On Interval Routing in Random Meshes*

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Abstract

In this paper we consider routing with compact tables in random meshes. More precisely, we study interval routing on random graphs $\mathbb{G}(B,p)$ obtained from a base graph B by independently removing each edge of B with a failure probability $1-p$. We focus on additive stretched routing for n-node random graphs for which the base B is a square mesh and with $p = 0.5$, that is the percolation model at the critical phase. We show a lower bound of $\Omega(\sqrt{\log n}/(\delta+1))$ on the number of intervals required per edge for every additive stretch $\delta \geqslant 0$. On the other side, our experimental results show that the size of the largest biconnected component is $\Theta(n^{0.827})$, and thus that there exists a trivial shortest-path routing scheme using at most $O(n^{0.827})$ intervals per edge. On the other hand we prove that the size of the largest biconnected component is $\Omega(n^c)$ for some constant $c > 0$.

1 Introduction

Research in the area of routing algorithms on computer networks became of great interest by many researchers in recent years. This topic is interesting

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both from the theoretical and also from the application point of view. One aspect of research in this field is to compact routing tables, by maintaining the smallest amount of routing information (or knowledge) locally in each router while guaranteeing that the routes are near the shortest paths.

Many results concern the design of *universal* routing strategies in the sense that they are applicable to all networks. In particular, the proposed schemes give trade-offs between the memory requirements (the size of the local routing tables) and the stretch factor, namely the maximum ratio between the length of the route between any two nodes and their distance in the network.

Above strategies apply to all the networks, however is it natural to wonder whether other more efficient techniques can be applied on realistic networks. Although there is still no answer to the question "what a realistic network is?", many models consider that such networks are based on some structured underlying topology (which is certainly not the complete network) with some random extra connections or some random link failures (cf. the augmented grid Kleingberg's model of Small World[1]).

2 Model and Problem Statement

2.1 Reliability Networks

In this paper we consider a point-to-point communication network modeled by a simple connected graph $G = (V, E)$, where V is the set of nodes (or processors or routers) and E is a set of edges (or bidirectional communication links). We focus on random graphs $\mathbb{G}(B, p)$ obtained from a graph B, with node set $\{1, \ldots, |V(B)|\}$, by independently removing each edge of B with a failure probability $1 - p$. So $G \in \mathbb{G}(B, p)$ is a uniform (labeled) random spanning subgraph of B as $V(G) = V(B)$ and $E(G) \subseteq E(B)$. More precisely,

$$\Pr(G) \ = \ p^{|E(G)|}(1-p)^{|E(B)|-|E(G)|} \ .$$

The graph B is called the *base* graph, and the value $1-p$ the *failure probability*. This model, called the *random B-graph model*, appears in Bollobás[2] and is described in more detail by Janson, Łuczak and Ruciński[3]. The random

[1] *The Small-World Phenomenon: An Algorithmic Perspective*, STOC '00.
[2] *Random Graphs*, Cambridge Studies in Advanced Math. 73, 2001.
[3] *Random Graphs*, Wiley-Interscience Publication, 2000.

B-graph is a natural generalization of the binomial random graph model of Erdös-Rényi, denoted hereafter $\mathbb{G}(n,p)$, for which $B = K_n$ is the complete graph on n nodes. The motivation of using random B-graph model comes from the several areas of computer science. This model is also called *reliability network model* because it describes computer (or processor) interconnection network with faulty communication links satisfying probabilistic distribution. The random B-graph model based on infinite square mesh belongs to percolation theory (cf. Grimmett[4]). This model can be generalized further by allowing different probabilities of failure at different edges. It is also related to other problems of computer science such as grid-computing, fault-tolerant distributed computing, effective data structures, etc.

2.2 Routing Tables and Interval Routing

Our paper is concerning *additive* stretched routing schemes, rather than multiplicative stretch (or stretch factor). A path of a graph is δ*-stretched* if the length of the path is at most the length of a shortest path between its extremities plus δ. A δ*-stretched routing scheme* is a scheme for which all the routes are δ-stretched paths. It is provable that even a small additive stretch allows better optimizations for spanner construction (Elkin and Peleg[5]), distance computation, and yields also compact routing tables (Dourisboure and Gavoille[6]).

We focus on *interval routing scheme*, a particular way of implementing standard routing tables. Recall that a routing table on a graph G consists of the set of distinct addresses ranging in $\{1,\ldots,|V(G)|\}$ and a set of *local routing tables* associated with each node. When a source u sends a message to a destination v, it attaches to the message the address of v, say the integer i, and transmits the message through the output port number q which is computed by u by looking at the ith entry of its local table. So the route is computed in a distributed fashion by all the nodes along the route between u and v. Obviously it is required that for every source-destination pair u, v a route connects u to v. Interval routing implements local routing tables as follows: u stores a d entry tables, d being the degree of u. The qth entry is the list of destination addresses v for which the route from u to v uses output port number q. If for all the nodes, the destination addresses using the same

[4] *Percolation*, Springer, 1999.

[5] $(1+\epsilon,\beta)$*-Spanner Constructions for General Graphs*, STOC '01.

[6] *Improved Compact Routing Scheme for Chordal Graphs*, DISC '02, LNCS 2508.

output port can be grouped into k sets of consecutive integers (consecutive modulo n), we say that the routing scheme is a *k-interval routing scheme* (k-IRS for short). For more precise formulations and other details see surveys of Gavoille[7] and Ružička[8].

The main difficulty in the design of interval routing schemes for a given graph is to find out a suitable address assignment for the nodes and a suitable system of routes for all the pairs of nodes such that the number of intervals per output port (equivalently per outgoing edge) is minimum while keeping the routes near-shortest paths. Whenever shortest paths are required, the problem to decide whether a graph supports a 1-IRS is already NP-complete (see Eilam, Moran and Zaks[9]. A δ-stretched k-IRS, denoted by (k, δ)-IRS, is simply a k-IRS that is a δ-stretched routing scheme. A $(k, 0)$-IRS is also called a shortest-path k-IRS.

3 Previous Works on Random Graphs

The main advantage of k-IRS concerns the size of the memory requirements. In an n-node graph supporting a k-IRS, a node of degree d has to store $O(kd \log n)$ bits of information whereas $O(n \log d)$ bits are required for a standard routing table implementation. In particular, interval routing is efficient for structured graphs like cycles, complete graphs, meshes, trees, outerplanar graphs, tori, hypercubes, etc. All these graphs support shortest-path $O(1)$-IRS.

Flammini, van Leeuwen and Marchetti-Spaccamela[10] proved a non-constant lower bound on k for shortest-path k-IRS on random graphs of $\mathbb{G}(n,p)$, the Erdös-Rényi model. It is proved therein that, with high probability, a graph $\mathbb{G}(n,p)$ requires $\Omega(n^{1-1/\Theta\sqrt{\log n}})$ intervals per outgoing edge for some specific value of p, namely for $p = 1/n^{1-1/\Theta\sqrt{\log n}}$. On the other hand, Gavoille and Peleg[11] proved that almost all graphs (that is graphs of $\mathbb{G}(n,p)$ for $p = 0.5$ and with high probability) support a shortest-path 2-IRS. Actually, they

[7] *A Survey On Interval Routing*, Theoret. Comp. Sci. 245, 2000.

[8] *On Efficiency of Path Systems Induced by Routing and Communication Schemes*, Computing and Informatics, 20, 2001.

[9] *The Complexity of the Characterization of Networks Supporting Shortest-Path Interval Routing*, SIROCCO '97.

[10] *The Complexity of Interval Routing on Random Graphs*, The Comp. J., 41, 1998.

[11] *The Compactness of Interval Routing for Almost All Graphs*, SIAM Journal on Computing, 31, 2001.

constructed a routing scheme such that every node has at most $O(\log^3 n)$ outgoing edges with 2 intervals, all the other one having 1 interval, leaving open the question of whether almost all graphs support shortest-path 1-IRS. Finally, Nehéz[12] proved a preliminary lower bound of $k = \Omega(\sqrt{\log n})$ for shortest-path k-IRS on random tori.

4 Our Results

Our contribution is a lower bound on k for (k, δ)-IRS on random n-node square meshes with constant failure probability. We show that asymptotically almost surely, $k = \Omega(\sqrt{\log n})$, for every additive stretch $\delta \geqslant 0$. This result has been generalized by the same authors to higher dimensional meshes[13].

For upper bounds, we have studied random square meshes with $p = 0.5$, that is the percolation threshold probability. Recall that when the size of the mesh becomes infinite, $p = 0.5$ is precisely the probability where the mesh contains a unique infinite connected components, cf. Grimmett. Unfortunately, as many interesting problems in Percolation Theory (and as suggested by Benjamini and Schramm[14], we are reduced to make experiments. Based on the expected size of the largest biconnected components of a random square meshes, our results suggest that random square n-node meshes support shortest-path k-IRS with $k = O(n^{0.827})$. However, we can show that almost surely the size of the largest biconnected components is at least n^c for some constant $c > 0$.

To prove Result 1 we combine probabilistic and graph-theoretic proof techniques covering also an inductive construction of a proper "worst-case" graph. Result 2 is obtained by some computer experiments and simulations, while the lower bound on the largest biconnected components is obtained by constructing a cycle of length n^c observing that with constant probability, a random square mesh contains a path connecting a node of the first column to a node of the last column.

[12] *The Compactness Lower Bound of Shortest-path Interval Routing on $n \times n$ Tori with Random Faulty Links*, KAM-DIMATIA Series, Charles University, Praha, 582, 2002.

[13] *Interval Routing in Reliability Networks*, SIROCCO '03.

[14] *Percolation beyond Z^d: many questions and a few answers*, Electronic Communications in Probability, 1, 1996.