Lower Bounds for Interval Routing on 3-Regular Networks

Cyril Gavoille[1] and Stéphane Pérennès[2]

[1]LIP, CNRS
École Normale Supérieure de Lyon*
69364 Lyon cedex 07, France
(*gavoille@ens-lyon.fr*)

[2]I3S, CNRS
Université de Nice-Sophia Antipolis**
Simon Fraser University
06903 Sophia Antipolis, France
(*sp@essi.fr*)

Abstract. Interval Routing was introduced to reduce the size of the routing tables: a router finds the direction to forward a message by determining the interval that contains the destination address of the message, each interval being associated to one particular direction. In this paper, we give lower bounds for the minimum number of intervals per edge needed to achieve shortest path routing for the class of 3-regular networks. We prove a tight lower bound of $\Theta(n)$ intervals for a 3-regular network of order at most n. Moreover, for the particular case of 3-regular planar networks, we establish a lower bound of $\Omega(\sqrt{n})$ intervals.

1 Introduction

Table routing is a standard solution to the shortest path routing problem for arbitrary networks. At each router in the network is stored a table listing for each possible destination the output port that should be used to send a message along a shortest path. This solution guarantees shortest paths but requires $O(n \log d)$ bits of memory per node, for an n-router network of maximum degree d. The *compact routing problem* is to implement routing schemes that uses a minimum memory size on each router. The interval routing scheme was introduced by Santoro and Khatib in [19] to improve the space requirements of routing tables.

In [16], we showed that it is not possible to find a "compact" routing scheme for all networks, *i.e.*, universal routing schemes that produce in each router compact data structures for the routing information. We showed that a constant fraction of the routers of the network need to store $\Omega(n \log d)$ bits of information

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independantly from the routing scheme used. This implies that the compact routing problem could be tackled with two dual approaches:

- find the best routing scheme for a given (small) class of networks,
- find all networks requiring small memory requirements for a given routing scheme.

The second approach consists in graph characterization. For the interval routing scheme, this approach has been investigated in [1, 5, 8]. The first approach consists in finding lower and upper bounds for given classes of graphs. The state of the art in the compact routing problem is presented in [7] for the general case, and in [11, page 88] for graphs of maximum degree d. In this paper we study the case of 3-regular networks for the shortest path[3] interval routing scheme.

The model of network is a symmetric digraph (drawn as an undirected graph), each arc represents a link. Graphs are finite, connected, loop-less and without multi-arc. In order to sharpen drawings, we depict bidirectional links with a single edge between two neighbor nodes. The basic idea of the interval routing is the following: (1) each nodes is labeled with an unique integer taken from the set $\{1, \ldots, n\}$, where n is the number of nodes of the network; (2) to each arc e is assigned a set I_e containing all the labels of the destinations using the arc e to receive messages routed along shortest paths. Moreover, the sets assigned to arcs emanating from the same node must be pairwise disjoint, and their union must cover $\{1, \ldots, n\}$. The interval routing coding strategy consists in representing every destination set I_e by an union of disjoint intervals. An "interval" is a subset of consecutive integers taken from $\{1, \ldots, n\}$, where n and 1 are considered consecutive. The routing is performed as follows: upon reception of a message M in a node labeled x, the label of the destination, y, attached to the header of M is first compared with x, to check if the message has arrived at its destination. If not, then M and its header are forwarded trough the unique arc e incident to x such that $y \in I_e$.

The goal of the interval routing is to find a labeling of nodes and an assignment of destination sets that minimizes the number of intervals on each arc. If, for a graph G, this number of intervals is at most k per arc then the memory space of a router of G of maximum degree d is at most $O(dk \log n)$ bits. Indeed, it is sufficient to store the end-points of each interval. The *compactness* of a graph G, denoted by $\textsc{Irs}(G)$, is defined as the smallest integer k such that each destination set of G is composed of at most k intervals. Since the interval routing was introduced in order to reduce the memory space of routers, compactness is an important parameter to consider.

The general study on the compactness of a graph of order n has been investigated by many authors, see in particular [18] and [4]. Unfortunately, Gavoille and Guévremont in [14], build graphs having a compactness of at least $n/12$ intervals, proving uneventful that interval routing scheme is not compact for all graphs of order n. Study of compactness for smallest classes of graphs, like bounded degree graphs, was first proposed in [13]. Their lower bound has been

[3] It means that the routing induces only shortest paths.

improved in [17], where Kranakis and Krizanc proved that some bounded degree graphs have a compactness of $\Omega(n/\log n)$ intervals.

In [16] we have proved a tight lower bound of the space requirement for any *universal* shortest path routing scheme. We showed that there exist graphs where the total space required to store the routing information of all the nodes is $\Omega(n^2 \log d)$ bits, where d is the maximum degree of the graphs. In particular if d is constant this implies a lower bound of $\Omega(n)$ bits per router for the local routing information. Therefore, we can derive a trivial lower bound of $\Omega(n/\log n)$ (that is not a new result [17]) for the compactness of a bounded degree graph, since a router using interval routing scheme on a graph of compactness k and of bounded degree cannot store more than $O(k \log n)$ bits of information. In this paper, we show that this lower bound is not tight by proving that the maximum compactness of a 3-regular graph of order at most n is $\Theta(n)$. Moreover, this number of intervals can be required on $\Omega(n)$ edges, that is asymptotically the total number of edges of the graph. This tight lower bound shows that the worst case for interval routing does not depend on the maximum degree of the graph, whereas the memory requirements of universal routing schemes do [16]. We also show a lower bound of $\Omega(\sqrt{n})$ for 3-regular graphs that are planar. Note that to prove lower bounds on the compactness of 3-regular graphs is stronger than to prove lower bounds for larger classes of graphs, like bounded degree graphs or even unbounded degree graphs.

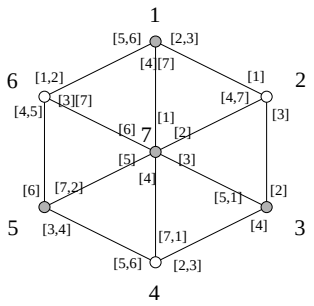
In the next section, we introduce definitions and tools to prove the lower bounds of the paper. In Section 3, we prove a tight $\Theta(n)$-lower bound of compactness of 3-regular graphs. In Section 4, we give a worst-case construction establishing a $\Omega(\sqrt{n})$-lower bound for 3-regular planar graphs.

2 Statement of the Problem

2.1 A simple example

Let us consider the following example of interval routing on a graph G_0 of 7 vertices depicted on Figure 1. Nodes are labeled by integers from 1 to 7, and intervals are assigned to each arc. If the node 5 sends a message to node 1, the message will successively be forwarded along the arc $(5,7)$ then along $(7,1)$, because $1 \in I_{(5,7)} = [7,2] = \{7,1,2\}$ and $1 \in I_{(7,1)} = [1] = \{1\}$. Each destination set I_e, e arc of G_0, is composed of at most two intervals of consecutive labels. Therefore, $\textsc{Irs}(G_0) \leq 2$. However, another vertex-labeling and another choice of shortest paths might decrease this number. For example, there exists two shortest paths from 5 to 1, but one would force to add one more interval on the arc $(5,6)$: $I_{(5,6)} = [6] \cup [1]$. We will see in the next paragraph a simple proof showing that the compactness of G_0 cannot be 1, and thus $\textsc{Irs}(G_0) = 2$.

It should be borne in mind that the problem of finding the compactness of a graph is NP-complete [3]. More precisely, Flammini in [2], showed that to determine if a graph has a compactness 2 is already NP-complete. Note that for every graph G, $\textsc{Irs}(G) \leq n/2$, since any destination set I_e cannot contain more than $n/2$ non consecutive integers.



$$M_0 = \begin{array}{|ccc|c} \multicolumn{3}{c}{(6,7)\ (2,7)\ (4,7)} & \\ \hline 1 & 1 & 1 & 7 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \\ \hline \end{array}$$

Fig. 1. An example of interval routing with at most 2 intervals per link. On the right, a matrix of constraints of this graph.

2.2 Matrices of constraints

The key of our proof is to use some graphs built from a boolean matrix. This technic, already used in [14] and in [4], is related to the *matrices of constraints*. Informally, for every triple of vertices (u, v, w) of a graph G, where u and v are adjacent and $u \neq w$, three cases may occur for shortest paths from u to w:

1. Every shortest path from u to w uses the arc (u, v).
2. Every shortest path from u to w does not use the arc (u, v).
3. There are shortest paths from u to w that use arc (u, v) and there are shortest paths from u to w that do not use the arc (u, v), depending on the choice of the shortest path routing function.

For the first two cases, the routing decision does not depend on the shortest path routing function. So, we say that arc (u, v) forms a *constraint* for the vertex w. A *matrix of constraints* of a graph G is a $p \times q$ boolean matrix $M = (m_{i,j})$ whose rows are labeled with some vertices of G $\{v_1, \ldots, v_p\}$, and whose columns are labeled with some arcs $\{e_1, \ldots, e_q\}$ of G such that:

$$\begin{cases} m_{i,j} = 1 \Leftrightarrow \text{every shortest path from the tail of } e_j \text{ to } v_i \text{ uses the arc } e_j. \\ m_{i,j} = 0 \Leftrightarrow \text{no shortest path from the tail of } e_j \text{ to } v_i \text{ uses the arc } e_j. \end{cases}$$

For example, the matrix M_0 of Figure 1 is a matrix of constraints of the graph draw on Figure 1: $I_{(6,7)}$ must contain 7 and 3, but 5 and 1; $I_{(2,7)}$ must contain 7 and 5, but 3 and 1; $I_{(4,7)}$ must contain 7 and 1, but 3 and 5. Note that a matrix of constraints of a given graph is not unique in general.

The main interest of matrices of constraints is for proving lower bound on the compactness for particular graphs. Assume that M is a matrix of constraints for a graph G. It is not difficult to see that the maximum number of blocks of consecutive ones in a column of M, minimized over all row permutations of M, is a lower bound on the compactness of G. This fact was first remarked in [4], then developed and generalized in [14]. A natural way of defining the *compactness of a boolean matrix* M, denoted by $\mathrm{I}(M)$, is as the maximum (taken over all the columns of M) of the number of blocks of consecutive ones in a column of M,

minimized over all row permutations of M. For every matrix of constraints M of G, we have of course $\textsc{Irs}(G) \geq \mathrm{I}(M)$. For instance, $\mathrm{I}(M_0) = 2$, because whatever the row permutation of M_0 there always exists a column that needs two blocks of consecutive ones. Therefore, $\textsc{Irs}(G_0) \geq 2 = \mathrm{I}(M_0)$.

3 A Tight Lower Bound for 3-Regular Graphs

In this section, we will build a 3-regular graph of order at most n having a matrix of constraints of compactness $\Theta(n)$.

3.1 Graph construction

Our construction is based on the construction given in [16]. For every integer $p \geq 2$, we define H_p by induction as follows (see Figure 2 for H_2). In the graph H_p, we distinguish a set of arcs $\{e_1, \ldots, e_p\}$ and a set of 2^p vertices B_p (drawn in gray on Figure 2). Each vertex of B_p is denoted by an unique binary word of p bits. We denote by A_p the set of tails of the arcs e_i, for $i \in \{1, \ldots, p\}$. H_{p+1} is composed of: a copy of H_p; two complete binary trees of 2^{p-1} leaves; 2^p intermediary vertices forming a set B'_p; 2^{p+1} news vertices forming the set B_{p+1} (that is also the set of all binary words of length $p+1$ bits). The roots of the two trees are connected by an edge, and one of its arcs is e_{p+1}. Vertices of B_p and vertices of B'_p are connected by a matching. Each vertex $x \in B'_p$ is connected to the vertices $x0$ and $x1$ of B_{p+1} (x being a binary word of p bits). Moreover, every vertex $w \in B_{p+1}$ is connected to a leaf of the first tree if $w = x0$, and to a leaf of the second tree otherwise, and every leaf of the two trees has a maximum of two neighbors in B_{p+1}.

The graph H_p has an important property: for any vertex $a_i \in A_p$, $i \in \{1, \ldots, p\}$, and for any vertex $w \in B_p$, every shortest path from a_i to w has to use the arc e_i if and only if the i-th bit of w is 1. This fact can be trivially shown by induction on p. Clearly, this property implies that H_p has a particular matrix of constraints that is the $2^p \times p$ boolean matrix whose i-th row is the integer $i-1$ written in binary. Indeed, each arc of $\{e_1, \ldots, e_p\}$ forms a constraint for all vertices of B_p. For instance, in the graph H_3 drawn on Figure 2, the arcs e_1, e_2 and e_3 form constraints for all the vertices of B_3.

The idea is to build a new graph by connecting two copies of H_p by a perfect matching between vertices of the sets B_p. The perfect matching is defined as a permutation π of the set $\{1, \ldots, 2^p\}$. This new graph is denoted by $G_{p,\pi}$. An example of the $G_{p,\pi}$ graph is depicted[4] on Figure 3 for a particular permutation π.

For every integer p and every permutation π of $\{1, \ldots, 2^p\}$, we denote by $M_{p,\pi}$ the $(2^p) \times (2p)$ boolean matrix defined as follows: the i-th row of $M_{p,\pi}$ is

[4] Recall that graphs are symmetric digraphs and depicted as undirected. However, some arcs are marked on Figure 2 just to distinguish in the discussion the tail and the head.

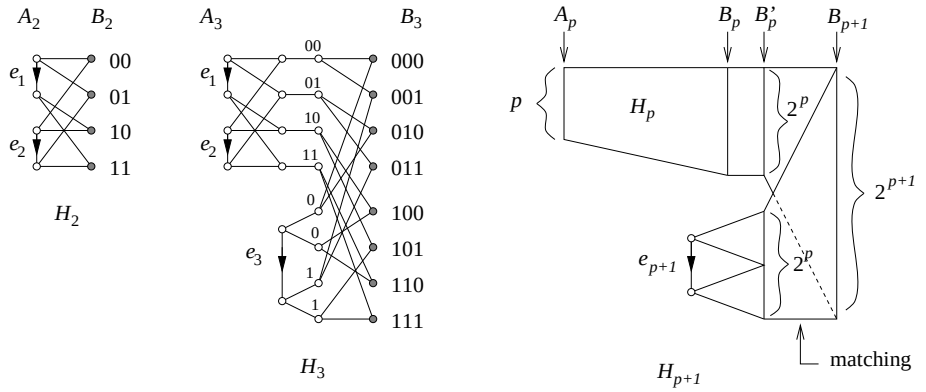


Fig. 2. The recursive construction of the graph H_p.

composed of $2p$ bits, the first p are the binary writing of $i-1$ and the last p bits are the binary writing of $\pi(i)-1$.

Example: $\pi_3 = (7,5,1,2,8,4,3,6)$ (see Figure 3).

Lemma 1. *Let $p \geq 2$ and π be a permutation of $\{1,\ldots,2^p\}$. The graph $G_{p,\pi}$ satisfies:*

1. *$G_{p,\pi}$ is 3-regular and of order $5\cdot 2^{p+1} - 4p - 16$*
2. *$G_{p,\pi}$ possesses $M_{p,\pi}$ as matrix of constraints*

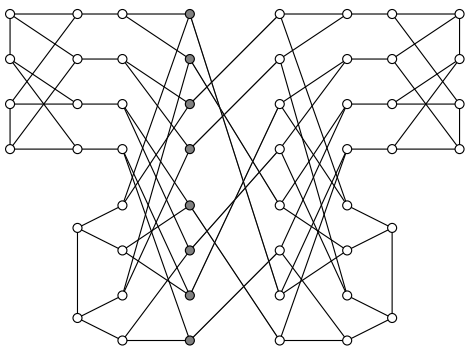
Proof. By construction, $G_{p,\pi}$ is 3-regular, and its order is $5\cdot 2^{p+1} - 4p - 16$. Let B_1 be the set B_p of the first copy of H_p (grayed vertices on Figure 3), and A'_p and B'_p the sets of distinguished vertices of the second copy of H_p. $M_{p,\pi}$ is a matrix of constraints of $G_{p,\pi}$. Indeed, from the basic property of graphs H_p there exists a unique shortest path between the vertices of A'_p and the vertices of B'_p, and thus an unique shortest path between the vertices of A'_p and the vertices of the set B_1 (which uses the perfect matching between the two copies of the graph H_p). □

Now, let us show that this graph requires a large compactness in general.

3.2 A $\Theta(n)$-lower bound for compactness

We will first compute the compactness of matrices $M_{p,\pi}$. The matrix M_{3,π_3} on Figure 3 has 4 blocks of consecutive ones in its third column. However, it does not implies that $\mathrm{I}(M_{3,\pi_3}) = 4$, because rows of the matrix can be permuted to decrease this number, corresponding for the graph G_{3,π_3} (on Figure 3) to a possible efficient vertex-labeling. For instance, exchanging the 6-th and 7-th row of M_{3,π_3} turns out $\mathrm{I}(M_{3,\pi_3}) \leq 3$. However, we have the following lower bound for an arbitrary $M_{p,\pi}$ matrix:

$$M_{3,\pi_3} = \begin{bmatrix} 000 & 110 \\ 001 & 100 \\ 010 & 000 \\ 011 & 001 \\ 100 & 111 \\ 101 & 011 \\ 110 & 010 \\ 111 & 101 \end{bmatrix}$$



G_{3,π_3}

Fig. 3. An example of the $G_{p,\pi}$ construction for $p = 3$ and $\pi = \pi_3$.

Lemma 2. *For every $p \geq 1$, there exists a permutation π such that $\mathrm{I}(M_{p,\pi}) \geq k$ with*

$$\left(\sum_{i=0}^{k} \binom{2^p}{2i}\right)^{2p} \geq (2^p)! \tag{1}$$

Proof. Let p be an integer and let $\mathcal{A}$ be the set of matrices $M_{p,\pi}$ for all permutations π of $\{1, \ldots, 2^p\}$. Let $k = \max\{\mathrm{I}(M) \mid M \in \mathcal{A}\}$, that is the maximum compactness of matrices of $\mathcal{A}$. Let us prove that k satisfies Equation 1. Let f be a function that returns, for every matrix $M \in \mathcal{A}$, a matrix $M' = f(M)$ that is obtained from M by rows permutation such that every column of M' has at most k blocks of consecutive ones. By definition of k, such a function f exists. We denote by $\mathcal{B}$ the set of $(2^p) \times (2p)$ boolean matrices for which all the columns are composed of at most k blocks of consecutive ones. By definition $f(\mathcal{A}) \subset \mathcal{B}$, that implies $|f(\mathcal{A})| \leq |\mathcal{B}|$. Clearly, f is a bijective function because for every matrix $M' \in f(\mathcal{A})$ we can permute the rows of M' in order to obtained the matrix $f^{-1}(M')$, where the first p bits of the rows is the sequence of binary words 0...00, 0...01, 0...10, 0...11, and so on, until 1...11. Therefore, $|f(\mathcal{A})| = |\mathcal{A}| = (2^p)! \leq |\mathcal{B}|$. On the other hand, there exists $\binom{2^p}{2i}$ ways to have exactly i blocks of consecutive ones in a binary column of length 2^p, and thus at most $\sum_{i=0}^{k} \binom{2^p}{2i}$ ways to have at most k blocks of consecutive ones in a column. So, there is at most $(\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$ ways to have a matrix having at most k blocks of consecutive ones on each of its $2p$ columns. In sequel, $|\mathcal{B}| \leq (\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$, and finally k must satisfy $(2^p)! \leq (\sum_{i=0}^{k} \binom{2^p}{2i})^{2p}$. □

Corollary 3. *For every $p \geq 3$, there exists a permutation π such that $\mathrm{I}(M_{p,\pi}) = \Omega(2^p)$.*

In a sense this result is optimal because for every p and for every π, $\mathrm{I}(M_{p,\pi}) \leq 2^p/2$, since a matrix of 2^p rows cannot require more than $2^p/2$ intervals per column.

Proof. Applying Lemma 2, for every $p \geq 3$ there exists a permutation π such that Equation 1 is satisfied. Let $k = \mathrm{I}(M_{p,\pi})$. If $2k \geq 2^p/2$ then $\mathrm{I}(M_{p,\pi}) = \Omega(2^p)$. Otherwise, $\sum_{i=0}^{k} \binom{2^p}{2i} \leq (k+1)\binom{2^p}{2k}$, since in this case $\binom{2^p}{2k}$ is the maximum term of the sum. Moreover, since $k < 2^{p-2}$, we get $(\sum_{i=0}^{k} \binom{2^p}{2i})^{2p} \leq 2^{2p(p-2)}\binom{2^p}{2k}^{2p}$. For any two integers n and p, $(n/e)^n \leq n!$ and $\binom{n}{p} \leq (ne/p)^p$ (where $e = 2.718281828...$). Therefore, by Equation 1, k must satisfy:

$$\left(\frac{2^p}{e}\right)^{2^p} \leq 2^{2p(p-2)}\left(\frac{2^p e}{2k}\right)^{4kp} \quad \Rightarrow \quad 2^p \log_2\left(\frac{2^p}{e}\right) \leq 2p(p-2) + 4kp\log_2\left(\frac{2^p e}{2k}\right)$$

Let us define α by $\alpha = 2^p/(2k)$, and let us show that α is upper-bounded by a constant. The latter equation implies:

$$2^p(p - \log_2 e) \leq 2p(p-2) + \frac{2p \cdot 2^p}{\alpha}\log_2(\alpha e) \Rightarrow l(p) = \frac{1}{2} - \frac{\log_2 e}{2p} - \frac{p-2}{2^p} \leq \frac{\log_2(\alpha e)}{\alpha}$$

The left term of the latter inequality, $l(p)$, is an increasing function as p (from $p \geq 3$). In particular for every $p \geq 3$, this implies that α must satisfy $0.134 < l(3) \leq l(p) \leq (\log_2(\alpha e))/\alpha$, implying $\alpha < 54$. Therefore, for every $p \geq 3$, $k = 2^p/(2\alpha) > 2^p/108 = \Omega(2^p)$. □

For such matrices of large compactness, one can check that $\Omega(p)$ columns require $\Omega(2^p)$ blocks of consecutive ones.

We can now derive the main Theorem of this section:

Theorem 4. *There exists a 3-regular graph G of order at most n such that* $\mathrm{IRS}(G) = \Theta(n)$.

Proof. $\mathrm{IRS}(G) \leq n/2$. Applying Lemma 1 and Corollary 3, there exists a 3-regular graph $G_{p,\pi}$ such that $\mathrm{IRS}(G_{p,\pi}) \geq \mathrm{I}(M_{p,\pi}) = \Omega(2^p)$. The order of $G_{p,\pi}$ is $5 \cdot 2^{p+1} - 4p - 16$. In particular, for every integer n large enough there exists p such that $5 \cdot 2^{p+1} \leq n < 5 \cdot 2^{p+2}$, $G_{p,\pi}$ is of order at most n and of compactness at least $\mathrm{I}(M_{p,\pi}) = \Omega(n)$. □

3.3 Large compactness required by $\Theta(n)$ edges

Even though the local memory requirement seems to be more important in practice for the concrete design of routing chips, the number of routers requiring a large compactness can give a theoretical information about the "routing complexity" of the network. We show that not only the compactness of a 3-regular graph can be very large, but also it can be required by many edges of the graph.

We have already seen that $\Omega(p)$ columns of some matrices $M_{p,\pi}$ require each $\Omega(2^p)$ intervals. Actually, it means that $\Omega(p) = \Omega(\log n)$ arcs of the graph $G_{p,\pi}$ require $\Omega(n)$ intervals. Using the idea devellopped in [16], we can proved that the lower bound of $\Omega(n)$ intervals can be required on $\Omega(n)$ arcs, for another graph construction. In the following we say that a vertex u is a *constrained vertex* for a subset of vertices B if there exists an arc (u, v) such that for every vertex $w \in B$, (u, v) forms a constraint for w (see also Paragraph 2.2 for definition).

Lemma 5. *Let G be a 3-regular graph of order n and A be a set of p constrained vertices for a subset of vertices B of G. Each constrained vertex $x \in A$ can be "split" in $\Theta(n/p)$ other vertices forming a new graph G' such that every new split vertex has the same constraints as x for the set B. Moreover, G' is 3-regular and is of order $\Theta(n)$.*

The proof of this Lemma can easily be derived from [16, Lemma 6].

Theorem 6. *There exists a 3-regular graph of order at most n having a compactness of $\Omega(n)$, required by $\Theta(n)$ edges.*

Proof. Let G be the graph of Theorem 4 of order at most n and having $p = \Omega(\log n)$ arcs requiring $\Omega(n)$ intervals each one. Therefore, by the graph transformation of Lemma 5, there exists a 3-regular graph of order at most $n' = \Theta(n)$ and of compactness $\Omega(n) = \Omega(n')$. This number is independently required by $\Theta(n/p)$ groups of p arcs, thus required by a total of $\Theta(n) = \Theta(n')$ arcs. □

4 Lower Bound for Planar Graphs

4.1 Compactness of 3-regular planar graphs

After the study of graphs of bounded degree, we investigate another class of sparse graphs that includes a large range of potential topologies for real networks: the class of planar graphs. In this section we give a lower bound for the maximum compactness of a planar graph of order n. General study of compact routing for planar graphs is principally due to Frederickson and Janardan [8, 9, 10]. They showed in [8] that every outer-planar graph is of compactness 1, that is, a planar graph whose all its vertices lie on a face. The best known lower bound is due to Tse and Lau in [20]. They built a planar graph of maximum degree $\Omega(\sqrt{n})$, and of compactness at least $\sqrt{n}/8.5$ intervals[5] on at least one edge, improving the previous $\Omega(\sqrt[3]{n})$-lower bound of Kranakis, Krizanc and Ravi [18]. We tighten this lower bound for a 3-regular planar graph with the following theorem:

Theorem 7. *For every integer n large enough, there exists a 3-regular planar graph G of order at most n such that*

$$\textsc{Irs}(G) > \frac{\sqrt{n}}{5.6}$$

Moreover, we will see that $\Omega(\sqrt{n})$ intervals can be required on $\Omega(\sqrt{n})$ edges. Our proof is constructive and uses matrices of constraints. For every integer $m \geq 2$, we give the construction of a 3-regular planar graph denoted by P_m.

Description of the P_m graph.

Let $m \geq 2$ be an integer. Roughly speaking, the P_m graph is composed of a $m \times m$ grid with two trees: one connected to a vertical border, and the other

[5] The exact value is $\sqrt{n/72}$.

connected to the horizontal border of the grid. In order to have a 3-regular graph, each vertex is removed by small 3-regular structures. Each one is composed of smaller structures namely A, B, C, D, and E as depicted on the right side of Figure 4. They are arranged by their free links. The left side of Figure 4 represents the general topology of the P_m graph with the arrangement of the structures $A, \ldots, E$. The structures $A, \ldots, E$ can be swiveled round in the plane to join their free links. Figure 5 shows an example of the graph P_m for $m = 2$.

Some vertices and some arcs of P_m are distinguished: we say that u is a *gray-vertex* if u is one of the vertices of the structures A, B, or of C drawn in gray on Figure 4. Moreover, we say that e is a *vertical-arc* (respectively *horizontal-arc*) if e is an arc distinguished on the structure D connected on the vertical border (respectively horizontal border) of P_m. Let us recall the every two neighbors are connected by an edge that is composed of two symmetric arcs. The structure D is an arrangement of k blocks of length 3 (two triangles sharing an edge) and must be of minimum length such that $L \geq 2m - 3$ and $L = 3k - 1$. Thus the length of D is such that $2m-3 \leq L \leq 2m-1$. The number of blocks for a length L is $k = \lceil (L+1)/3 \rceil$. Therefore, for every $m \geq 2$, the total number of vertices of D is at most $8k = 8\lceil (L+1)/3 \rceil \leq 8\lceil 2m/3 \rceil < 16m/3 + 8$.

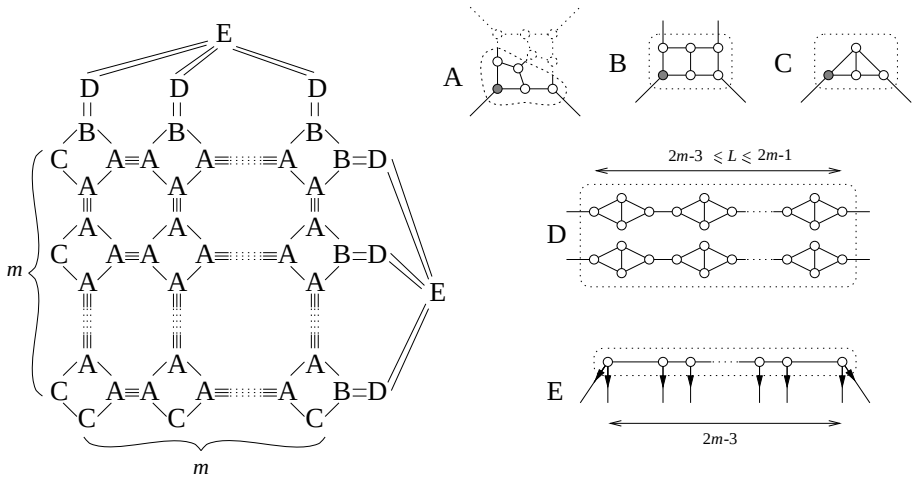


Fig. 4. Construction of the P_m graph.

Lemma 8. *For every integer $m \geq 2$,*

- *P_m is 3-regular and planar*
- *P_m is of order at most $92m^2/3 - 10m + 56$*
- *$\textsc{Irs}(P_m) \geq m$*

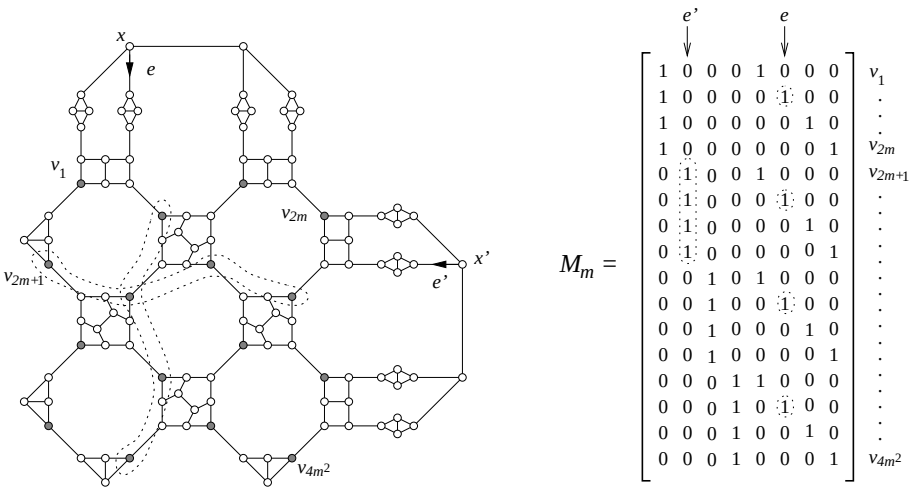


Fig. 5. The 3-regular planar graph P_m with compactness at least m of order $\Theta(m^2)$, here for $m = 2$.

Proof. Let $m \geq 2$. By construction, P_m is planar and 3-regular. We denote by A, B, ..., E the order of basic structures of P_m on Figure 4. Note that $A = 5$, $B = 6$, $C = 4$, $D < 16m/3 + 8$, and $E = 2m - 2$. Referring to the left side of Figure 4, the order of P_m is:

$$n = 2E + 2m(D + B + C) + 4(m-2)^2 A + 6(m-2)A + 8A < \frac{92}{3}m^2 - 10m + 56.$$

To prove the third point, we will show that P_m possesses a matrix of constraints, denoted M_m, such that $\mathrm{I}(M_m) \geq m$. M_m is composed of: (1) $4m^2$ gray-vertices $v_1, \ldots, v_{4m^2}$ that we assume indexed first by columns then by rows; (2) horizontal-arcs (the first $2m$ columns) and vertical-arcs (the last $2m$ ones). Figure 5 shows an example of the graph P_m and its matrix of constraints M_m. M_m is a matrix of constraints of P_m because one can easily check that P_m satisfied the following property: for every gray-vertex y and every vertical-arc e of tail x, a shortest path from x to y in P_m uses the arc e if and only if x and y belong to the same "column". "Column" means the same subset of gray-vertices delimited by a dotted line depicted on Figure 5 (the situation is symmetric for vertical-arcs).

First assume that x and y are not on the same column. Let us call P' any shortest path from x to y that uses an arc e' different of e but incident to e. If there exists a shortest path P from x to y using e it has to cross two structures A. It implies that the length of P is at least one more than the length of P' (for P one must pay 2 for structures AA, whereas for P' one pays only 1 for the arc e').

Now, if x and y are located on the same column, a path using the horizontal copy of the structure E cannot be shorter, because of length of the structures

D. Indeed, suppose that x is the tail of the most left vertical-arc. x is connected to two vertical-arcs, e_1 and e_2 (assume that e_2 is the left most arc). Finally, let L denote the length of the structure D. The length of any path from x to the gray-vertex v_{4m^2-1} traveling e_1 has of length at most $L + 6m + 1$, whereas the length of any path using e_2 and the horizontal copy of the structure E is of length at least $3L + 2m + 8$. It implies that the length of D, *viz* L, must be such that $L + 6m + 1 < 3L + 2m + 8$, implying $L > 2m - 7/3$, and thus $L \geq 2m - 3$. Therefore M_m is a matrix of constraints of P_m.

Let us show that $\mathrm{I}(M_m) \geq m$. Consider each row of M_m as a binary word of length $4m$ bits. Each row is unique and has exactly 2 bits set to 1. It follows that any two rows of M_m differ in at least 2 places. The result of [4, Lemma 5] states that compactness of any $p \times q$ boolean matrix M such that any two rows differ of at least r places satisfies $\mathrm{I}(M) \geq rp/(2q)$. Therefore, we can give a trivial lower bound on the compactness of M_m that is $\mathrm{I}(M_m) \geq 2 \cdot 4m^2/(2 \cdot 4m) = m$, that ends the proof. □

It is easy to find a labeling of P_m to state that $\textsc{Irs}(P_m) = m + O(1)$ ($2m + O(1)$ is trivial, $m + O(1)$ can use a Z-like labeling of gray-vertices). Moreover, the total number of blocks of consecutive ones on $4m$ columns of M_m is $4m^2$. Since there is at most $m + O(1)$ intervals on every arcs, it follows that P_m possesses $\Omega(m)$ arcs with $\Omega(m)$ intervals each.

Proof of Theorem 7.

Let $m = \lfloor x_0 \rfloor$, where x_0 is the positive solution of the equation $92x^2/3 - 10x + 56 = n$ ($x_0 = (\sqrt{276n - 5127} + 5)/92$). For every n large enough, $m \geq 2$ and $92m^2/3 - 10m + 56 \leq n$. Applying Lemma 8, the graph P_m has at most n vertices and verifies $\textsc{Irs}(P_m) \geq m$. For every n large enough this yields $\textsc{Irs}(P_m) > x_0 - 1 > \sqrt{n}/5.6$, since $(\sqrt{276}/92)^{-1} \approx 5.537$. □

If we relax the constraint on the degree, we can improve a little the multiplicative constant to find a better lower bound for the compactness of a planar graph. However, this new lower bound is still in $\Omega(\sqrt{n})$.

Theorem 9. *There exists a planar graph G of order at most n and of maximum degree $\Theta(\sqrt{n})$ such that*

$$\textsc{Irs}(G) > \frac{\sqrt{n}}{2.6}$$

Proof. Consider, for every $m \geq 1$, the graph P'_m composed of a $(2m) \times (2m)$ grid and of two isomorphic trees T whose leaves are respectively connected to an horizontal and to a vertical border of the grid. T is a complete bipartite graph $K_{1,2m}$ where the root is the vertex of degree $2m$ and such that each edge is subdivided in $L - 1$ vertices, for a given integer $L \geq 1$. Therefore the distance between the root of T and its leaves is L. Now, let us consider the $4m^2$ vertices of the grid and the $4m$ arcs whose the tail is the root of the horizontal and the vertical tree. One can check that the boolean matrix M_m (defined in the proof of Lemma 8) is a matrix of constraints of P'_m if L satisfies $3L > 2m$. Therefore, for $L = \lfloor 2m/3 \rfloor + 1$, P'_m is of order $n = 4m^2 + 4m \lfloor 2m/3 \rfloor + 2 = 20m^2/3 + O(m)$,

and has M_m as matrix of constraints. Since $\text{IRS}(P'_m) \geq \text{I}(M_m) \geq m$, it implies for every integer n large enough that $\text{IRS}(P'_m) > \sqrt{n}/2.6$ ($\sqrt{20/3} \approx 2.581$). The maximum degree of P'_m is $2m = \Theta(\sqrt{n})$. □

4.2 Large compactness required by many edges

Like in Paragraph 3.3, we will see that the compactness of a planar graph of order n can be required by $\Theta(n)$ edges, that is also asymptotically the total number of edges of the graph.

Theorem 10. *There exists a planar graph of order at most n and of maximum degree $\Theta(\sqrt{n})$ having a compactness $\Omega(\sqrt{n})$, required by $\Theta(n)$ edges.*

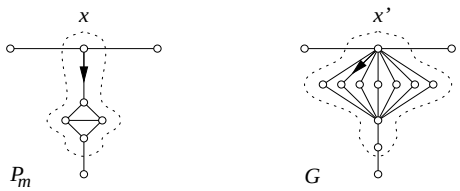


Fig. 6. The "planar-splitting" transformation shows that a large compactness can be required by $\Theta(n)$ edges.

Proof. Let $m = \lfloor c\sqrt{n} \rfloor + 2$, for some suitable constant c. By applying Lemma 8, the graph P_m is 3-regular, planar, and has $\Theta(m^2)$ vertices. We already have seen that P_m requires $\Omega(m)$ intervals on $\Omega(m)$ arcs (the distinguished vertical and horizontal-arcs). From P_m we build a new graph G, by transforming each vertical-arc (and horizontal-arc) of tail x by a $K_{2,m}$ as it is shown on Figure 6. Notice that, since $m \geq 2$, the structure D of P_m is composed of at least $\lceil (L+1)/3 \rceil \geq \lceil (2m-3)/3 \rceil \geq 1$ blocks of length 3 (see the description of the graph P_m). G is planar, is of maximum degree $2m+2$ (due to the vertex x'). The total number of vertices added is $O(m^2)$, thus G has $\Theta(m^2)$ vertices, that is less than n for a suitable constant c. The constraints from x' to any gray-vertex of G remain those of the vertex x in P_m, since the distances are preserved. Therefore, m independent groups of $\Omega(m)$ arcs require each $\Omega(m)$ intervals, so a total of $\Omega(m^2) = \Omega(n)$ arcs requiring each $\Omega(m) = \Omega(\sqrt{n})$ intervals. □

Theorem 10 shows that the $\Omega(\sqrt{n})$-lower bound can be required by $\Theta(n)$ edges, unfortunately for an unbounded degree planar graph. The following theorem, presented in [12], generalizes this result. It allows to derive a tradeoff between the compactness, the number of edges requiring such a compactness and the maximum degree.

Theorem 11 [12]. *For every integer $d \geq 3$ and for every integer k such that $d-1 \leq k \leq \sqrt{n \log_d n}$, there exists a planar graph of order at most n and of*

degree at most d having a compactness $\Omega(m)$, required by $\Omega(k \cdot m)$ edges, where $m = k/\log_d k$.

Theorem 10 is therefore inferred from an application of Theorem 11 with $k = d = \Theta(\sqrt{n})$.

Combined with Theorem 11, the following proposition will allow to state a similar result for 3-regular planar graphs:

Proposition 12. *Every graph G of order n and of maximum degree 3 can be transformed in a 3-regular graph G' of order at most $6n$ such that* $\textsc{Irs}(G') \geq \textsc{Irs}(G)$. *Moreover, G' is planar if G is.*

In particular, by plugging $d = 3$ and $k = \sqrt{n}$ in Theorem 11, it turns out that there exists a 3-regular planar graph having a compactness $\Omega(\sqrt{n/\log n})$, required by $\Theta(n)$ edges.

Proof. For every vertex of degree 1 or 2 of G, we add the planar structure drawn on Figure 7 to transform it in a 3-regular graph G'. For each vertex, the number of added vertices is bounded by 5. Thus, G' is 3-regular and is obtained by adding at most $5n$ vertices. Moreover, G' is planar if G is. All the shortest paths in G' between any two vertices of G are clearly wholly contained in G. Thus G is a *subgraph of shortest paths* of G'. Recall that B is a subgraph of shortest paths of A if B is a subgraph of A and if all the shortest paths between any two vertices of B in A are wholly contained in B. Therefore, by application of [6, Theorem 2], the compactness of G' is greater or equal than the compactness of G. □

Fig. 7. A 3-regular planar graph transformation.

In [12], we showed that the $\Omega(\sqrt{n})$-lower bound for the compactness of planar graphs holds for 4-regular, 5-regular, and bounded degree planar graphs whose all its faces are triangles as well.

5 Conclusion

We proved a tight lower bound of the compactness for 3-regular networks, showing that compactness could not depend on the maximum degree of the graph in general. We gave also a new lower bound of $\Omega(\sqrt{n})$ intervals for 3-regular planar graphs and bounded degree planar graphs with only triangles. In [8], Frederickson and Janardan showed that every planar graph G composed of p *planes*

satisfied $\text{IRS}(G) \leq 3p/2$. The number of planes of G is defined as being the smallest number of faces needed to cover all the nodes, taken over all possible plane embedding of G. For example, the 1-plane graphs are precisely the outer-planar graphs. Unfortunately the number of planes of a graph of order n can raise up to $\Theta(n)$. It would be interesting to know if a better lower bound can be found for planar graphs. For the series-parallel graphs, which is a subclass of planar graphs including the outer-planar graphs, Gavoille and Luccio [15] have proved that compactness can rise up to $\Omega(\sqrt{n})$, even for bounded degree series-parallel graphs.

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