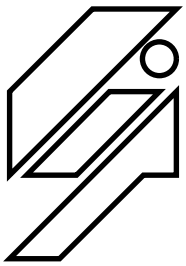

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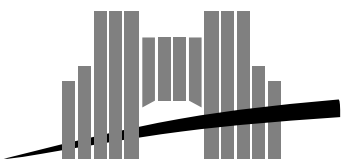

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Abstract

Interval Routing was introduced to reduce the size of the routing tables: a router finds the direction to forward a message by determining the interval that contains the destination address of the message, each interval being associated to one particular direction. In this paper, we give lower bounds for the minimum number of intervals per edge needed to achieve shortest path routings for the class of 3-regular networks. We prove a tight lower bound of $\Theta(n)$ intervals for a 3-regular network of order at most n. Moreover, for the particular case of 3-regular planar networks, we establish a lower bound of $\Omega(\sqrt{n})$ intervals.

Keywords: compact routing in distributed networks, shortest path interval routing, 3-regular graphs, planar graphs

Résumé

Le routage par intervalles a été introduit pour réduire la taille des tables de routage : un routeur trouve la direction pour communiquer un message en déterminant l'intervalle contenant l'adresse de destination du message, chaque intervalle étant associé à une direction particulière. Dans cet article, nous donnons une borne inférieure du nombre minimum d'intervalles par arête nécessaire pour effectuer un routage de plus courts chemins pour la classe des réseaux 3-réguliers. Nous prouvons une borne inférieure optimale de $\Theta(n)$ intervalles pour un réseau 3-régulier d'ordre au plus n. De plus, pour la classe des réseaux 3-réguliers et planaires, nous établissons une borne inférieure de $\Omega(\sqrt{n})$ intervalles.

Mots-clés: routage compact dans réseaux distribués, routage par intervalles et de plus courts chemins, graphes 3-réguliers, graphes planaires

Lower Bounds for Shortest Path Interval Routing*

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Abstract

Interval Routing was introduced to reduce the size of the routing tables: a router finds the direction to forward a message by determining the interval that contains the destination address of the message, each interval being associated to one particular direction. In this paper, we give lower bounds for the minimum number of intervals per edge needed to achieve shortest path routings for the class of 3-regular networks. We prove a tight lower bound of $\Theta(n)$ intervals for a 3-regular network of order at most n. Moreover, for the particular case of 3-regular planar networks, we establish a lower bound of $\Omega(\sqrt{n})$ intervals.

1 Introduction

Table routing is a standard solution to the shortest path routing problem for arbitrary networks. At each router in the network is stored a table listing for each possible destination the output port that should be used to send a message along a shortest path. This solution guarantees shortest paths but requires $O(n \log d)$ bits of memory per node, for a n-router network of maximum degree d. The *compact routing problem* implements routing schemes that uses a minimum memory size on each router. The interval routing scheme was introduced by Santoro and Khatib in [14] to improve the space requirements of routing tables.

In [10], we showed that there is not possible to find a "compact" routing scheme for all networks, *i.e.*, universal routing schemes that produce in each router compact data structures for the routing information. We showed that some routers need to store $\Omega(n \log d)$ bits of information independantly from the routing scheme used. This implies that the compact routing problem could be tackled with two dual approaches :

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- find the best routing scheme for a given class and for a small class of networks,
- find all networks requiring small memory requirements for a given routing scheme.

The second approach consists in graph characterization. For the interval routing scheme, this approach has been investigated in [1, 3, 6]. The first approach consists in finding lower and upper bounds for given classes of graphs. The state of the art in the compact routing problem is presented in [5] for the general case, and in [7, page 88] for graphs of maximum degree d. In this paper we study the case of 3-regular networks for the shortest path interval routing scheme.

The model of networks is a symmetric digraph (drawn as an undirected graph), each arc represents a link. Graphs are finite, connected, loop-less and without multi-arc. In order to sharpen drawings, we depict bidirectional links with a single edge between two neighbor nodes. The basic idea of the interval routing is the following: (1) each nodes is labeled with an unique integer taken from the set $\{1, \ldots, n\}$, where n is the number of nodes of the network; (2) to each arc e is assigned a set I_e containing all the labels of the destinations using the arc e to receive messages routed along shortest paths. Moreover, the sets assigned to arcs emanating from the same node must be pairwise disjoint, and their union must cover $\{1, \ldots, n\}$. The interval routing coding strategy consists in representing every destination set I_e by an union of disjoint intervals. An "interval" is a subset of consecutive integers taken from $\{1, \ldots, n\}$, where n and 1 are considered consecutive.

The goal of the interval routing is to find a labeling of nodes and an assignment of destination sets that minimizes the number of intervals on each arc. If, for a graph G, this number of intervals is at most k per arc then the memory space of a router of G of maximum degree d is at most $O(dk \log n)$ bits. Indeed, it is sufficient to store the end-points of each interval. The *compacity* of a graph G, denoted by $\textsc{Irs}(G)$, is defined as the smallest integer k such that each destination set of G is composed of at most k intervals. Since the interval routing was introduced in the order to reduce the memory space of routers, compacity is an important parameter to consider.

The general study on the compacity of a graph of order n has been investigated by many authors, see in particular [13] and [2]. Unfortunately, Gavoille and Guévremont in [9], build graphs having a compacity of at least $n/12$ intervals, proving uneventful that interval routing scheme is not compact in general. Study of compacity for smallest classes of graphs was first proposed in [8]. Their lower bound has been improved in [12], where Kranakis and Krizanc proved that some bounded degree graphs have a compacity of $\Omega(n/\log n)$ intervals.

In [10] we have proved a tight lower bound for any *universal* shortest path routing scheme. We showed that there exist graphs where the total space required to store the routing information of all nodes is $\Omega(n^2 \log d)$ bits, where d is the maximum degree of the graphs. In particular if d is constant this implies a lower bound of $\Omega(n)$ bits per router for the local routing information. Therefore, we have a trivial lower bound of $\Omega(n/\log n)$ (that is not a new result) for the compacity of a bounded degree graph, since a router using interval routing scheme on a graph of compacity k and of degree d cannot store more than $O(dk \log n)$ bits of information. In this paper, we show that this lower bound is not tight by proving that the maximum compacity of a 3-regular graph of order at most n is $\Theta(n)$. Moreover, this number of intervals can be required on $\Omega(n)$ edges, that is asymptotically the total number of edges of the graph. This tight lower bound shows that the worst case for interval routing does not depend on the maximum degree of the graph, whereas the memory requirements of universal routing schemes do.

2 A Tight Lower Bound for 3-Regular Graphs

2.1 Definitions

The key of our proof is to use some graphs built from a boolean matrix. This technic, already used in [9] and in [2], is related to the *matrices of constraints*. Informally, for every triple of vertices (u, v, w) of a graph G, where u and v are adjacent and $u \neq w$, three cases may occur for shortest paths:

1. Every shortest path from u to w uses the arc (u, v).
2. Every shortest path from u to w does not use the arc (u, v).
3. There are shortest paths from u to w that use arc (u, v) and there are shortest paths from u to w that do not use the arc (u, v), depending on the choice of the shortest path routing function.

For the first two cases, the arc (u, v) forms a *constraint* for the vertex w. A *matrix of constraints* of a graph G is a $p \times q$ boolean matrix $M = (m_{i,j})$ whose rows are labeled with some vertices of G $\{v_1, \ldots, v_p\}$, and whose columns are labeled with some arcs $\{e_1, \ldots, e_q\}$ of G such that:

$$\begin{cases} m_{i,j} = 1 \text{ if and only if every shortest path from the tail of } e_j \text{ to } v_i \text{ uses the arc } e_j. \\ m_{i,j} = 0 \text{ if and only if no shortest path from the tail of } e_j \text{ to } v_i \text{ uses the arc } e_j. \end{cases}$$

The main interest of matrices of constraints is for proving lower bound on the compacity for particular graphs. Assume that M is a matrix of constraints for a graph G. It is not difficult to see that the maximum number of blocks of consecutive ones in a column of M, minimized over all row permutations of M, is a lower bound on the compacity of G. This fact was first remarked by [2], then developped and generalized in [9]. A natural way of defining the *compacity of a boolean matrix* M, denoted by $\mathrm{I}(M)$, is as the maximum (taken over all the columns of M) of the number of blocks of consecutive ones in a column of M, minimized over all row permutations of M. For every matrix of constraints M of G, we have of course $\textsc{Irs}(G) \geq \mathrm{I}(M)$.

In the next paragraphs, we will build a 3-regular graph of order at most n having a matrix of constraints of compacity $\Theta(n)$.

2.2 Graph construction

Our construction is based on the construction given in [10]. For every integer $p \geq 2$, we define H_p by induction as follows (see Figure 1 for H_2). In the graph H_p, we distinguish a set of arcs $\{e_1, \ldots, e_p\}$ and a set of 2^p vertices B_p (drawn in gray on Figure 1). Each vertex of B_p is denoted by an unique binary word of p bits. We note A_p the set of tails of the arcs e_i, for $i \in \{1, \ldots, p\}$. H_{p+1} is composed of a copy of H_p, of two complete binary trees of 2^{p-1} leaves, of 2^p intermediary nodes forming a set B'_p, and finally by 2^{p+1} news nodes forming the set B_{p+1} (that is also the set of all binary words of length $p + 1$ bits). The roots of the two trees are connected by an edge, whose e_{p+1} is an arc. Vertices of B_p and vertices of B'_p are connected by a matching. Each vertex $x \in B'_p$ is connected to the vertices of the form $x0$ and $x1$ of B_{p+1} (x being a binary word of p bits). Moreover, every vertex $w \in B_{p+1}$ is connected to a leaf of the first tree if $w = x0$, and to a leaf of

the second tree otherwise, such that every leaf of the trees have a maximum of two neighbors in B_{p+1}.

The graph H_p possesses an important property: for any vertex $a_i \in A_p$, $i \in \{1, \ldots, p\}$, and for any vertex $w \in B_p$, every shortest path from a_i to w has to use the arc e_i if and only if the i-th bit of w is 1. This fact can be easily showed by induction on p. Clearly, this property implies that H_p has a matrix of constraints that is the $2^p \times p$ boolean matrix whose the i-th row if the binary written of $i-1$. Indeed, each arc of $\{e_1, \ldots, e_p\}$ forms a constraint for all vertices of B_p.

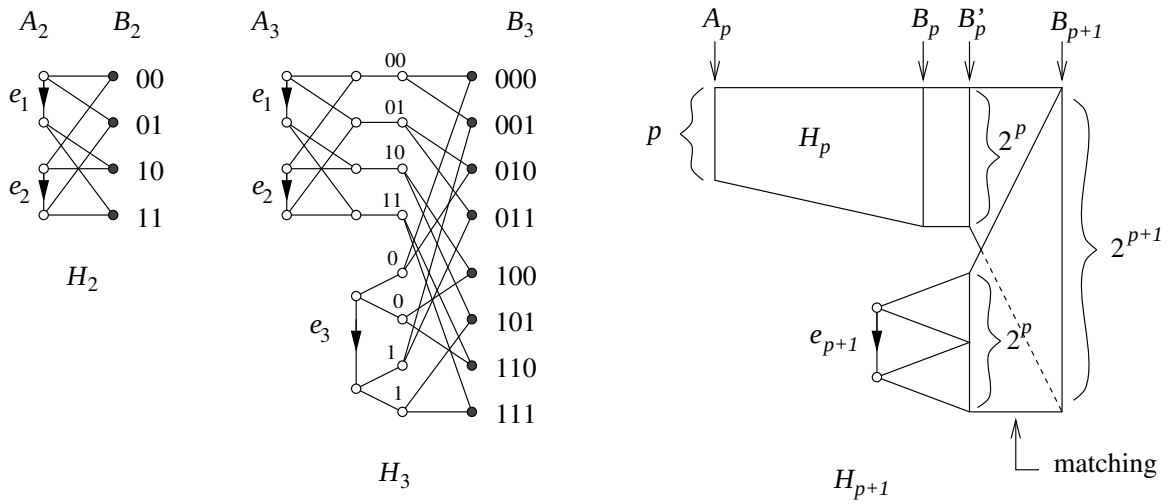

Figure 1: The recursive construction of the graph H_p.

The idea is to build a new graph by connecting two copies of H_p by a perfect matching between vertices of the sets B_p, the perfect matching being defined by a permutation π of the set $\{1, \ldots, 2^p\}$. This new graph is denoted by $G_{p,\pi}$. An example of the $G_{p,\pi}$ graph is depicted[1] on Figure 2 for a particular permutation π.

For every integer p and every permutation π of $\{1, \ldots, 2^p\}$, we denote by $M_{p,\pi}$ the $(2^p) \times (2p)$ binary matrix defined as follows: the i-th row of $M_{p,\pi}$ is composed of $2p$ bits, the first p ones being the binary writing of $i-1$ and the last p ones being the binary writing of $\pi(i)-1$.

Example: $\pi_3 = (7, 5, 1, 2, 8, 4, 3, 6)$.

$$M_{3,\pi_3} = \begin{bmatrix} 0\,0\,0 & 1\,1\,0 \\ 0\,0\,1 & 1\,0\,0 \\ 0\,1\,0 & 0\,0\,0 \\ 0\,1\,1 & 0\,0\,1 \\ 1\,0\,0 & 1\,1\,1 \\ 1\,0\,1 & 0\,1\,1 \\ 1\,1\,0 & 0\,1\,0 \\ 1\,1\,1 & 1\,0\,1 \end{bmatrix} \tag{1}$$

[1]Recall that graphs are symmetric digraphs. However, some arcs are marked on Figure 1 just to distinguish head and tail of the arcs.

Lemma 1 *Let $p \geq 2$ and π be a permutation of $\{1, \ldots, 2^p\}$. The graph $G_{p,\pi}$ satisfies:*

1. *$G_{p,\pi}$ is 3-regular and of order $5 \cdot 2^{p+1} - 4p - 16$*
2. *$G_{p,\pi}$ possesses $M_{p,\pi}$ as matrix of constraints*

Proof. $G_{p,\pi}$ is 3-regular, and its order is $5 \cdot 2^{p+1} - 4p - 16$. Let B_1 be the set B_p of the first copy of H_p (grayed vertices on Figure 2), and A'_p and B'_p the sets of distinguished vertices of the second copy of H_p. $M_{p,\pi}$ is a matrix of constraints of $G_{p,\pi}$. Indeed, from the basic property of graphs H_p there exists a unique shortest path between vertices of A'_p and of B'_p, and thus an unique shortest path between vertices of A'_p and vertices of the set B_1 (that uses the perfect matching between the two copies of the graph H_p). □

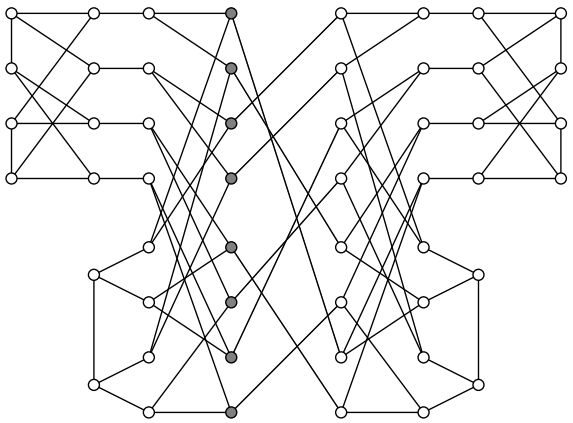

Figure 2: An example of the $G_{p,\pi}$ construction for $p = 3$ and $\pi = \pi_3$ (see definition of π_3 on page 4).

Remark. The $G_{p,\pi}$ graph has only $3n/2$ edges for an order n. However, $G_{p,\pi}$ cannot be planar in general because the matching π between the two copies of the graph H_p.

Now, let us show that this graph requires a large compacity in general.

2.3 The $\Theta(n)$ lower bound

The matrix M_{3,π_3} of the equation 1 has 4 blocks of consecutive ones in its third column. However, it does not implies that $\mathrm{I}(M_{3,\pi_3}) = 4$, because rows of the matrix can be permuted to decrease its compacity, corresponding for the graph G_{3,π_3} (on Figure 2) to a possible efficient labeling of its vertices.

Lemma 2 *For every $p \geq 1$, there exists a permutation π such that $\mathrm{I}(M_{p,\pi}) \geq k$ with*

$$\left(\sum_{i=0}^{k} \binom{2^p}{2i} \right)^{2p} \geq (2^p)! \tag{2}$$

Proof. Let p be an integer and let $\mathcal{A}$ be the set of matrices $M_{p,\pi}$ for all permutations π of $\{1, \ldots, 2^p\}$. Let $k = \max\{\mathrm{I}(M) \mid M \in \mathcal{A}\}$, that is the maximum compacity of matrices of $\mathcal{A}$. Let

us prove that k satisfies Equation 2. Let f be a function that returns, for every matrix $M \in \mathcal{A}$, a matrix $M' = f(M)$ that is obtained from M by rows permutation such that every column of M' has at most k blocks of consecutive ones. By definition of k, such a function f exists. We denote by $\mathcal{B}$ the set of $(2^p) \times (2p)$ binary matrices for which all the columns are composed of at most k blocks of consecutive ones. By definition $f(\mathcal{A}) \subset \mathcal{B}$, that implies $|f(\mathcal{A})| \leq |\mathcal{B}|$. Clearly, f is a bijective function because for every matrix $M' \in f(\mathcal{A})$ we can permute the rows of M' in order to obtained the matrix $f^{-1}(M')$, where the first p bits of the rows are the binary words 0...00, 0...01, 0...10, 0...11, and so on. Therefore, $|f(\mathcal{A})| = |\mathcal{A}| = (2^p)! \leq |\mathcal{B}|$. On the other hand, there exists $\binom{2^p}{2i}$ ways to have exactly i blocks of consecutive ones in a binary column of length 2^p, and thus at most $\sum_{i=0}^{k} \binom{2^p}{2i}$ ways to have at most k blocks of consecutive ones in a column. So, there is at most $(\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$ ways to have a matrix having at most k blocks of consecutive ones on each of its $2p$ columns. In sequel, $|\mathcal{B}| \leq (\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$, and finally k must satisfied $(2^p)! \leq (\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$. □

Corollary 1 *For every $p \geq 3$, there exists a permutation π such that* $\mathrm{I}(M_{p,\pi}) = \Omega(2^p)$.

In a sense this result is optimal because for every p and for every π, $\mathrm{I}(M_{p,\pi}) \leq 2^p/2$.

Proof. Applying Lemma 2, for every $p \geq 3$ there exists a permutation π such that Equation 2 is satisfied. Let $k = \mathrm{I}(M_{p,\pi})$. If $2k \geq 2^p/2$ then $\mathrm{I}(M_{p,\pi}) = \Omega(2^p)$. Otherwise, $\sum_{i=0}^{k} \binom{2^p}{2i} \leq (k+1)\binom{2^p}{2k}$, since in this case $\binom{2^p}{2k}$ is the maximum term of the sum. Moreover, since $k < 2^{p-2}$, we get $(\sum_{i=0}^{k} \binom{2^p}{2i})^{2p} \leq 2^{2p(p-2)} \binom{2^p}{2k}^{2p}$. For every integer n and p, $(n/e)^n \leq n!$ and $\binom{n}{p} \leq (ne/p)^p$ (where $e = 2.718281828...$). Therefore, by Equation 2, k must satisfied:

$$\left(\frac{2^p}{e}\right)^{2^p} \leq 2^{2p(p-2)} \left(\frac{2^p e}{2k}\right)^{4kp} \quad \Rightarrow \quad 2^p \log_2 \left(\frac{2^p}{e}\right) \leq 2p(p-2) + 4kp \log_2 \left(\frac{2^p e}{2k}\right)$$

Let us define α by $\alpha = 2^p/(2k)$, and let us show that α is upper-bounded by a constant. Latter equation implies:

$$2^p(p - \log_2 e) \leq 2p(p-2) + \frac{2p \cdot 2^p}{\alpha} \log_2(\alpha e) \quad \Rightarrow \quad l(p) = \frac{1}{2} - \frac{\log_2 e}{2p} - \frac{p-2}{2^p} \leq \frac{\log_2(\alpha e)}{\alpha}$$

Since for every $p \geq 3$ the function $l(p)$ is an increasing function as p, this implies that α must satisfy $0.134 < l(3) \leq (\log_2(\alpha e))/\alpha$, implying $\alpha < 54$. Therefore, for every $p \geq 3$, $k = 2^p/(2\alpha) > 2^p/108 = \Omega(2^p)$. □

Using the Stirling approximation of $n!$, we can tighten the upper bound on the constant α and we can show that for some permutation π and for sufficiently large integer p, $\mathrm{I}(M_{p,\pi}) \geq 2^p/19$. However, we think that, with a constructive proof of Corollary 1, this lower bound can be improved.

Note that matrices requiring a large compacity are not so particular. Indeed, a simple counting argument shows that "almost all" the matrices $M_{p,\pi}$ require $\Omega(2^p)$ intervals (for example a permutation chosen randomly among the $(2^p)!$ ones). Moreover, for such matrices of large compacity, one can check that $\Omega(p)$ columns require $\Omega(2^p)$ blocks of consecutive ones.

We can now derive the main Theorem of this section that is:

Theorem 1 *There exists a 3-regular graph G of order at most n such that* $\mathrm{IRS}(G) = \Theta(n)$.

Proof. Clearly, $\textsc{Irs}(G) \leq n/2$ for every graph of order n. Applying Lemma 1 and Corollary 1, there exists a 3-regular graph $G_{p,\pi}$ such that $\textsc{Irs}(G_{p,\pi}) \geq \mathrm{I}(M_{p,\pi}) = \Omega(2^p)$. The order of $G_{p,\pi}$ is $5 \cdot 2^{p+1} - 4p - 16$. In particular, for every integer n large enough there exists p such that $5 \cdot 2^{p+1} \leq n < 5 \cdot 2^{p+2}$, $G_{p,\pi}$ is of order at most n and of compacity at least $\mathrm{I}(M_{p,\pi}) = \Omega(n)$. □

We have already seen that $\Omega(p)$ columns of some matrices $M_{p,\pi}$ require each $\Omega(2^p)$ intervals. Actually, it means that $\Omega(p) = \Omega(\log n)$ arcs of the graph $G_{p,\pi}$ require $\Omega(n)$ intervals. Using the idea devellopped in [10], we can proved that the lower bound of $\Omega(n)$ intervals can be required on $\Omega(n)$ arcs, for another graph construction. In the following we say that a vertex u is a *constrained vertex* for a subset of vertices B if there exists an arc (u, v) such that for every vertex $w \in B$, (u, v) forms a constraint for w (see Paragraph 2.1). For example, Figure 1 shows that each vertex of the set A_3 is a constrained vertices for the set B_3.

Lemma 3 *Let G be a 3-regular graph of order n and A be a set of constrained vertices for a subset of vertices B of G. If $|A| = O(\log n)$ then each constrained vertex $x \in A$ can be "split" in $\Theta(n/\log n)$ other vertices forming a new graph G' such that every new split vertex has the same constraints as x for the set B. Moreover, G' is 3-regular and is of order $\Theta(n)$.*

The proof of this Lemma can easily be derived from [10, Lemma 6]. An illustration of this Lemma is shown on Figure 3.

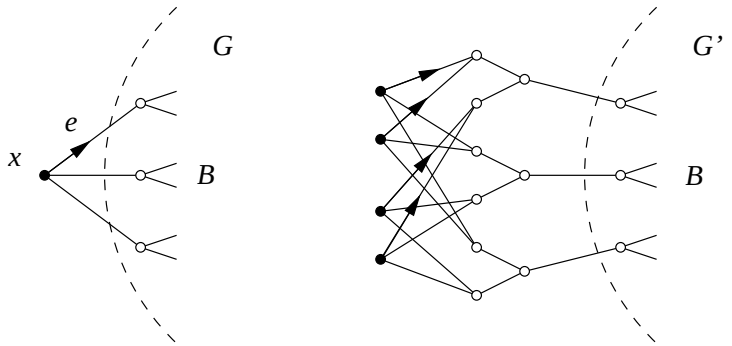

Figure 3: The "splitting" transformation shows that a large compacity can be required by $\Omega(n)$ arcs.

Theorem 2 *There exists a 3-regular graph of order at most n that has a compacity of $\Omega(n)$, and such that this number of intervals is required by $\Omega(n)$ arcs.*

Proof. Let G be the graph of Theorem 1 of order at most n and having $\Omega(\log n)$ arcs requiring $\Omega(n)$ intervals each one. Therefore, by transformation of Lemma 3, there exists a 3-regular graph of order at most $n' = \Theta(n)$ and of compacity $\Omega(n) = \Omega(n')$, this number being required by $\Theta(n/\log n) \cdot \Omega(\log n) = \Omega(n) = \Omega(n')$ arcs. □

In [8], it was introduced the parameter $\textsc{Irs}(n, d)$ denoting the maximum of the compacity over all graphs of order n and of maximum degree d. The following theorem generalizes the results given in [9] for bounded and for unbounded graphs:

Theorem 3 *Let ε be a real constant such that $0 < \varepsilon < 1$. For every sufficiently large integer n and for every integer d, $3 \leq d \leq \varepsilon n$,*

$$\textsc{Irs}(n, d) = \Theta(n).$$

Note that $\textsc{Irs}(n, 2) = 1$.

Proof. From Theorem 2, it is not difficult to build a graph of order exactly n and of maximum degree d, for every n and every $d \leq \varepsilon n$. We start from the 3-regular graph built in Theorem 2 with order at most $\lfloor (1-\varepsilon)n \rfloor = \Theta(n)$ and we connect at any vertex the root of a $K_{1,d}$ graph (the complete bipartite graph). Clearly, this graph has a maximum degree d. Since $d \leq \varepsilon n$, the order of this construction is at most $\lfloor (1-\varepsilon)n \rfloor + \varepsilon n \leq n$. Eventually, we can add a path to complete the order of the graph to obtain exactly n vertices. The compacity of this graph is greater or equal than the compacity of the initial graph (because the initial graph is a *subgraph of shortest path*, see [4, Theorem 2]). Therefore, applying Theorem 2, its compacity is $\Omega((1-\varepsilon)n) = \Omega(n)$, since ε is a constant. □

3 Lower Bound for Planar Graphs

After the study of graphs of bounded degree, we investigate the class of planar graphs that includes a large range of potential topologies for real networks. In this section we give a lower bound for the maximum compacity of a planar graph. General study of compact routing for planar graphs is principally due to Frederickson and Janardan. They showed in [6] that every outer-planar graph is of compacity 1. The best known lower bound is due to Kranakis, Krizanc and Ravi in [13]. They built a planar graph of order at most n and of maximum degree $\Omega(\sqrt[3]{n})$ of compacity $\Omega(\sqrt[3]{n})$ intervals on at least one arcs. We improved this lower bound for a 3-regular planar graph with the following theorem:

Theorem 4 *For every n large enough, there exists a 3-regular planar graph G of order at most n such that*

$$\textsc{Irs}(G) > \frac{\sqrt{n}}{5.6}$$

Moreover, we will see that $\Omega(\sqrt{n})$ intervals can be required on $\Omega(\sqrt{n})$ arcs. Our proof is constructive and uses matrices of constraints. For every integer $m \geq 2$, we give the construction of a 3-regular planar graph denoted by P_m.

Description of the P_m graph.

Let $m \geq 2$ be an integer. Roughly speaking, the P_m graph is composed of a $m \times m$ grid with two trees : one connected to a vertical border, and the other connected to the horizontal border of the grid. In order to have a 3-regular graph, each node is removed by small 3-regular structures. Each one is composed of smaller structures namely A, B, C, D, and E as depicted on the right side of Figure 4. They are arranged by their free links. The left side of Figure 4 represents the general topology of the P_m graph with the arrangement of the structures $A, \ldots, E$. The structures

$A, \ldots, E$ can be swiveled round in the plane to join their free links. Figure 5 shows an example of the graph P_m for $m = 2$.

Some vertices of P_m are distinguished: we say that x is a *gray-vertex* if x is one of the vertices of the structures A, B, or of C drawn in gray on Figure 4. Moreover, we say that e is a *vertical-arc* (respectively *horizontal-arc*) if e is an arc distinguished on the structure D connected on the vertical border (respectively horizontal border) of P_m. Let us recall the every two neighbors are connected by an edge that is composed of two symmetric arcs. Note that the structure D is an arrangement of blocks of length 3 (two triangles sharing an edge) and must be of length $L \geq 2m-5$. Thus the length of D is such that $2m - 6 < L < 2m - 2$. The number of blocks for a length L is $k = \lceil (L+1)/3 \rceil$. Therefore, for every $m \geq 2$, the total number of vertices of D is at most $8k = 8\lceil (L+1)/3 \rceil \leq 8\lceil (2m-2)/3 \rceil \leq 16m/3 + 3$.

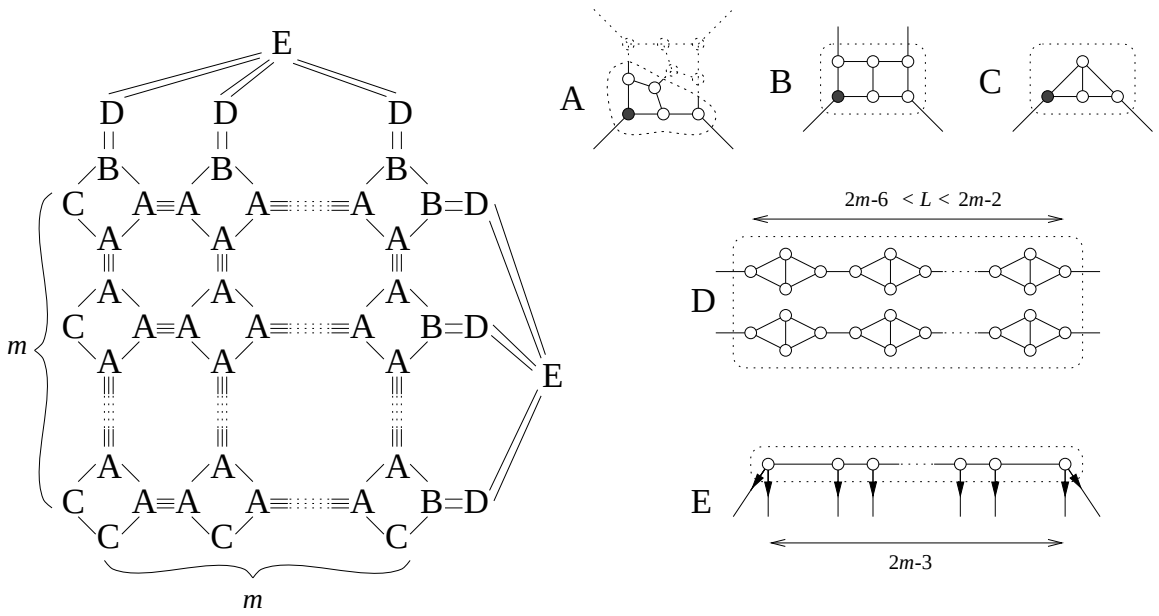

Figure 4: Construction of the P_m graph.

Lemma 4 *For every integer $m \geq 2$,*

- *P_m is 3-regular and planar*
- *P_m is of order at most $\frac{92}{3}m^2 - 20m + 56$*
- $\textsc{Irs}(P_m) \geq m$

Proof. Let $m \geq 2$. By construction, P_m is planar and 3-regular. We denote by A, B, ...the order of basic structures of P_m of Figure 4. Note that $A = 5$, $B = 6$, $C = 4$, $D \leq 16m/3 + 3$, and $E = 2m - 2$. Referring to the left side of Figure 4, the order of P_m is:

$$n = 2E + 2m(D + B + C) + 4(m-2)^2 A + 6(m-2)A + 8A \leq \frac{92}{3}m^2 - 20m + 56.$$

To prove the third point, we will show that P_m possesses a matrix of constraints, denoted M_m, such that $\mathrm{I}(M_m) \geq m$. M_m is composed of : (1) $4m^2$ gray-vertices $v_1, \ldots, v_{4m^2}$ that we assume

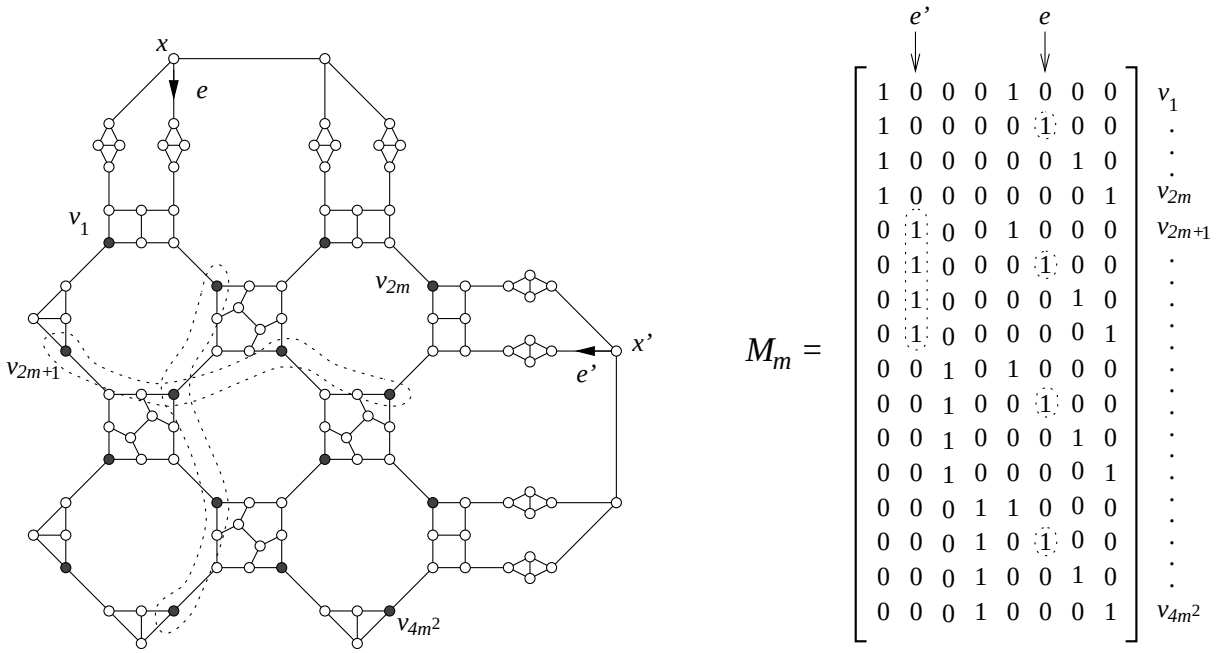

Figure 5: The 3-regular planar graph P_m with compacity at least m, here for $m = 2$.

indexed first by columns then by rows; (2) horizontal-arcs (the first $2m$ columns) and vertical-arcs (the last $2m$ ones). Figure 5 shows an example of the graph P_m and its matrix of constraints M_m. M_m is a matrix of constraints of P_m because one can easily check that P_m satisfied the following property: for every gray-vertex y and every vertical-arc e of tail x, a shortest path from x to y in P_m uses the arc e if and only if x and y belong to the same "column". "Column" means the same subset of gray-vertices like it is depicted by a doted line on Figure 5 (the situation is symmetric for vertical-arcs).

First assume that x and y are not on the same column. Let us call P' any shortest path from x to y that uses an arc e' different of e but incident to e. If there exists a shortest path P from x to y using e it has to cross two structures A. It implies that the length of P is at least one more than the length of P' (for P one must pay 2 for structures AA, whereas for P' one pays only 1 for the arc e').

Now, if x and y are located on the same column, a path using the horizontal copy of the structure E cannot be shorter, because of length of the structures D. Indeed, suppose that x is the tail of the most left vertical-arc. x is connected to two vertical-arcs, e_1 and e_2 (assume that e_2 is the left most arc). Finally, let L denote the length of the structure D. The length of any path from x to the gray-vertex v_{4m^2-1} traveling e_1 has of length at most $L + 6m$, whereas the length of any path using e_2 and the horizontal copy of the structure E is of length at least $3L + 2m + 11$. It implies that the length of D, *viz* L, must be such that $L + 6m < 3L + 2m + 11$, implying $L \geq 2m - 5$. Therefore M_m is a matrix of constraints of P_m.

Let us show that $\mathrm{I}(M_m) \geq m$. Consider each row of M_m as a binary word of length $4m$ bits. Clearly, each row has exactly 2 bits set to 1. It follows that any two rows of M_m differ in at least 2 places. By application of Lemma 4.2 of [7], we can give a trivial lower bound on the compacity of M_m that is $\mathrm{I}(M_m) \geq 2 \cdot 4m^2/(2 \cdot 4m) = m$ (the total number of blocks of consecutive ones being

$4m^2$), that ends the proof. □

It is easy to find a labeling of P_m to state that $\text{IRS}(P_m) = m + O(1)$ ($2m + O(1)$ is trivial, $m + O(1)$ can use a Z-like labeling of gray-vertices). Moreover, since the total number of blocks of consecutive ones on $4m$ arcs is $4m^2$ and since there is at most $m + O(1)$ intervals on every arcs, it means that P_m possesses $\Omega(m)$ arcs with $\Omega(m)$ intervals each.

Proof of Theorem 4.

Let $m = \lfloor x_0 \rfloor$, where x_0 is the positive solution of the equation $92x^2/3 - 20x + 56 = n$ ($x_0 = (\sqrt{69n - 3639} + 15)/46$). For every n large enough $m \geq 2$, and $92m^2/3 - 20m + 56 \leq n$. Applying Lemma 4, the graph P_m has at most n vertices and is such that $\text{IRS}(P_m) \geq m$. For every n large enough this yields $\text{IRS}(P_m) > x_0 - 1 \geq \sqrt{n}/5.6$, since $(\sqrt{69}/46)^{-1} \approx 5.537$. □

If we relax the 3-regular constraint, then we can derive a better lower bound for the compacity of a planar graph. In particular, we have the following result that shows that 4-regular planar graphs can require a large compacity.

Theorem 5 *There exists a 4-regular planar graph of order at most n having a compacity $\Omega(\sqrt{n})$.*

Note that no d-regular planar graph exists for $d \geq 6$.

Proof. The technique used is similar to the one of Theorem 4. We consider the graph P'_m, built for every $m \geq 2$, drawn on Figure 6. Note that, like the graph P_m, the graph is composed of a structure D of length $L = 2m + O(1)$ that preserves the constraints between the distinguished arcs and the gray-vertices. One can easily check that the boolean matrix M_m, defined in the proof of Theorem 4, is a matrix of constraints of P'_m as well. □

Janardan and Siu Cheng in [11, Lemma 1] showed that every planar graph G composed of p *planes* satisfied $\text{IRS}(G) \leq 3p/2$. The number of planes of G is defined as being the smallest number of faces needed to cover all the nodes, taken over all possible plane embedings of G. For example, the 1-plane graphs are precisely the outer-planar graphs. Unfortunately the number of planes of a graph of order n can raise up to $\Theta(n)$.

4 Conclusion

We proved a tight lower bound of the compacity for 3-regular networks. We gave also a new lower bound of $\Omega(\sqrt{n})$ intervals for 3-regular (and 4-regular) planar graphs. It would be interesting to know if a best lower bound can be found for planar graphs (not only for bounded degree graphs).

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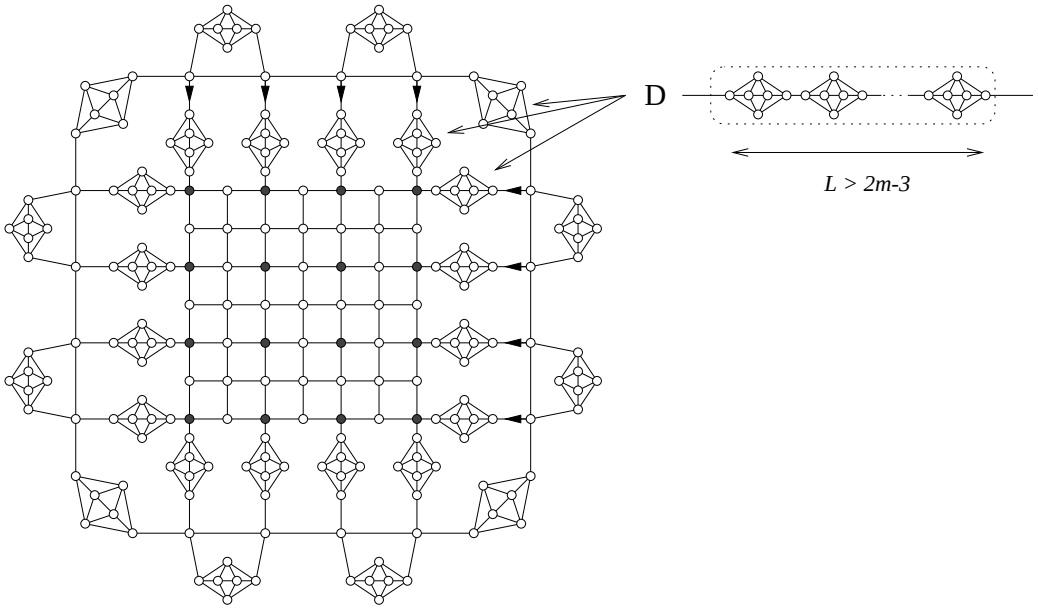

Figure 6: A 4-regular planar graph P'_m of order $\Theta(m^2)$ and of compacity at least m (here $m = 2$).

References

[1] E. M. Bakker, J. van Leeuwen, and R. B. Tan, *Linear interval routing*, Algorithms Review, 2 (1991), pp. 45–61.

[2] M. Flammini, J. van Leeuwen, and A. M. Spaccamela, *Lower bounds on interval routing*, Tech. Rep. 69, Università di L'Aquila, Dipartimento di matematica Pura ed Applicata, Oct. 1994.

[3] P. Fraigniaud and C. Gavoille, *A characterization of networks supporting linear interval routing*, in 13^{th} Annual ACM Symposium on Principles of Distributed Computing (PODC), ACM PRESS, ed., Aug. 1994, pp. 216–224.

[4] ——, *Optimal interval routing*, in Parallel Processing: CONPAR '94 - VAPP VI, B. Buchberger and J. Volkert, eds., vol. 854 of Lecture Notes in Computer Science, Springer-Verlag, Sept. 1994, pp. 785–796.

[5] ——, *Local memory requirement of universal routing schemes*, in 8^{th} Annual ACM Symposium on Parallel Algorithms and Architecture (SPAA), ACM PRESS, ed., June 1996. To appear.

[6] G. N. Frederickson and R. Janardan, *Designing networks with compact routing tables*, Algorithmica, 3 (1988), pp. 171–190.

[7] C. Gavoille, *Complexité mémoire du routage dans les réseaux distribués*, PhD thesis, École Normale Supérieure de Lyon, 46, allée d'Italie, Jan. 1996.

[8] C. Gavoille and E. Guévremont, *On the compactness of bounded degree graphs for shortest path interval routing*, in 2^{nd} Colloquium on Structural Information & Communication Complexity (SIROCCO), L. M. Kirousis and E. Kranakis, eds., Carleton University Press, June 1995, pp. 113–121.

[9] ——, *Worst case bounds for shortest path interval routing*, Research Report 95-02, LIP, École Normale Supérieure de Lyon, 69364 Lyon Cedex 07, France, Jan. 1995.

[10] C. Gavoille and S. Pérennès, *Memory requirement for routing in distributed networks*, in 15^{th} Annual ACM Symposium on Principles of Distributed Computing (PODC), ACM PRESS, ed., May 1996. To appear.

[11] R. Janardan and W. Siu Cheng, *Efficient distributed algorithms for single-source shortest paths and related problems on plane networks*, in Distributed Algorithms, 4^{th} International Workshop (WDAG), G. Goos, J. Hartmauis, J. v. Leeuwen, and N. Santoro, eds., vol. 486 of Lecture Notes in Computer Science, Springer-Verlag, Sept. 1990, pp. 133–150.

[12] E. Kranakis and D. Krizanc, *Lower bounds for compact routing*, in 13^{th} Annual Symposium on Theoretical Aspects of Computer Science (STACS), Feb. 1996.

[13] E. Kranakis, D. Krizanc, and S. S. Ravi, *On multi-label linear interval routing schemes*, in 19^{th} International Workshop on Graph - Theoretic Concepts in Computer Science - Distributed Algorithms (WG), vol. 790 of Lecture Notes in Computer Science, Utrecht, June 1993, Springer-Verlag, pp. 338–349.

[14] N. Santoro and R. Khatib, *Labelling and implicit routing in networks*, The Computer Journal, 28 (1985), pp. 5–8.