

Shorter Implicit Representation for Planar Graphs and Bounded Treewidth Graphs

C. Gavoille A. Labourel

University of Bordeaux (LaBRI)

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The problem: implicit graph representation

Problem

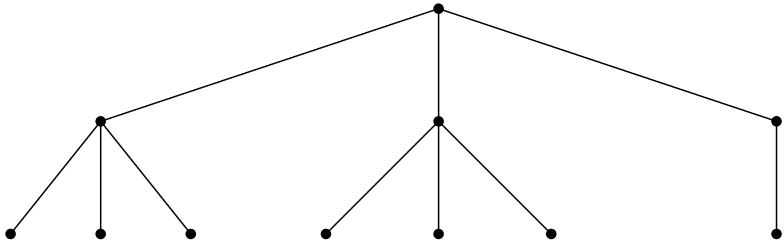
Give local information to vertices of graphs of a given graph family \mathcal{F} such that adjacency between two vertices can be tested using only the local information.

Local information are viewed as binary labels.

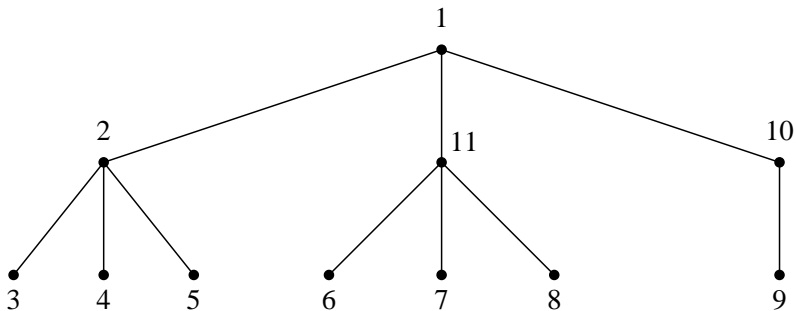
Goal

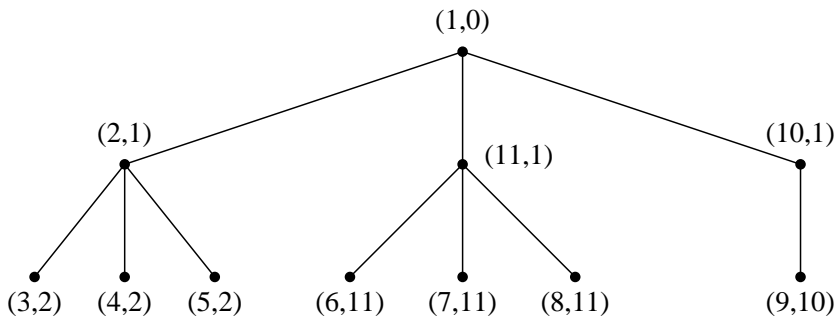
Minimize the size of the labels and also the time complexity of adjacency test and labeling.

An example of a small scheme for trees ($2 \log n$ bits labels)



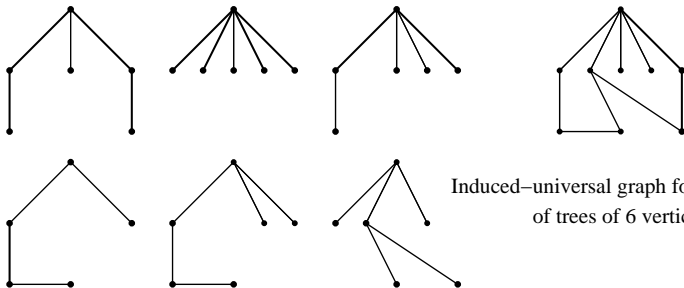
An example of a small scheme for trees ($2 \log n$ bits labels)



An example of a small scheme for trees ($2 \log n$ bits labels)

Induced universal graphs

A graph is \mathcal{F} -universal if it contains all graphs of \mathcal{F} as induced subgraphs.



Trees of 6 vertices

Induced-universal graph for the family of trees of 6 vertices

The link between adjacency scheme and induced-universal graphs

[Kannan, Naor et Rudich 1992]: adjacency scheme for \mathcal{F} with labels of λ bits $\Leftrightarrow \mathcal{F}$ -induced-universal graphs with 2^λ vertices.

\Rightarrow Any result for adjacency labeling scheme implies a result for induced-universal graphs and conversely.

Our result

Theorem

The family of n -vertex planar graphs enjoys an adjacency labeling scheme with labels of $2 \log n + O(\log \log n)$ bits.

Labels: [Chung 1994, AR 2002]

$(3 + o(1)) \log n$ bits $\rightarrow 2 \log n + O(\log \log n)$

Universal graphs: $\tilde{O}(n^3) \rightarrow \tilde{O}(n^2)$

Key lemma

Lemma

The family of n -vertex treewidth- k graphs enjoys an adjacency labeling scheme with labels of $\log n + O(k \log \log(n/k))$ bits.

Labels: [KNR 1992] $O(k \log n)$ bits $\rightarrow \log n + O(k \log \log(n/k))$

Universal graphs: $n^{O(k)} \rightarrow n \cdot \log^{O(k)}(n/k)$

Edge partition of planar graphs

Theorem (Gonçalves 2006)

Every planar graph can be edge partitioned into two outerplanar graphs (that have treewidth at most two).

Principle

Use a scheme of λ bits for the two outerplanar graphs and concatenate the labels to obtain a scheme of 2λ bits for planar graphs.

Edge partition of bounded genus graphs

Theorem (Ding, Oporowski, Sanders et Vertigan 2000)

Every bounded genus graph can be edge partitioned into two bounded treewidth graphs ($\leq 6g + 3$ for graphs of genus g).

Labels: $O(\sqrt{g} \log n) \rightarrow 2 \log n + O(g \log \log(n/g))$

Edge partition of graphs with no K minor

Theorem (DeVos, Ding, Oporowski, Sanders, Reed, Seymour et Vertigan 2004)

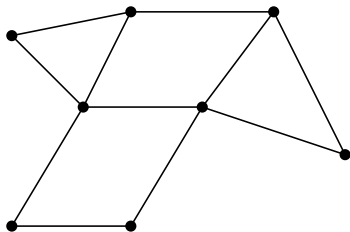
Every graph with no K minor can be edge partitioned into two graphs of treewidth depending only on K .

Labels: $2 \log n + O(\log \log n)$ bits for fixed K

Ex: Planar graphs exclude K_5

Treewidth

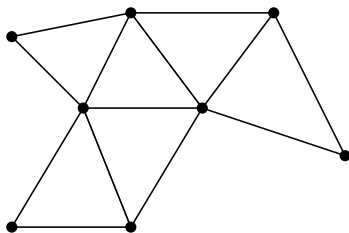
A graph of treewidth k is a subgraph of a graph with no induced cycle of size > 3 and with max clique size equal to $k + 1$.



Example of a graph of treewidth 2.

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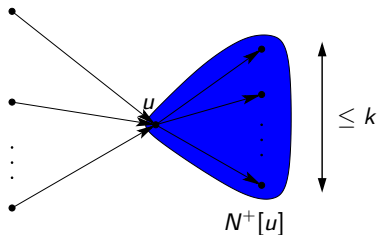
Example of a graph of treewidth 2.

k -orientation

k -orientation: orientation of edges of the graph such that out degree is at most k .

$N^+[u]$: u plus its out neighborhood.

u and v are adjacent $\Leftrightarrow u \neq v$ and $(u \in N^+[v]$ or $v \in N^+[u])$.



Adjacency test

The idea is to store for each vertex v identifiers of vertices of $N^+[v]$.

Using a naive approach, this can be done with $(k + 1) \log n$ bits.

Actually, we can achieve much more compact scheme with a special partition for bounded treewidth graphs.

Bidecomposition

Lemma (Bidecomposition)

For every n -vertex treewidth- k graph G , there exists a k -orientation and a vertex partition such that:

- ▶ *the parts are the nodes of a rooted binary tree T of depth $\leq \log(n/k)$;*
- ▶ *each part has $O(k \log n)$ vertices of G ;*
- ▶ *for every vertex $u \in V(G)$, all the parts of $N^+[u]$ belong to a path from the root to a leaf of T .*

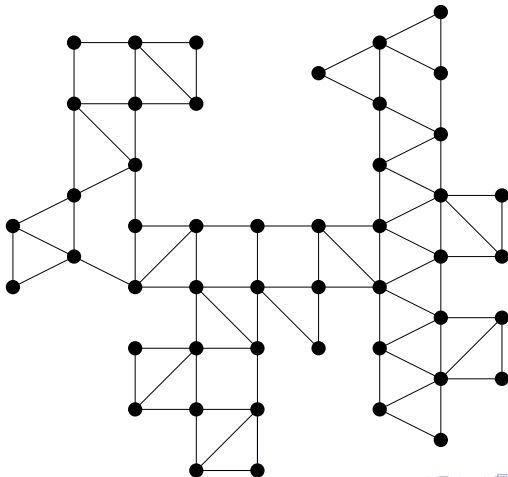
Outline of the bidecomposition

Step 1: Add edges to obtain a graph with no induced cycle of size > 3 .

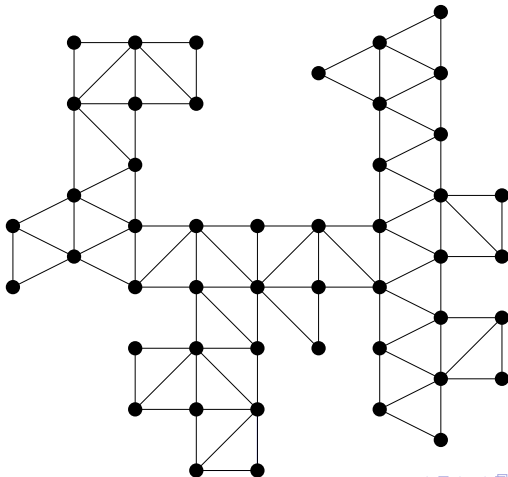
Step 2: Recursively cut the graph into two parts using a special separator.

Step 3: orient the augmented graph such that for every $u \in V(G)$ $N^+(u)$ induces a clique.

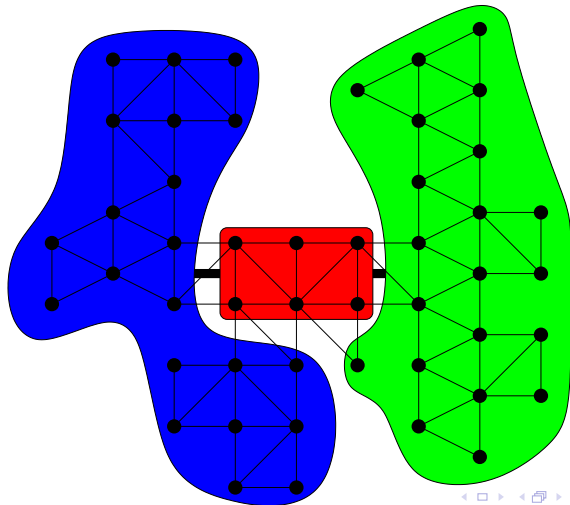
Example of bidecomposition



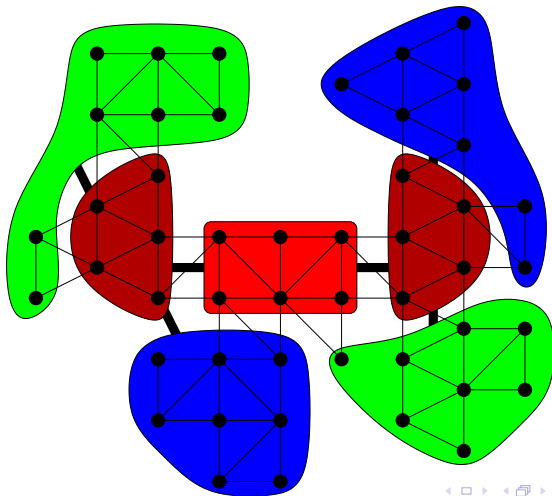
Example of bidecomposition



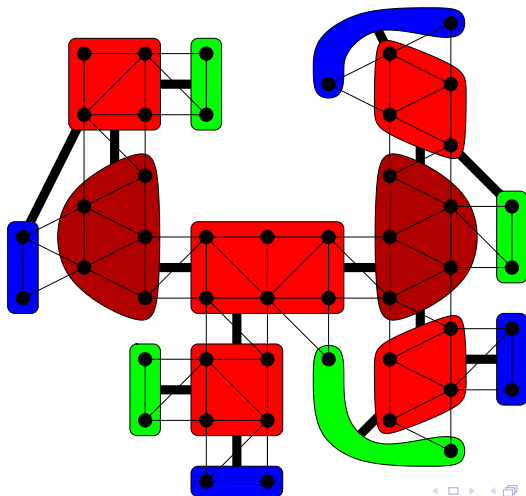
Example of bidecomposition



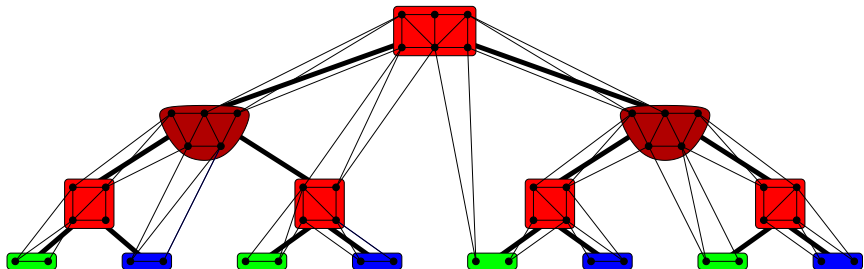
Example of bidecomposition



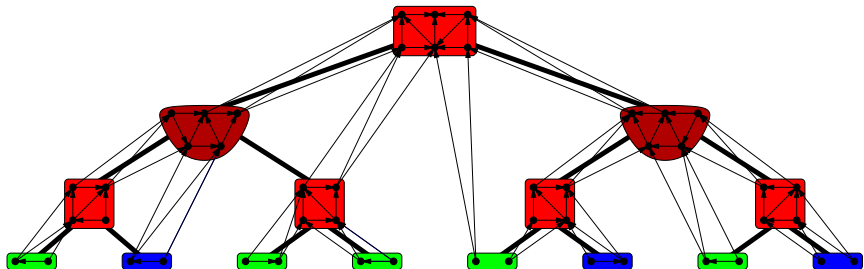
Example of bidecomposition



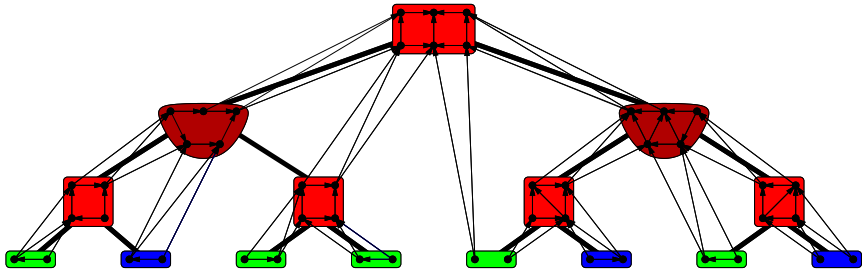
Example of bidecomposition



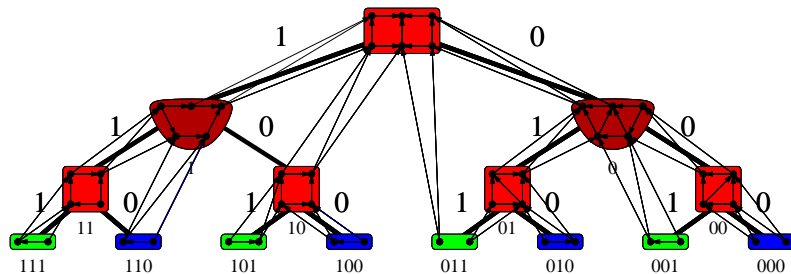
Example of bidecomposition



Example of bidecomposition



Encode using bidecomposition



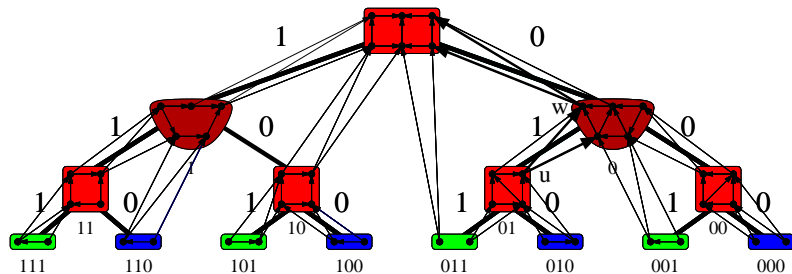
Encoding using bidecomposition

Identifier of u : path from the root to its part plus its position in the part.

Label of u : path from the root containing all the parts of $N^+[u]$ plus depth and position of each vertex of $N^+[u]$.

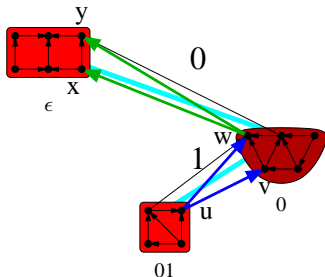
From the label of u , we can extract identifiers of all vertices of $N^+[u]$.

Encode using bidecomposition



Encoding using bidecomposition

$\text{id}(u) = ("01", 2)$
 $\text{id}(v) = ("0", 4)$
 $\text{id}(w) = ("0", 1)$
 $\text{id}(x) = (\epsilon, 6)$
 $\text{id}(y) = (\epsilon, 3)$



label of w
branch "0"

$\text{depth}(w) = 1$ $\text{pos}(u) = 1$
 $\text{depth}(x) = 0$ $\text{pos}(x) = 6$
 $\text{depth}(y) = 0$ $\text{pos}(y) = 3$

label of u
branch "01"

$\text{depth}(u) = 2$ $\text{pos}(u) = 2$
 $\text{depth}(v) = 1$ $\text{pos}(v) = 4$
 $\text{depth}(w) = 1$ $\text{pos}(w) = 1$

Open problems

- ▶ Does there exist a scheme of $(1 + o(1)) \log n$ bits for planar graphs (Universal planar graph $\tilde{O}(n)$)?
- ▶ Does there exist a scheme of $\log n + O(k)$ bits for graphs of treewidth k ?

Perspectives

In our model, adding a vertex in the graph generally implies a complete relabeling of the graph to maintain the encoding.

Dynamic encoding

An adjacency labeling scheme that maintains the labels with $O(\log n)$ amortized message complexity, per topology change.

Thank you