

Subquadratic additive spanners for directed graphs

Cyril Gavoille¹
*Quentin Godfroy*¹
Laurent Viennot²

¹LaBRI
Université Bordeaux I

²INRIA – LIAFA
Université Paris VII

JGA 2010

Part I: Introduction to problem



General background

$G = (V, E)$ some weighted graph, $\#V = n$, $\#E = m$.
Potentially $m = n^2$. Can we spare some edges while keeping:

- connectivity ?
- some distance property ?

Spanners

Introduced by Peleg and Schäffer in 89 for undirected graphs.

Let δ be a metric

Definition

H covering subgraph of G is an (α, β) -spanner of G if for any $u, v \in V$,

$$\delta_H(u, v) \leq \alpha \cdot \delta_G(u, v) + \beta$$

(α, β) is called the **stretch**.

An **additive spanner** is a spanner where α is as small as possible, possibly 1.

What counts for a spanner ?

Three elements to determine the quality:

- Number of edges: less is better.
- The **stretch**: smaller is better.
- Computation time.

For some stretch, the less edges the better.

Undirected graphs

fast survey

Let $\delta(u, v) = d(u, v)$ be the length of a smallest path between u and v , and G an undirected graph

Easy to do:

- $(2k - 1, 0)$ for all k size $O(n^{1+1/k})$ (possibly optimal)
- $(1, 2)$ size $O(n^{3/2})$ (optimal)

Less easy:

- $(1, 6)$ size $O(n^{4/3})$ (optimal)

Directed graphs

Cowen 99: bounded stretch and $\delta = d \rightarrow n^2$ edges.

Definition

Roundtrip distance: $\delta = d^{\leftrightarrow}(u, v) = d(u \rightarrow v) + d(u \leftarrow v)$

It respects the **triangular inequality** (crutial).

Roditty et alii 08:

- $(2k + \epsilon)$ -spanner with $O((k^2/\epsilon)n^{1+1/k} \log(nW))$ edges
- $(2^k - 1, 0)$ -spanner with $\tilde{O}(n^{1+1/k})$ edges

This work:

- $(2, 8W)$ -spanner with $O(n^{3/2})$ edges (2 is optimal).

It is an adaptation from Gavoille, G. and Viennot 10 which presents a spanner which preserves cycles in undirected graphs, itself an adaptation from the (1, 2) construction.

Part II: Spanner construction

Construction

Tools

- $\text{Intree}(u)$ MST following inward edges
- $\text{Outtree}(u)$ MST following outward edges
- $B^{\leftrightarrow}(u, 4)$ subgraph obtained by collating all edges and nodes participating in a **directed** cycle of length less than 4 passing through u .
- $B^2(u, 4)$ subgraph obtained by collating all edges and nodes participating in an **undirected** cycle of length less than 4 passing through u .
- $N(u)$ neighbours of u in the undirected graph.

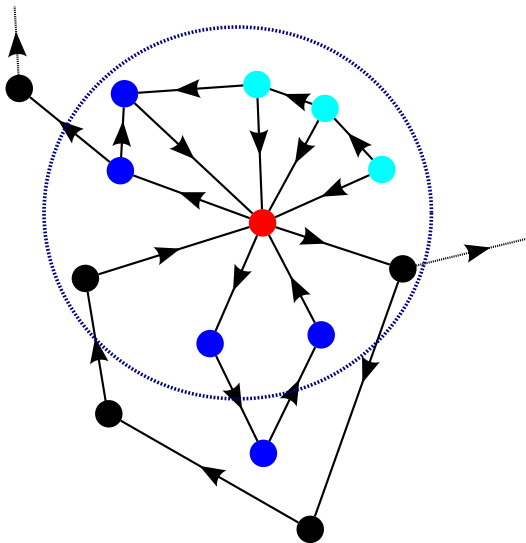


Figure: $N(u)$, $B^2(u, 4)$ and $B^{\leftrightarrow}(u, 4)$

Construction

Algorithm

The scheme works in weighted graphs, we present here the unweighted case for simplicity. The loop is done from larger to smaller balls.

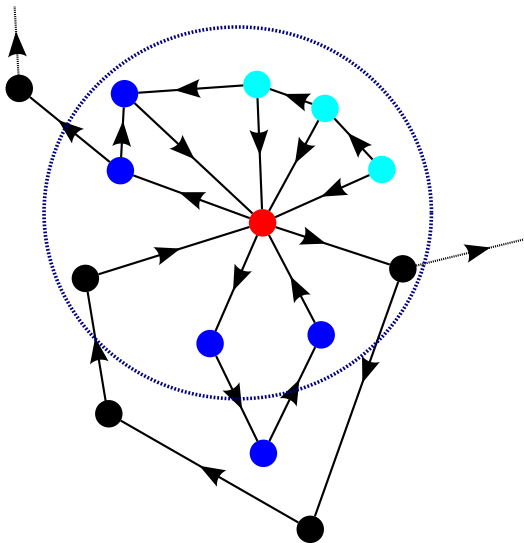
```
H = (V, \emptyset);
G_1 = G;
tant que  $\exists u$  s.t.  $N(u) \cap B^2(u, 4) \geq \sqrt{n}$  faire
  E(H) = E(H) \cup \text{Intree}(u) + \text{Outtree}(u);
  G_1 = G_1 \setminus (N(u) \cap B^2(u, 4));
  E(G_1) = E(G_1) \setminus \{u - x \text{ s.t. } x \in B^2(u, 4)\};
E(H) = E(H) \cup E(G_1);
```

Figure: building algorithm

Algorithm

Find a ball

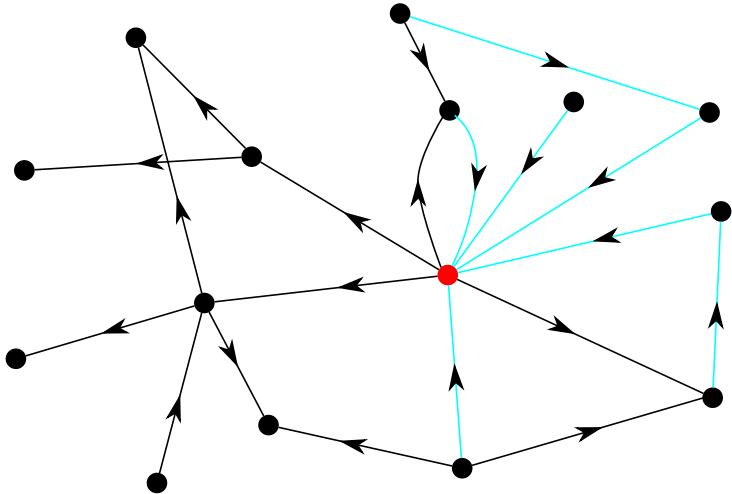
Find a big ball (in cyan & blue):



Algorithm

Add intree

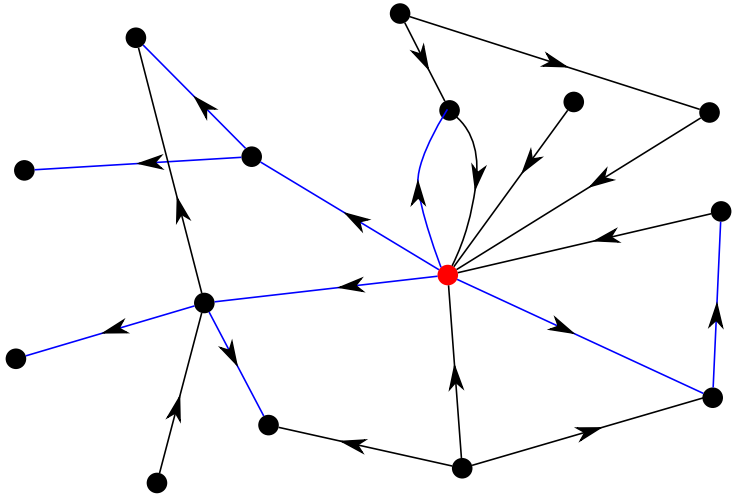
Add intree to everybody in H :



Algorithm

Add outtree

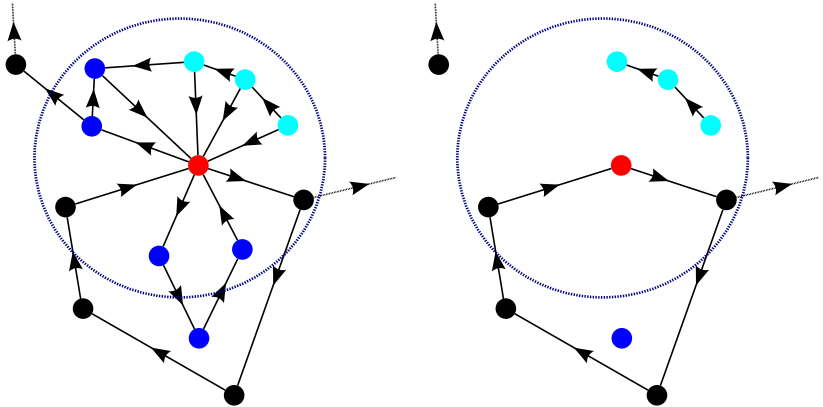
Add outtree to everybody in H :



Algorithm

Remove what's removable

Remove what's removable from G_1 :



The stretch

analysis of stretch

Let a, b a pair of nodes, P_G a cycle which realises $d^{\leftrightarrow}(u, v)$.

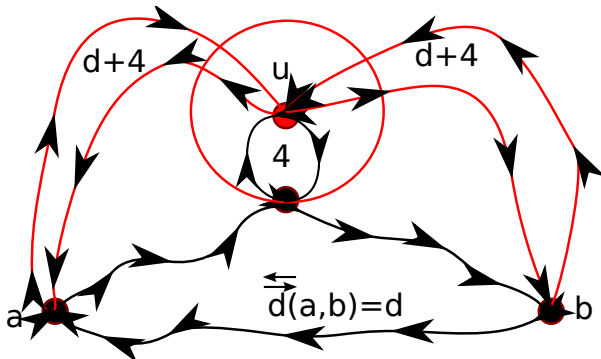
Two cases are possible:

- 1 P_G exists in H : nothing to do !
- 2 P_G does not exist in H .

In case 2, let u be the first center which removes edges.

The stretch

Drawing



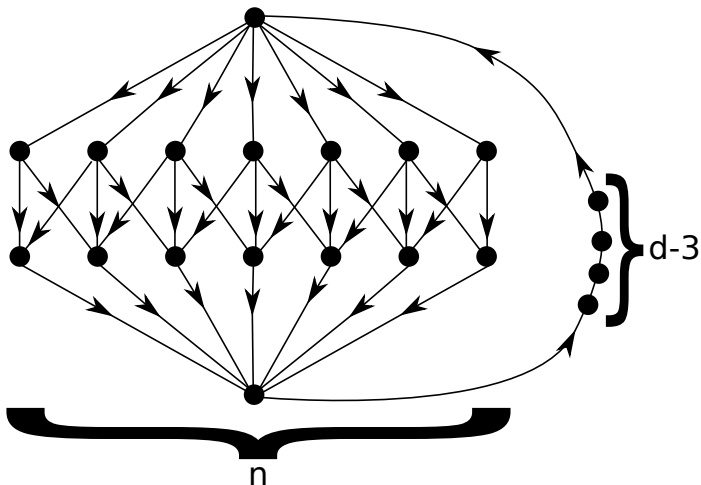
There is a bound on the size of the path in the roundtrip tree, thanks to the path between a and b .

Number of edges

- Unions of In/Out-trees: $O(2n \cdot \sqrt{n}) = O(n^{3/2})$.
- Graph left after while loop, two components:
 - Union of edges incident to some u in $B^2(u, 4)$: G' .
 - The rest G'' .
- G' is composed of vertices with degree less than \sqrt{n} : $O(n \cdot \sqrt{n})$.
- G'' has cycles of length no less than 5: $O(n^{1+1/2})$.

Is it optimal ?

The factor 2 in $(2, 8)$ is optimal:



Removing one edge from the graph multiplies by two for some pair of nodes in the bipartite part.

Summary

- We showed how to make a $(2, 8)$ -roundtrip spanner with $O(n^{3/2})$ edges.
- We showed the bound on multiplicative constant for such spanners

- Future prospects:
 - Improve the 8 ?
 - What if δ does not respect triangular inequality ?