Approximate Distance Labeling Schemes

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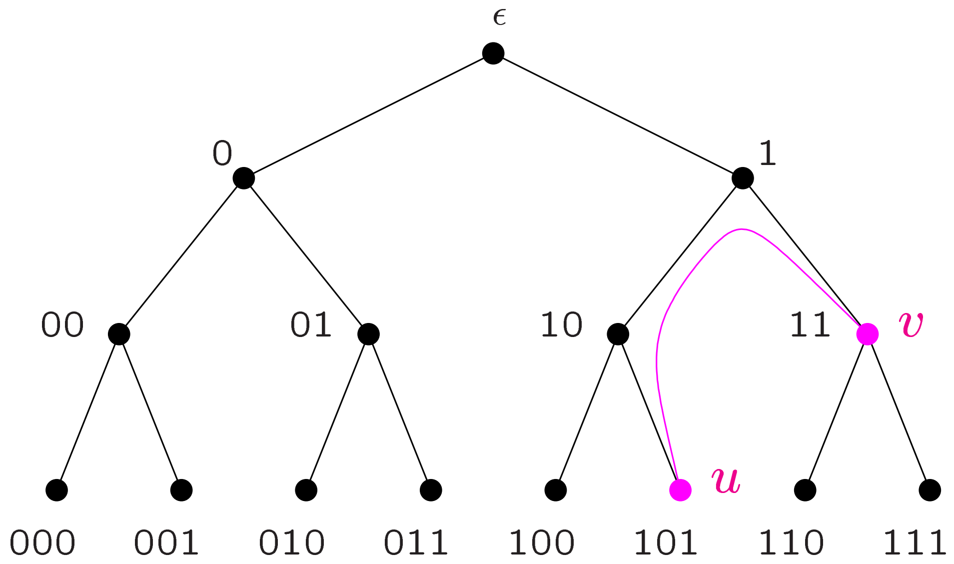
Problem:

How to label the nodes of a graph G in such a way that the distance between any two nodes u, v of G can be computed by inspecting the labels of u and v only.

Ex: G = complete binary tree with $n = 2^{h+1} - 1$ nodes



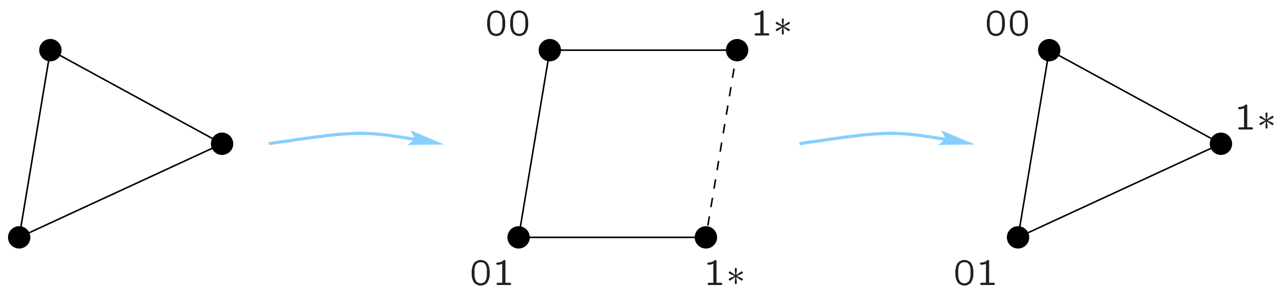
Labeling: node x of level i labeled with $L(x) \in \{0,1\}^i$, $i \leqslant h \leqslant \log n$

Distance decoder: $L(u) = \mathbf{1}01$, $L(v) = \mathbf{1}1$, $\text{prefix}(L(u), L(v)) = \mathbf{1}$

$$\begin{aligned} d_G(u,v) &= \text{length}(L(u)) + \text{length}(L(v)) - 2 * \text{length}(\text{prefix}(L(u), L(v))) \\ &= 3 + 2 - 2 * 1 = 3 \end{aligned}$$

General Case

Embed G in the "Squashed Cube"



$q = 2$

Labeling: $L(x) \in \{0, 1, *\}^q$

Distance: Hamming distance with $d(0, *) = d(1, *) = d(*, *) = 0$

Graham-Pollak '78: The *squashed-cube dimension* of a connected graph G is at least r, where r is the maximum of the number of positive eigenvalues and number of negative eigenvalues of the distance matrix of G.

Definition

Distance Labeling Scheme for $\mathcal{F}$ (family of connected graphs), is a pair $\langle L, f \rangle$ such that: $\forall G \in \mathcal{F}$, $\forall u \neq v \in V(G)$

(Labeling) $\quad L(u, G) \in \{1, \ldots, 2^k\}$

(Decoder) $\quad f(L(u, G), L(v, G)) = d_G(u, v)$

Goals:

- Minimize k (the size in bits of the labels)
- Find a low worst-case time complexity decoder f (constant time)

Results

$\mathcal{F}$ = connected n-node graphs

Winkler '83: The squashed-cube dimension of every graph $\leqslant n-1$

$\Rightarrow$

Labels $= n \cdot \log_2(3) \approx 1.58n$ bits

Time $= \Theta(n)$

Tabulation of $d_G(u,\cdot)$ | Gavoille-Peleg-Pérennes-Raz '00

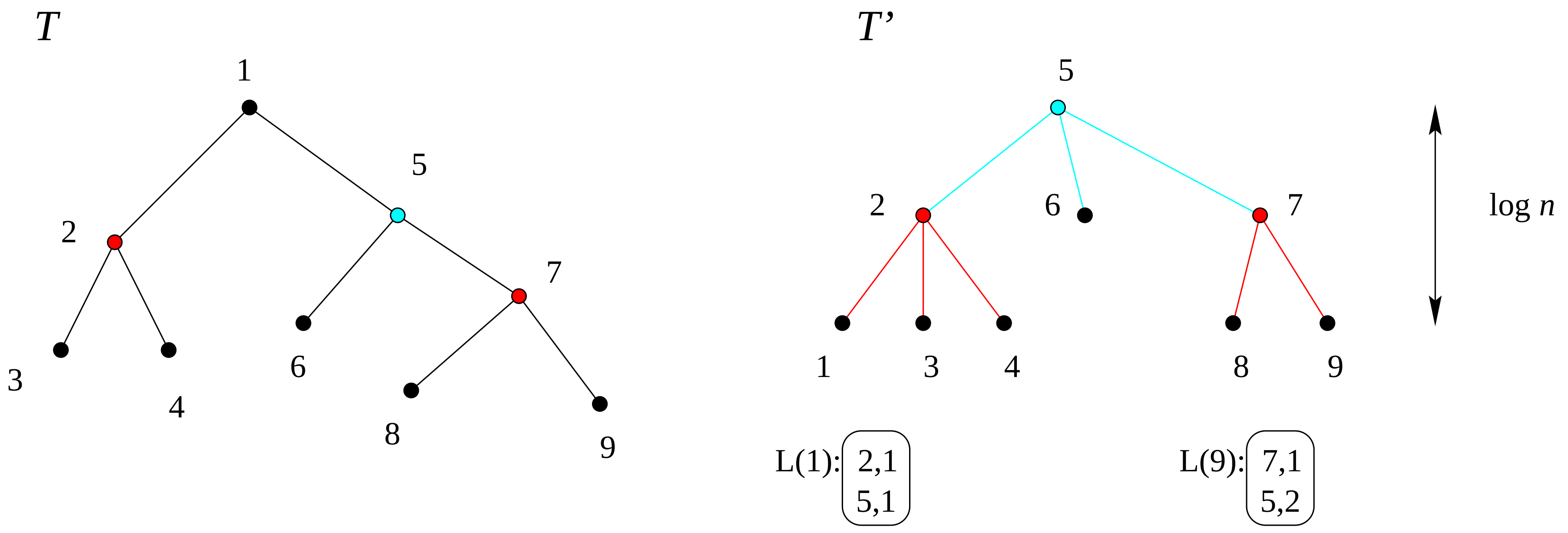
Labels $= n \log n$ bits | Labels $= 9n$ bits

Time $= O(1)$ | Time $= O(\log\log n)$

Lower bound: Labels $\geqslant n/2$ bits

Results (cont'd)

$\mathcal{F}$ = n-node trees, Peleg '99



Labels $= 2\log^2 n$ bits

Time $= O(\log n)$

Lower bound: Labels $\geqslant \frac{1}{8}\log^2 n$ (GPPR '00)

Approximate Distance Labeling

$\langle L, f\rangle$ is (s, r)-approximate if $\forall G \in \mathcal{F}$, $\forall u \neq v \in V(G)$,

$$d_G(u,v) \;\leqslant\; f(L(u,G), L(v,G)) \;\leqslant\; s \cdot d_G(u,v) + r$$

multiplicative s-approximation $=$ $(s, 0)$-approximation
additive r-approximation $=$ $(1, r)$-approximation

Peleg '99: $\mathcal{F}$ = connected n-node graphs

Labels $= O(\log^3 n)$ $\qquad s = 2\log n$

Time $= O(\log n)$

Some Results

Gavoille-Katz-Katz-Peleg '00

$\mathcal{F}$ = n-node trees

Labels $= O(\log n \cdot \log\log n)$ $\qquad$ $s = 1 + 1/\log n$

Time $= O(1)$

Lower bound: $s \leqslant 1 + 6/\log n \Rightarrow$ Labels $= \Omega(\log n \cdot \log\log n)$

$\mathcal{F}$ = connected planar n-node graphs

Labels $= O(n^{1/3} \log n)$ $\qquad$ $s = 3$

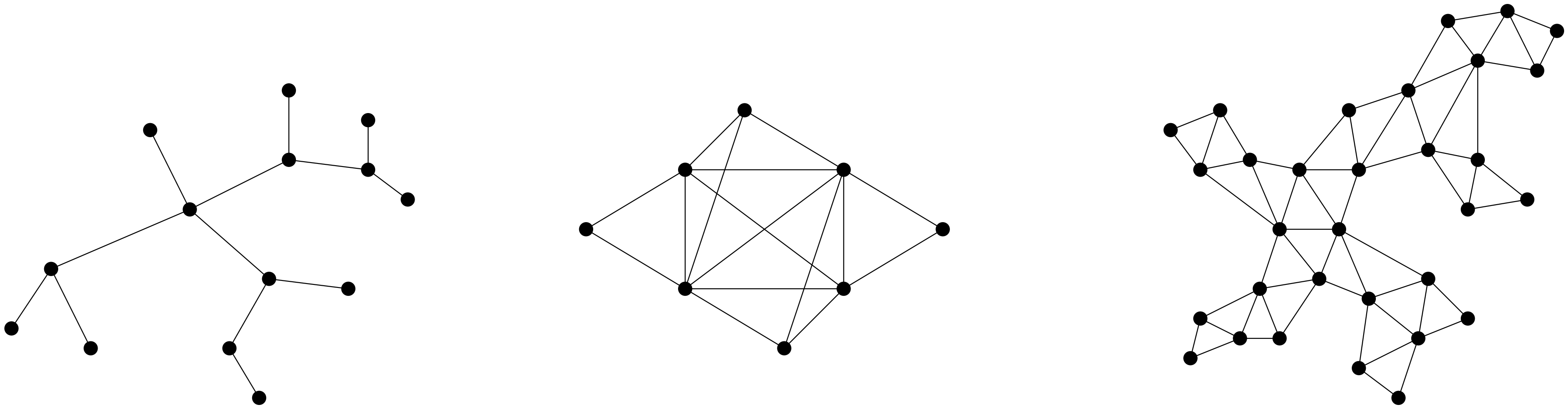
Time $= O(1)$

(Lower bound: $s = 1 \Rightarrow$ Labels $= \Omega(n^{1/3})$)

$\mathcal{F}$ = connected n-node chordal graphs

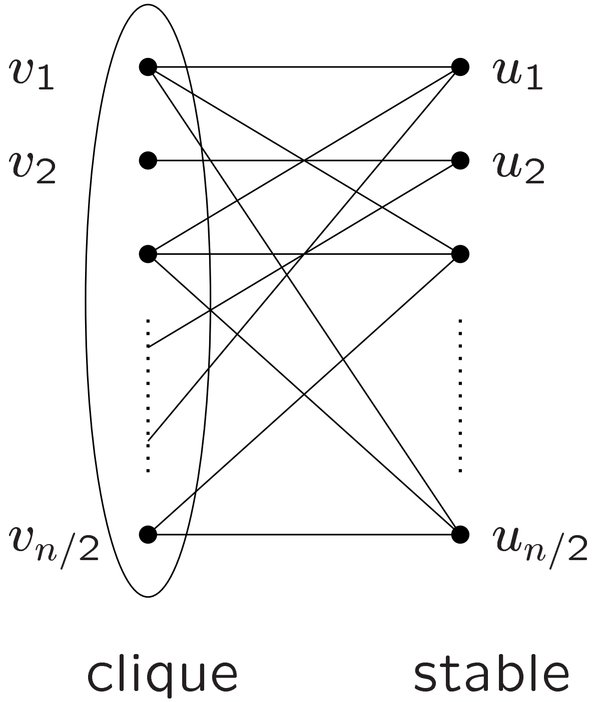
(Gavoille-Katz-Katz-Paul-Peleg '00)

chordal $\Leftrightarrow$ no induced cycle of length greater than 3



k-chordal $\Leftrightarrow$ no induced cycle of length greater than k

Lower Bound for Chordals



$\mathcal{S}$ = split-graphs $\subset$ chordals

#split-graphs = $2^{n^2/4}$

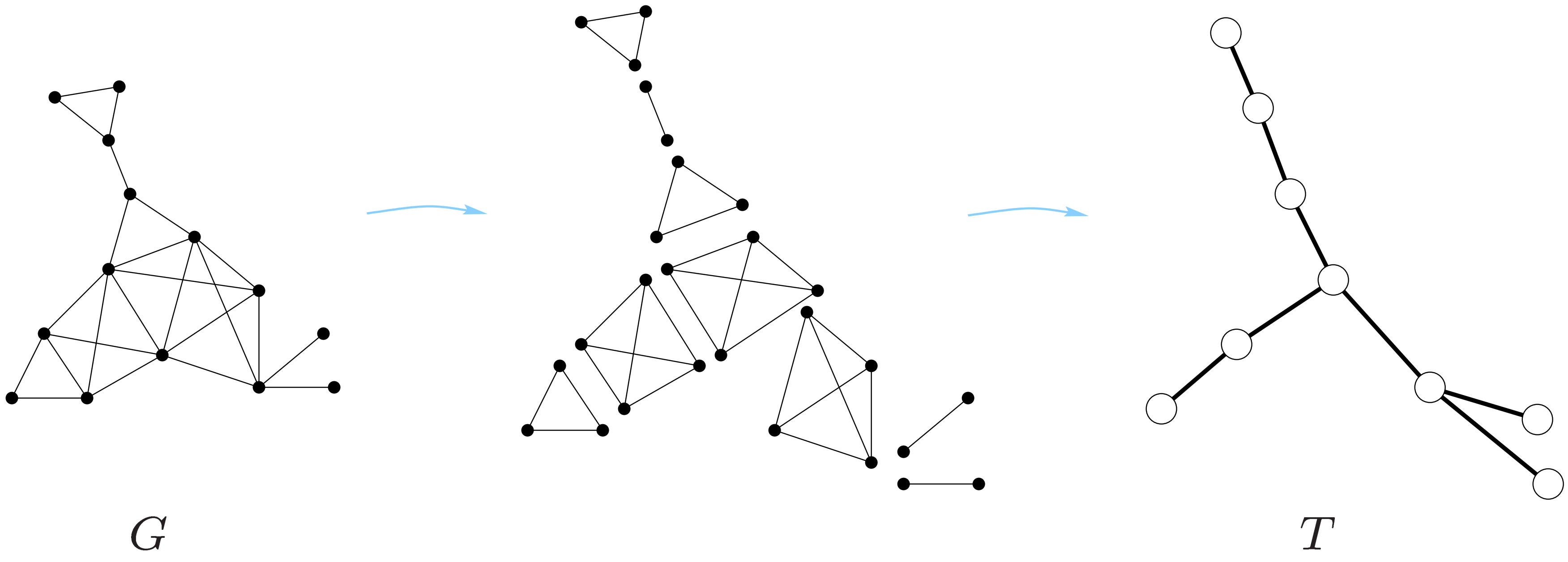
Assume $\langle L, f \rangle$ is a multiplicative s-approximation on chordals, for any $1 \leqslant s < 2$. Then, $\exists G_0 \in \mathcal{F}$,

$$\text{“}L(u_1, G_0); L(u_2, G_0); \ldots; L(v_1, G_0); \ldots; L(v_{n/2}, G_0)\text{”} \quad \geqslant \quad \log |\mathcal{S}|$$

$\Rightarrow \exists w \in V(G_0) \geqslant \frac{1}{n} \log |\mathcal{S}|$. Thus $1 \leqslant s < 2 \Rightarrow$ Labels $\geqslant n/4$ bits

Upper Bound

G chordal $\Leftrightarrow$ has a tree-decomposition T whose each node induces a clique in G



Tree-decomposition: Robertson-Seymour

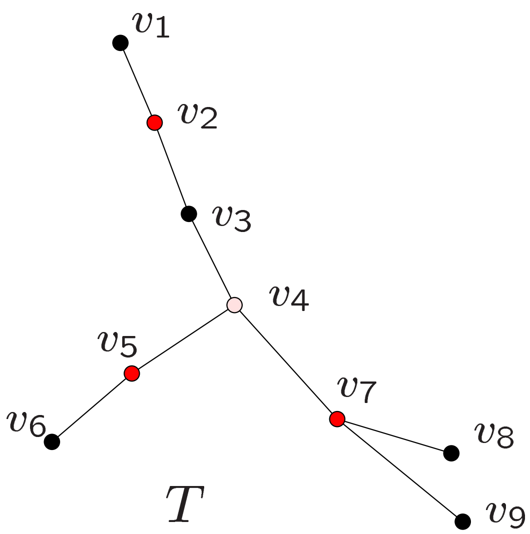
R1. $\bigcup_{v_i \in V(T)} v_i = V(G)$

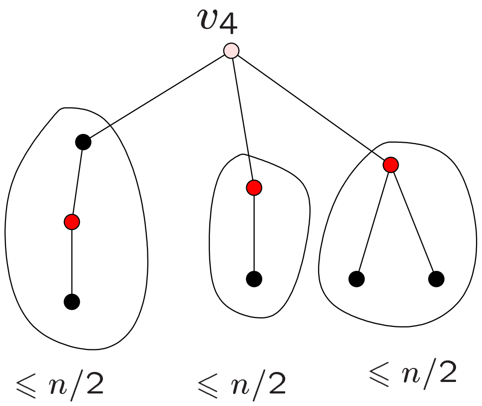
R2. Every edge of G is contained in at least one node of T

R3. The set of nodes of T containing a node of G induces a subtree

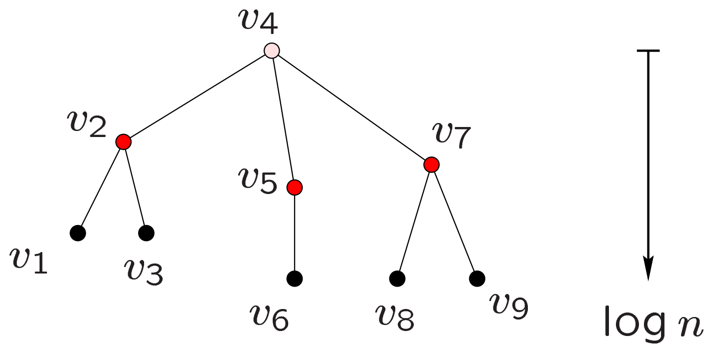
A Scheme for Chordals

R2 + R3 $\Rightarrow$ each node of T is a separator of G





1st separator: v_4



recursivement: T'

$\forall a \in v_1$, $L(a,G) = \langle v_2, d_G(a,v_2)\rangle$
$\langle v_4, d_G(a,v_4)\rangle$

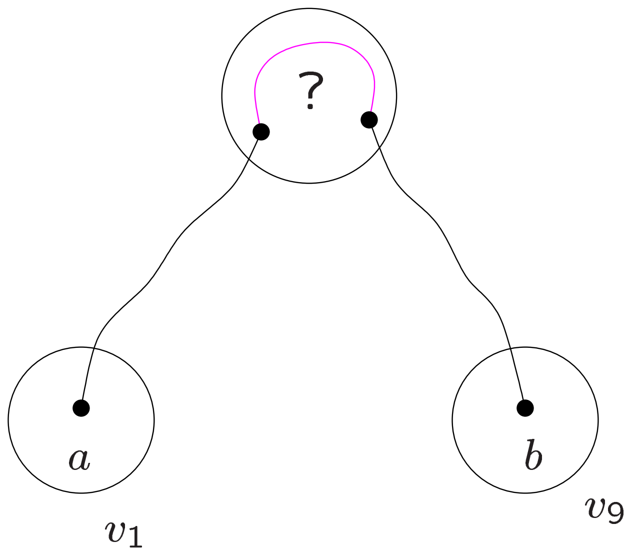
$\forall b \in v_9$, $L(b,G) = \langle v_7, d_G(b,v_7)\rangle$
(label once) $\langle v_4, d_G(b,v_4)\rangle$

Distance decoder:
Find the least common ancestor in T', and sum the distances

Labels $= 2\log^2 n$ bits

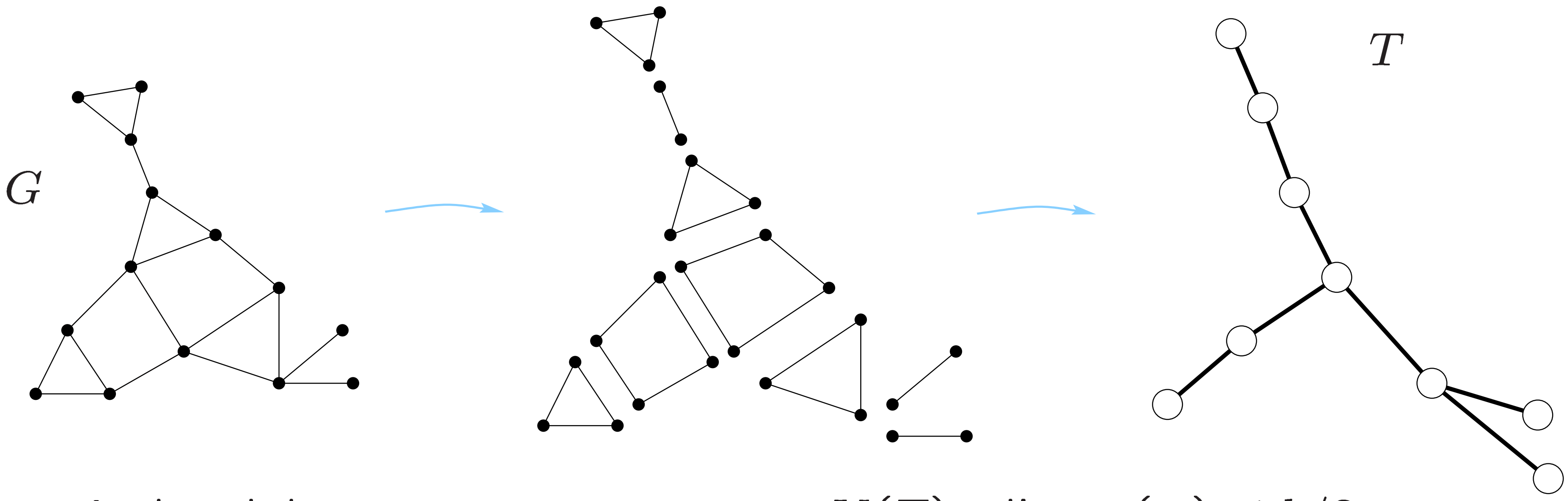
A Scheme for Chordals (Cont'd)

$r = 1$, additive 1-approximation



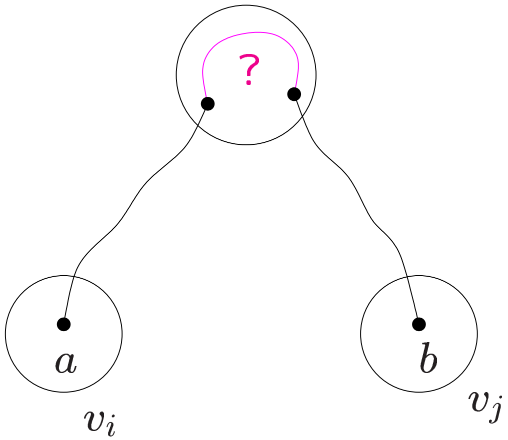
Time $= O(\log\log n)$
($O(1)$ in $O(\log n)$-word RAM model with integer division)

A Scheme for k-chordals



4-chordal

$v_i \in V(T)$, $\mathrm{diam}_G(v_i) \leqslant k/2$

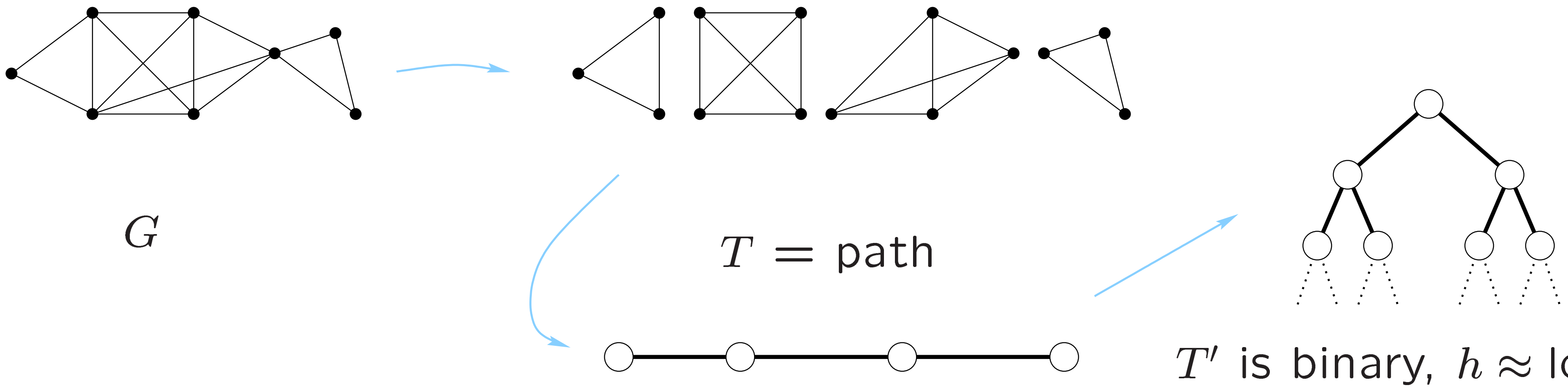


Labels $= 2\log^2 n$ bits

$r = \lfloor k/2 \rfloor$ (additive)

Time $= O(1)$

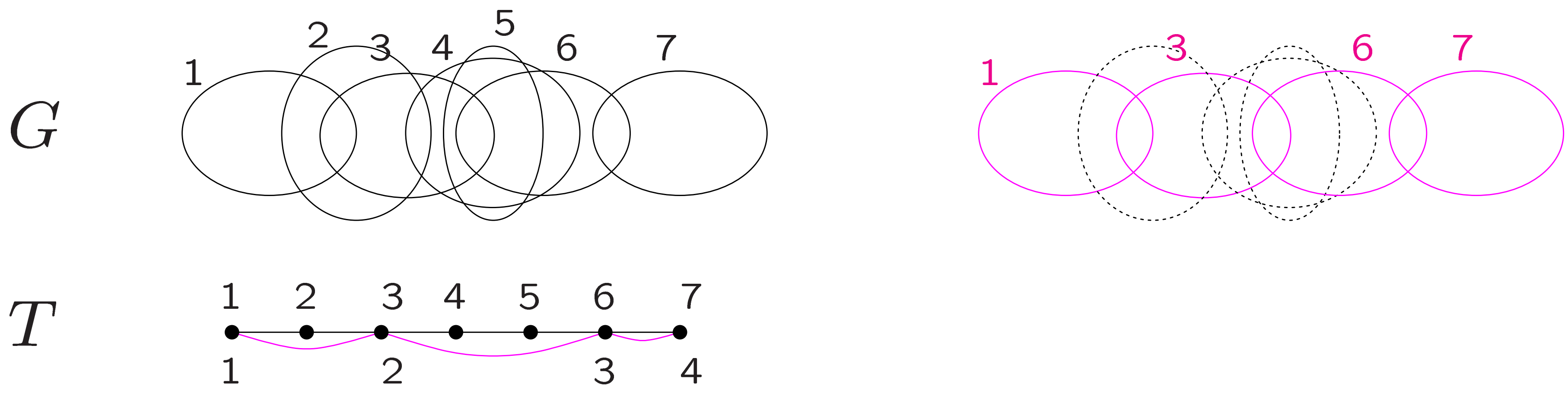
Interval Graphs



T' is binary, $h \approx \log n$

$\Rightarrow \Omega(\log^2 n)$ bits

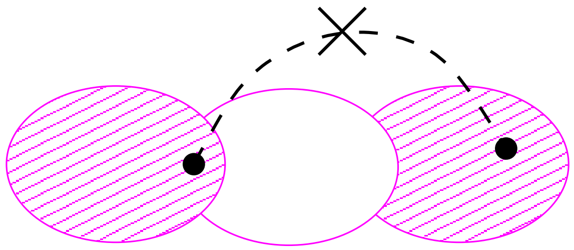
Another Approach for Interval Graphs



$u \to r(u)$, the index of the red clique just following u's clique

$u \to z(u)$, $z(u) = 0$ if $u \in$ red clique, 1 otherwise (the clique of u = the lowest clique containing u)

Properties of the Red Cliques



$\underline{r(u) < r(v)}$:

$$r(v) - r(u) + z(u) \leqslant d_G(u, v) \leqslant r(v) - r(u) + z(u) + 1$$

$(r(u) = r(v) \Rightarrow 1 \leqslant d_G(u, v) \leqslant 2$, so error 1 if return 1)

$$L(u, G) = \langle r(u), z(u) \rangle, \text{ and } r(u) \leqslant D = \text{diam}(G)$$

Labels $= \log D + O(1)$ bits

Additive 1-approximation

Time $= O(1)$

AT-free Graphs

Includes:

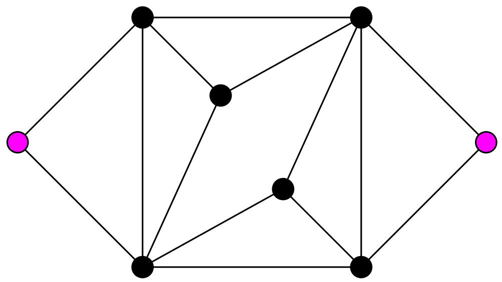
permutation, bounded tolerance, co-comparability graphs, ...

Similar approach using dominating pairs,

Labels $= \log D + O(1)$ bits

$r = 2$ (AT-free $\subset$ 5-chordal)

Time $= O(1)$



Open Problems

Interval graphs

$s = 1$ and $r = 0 \Rightarrow$ Labels = ?

$$\Omega(\log n) \ \ldots \ O(\log^2 n)$$

Permutation graphs

$s = 1$ and $r = 1$ or $0 \Rightarrow$ Labels = ?

$$\Omega(\log n) \ \ldots \ O(\log^2 n)$$

Planar graphs

$s = 1$ and $r = 0 \Rightarrow$ Labels = ?

$$\Omega(n^{1/3}) \ \ldots \ O(\sqrt{n} \log n)$$