From fault tolerant spanners to multipath spanners

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Part I: Spanners
$G = (V, E)$ some weighted graph, $\#V = n$, $\#E = m$. Potentially $m = n^2$. Can we spare some edges while keeping:

- connectivity?
- some distance property?
Spanners

Introduced by Peleg and Schäffer in 89 for undirected graphs.

Let $d(u, v)$ be the distance: length of smallest path between $u$ and $v$.

**Definition**

$H$ covering subgraph of $G$ is an $(\alpha, \beta)$-spanner of $G$ if for any $u, v \in V$,

$$d_H(u, v) \leq \alpha \cdot d_G(u, v) + \beta$$

($\alpha, \beta$) (or only $\alpha$ if $\beta = 0$) is called the stretch.

- A **multiplicative** spanner is a spanner where $\beta = 0$.
- An **additive** spanner is a spanner where $\alpha$ is as small as possible, possibly 1.
What counts for a spanner?

Three elements to determine the quality:
- Number of edges: less is better.
- The stretch: smaller is better.
- Computation time.

For some stretch, the less edges the better.
Why spanners?

Fundamental object related to:

- Compact routing
- Distance oracles
- Distributed distance guessing

One can view them as a possible extension of spanning trees.
Some spanners

- $(2k - 1, 0)$-spanner with $O(n^{1+1/k})$ edges.
- $(1, 2)$-spanner with $O(n^{3/2})$ edges.
- $(1, 6)$-spanner with $O(n^{4/3})$ edges.
- $(k, k - 1)$-spanner with $O(n^{1+1/k})$ edges.
- $(1 + \epsilon, 4)$ with $O(1/\epsilon \cdot n^{4/3})$ edges.

Probably optimal.
Part II: Multipath spanners
Goals for improvement

We would like to:

- Improve bandwidth
- Be tolerant to faulty vertices/links
- Minimize delays
The standard spanner definition is oblivious to the metric used.

**Definition 1**
\[ d^p(a, b) \] is the minimum cost of a subgraph containing \( p \) vertex-disjoint paths between \( a \) and \( b \).

We plug \( d^p \) into the definition:

**Definition 2**
\( H \) is a \( p \)-multipath \((\alpha, \beta)\)-spanner of \( G \) if \( \forall u, v : \)

\[
d_H^p(u, v) \leq \alpha \cdot d_G^p(u, v) + \beta
\]

**Question**
Can we construct such things?
We have bad news:

\[ d_2(u, v) \text{ & } d_2(v, w) \text{ finite, yet } d_2(u, w) = \infty \]

Figure: \( d_2(u, v) \text{ & } d_2(v, w) \text{ finite, yet } d_2(u, w) = \infty \)

And most of spanner algorithms make use of triangular inequality.
Fault-tolerant spanners
Close cousin built for vertex fault tolerance

Definition
A $f$-fault-tolerant spanner is a spanner that can tolerate $f$ faults and still be a spanner on the remaining graph.

Question
Is a $p - 1$-fault tolerant $s$-spanner a $p$-multipath spanner?

Answer
In weighted graphs: NO if it is not a spanner of the same graph with all edges weights equal to 1.
Figure: Cycle of $n$ hops, weight $s/n$. Blue paths of weight $s$. Stretch unbound for $a^2$ but bounded for 1-fault.
Node disjoint spanners

Theorem

Theorem 1
A $s$-hop $p - 1$-fault tolerant spanner is a $p$-multipath $sp \cdot O(1 + p/S)^s$-spanner.

Bonus
It is buildable in the LOCAL model in no additional time.

Idea
Do every possible fault on a path, and recursively.

That explains why we need bounded hop spanners.
Theorem 2
Every graph has a spanner of stretch $(2, O(1))$ and $O(n^{3/2})$ edges for the $d^2$ metric.

Idea
The proof is inspired by the (2, 1) spanner algorithm: remove highly connected nodes, and replace them with an efficient structure.
We define a new type of balls:

**Definition**
\( \tilde{B}(uv, r) \): all \( x \) such that \( d^2(uv, x) \leq r \).

**Implication**
\( x, y \in \tilde{B}(uv, r) \implies d^2(x, y) \) bounded.
Algorithm

1. While $\exists \tilde{B}(uv, 4) \cap (N(u) \cup N(v)) > \sqrt{n}$ do:
   1. Add 2-tree $\bigcup \text{BFS}(u, 2) \cup \text{BFS}(v, 2)$
   2. $G = G \setminus B \cap N$...

2. $H = H \cup G$

Must check:
- Number of edges
- Stretch
Useful result:

**Lemma**
A graph with every \( \tilde{B}(uv, 2k) \cap N(u) \leq n^{1/k} \) has \( O(n^{1+1/k}) \) edges.

And:
- “While” executes itself \( n/\sqrt{n} \) times.
- “While” adds \( O(n) \) edges at each step.
- Apply lemma to remaining graph \( G \) before step 4 and \( k = 2 \).

So \( H \) has \( O(n^{3/2}) \) edges.
Let two nodes \(a, b\), and a cycle \(P\) which realize \(d^2(a, b)\).

Two cases are possible:
- The path is in \(H\): nothing to prove
- It is not: some edge is missing. Take the first one.
• One of the vertices $t$ is in vicinity of the root edge of some $\tilde{B}(uv, 4)$
• Because of BFS it is covered by another node to $u$
• $d^2(t, u) \leq 6$
It goes the same for every nodes of edges missing. The closest from $a$ and $b$ are the important ones.

We use these cycles to bound $d^2(a, uv)$ and $d^2(b, uv)$ in $G$, and so in $H$ because of the 2-tree. We use menger theorem to finally bound $d^2(a, b)$ in $H$. 
Conclusion

Summary

• We showed that fault-tolerant spanners existed, and can be built in a local setting.
• We showed how to construct a \((2, O(1))\)-bipath-spanner.

What next?

• factor 2 mandatory for 2-paths?
• Other special constructions?
• Improve the stretch of the general result?