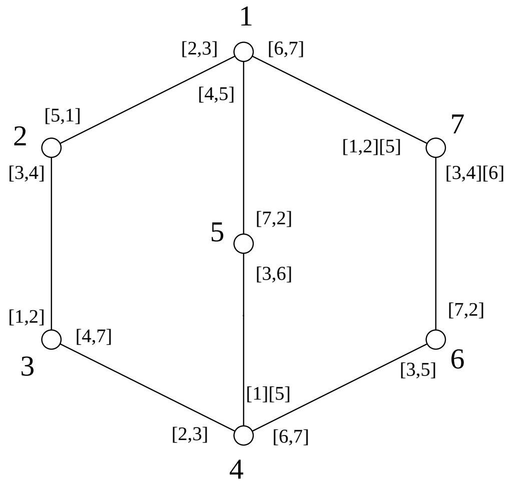

Figure 1: A 2-IRS for a graph G.

The compactness of many graph classes has been studied. Its value is 1 for trees [17], outerplanar graphs [6], hypercubes and meshes [18], r-partite graphs [12], interval graphs [16], and unit-circular graphs [5]. It is 2 for tori [18], at most 3 for 2-trees [16], and at most $2\sqrt{n}$ for chordal rings on n nodes [15] (see [7] for a survey of recent State-of-the-Art). Finally, it has been proved that compactness $\Theta(n)$ might be required [9].

The next section presents the results of the paper. In Section 3 we prove that $n/4 + o(n)$ intervals are always sufficient, and in Section 4 that $n/4 - o(n)$ intervals might be required. We conclude in Section 5.

2 The Results

Clearly, the compactness of a graph cannot exceed $n/2$, since any set $\mathcal{I}(e) \subset \{1, \ldots, n\}$ containing more than $n/2$ integers must contain at least two consecutive integers, which can be merged into a same interval. On the other hand it has been proved in [9] that for every $n \geq 1$ there exists a n-node graph of compactness at least $n/12$, and $n/8$ for every n power of 2.

In this paper we close this gap, by showing that $n/4$ is asymptotically a tight bound for the compactness of n-node graphs. More specifically:

Theorem 1 *Every n-node graph G satisfies*

$$\mathrm{IRS}(G) < \frac{n}{4} + \frac{1}{4}\sqrt{2n \ln (3n^2)}.$$

Theorem 2 *For every sufficiently large integer n, there exists an n-node graph G such that*

$$\mathrm{IRS}(G) > \frac{n}{4} - 1.72\, n^{2/3} \ln^{1/3} n.$$

Moreover, G has diameter 2, maximum degree at most $n/2$, and fewer than $1.15\, n^{5/3} \ln^{1/3} n$ edges, and every single k-IRS on G with $k < \mathrm{IRS}(G)$ contains some routing path of length at least 3.

We later show that both the upper and the lower bounds hold even if the single and/or shortest path assumptions are relaxed.

Theorem 1 improved directly the results of [5, Theorem 11], of [3, Theorem 2], and also a result of [2, Theorem 9].

The lower bound is proved using Kolmogorov complexity. As a result, only the existence of such a worst-case graph G can be proved. Moreover, the bound gives an asymptotic bound since the Kolmogorov complexity is defined up to a constant. This is in contrast to the technique of [9], which gave explicit recursive constructions of worst-case graphs of compactness $n/12$, for every $n \geq 1$.

3 The Upper Bound

The basic idea for the upper bound, and partially for the lower bound, is to give a boolean matrix representation $M(\mathcal{R})$ for a given k-IRS $\mathcal{R} = (\mathcal{L}, \mathcal{I})$ on a graph $G = (V, E)$. Recall that for each arc e, $\mathcal{I}(e)$ is the set of addresses that labels the arc e. Let u_e be the characteristic sequence of the subset $\mathcal{I}(e)$ in $\{1, \ldots, n\}$, namely, the ith element of u_e is 1 if $i \in \mathcal{I}(e)$, and 0 otherwise. It is easy to see that there is a one-to-one correspondence between the intervals of $\mathcal{I}(e)$ and the blocks of consecutive ones in u_e. The number of blocks of consecutive ones in u_e

can be seen as the occurence number of 01-sequences[3] in the binary vector u_e. By collecting all the u_e's sequences in order to form a boolean matrix $M(\mathcal{R})$ of dimensions $n \times 2|E|$, the problem of finding a node-labeling $\mathcal{L}$ of G such that each set $\mathcal{I}(e)$ is composed of at most k intervals is equivalent to the problem of finding a row permutation of $M(\mathcal{R})$ such that every column has at most k blocks of consecutive ones.

Throughout this section, M denotes a boolean matrix of n rows and p columns. For every column u of M, and for every row permutation π, we denote by $c(u, \pi)$ the number of blocks of consecutive ones in the column u under π. For every matrix M, define the *compactness* of M, denoted $\mathrm{comp}(M)$, as the smallest integer k such that there exists a row permutation π of M satisfying, for every column u of M, $c(u, \pi) \leq k$.

The following theorem is the key of the proof of Theorem 1.

Theorem 3 *Let M be an $n \times p$ boolean matrix, $p < e^{n/2}/n$, let u be a column of M, and let $A_u(k) = \{\pi \mid c(u, \pi) = k\}$ be the set of row permutations of M that provides k blocks of consecutive ones for the column u. Then for every integer k in the range $n/4 + (1/4)\sqrt{2n \ln{(pn)}} < k \leq n/2$,*

$$|A_u(k)| < \frac{4n!}{pn} .$$

Proof. Let us consider a column u of M and an integer k. Let a (respectively, b) be the number of 0's (resp., 1's) of u. Clearly, if $a < k$ or $b < k$, the theorem holds because in this case $A_u(k) = \emptyset$. Hence suppose $a, b \geq k$, with $a + b = n$. There are $a!$ permutations of the rows $\{x_1, \ldots, x_a\}$ containing 1, and $b!$ permutations of the rows $\{y_1, \ldots, y_b\}$ containing 0 in u, and each such pair of permutations creates a different and disjoint set of permutations in $A_u(k)$. Moreover, each of the $a!$ permutations needs to be broken into k non-empty blocks, which can be done in $\binom{a}{k}$ ways, and similarly for the $b!$ permutations of the rows $\{y_1, \ldots, y_b\}$. Each partitioned pair can be merges, alternating a block of 1's and a block of 0's, in order to yield a permutation in $A_u(k)$. Overall, $|A_u(k)| = a!\binom{a}{k}b!\binom{b}{k}$, and we need to show that

$$a!\binom{a}{k}b!\binom{b}{k} \; < \; \frac{4n!}{pn} . \tag{1}$$

Using Formula (9.91) of [11] on page 481, derived from Stirling's formula, we have for every $n \geq 1$,

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} \; < \; n! \; < \; \gamma \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \; ,$$

[3]If u_e does not contain any 0, u_e is composed of exactly one block of consecutive ones.

where $\gamma = e^{1/12-1/360+1/1260} \approx 1.08$. Thus

$$\frac{4n!}{pn} > \frac{n^n e^{-n} 4\sqrt{2\pi n}}{pn} = e^{-n}\, 4\sqrt{2\pi}\, \frac{n^{n-1/2}}{p} . \qquad (2)$$

From Stirling's bound, for every k in the range $0 < k < a$,

$$\binom{a}{k} < \left(\frac{a}{k}\right)^k \left(\frac{a}{a-k}\right)^{a-k} \frac{\gamma}{\sqrt{2\pi}} \sqrt{\frac{a}{k(a-k)}} . \qquad (3)$$

This bound cannot apply for $k = a$. Let us first handle the extremal cases.

Claim 3.1 *Inequality (1) holds for $a = k$, or $b = k$, for every integer k, $0 \leq k \leq n/2$.*

Proof. In both cases assumed in the claim, Inequality (1) is equivalent to

$$k!(n-k)!\binom{n-k}{k} = \frac{(n-k)!^2}{(n-2k)!} < \frac{4n!}{pn^{3/2}} . \qquad (4)$$

The ratio $(n-k)!^2/(n-2k)!$ increases for $0 \leq k \leq n/2$. Indeed, in this range $n-k \geq n-2k$, hence $(n-k)! \geq (n-2k)!$, and therefore $(n-k)!^2 \geq (n-2k)!$. It is thus sufficient to prove Inequality (4) for $k = n/2$, in which case it becomes

$$\left(\frac{n}{2}\right)!^2 < \frac{4n!}{pn} . \qquad (5)$$

Using Stirling's bound, $(n/2)!^2 < (n/2)^n e^{-n} \gamma^2 2\pi n$, and simplifying with the lower bound of Inequality (2), we get that to prove Inequality (5) it suffices to prove

$$\left(\frac{n}{2}\right)^n < c\,\frac{n^{n-3/2}}{p} , \quad \text{or} \quad pn^{3/2} < c\,2^n, \quad \text{where } c = \frac{4}{\gamma^2\sqrt{2\pi}} \approx 1.36 .$$

This last inequality is satisfied for every $n \geq 1$, since $p < e^{n/2}/n$, and $e^{n/2}\sqrt{n} < c\,2^n$ is equivalent to $n/2 < n(\ln 2 - (1/(2n))\ln n) + \ln c$, which is trivial because $(1/(2n))\ln n \leq (1/4)\ln 2$, and $(1-(1/4))\ln 2 \approx 0.51 > 1/2$ and moreover $\ln c > 0$. This completes the proof of Claim 3.1. $\square$

For the remainder of the proof, let us assume that $k < a, b$. Therefore, it is possible to apply the bound of Inequality (3), which gives

$$a!\binom{a}{k} b!\binom{b}{k} < a^a \left(\frac{a}{k}\right)^k \left(\frac{a}{a-k}\right)^{a-k} b^b \left(\frac{b}{k}\right)^k \left(\frac{b}{b-k}\right)^{b-k} \frac{ab\, e^{-n}\gamma^4}{k\sqrt{(a-k)(b-k)}} . \qquad (6)$$

Claim 3.2 *For every integers k, a, b and n such that $0 < k < a$ and $a + b = n$,*

$$\frac{ab}{k\sqrt{(a-k)(b-k)}} \;<\; \frac{3\sqrt{3}}{4}\sqrt{n}\ .$$

Proof. Set $b = n - a$, and let $f(a) = (a-k)(n-a-k)$. Claim 3.2 holds if

$$\frac{ab}{k\sqrt{f(a)}} \;<\; \frac{3\sqrt{3}}{4}\sqrt{n}\ .$$

Observing that $ab \le (a+b)^2/4 = n^2/4$, it suffices to prove that

$$\frac{n^2}{4k\sqrt{f(a)}} \;<\; \frac{3\sqrt{3}}{4}\sqrt{n}\ .$$

Let us lower bound the term $k\sqrt{f(a)}$. Noting that $f(a)$ is symmetric around the point $n/2$, let us assume without loss of generality that $a \le n/2$. In this range $f'(a) = n - 2a \ge 0$. So, in the desired range $f(a)$ attains its minimum where a is minimum, and thus $k\sqrt{f(a)} > k\sqrt{f(k)} = k\sqrt{n-2k}$. Let $f_2(k) = k\sqrt{n-2k}$. ${f_2}'(k) = \sqrt{n-2k} - k/\sqrt{n-2k}$, which is of the same sign that $n - 3k$. Hence in the range $0 < k < n/2$, $f_2(k)$ first decreases until its minimum at the point $n/3$, then increases between $n/3$ and $n/2$. So, $f_2(k) \ge f_2(n/3) = (n/3)^{3/2}$. Therefore

$$\frac{n^2}{4k\sqrt{f(a)}} \;<\; \frac{n^2}{4(n/3)^{3/2}} \;=\; \frac{3\sqrt{3}}{4}\sqrt{n}\ ,$$

which completes the proof of Claim 3.2. □

In view of Claim 3.2, Inequality (6) becomes

$$a!\binom{a}{k}b!\binom{b}{k} \;<\; a^a\left(\frac{a}{k}\right)^k\left(\frac{a}{a-k}\right)^{a-k} b^b\left(\frac{b}{k}\right)^k\left(\frac{b}{b-k}\right)^{b-k} e^{-n}\,\gamma^4\frac{3\sqrt{3}}{4}\sqrt{n}\ .$$

Simplifying and applying the lower bound of Inequality (2), we obtain that to prove Inequality (1) it suffices to show:

$$a^a\left(\frac{a}{k}\right)^k\left(\frac{a}{a-k}\right)^{a-k} b^b\left(\frac{b}{k}\right)^k\left(\frac{b}{b-k}\right)^{b-k} \;<\; \frac{16\sqrt{2\pi}}{\gamma^4\,3\sqrt{3}}\;\frac{n^{n-1}}{p}\ .$$

Noting that $16\sqrt{2\pi}/(\gamma^4\,3\sqrt{3}) \approx 5.57$, it remains to prove that

$$p^{-1}n^{n-1}k^{2k}(a-k)^{a-k}(b-k)^{b-k} - a^{2a}b^{2b} > 0. \qquad (7)$$

Assume that $k_0 < k < a \leq n/2 \leq b$, with $b = n - a$, and with $k_0 = n/4 + (1/4)\sqrt{2n \ln (pn)}$. The case $b \leq a$ is dual, and at most doubles the number of permutations (which is taken in account in the removing of the multiplicative constant 5.57 in Inequality (7)).

Let $f(a) = p^{-1} n^{n-1} k^{2k} (a-k)^{a-k} (n-a-k)^{n-a-k} - a^{2a} (n-a)^{2(n-a)}$.

To establish Inequality (7) and complete the proof, it remains only to show the following lemma.

Lemma 3.3 *$f(a) > 0$ in the range $k_0 < k < a \leq n/2$.*

Proof. Write $f(a) = \exp(A) - \exp(B)$, where

$$\begin{aligned} A &= -\ln p + (n-1)\ln n + 2k \ln k + (a-k)\ln (a-k) + (n-a-k)\ln (n-a-k), \\ B &= 2a \ln a + 2(n-a)\ln (n-a). \end{aligned}$$

Then $f(a) > 0$ if and only if $A - B > 0$. Letting $f_2(a) = A - B$, it remains to prove that $f_2(a) > 0$ in the range $k_0 < k < a \leq n/2$. The first derivative of f_2 is

$$f_2{}'(a) \;=\; \ln\left(\frac{a-k}{n-a-k}\right) + 2\ln\left(\frac{n-a}{a}\right).$$

Claim 3.4 *$f_2{}'(a) \leq 0$ in the range $k_0 < k < a \leq n/2$.*

Proof. It suffices to show that in the range specified in the claim,

$$\frac{a-k}{n-a-k}\left(\frac{n-a}{a}\right)^2 \leq 1,$$

or

$$f_3(a) \;\equiv\; (a-k)(n-a)^2 - (n-a-k)a^2 \;\leq\; 0.$$

This is shown by noting that $f_3(a)$ is increasing in this range, hence its maximum is attained at the point $a = n/2$, where $f_3(n/2) = 0$. To show that $f_3(a)$ is increasing, we need to show that $f_3{}'(a) = 6a^2 - 6an + n^2 + 2nk \geq 0$ in this range. This is shown by noting that $f_3{}'(a)$ is decreasing in this range, hence its minimum is attained at the point $a = n/2$, where $f_3{}'(n/2) = (2k - n/2)n \geq 0$ since $k > k_0 > n/4$. To show that $f_3{}'(a)$ is decreasing, we need to show that $f_3{}''(a) = 12a - 6n \leq 0$ in this range, which is trivial since $a \leq n/2$. This completes the proof of Claim 3.4. $\square$

It follows from Claim 3.4 that $f_2(a)$ is decreasing in this range, and hence its minimum is attained at $a = n/2$. Hence in this range,

$$f_2(a) \;\geq\; f_2\left(\frac{n}{2}\right) \;=\; p^{-1}n^{n-1}\left(\frac{n}{2}-k\right)^{n-2k} \;-\; \left(\frac{n}{2}\right)^{2n}.$$

Consequently, it remains to prove that $f_2(n/2) > 0$ in the desired range. Simplifying, we need to show that $k^{2k}(n-2k)^{n-2k}2^{n+2k} - pn^{n+1} > 0$ in the range $k_0 < k < n/2$. Writing $k = \alpha n$, we need to prove that

$$\begin{aligned}
\left(\alpha^{2\alpha}(1-2\alpha)^{1-2\alpha}2^{1+2\alpha}\right)^n n^n \;&>\; pn^{n+1}, \text{ or}\\
\alpha^{2\alpha}(1-2\alpha)^{1-2\alpha}2^{1+2\alpha} \;&>\; (pn)^{1/n}, \text{ or that}\\
2\alpha\log\alpha + (1-2\alpha)\log(1-2\alpha) + 1 + 2\alpha \;&>\; \frac{\log(pn)}{n},
\end{aligned}$$

in the range $k_0/n < \alpha < 1/2$ (the function log represents logarithm to base 2). Let $g(\alpha) = 2\alpha\log\alpha + (1-2\alpha)\log(1-2\alpha) + 1 + 2\alpha$. It remains to prove the following claim.

Claim 3.5 $g(\alpha) > \dfrac{\log(pn)}{n}$ *in the range* $\dfrac{k_0}{n} < \alpha < \dfrac{1}{2}$.

Proof. Note that $\dfrac{k_0}{n} = \dfrac{1}{4} + \dfrac{1}{4}\sqrt{\dfrac{2\ln(pn)}{n}}$. So, if $p < e^{n/2}/n$, then $k_0/n < 1/2$, thus the range for α is not empty. Moreover,

$$\begin{aligned}
g'(\alpha) \;&=\; 2\log\alpha - 2\log(1-2\alpha) + 2\ ,\\
g''(\alpha) \;&=\; \frac{2}{\alpha\ln 2} + \frac{4}{(1-2\alpha)\ln 2}\ ,\\
g'''(\alpha) \;&=\; -\frac{2}{\alpha^2\ln 2} + \frac{8}{(1-2\alpha)^2\ln 2}\ .
\end{aligned}$$

In the range $1/4 < \alpha < 1/2$, let us show that $g'''(\alpha) > 0$. This happens if

$$\begin{aligned}
\frac{8}{(1-2\alpha)^2\ln 2} \;&>\; \frac{2}{\alpha^2\ln 2}\ , \text{ or}\\
4\alpha^2\ln 2 \;&>\; (1-2\alpha)^2\ln 2, \text{ or}\\
2\alpha \;&>\; 1-2\alpha,
\end{aligned}$$

which is trivial since $\alpha > 1/4$. Moreover $g(1/4) = g'(1/4) = 0$, and $g''(1/4) = 16/\ln 2$. Thus we have the following bound for $g(\alpha)$:

$$g(\alpha) \;>\; \frac{g''(1/4)}{2!}\left(\alpha - \frac{1}{4}\right)^2 \;=\; \frac{8}{\ln 2}\left(\alpha - \frac{1}{4}\right)^2.$$

So, it suffices to take α such that,

$$\begin{aligned}\frac{8}{\ln 2}\left(\alpha-\frac{1}{4}\right)^2 &> \frac{\log(pn)}{n}, \text{ or}\\ \alpha &> \frac{1}{4}+\sqrt{\frac{\ln 2}{8}\frac{\log(pn)}{n}} = \frac{1}{4}+\frac{1}{4}\sqrt{\frac{2\ln(pn)}{n}} = \frac{k_0}{n},\end{aligned}$$

to complete the proof of Claim 3.5. □

This completes also the proof of Lemma 3.3, and subsequently of Theorem 3. □ □

Corollary 3.6 *Let M be an $n \times p$ boolean matrix, $p < e^{n/2}/n$. Then*

$$\mathrm{comp}(M) < \frac{n}{4}+\frac{1}{4}\sqrt{2n\ln(pn)}\ .$$

Proof. We need to show that there exists a row permutation π of M, such that $c(u,\pi) < n/4 + (1/4)\sqrt{2n\ln(pn)}$ for every column u of M. Let us set $k_0 = n/4+(1/4)\sqrt{2n\ln(pn)}$. A permutation π is said to be "bad" if there exists a column u of M such that $c(u,\pi) > k_0$. Let B_u be the set of bad permutations for the column u, i.e.,

$$B_u = \bigcup_{k_0<k\le n/2} A_u(k).$$

The entire set of bad permutations for M is $B = \bigcup_u B_u$, over all the p columns of M. Theorem 3 implies that for every u,

$$|B_u| < \frac{(n/2-k_0)4n!}{pn} < \frac{n!}{p}\ ,$$

because $n/2 - k_0 < n/4$. It follows that $|B| < n!$. Therefore, there is at least one "good" permutation for the rows of M, i.e., a permutation providing at most $\lfloor k_0 \rfloor$ blocks of consecutive ones for each of the columns. We conclude by remarking that $\lfloor k_0 \rfloor < k_0$, since $\ln(pn)$ cannot be an integer for integer $pn > 1$. □

Proof of Theorem 1: Let us consider any node labeling $\mathcal{L}$ of V, and any routing function $\mathcal{R} = (\mathcal{L},\mathcal{I})$ on G, e.g., a single shortest path routing function. Form the $n \times p$ boolean matrix $M(\mathcal{R})$ as explained earlier. By Corollary 3.6 (which is clearly applicable as $p < e^{n/2}/n$), there exists a row permutation π such that $c(u,\pi) < n/4+(1/4)\sqrt{2n\ln(pn)}$ for every column u of M. Permute the labeling of the nodes of V according to π, to obtain a labeling $\mathcal{L}'$ such that the resulting interval routing scheme, $\mathcal{R}' = (\mathcal{L}',\mathcal{I})$, is a q-IRS for

$$q < \frac{n}{4}+\frac{1}{4}\sqrt{2n\ln(pn)}\ , \tag{8}$$

namely, $\mathcal{R}'$ has fewer than q intervals on each arc. Let us show that only $p \leq 3n$ arcs has to be considered.

In the case of single IRS, each destination is assigned to a unique set $\mathcal{I}(e)$ in each node. For each node of degree three or less, we consider all its outgoing edges. Consider a node x of degree greater than three, and let I, J, K be the three largest cardinality sets assigned to outgoing edges of x. Assume that the nodes are relabeled using the permutation π in such a way that all the sets I, J, K are composed respectively of i, j, and k intervals. We remark that $i + j + k < 3n/4 + o(n)$ by Corollary 3.6. Hence all the other sets share at most $n/4$ intervals, and do not need to be considered.

We complete the proof by plugging $p = 3n$ in Inequality (8). $\square$

Remark. The parameter p of Inequality (8) represents the total number of arcs we are required to consider. For graphs with fewer edges one can choose $p = 2|E|$, which is better than $3n$ only for graphs of average degree at most 3. Note that there exists some 3-regular graphs of compactness $\Theta(n)$ [10].

Here we give another application of Theorem 3.

Corollary 3.7 *Let M be an $n \times p$ boolean matrix, $p < e^{n/2}/n^2$, and let π be an arbitrary row permutation of M. With probability at least $1 - n^{-1}$, $c(u, \pi) < n/4 + (1/4)\sqrt{2n \ln(pn^2)}$ for every column u of M.*

Proof. Let M be an $n \times p$ boolean matrix with $p < e^{n/2}/n^2$. Build from M a matrix M' composed of all the p columns of M and completed by $(n-1)p$ other columns, each filled up with 0's. M' has dimensions $n \times pn$. Clearly, the set of "bad" permutations for M' and M is the same. The total set of bad permutations for M' is

$$B = \bigcup_u B_u(k) = \bigcup_u \left(\bigcup_{k_0 < k \leq n/2} A_u(k) \right),$$

where the union is taken over all the pn columns u of M', and $k_0 = n/4 + (1/4)\sqrt{2n \ln(pn^2)}$. Theorem 3 implies that $|B| < n!/n$, noting that $pn < e^{n/2}/n$. We conclude that the number of "good" permutations for M' (hence for M), i.e., providing at most $\lfloor k_0 \rfloor$ blocks of consecutive ones for all the columns, is at least $n! - n!/n$, which is a fraction of $1 - n^{-1}$ of all the row permutations of M. The proof is completed by remarking that $\lfloor k_0 \rfloor < k_0$, for every integer $pn^2 > 1$. $\square$

Therefore, to have a labeling with fewer than $n/4 + O(\sqrt{n \log n})$ intervals on all the edges of G, it suffices to fix a node labeling, and a routing function on G, then to randomly permute the n labels of nodes by choosing a random permutation π of $\{1, \ldots, n\}$.

Note that the previous algorithm applies not only to single shortest path routing schemes, but also to any routing scheme implementable by using interval routing schemes. Thus for every IRS on every graph we can relabel the nodes in order to have at most $n/4 + O(\sqrt{n \log n})$ intervals per arc. It is still unknown whether there exists a polynomial time deterministic IRS construction algorithm that guarantees at most $n/4 + o(n)$ intervals per edges.

We do not know whether the upper bound is reached for certain graphs. However, it is well-known that some small graphs have compactness strictly greater than $n/4$. In [9], it is shown that the example depicted on Figure 1 with 7 nodes and 8 edges, has compactness 2, whereas all graphs of order at most 6 have compactness 1. Note also that the compactness of the Petersen graph is 3, whereas its order is 10, and its size 15.

4 The Lower Bound

The lower bound idea is based on a representation similar to the one used in the upper bound, namely, a boolean matrix M representation of the k-IRS on G. However, this time we need to show that no row permutation of M yields fewer than k blocks of consecutive ones on all the columns. Furthermore, this must be shown for every choice of shortest routing paths. For instance, every $\sqrt{n} \times \sqrt{n}$ grid has compactness 1, using the standard node labeling and single-bend YX-routing paths. Clearly, a different choice of shortest routing paths would increase the number of intervals per edge. That is why we use smaller matrices, say of dimensions $|W| \times |A|$, by considering only a subset of nodes, W, and a subset of arcs, A, where the shortest paths between the tails of the arcs of A and the nodes of W are all unique.

Our worst-case graph construction is a function of a boolean matrix M, denoted G_M. For every $p \times q$ boolean matrix M, define the graph G_M as follows. For every $i \in \{1, \ldots, p\}$, associate with the ith row of M a vertex v_i. For every $j \in \{1, \ldots, q\}$, associate with the jth column of M a pair of vertices $\{a_j, b_j\}$ connected by an edge. In addition, for every $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$, if $m_{i,j} = 0$ we add to G_M an edge connecting v_i to a_j, and otherwise we connect v_i to b_j. Note that the graph obtained from G_M by contracting the edges $\{a_j, b_j\}$, $j \in \{1, \ldots, q\}$ is a complete bipartite graph $K_{p,q}$. It is easy to see that the shortest path from any a_j to any v_i is unique, and is determined by the entry $m_{i,j}$ of M.

For integers p, q, let $\mathcal{M}$ be the collection of $p \times q$ boolean matrices having

$\lfloor p/2 \rfloor$ 1-entries per column. Let $\mathcal{M}_1$ be the subset of matrices of $\mathcal{M}$ such that all the rows are pairwise non complementing, and let $\mathcal{M}_2$ be the subset of matrices of $\mathcal{M}$ such that for every pair of columns the $2 \times p$ matrix composed of the pair of columns contains the sub-matrix[4] $\left[\begin{smallmatrix}0&0&1&1\\0&1&0&1\end{smallmatrix}\right]$ up to column permutation. We next use a direct consequence of a result proved recently in [8]. In the following, $f(n) \sim g(n)$ means that $f(n) = g(n) + o(g(n))$.

Lemma 4.1 (Gavoille, Gengler [8]) *Let p, q be two sufficiently large integers. If $p = o(2^{q/2})$, and $q = o(2^{p/4})$, then $|\mathcal{M}_1 \cap \mathcal{M}_2| \sim |\mathcal{M}|$.*

Throughout the remainder of the paper, we set $\mathcal{M}_0 = \mathcal{M}_1 \cap \mathcal{M}_2$. We will see later that the graphs G_M built from the matrices $M \in \mathcal{M}_0$ have diameter 2 exactly. Furthermore, almost all matrices are in $\mathcal{M}_0$.

We will see that the compactness of M is a lower bound of the compactness of G_M. Here we give a lower bound of the compactness of matrices of $\mathcal{M}_0$.

Lemma 4.2 *For every sufficiently large integers p, q such that $3 \log p \leq q \leq 2^{p/5}$, there exists a $p \times q$ boolean matrix M of $\mathcal{M}_0$ of compactness*

$$\operatorname{comp}(M) \;\geq\; \frac{p}{4} - \sqrt{\frac{3p^2}{2q} \ln p + O\left(\frac{p^2}{q} + p \log p\right)}\;.$$

Proof. We use a counting argument which can be formalized using Kolmogorov complexity (see [14] for an introduction). Basically, the Kolmogorov complexity of an individual object X is the length (in bits) of the smallest program, written in a fixed programming language, which prints X and halts. A simple counting argument allows us to argue that no program of length less than K can print a certain X_0 taken from a set of more than 2^K elements.

Let us begin by showing that the claim of the lemma holds for some matrices of $\mathcal{M}$. For every $M \in \mathcal{M}$, we define $\operatorname{cl}(M)$ to be the subset of the matrices of $\mathcal{M}$ obtained by row permutation of M. We claim that there exists a matrix $M_0 \in \mathcal{M}$ such that all the matrices of $\operatorname{cl}(M_0)$ have Kolmogorov complexity at least $C = \log |\mathcal{M}| - \log(p!) - 3 \log p$. Indeed, consider a matrix $M_0 \in \mathcal{M}$ for which there exists a matrix $M_0' \in \operatorname{cl}(M_0)$ of Kolmogorov complexity $C' < C$. Then M_0 may be described by an ordered pair (i_0, M_0'), where i_0 is the index of the row permutation of M_0' into M_0. Such an integer can be coded, in a self-delimiting way, by $\log(p!) + 2 \log p + O(1)$ bits. ($2 \lceil \log p \rceil$ bits are sufficient to describe p, thus the length of any $i_0 \leq p!$, in a self-delimiting way.) Hence the Kolmogorov

[4] A is a sub-matrix of B if A can be obtained from B by removing some columns and rows in B.

complexity of M_0 is at most $C' + \log(p!) + 2\log p + O(1) < \log|\mathcal{M}|$. By the counting argument mentioned earlier, it is impossible for all matrices $M_0 \in \mathcal{M}$ to have such low complexity.

The class $\mathcal{M}$ is of size $|\mathcal{M}| = \binom{p}{\lfloor p/2 \rfloor}^q = \Theta(2^p/\sqrt{p})^q$. By Stirling's formula $\log|\mathcal{M}| = pq - O(q\log p)$, and $\log(p!) = p\log p - O(p)$. Hence,

$$C \;=\; \log|\mathcal{M}| - \log(p!) - 3\log p \;=\; pq - p\log p - O(p + q\log p). \qquad (9)$$

All the matrices of $\mathcal{M}$ have q columns, each one of Kolmogorov complexity bounded by $p + O(1)$. Therefore there exists a matrix M_0 such that every matrix in $\mathrm{cl}(M_0)$ has a column of Kolmogorov complexity at least

$$\frac{C}{q} - 2\log p \;=\; p - \frac{p\log p}{q} - O\left(\frac{p}{q} + \log p\right)\ . \qquad (10)$$

The term $2\log p$ codes the length of the description of such a column in a self-delimiting way. Define a *deficiency function* as a $\mathbb{N} \mapsto \mathbb{N}$ function such that it is possible to retrieve n and $\delta(n)$ from $n - \delta(n)$ by a self-delimiting program of constant size. From [14, Theorem 2.15, page 131], every binary string of length p bits and of Kolmogorov complexity at least $p - \delta(p)$ contains at least

$$\frac{p}{4} - \sqrt{(\delta(p) + c)p\,\frac{3}{2}\ln 2} \qquad (11)$$

occurrences of 01-sequences, for any deficiency function δ, and some constant c depending on the definition of the Kolmogorov complexity. Since each 01-sequence in a binary string necessarily starts a new block of consecutive ones, we get a lower bound on the number of blocks of consecutive ones for such strings.

By choosing for δ the function $\delta(p) = (p/q)\log p + O(p/q + \log p)$, and by Inequality (11), it follows that M_0 has compactness

$$\mathrm{comp}(M_0) \;\geq\; \frac{p}{4} - \sqrt{\frac{3p^2}{2q}\ln p + O\left(\frac{p^2}{q} + p\log p\right)}\ . \qquad (12)$$

Finally, let us show that the result of the lemma, shown for some matrices in $\mathcal{M}$, holds also for the compactness of some matrices of $\mathcal{M}_0$. From Lemma 4.1, because $p = o(2^{q/2}) = o(p^{3/2})$, we get that $|\mathcal{M}_1| \sim |\mathcal{M}|$. Similarly, $q = O(2^{p/5}) = o(2^{p/4})$, thus $|\mathcal{M}_2| \sim |\mathcal{M}|$. Since $|\mathcal{M}_0| \leq |\mathcal{M}|$, and $|\mathcal{M}_0| \geq |\mathcal{M}| - (|\mathcal{M}| - |\mathcal{M}_1| + |\mathcal{M}| - |\mathcal{M}_2|) \geq |\mathcal{M}| - o(|\mathcal{M}|)$, it follows that $|\mathcal{M}_0| \sim |\mathcal{M}|$. Clearly it implies that $\log|\mathcal{M}_0| = \log|\mathcal{M}| + o(1)$, and thus Inequalities (9), (10), (11), and (12) hold for $\mathcal{M}_0$ as well, which completes the proof. $\square$

Remark. The proof of Lemma 4.2 is nonconstructive. As a result, it can prove only the existence of such a worst-case graph G_M.

We are now ready to prove Theorem 2.

Proof of Theorem 2: Let $M \in \mathcal{M}_0$ be a matrix satisfying Lemma 4.2, and consider the graph G_M, built from M. Let us show that the diameter of G_M is 2. For any two nodes x, y, denote by $\mathrm{dist}(x, y)$ the distance between x and y in G_M. The distance between any a_j (or b_j) and any v_i is at most 2 (since a_j and b_j are adjacent). The fact that $M \in \mathcal{M}_1$ implies that rows of M are pairwise non complementing. Thus for every i, i' there exists some j such that $m_{i,j} = m_{i',j}$, which implies $\mathrm{dist}(v_i, v_{i'}) = 2$.

Since $M \in \mathcal{M}_2$, M has the following property: for any two columns j, j' there exists some i_1, i_2, i_3 and i_4 such that $m_{i_1,j} = 1 = m_{i_1,j'}$, $m_{i_2,j} = 0 = m_{i_2,j'}$, $m_{i_3,j} = 1 \neq m_{i_3,j'}$, and $m_{i_4,j} = 0 \neq m_{i_4,j'}$. Therefore in G_M, $\mathrm{dist}(a_j, a_{j'}) = 2$, $\mathrm{dist}(b_j, b_{j'}) = 2$, $\mathrm{dist}(a_j, b_{j'}) = 2$, and $\mathrm{dist}(b_j, a_{j'}) = 2$. It follows that G_M is of diameter 2.

Let $\mathcal{R} = (\mathcal{L}, \mathcal{I})$ be any interval routing scheme on G_M.

Claim 4.3 *For every arc (a_j, b_j), and for every vertex v_i, if $m_{i,j} = 1$ and $\mathcal{L}(v_i) \notin \mathcal{I}(a_j, b_j)$, or if $m_{i,j} = 0$ and $\mathcal{L}(v_i) \in \mathcal{I}(a_j, b_j)$, then $\mathcal{R}$ builds a path of length at least* 3.

Proof. Any "wrong" decision of $\mathcal{R}$ in routing from a_j to v_i (meaning, any decision to start the route from a_j to v_i on any outgoing arc of a_j other than the arc (a_j, b_j)) results in a route that goes through some vertex $v_{i'} \neq v_i$ before it reaches v_i. The claim now follows from the fact that there is no path shorter than two hops between any two vertices $v_i \neq v_{i'}$. $\square$

Let $k = \mathrm{comp}(M)$ be the compactness of M.

Claim 4.4 $\mathrm{IRS}(G_M) \geq k$.

Proof. The claim is proved by showing that if there is an IRS $\mathcal{R}$ that uses no more than $k - 1$ intervals per arc, then $\mathcal{R}$ builds some path of length at least 3. Since G_M is of diameter 2,this implies that $\mathcal{R}$ is not a shortest paths scheme.

Given Claim 4.3, it suffices to prove that if there is an IRS $\mathcal{R}$ that uses no more than $k-1$ intervals per arc, then $\mathcal{R}$ must make the wrong decision for some v_i and (a_j, b_j).

Let j_0 be a column of M composed of at least k blocks of consecutive ones. Such a column exists because the compactness of M is k. Let us consider the tuple $u = (u_1, \ldots, u_p)$ defined by setting $u_i = 1$ if $\mathcal{L}(v_i) \in \mathcal{I}(a_{j_0}, b_{j_0})$, and $u_i = 0$ otherwise, for every $i \in \{1, \ldots, p\}$, Since $\mathcal{I}(a_{j_0}, b_{j_0})$ is composed of at most $k-1$ intervals, u is composed of at most $k-1$ blocks of consecutive ones. Thus the column j_0 and the tuple u differ in at least one place. Let i_0 be an index such that $m_{i_0,j_0} \neq u_{i_0}$. If $u_{i_0} = 1$, then $\mathcal{L}(v_{i_0}) \in \mathcal{I}(a_{j_0}, b_{j_0})$, and $m_{i_0,j_0} = 0$. If $u_{i_0} = 0$, then $\mathcal{L}(v_{i_0}) \notin \mathcal{I}(a_{j_0}, b_{j_0})$, and $m_{i_0,j_0} = 1$. Claim 4.4 now follows by applying Claim 4.3. □

The order of G_M is $n = p + 2q$. Let us choose $q = \left\lfloor cn^{2/3} \ln^{1/3} n \right\rfloor$, where $c = (3/2)^{1/3} \approx 1.14$. The maximum degree of G_M is $\max\{q, \lceil p/2 \rceil + 1\} < p/2 + 2 < n/2$. (In particular, nodes a_j are connected to b_j and to the v_i's corresponding to all the 0-entries of the jth column of M). The total number of edges in G_M is $|E| = pq + q < nq < 1.15\, n^{5/3} \ln^{1/3} n$. Replacing $p = n - 2q$, and applying Lemma 4.2, the compactness of M, k, satisfies

$$k \;\geq\; \frac{n}{4} - \frac{q}{2} - \sqrt{\frac{3(n-2q)^2}{2q} \ln{(n-2q)} + O\left(\frac{n^2}{q} + n\log n\right)} \; .$$

Noting that $O(n^2/q + n\log n) = O(n^{4/3})$, and that

$$\frac{3(n-2q)^2}{2q} \ln{(n-2q)} \;<\; \frac{3n^2}{2q} \ln n \; ,$$

we get

$$\begin{aligned} k \;&\geq\; \frac{n}{4} - \frac{c}{2} n^{2/3} \ln^{1/3} n - \sqrt{\frac{3n^2 \ln n}{2cn^{2/3} \ln^{1/3} n} + O(n^{4/3})} \\ &\geq\; \frac{n}{4} - \frac{c}{2} n^{2/3} \ln^{1/3} n - \sqrt{\frac{3}{2c} n^{4/3} \ln^{2/3} n} - O\left(n^{2/3}\right) \\ &\geq\; \frac{n}{4} - \left(\frac{c}{2} + \sqrt{\frac{3}{2c}}\right) n^{2/3} \ln^{1/3} n - O\left(n^{2/3}\right) \\ &\geq\; \frac{n}{4} - \left(\frac{3}{2}\right)^{4/3} n^{2/3} \ln^{1/3} n - O\left(n^{2/3}\right) \\ &>\; \frac{n}{4} - 1.72\, n^{2/3} \ln^{1/3} n \; . \end{aligned}$$

Therefore, we have shown that if $\mathcal{R}$ uses at most $k-1$ intervals per arc, $\mathcal{R}$ builds a route of length at least 3. It remains to show that this result holds also if $\mathcal{R}$ uses at most $\mathrm{IRS}(G_M) - 1$ intervals per arc.

Claim 4.5 *For every 2-connected graph G of girth g, if $k < \mathrm{IRS}(G)$ then the longest path of every (non shortest paths) single k-IRS is at least $\lfloor g/2 \rfloor + 1$.*

Proof. Let G be a graph as in the claim, and let $\mathcal{R}$ be a single k-IRS for G. Since $\mathrm{IRS}(G) > k$, there must exist two nodes x, y at distance d such that the routing specified by $\mathcal{R}$ from x to y is not along a shortest path. The routing path uses an alternative on a cycle between x and y. The length of this alternative path, l, satisfies $l + d \geq g$, which implies that $l \geq g/2$, because $d \leq g/2$. However $l = g/2$ is impossible, otherwise the message would use a shortest path, hence $l > g/2 \geq \lfloor g/2 \rfloor$, thus $l \geq \lfloor g/2 \rfloor + 1$. □

Clearly, the graph G_M is 2-connected and has no triangles, thus its girth is at least 4, and therefore any single k-IRS of G_M has a routing path of length at least 3, completing the proof of Claim 4.5. □

Remark. Theorem 2 is tight for the length of the longest path since it is proven in [13] that $\left\lceil \sqrt{n \ln n} \right\rceil + 1$ intervals per arc are sufficient to guarantee routes of length at most $\lceil 3D/2 \rceil$, where D is the diameter of the graph. Hence for the graphs considered here, which are of diameter 2, this yields paths of length at most 3. Using this IRS, G cannot have a routing path of length 4.

To our best knowledge, the "best" worst-case construction which does not use randomization remains that of [9], which yields graphs G with $\mathrm{IRS}(G) \geq n/8$, for every n power of 2.

Corollary 4.6 *For every sufficiently large integer n, and for every integer $D \geq 2$, $D = o(n)$, there exists an n-node graph G of diameter D such that*

$$\mathrm{IRS}(G) > \frac{n}{4} - o(n).$$

Proof. Take the worst-case n-node graph G of Theorem 2. G has diameter 2, hence it has a node x of eccentricity 2. Construct a new graph G' obtained from G by adding a path of length $D - 2$ to x. G' has diameter D exactly, and $n' = n + D - 2$ nodes. The proof of Theorem 2 applies on G' as well. It turns out that G' has compactness at least $n/4 - O(n^{2/3} \log^{1/3} n)$, that is $n'/4 - o(n')$, replacing n by $n' - D + 2 \geq n' - o(n)$. □

We conclude this section by showing that the lower bound can be applied to k-IRS that are not of shortest paths, and not single routing schemes.

A routing scheme $\mathcal{R}$ on G is of *stretch factor* s if for all nodes x, y, $x \neq y$, the routing path length from x to y is at most s times longer than the distance in G between x and y. In particular, a shortest path k-IRS is a routing scheme of stretch factor 1.

For every integer $\alpha \geq 1$, a routing scheme $\mathcal{R}$ on G is *α-adaptive* if for all nodes x, y, $x \neq y$, there exist $\min\{\alpha, \delta\}$ edge-disjoint routing paths between x and y, where δ is the total number of "possible" edge-disjoint routing paths between x and y in G having different first edges. A single shortest path k-IRS is a 1-adaptive routing scheme of stretch factor 1. A full-adaptive k-IRS on G is a Δ-adaptive routing scheme on G, where Δ is the maximum degree of G.

Since for G_M the shortest paths between the nodes a_j and v_i are unique, and since any wrong decision will route along paths of length at least $3/2$ times the distance, we have the following trivial lower bound.

Corollary 4.7 *For every sufficiently large integer n, for every s, $1 \leq s < 3/2$, and every integer $\alpha \geq 1$, there exists an n-node graph G such that no α-adaptive k-IRS of stretch factor s on G exists if*

$$k \leq \frac{n}{4} - 1.72\, n^{2/3} \ln^{1/3} n.$$

5 Conclusion

- Since the lower bound is based on Kolmogorov complexity of the labels of the edges, the resulting bound can be applied to every kind of edge-labeling based routing schemes. Moreover, the bounds can apply to adaptive (or multipath) routing schemes.
- It would be interesting to find tighter upper bounds for small values of n, and also to express these bounds as a function of other parameters and properties of the graphs under study, such as their maximum degree, planarity, genus, tree-width, and so on.

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