Interval Routing Schemes allow Broadcasting with Linear Message-Complexity*

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Abstract

The purpose of *compact routing* is to provide a labeling of the nodes of a network, and a way to encode the routing tables so that routing can be performed efficiently (e.g., on shortest paths) while keeping the memory-space required to store the routing tables as small as possible. In this paper, we answer a long-standing conjecture by showing that compact routing can also help to perform distributed computations. In particular, we show that a network supporting a shortest path *interval routing scheme* allows to broadcast with an $O(n)$ message-complexity, where n is the number of nodes of the network. As a consequence, we prove that $O(n)$ messages suffice to solve leader-election for any graph labeled by a shortest path interval routing scheme, improving therefore the $O(m+n)$ previous known bound.

Keywords: Compact routing, Interval routing, Broadcasting, Distributed computing.

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1 Introduction

This paper addresses a problem originally formulated by D. Peleg, and that can be informally summarized as follows: "Does networks supporting shortest path compact routing schemes present specific ability in term of distributed computation? E.g., broadcasting, leader election, etc." This paper answers by the affirmative, by showing that n-node networks supporting interval routing schemes [25, 26] (IRS for short), allow broadcasting with $O(n)$ message-complexity.

More formally, a network $G = (V, E)$ (in this paper, by *network*, we will always mean a connected undirected graph without loop and multiple edge) supports an IRS if the nodes of that network can be labeled from 1 to $n = |V|$ in such a way that the following is satisfied: given any node $x \in V$ of degree d, there is a set of d intervals $I_1, \ldots, I_d$ of $\{1, \ldots, n\}$, one for each edge $e_1, \ldots, e_d$ incident to x, such that (1) $V \setminus \{x\} \subseteq \cup_{i=1}^{d} I_i$, (2) $I_i \cap I_j = \emptyset$ for every $i \neq j$, and (3) $y \in I_i$ implies that there is shortest path from x to y passing through the edge e_i. So, IRS is a shortest path routing table having the property that the set of destination-addresses using a given link is a consecutive set of integers. IRS is a famous technique for the purpose of compact routing since a network of maximum degree Δ and supporting an IRS has its routing table of size $O(\Delta \log n)$ bits, to be compared with the $\Theta(n \log \Delta)$ bits of a table returning, for every destination, the output port corresponding to that destination. For more about IRS, we refer to [6, 10, 16, 18, 20, 21], and to the survey [15]. For more about compact routing in general, we refer to [11, 12, 13, 17, 19, 24].

On the other hand, broadcasting is the information dissemination problem which consists, for an arbitrary node of a network, to send a same message to all the other nodes. The message-complexity of broadcasting lies between $\Omega(n)$ and $O(m)$, $m = |E|$, since, on one hand, the reception of the message by every node but the source requires at least $n - 1$ messages, and, on the other hand, broadcasting can always be performed by flooding the network, that is, upon reception of the message, every node forwards that message through all its incident edges (apart the one on which it has received the message). Better upper and lower bounds can be derived as a function of the knowledge of the nodes of the network (e.g., see [1, 2, 4]), and of the maximal size of the message-headers (e.g., see [23]). In the following, the only knowledge of every node is its label in some IRS, and the intervals attached to its incident edges in the same IRS. The size of each message-header transmitted by our broadcasting protocol is $\lceil \log_2 n \rceil$ bits.

The relationship between IRS and broadcasting was previously investigated. For instances, van Leeuwen and Tan [26] proved that minimum spanning tree construction, and therefore broadcasting, and other related distributed problems such as leader-election can be solved by exchanging $O(n)$ messages in a ring labeled by an IRS, and $O(m + n)$ messages for arbitrary graphs labeled by an IRS. Recall that leader-election without any network knowledge requires $\Theta(n \log n)$ messages for a ring, and $\Theta(m + n \log n)$ messages for an arbitrary graph [14]. More generally, the question of how much a labeling can help in the solution of distributed problems was studied in [8, 9] in the framework of Sense of Direction [7]. It was shown in [5] that the message-complexity of the broadcast problem is $n - 1$ in the restricted class of networks supporting *all-shortest-path* IRS (an IRS where all the shortest paths are represented) with additional restrictions on the intervals (strictness and linearity). Finally, de la Torre, Narayanan and Peleg [3] showed that the same result holds in IRS networks satisfying the so-called *ssr-tree* property. (Informally, this property states that, for any node x, the set of paths induced by the IRS originated at x, and ending at all the other nodes, is a tree.) In this paper, we improve these results by showing that a network supporting standard shortest path IRS supports a broadcast protocol of message-complexity $\Theta(n)$.

Our result has many consequences on other problems, such as *leader-election* or *distributed spanning tree*. For instance Korach et al. [22] have shown that the leader-election problem can be solved using $(b(n)+n)(\log n + 1)$ messages where $b(n)$ is the message-complexity of broadcasting in an n-node networks. Therefore, n-node networks supporting shortest path IRS allow the leader-election problem to be solved with $O(n \log n)$ message-complexity (instead of $O(m + n \log n)$ for arbitrary networks). In fact, we prove that $O(n)$ messages suffice to solve leader-election for any graphs labeled by a shortest path IRS, improving therefore the $O(m+n)$ previous bound of van Leeuwen and Tan [26].

The paper is organized according to several hypotheses on the IRS. These hypotheses will be relaxed while going further and further in the paper. We can indeed distinguish two types of intervals: an interval $[a,b] = \{a,\ldots,b\}$ with $b \geq a$ is said to be *linear*, whereas an interval $[a,b]$ with $b < a$ refers to the set $\{a,\ldots,n,1,\ldots,b\}$, and is said to be *cyclic*. The class of networks supporting linear IRS (LIRS for short), that is such that all intervals of the IRS are linear, is strictly included in the class of networks supporting an IRS which possibly includes cyclic intervals. We can also distinguish the case $V \setminus \{x\} = \cup_{i=1}^{d} I_i$ for every node x, from the case in which x appears in the interval of one of its incident edges, for at least one node x. In the former case, the IRS is said to be *strict*. Hence we get four types of interval routing schemes: IRS, LIRS, strict IRS, and strict LIRS. The following section presents some preliminary results. Section 3 is dedicated to networks supporting strict LIRS. Section 4 relaxes the strictness requirement. Section 5 relaxes the linearity requirement, and presents our main result about IRS. Finally, Section 6 contains some remarks about related problems, specifically leader election and distributed spanning tree.

2 Preliminary results

In this section, we will present a distributed broadcast protocol for a network supporting an IRS. This protocol is very simple though very efficient since its message-complexity will be shown to be at most $O(n)$. It is called the up/down protocol. The second part of the section presents tools for the analysis of this protocol.

2.1 The up/down broadcast protocol.

Let ν be the source of the broadcast, identified by its label in the IRS. The source initiates the broadcast by sending two copies of the message, one destinated to node $\nu - 1$, and the other destinated to node $\nu + 1$. The latter copy is called the "up" copy whereas the former is called the "down" copy. (Obviously, if $\nu = 1$ or $\nu = n$, then the source sends only one copy.) There will be at most two copies of the message circulating in the network. Let us concentrate on the up copy destinated to $\nu + 1$. The message will eventually reach $\nu + 1$ by the shortest path set by the IRS. It may possibly cross intermediate nodes, but these nodes will just forward the message to its destination, without taking care of its content. Once $\nu + 1$ receives the message, it reads its content, modifies the header by replacing $\nu + 1$ by $\nu + 2$, and forwards it toward the node labeled $\nu + 2$. More generally, once a node labeled x, $\nu < x < n$, receives the message destinated to x, it reads its content, modifies the header by replacing x by $x + 1$, and forwards it toward node labeled $x + 1$. When the node labeled n receives the message, it reads its content, and removes it from the network. The same strategy is applied for the down copy, by replacing $x + 1$ by $x - 1$, $1 < x < \nu$, until it reaches node labeled 1 which reads its content and removes it from the network. Again, at

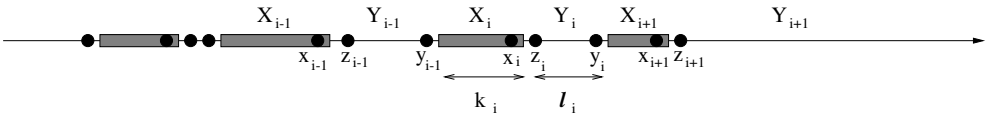

Figure 1: Notations

every given time, a copy of the message is destinated to only one specific node, called the *target*, and when a node different from that target receives the message, it does not take the opportunity to read its content, but just forwards the message to the target along the shortest path set by the IRS that leads to that target.

Clearly, the message-complexity of the up/down protocol is equal to $\sum_{i=1}^{n-1} d(i, i+1)$ where $d(x, y)$ is the distance between the node labeled x and the node labeled y in G. Unfortunately, up to the knowledge of the authors, there is no good bounds on $d(i, i+1)$ for networks supporting IRS. Let us just point out that it is known [5] that $d(i, i+1) = 1$ for networks supporting *all-shortest-paths* strict LIRS that is for networks such that the strict LIRS encodes *all* shortest paths. The class of networks supporting all-shortest-paths strict LIRS is very restricted, and, as we will see, only weaker results can be proved for (single shortest path) LIRS or IRS.

Assume that the source is labeled 1, and let $W = w_1, w_2, \ldots, w_\tau$ be the sequence of nodes visited by the message from its source w_1 (labeled 1) to its final destination w_τ (labeled n). The same node may appear several times in W. However, a node x can only appear once as a target node, and the possible other occurrences of node x correspond to steps of the protocol in which x is traversed by a message destinated to a target node $y \neq x$. Let us complement the sequence W by two virtual nodes, $w_0 = n + 1$, and $w_{\tau+1} = 0$. These two nodes are not target nodes. More precisely, W can be written as:

$$W = Y_0, X_1, Y_1, X_2, Y_2, \ldots, X_{\varrho-1}, Y_{\varrho-1}, X_\varrho, Y_\varrho \tag{1}$$

where X_i, $i = 1, \ldots, \varrho$, is a maximal sequence of consecutive target nodes, and Y_i, $i = 0, \ldots, \varrho$, is the sequence of non target nodes between the last node of X_i and the first node of X_{i+1}. In particular, $Y_0 = w_0$, and $Y_\varrho = w_{\tau+1}$. The nodes of the Y_i's are called *intermediate* nodes.

Notation. For every i, throughout the paper, we will always make use of the following notation (see Figure 1).

- $k_i = |X_i|$, and $\ell_i = |Y_i|$;
- x_i = the last node of X_i;
- y_i = the last node of Y_i; and
- z_i = the first node of Y_i.

2.2 Intermediate nodes sequences.

The next lemmas give some properties satisfied by the intermediate nodes in the Y_i's. These properties will be shown to be helpful to compute the message complexity of the up/down protocol.

Lemma 1 *Let G be a network supporting a strict LIRS. Let $u_0 = x, u_1, \ldots, u_k = x+1$ (resp. $v_0 = x, v_1, \ldots, v_k = x-1$) be the shortest path from node labeled x to node labeled $x+1$ (resp. to node labeled $x-1$) in a strict LIRS. Then $u_1 > u_2 > \ldots > u_{k-1} > x+1$ (resp. $v_1 < v_2 < \ldots < v_{k-1} < x-1$).*

Proof. Let I_i be the interval of edge (u_i, u_{i+1}) at u_i, $i = 0, \ldots, k-1$. We have $x+1 \in I_i$ by definition of the path $u_0 = x, u_1, \ldots, u_k = x+1$, and $u_{i+1} \in I_i$ because (u_i, u_{i+1}) is the unique shortest from u_i to u_{i+1}. On the other hand, $x \notin I_i$, $i > 0$, by definition of the path $u_0 = x, u_1, \ldots, u_k = x+1$, and $u_i \notin I_i$, $0 \leq i \leq k-1$, because the LIRS is strict. Therefore, since the intervals are all linear, $x+1 < u_{i+1} < u_i$ for $0 < i < k-1$. Similarly, one can show that $x-1 > v_{i+1} > v_i$ for $0 < i < k-1$. ■

As a direct consequence:

Corollary 1 *In a network supporting a strict LIRS, $d(i, i+1) \leq \min\{i, n-i\}$.*

This corollary does not directly yield a satisfactory upper bound on the message-complexity of the up/down protocol in strict LIRS networks since $\sum_{i=1}^{n-1} d(i, i+1) = O(n^2)$ according to this result. Lemma 1 is however important as it states that, in strict LIRS networks, once a target node x has received the message in the up/down protocol, no node of label smaller or equal to x will be visited anymore. Note that this is not enough to conclude that the message-complexity of the up/down protocol has a message-complexity $O(n)$ since nodes of large labels can be visited a non constant number of time in the up/down protocol. Nevertheless, Lemma 1 will be intensively used throughout this paper. The following results allows to understand the little difference between LIRS and strict LIRS.

Lemma 2 *Let G be a network supporting an LIRS. Let $u_0 = x, u_1, \ldots, u_k = x+1$ (resp. $v_0 = x, v_1, \ldots, v_k = x-1$) be the shortest path from node labeled x to node labeled $x+1$ (resp. to node labeled $x-1$) in an LIRS. Then, for every i, $k > i > 1$, we have $u_i > x+1$, and $u_i < u_j$ for every j, $1 < j \leq i-2$ (resp., for every i, $1 < i < k$, $v_i < x-1$, and $v_i > v_j$ for every j, $1 < j \leq i-2$).*

Proof. Let I_i be the interval of edge (u_i, u_{i+1}) at u_i, $i = 0, \ldots, k-1$. By definition, we have $x+1 \in I_i$, and $u_{i+1} \in I_i$. For $i \geq 1$, we also have $x \notin I_i$ (otherwise there would exist a shorter path from x to $x+1$). Thus $u_i > x+1$ for $i > 1$. More generally, if $i \geq 1$, then $u_j \notin I_i$ for every $j \leq i-1$. Therefore, since all the intervals are linear, $u_{i+1} < u_j$ for every j, $1 < j \leq i-1$. The result for the v_i's is obtained in a similar way. ■

Hence, we get a similar result as Corollary 1 for LIRS.

Corollary 2 *In a network supporting an LIRS, $d(i, i+1) \leq 1 + \min\{i, n-i\}$.*

The next two lemmas analyze the relationship between consecutive sequences of intermediate nodes. Their aim is to answer the following: given an intermediate node $w_i \in W$, where can we find another intermediate node $w_j \in W$, $j > i$, whose label is smaller, or at least not larger, than the label of w_i? (Lemmas 1 and 2 answer this question if w_i is not the last node of an intermediate sequence Y_r.)

For any set of nodes $S \subseteq V$, and any node $u \in V$, we denote by $d(u, S)$ the distance between u and S, that is $d(u, S) = \min_{v \in S} d(u, v)$.

Lemma 3 *Let G be a network supporting a strict LIRS, and let u be a node of G. Let $r \in \{1, \ldots, \varrho\}$, and assume $d(u, X_r) \leq \delta$. Assume that there exists $s \geq r$ such that $\sum_{i=r}^{s}(\ell_i - k_i) > \delta - 1$, and $\sum_{i=r}^{s'}(\ell_i - k_i) \leq \delta - 1$ for every s', $r \leq s' < s$. Let $\kappa = k_s + \sum_{i=r}^{s-1}(k_i - \ell_i) + \delta$, and let $Y_s = v_1, \ldots, v_{\ell_s}$. If $u > x$ for every $x \in \cup_{i=r}^{s} X_i$, and if, for every intermediate node $w \in \cup_{i=r}^{s-1} Y_i$, $w \geq u$, then $u > v_\kappa$, with $v_\kappa \in Y_s$.*

Proof. First note that $\sum_{i=r}^{s}(\ell_i - k_i) \geq \delta$ implies that $\kappa \leq \ell_s$. Note also that $\sum_{i=r}^{s-1}(\ell_i - k_i) \leq \delta - 1$ implies that $\kappa \geq k_s + 1 \geq 2$. Let us show by induction on i, that

$$d(u, y_i) \leq (\delta - 1) + \sum_{j=r}^{i}(k_j - \ell_j) \text{ for every } i = r, \ldots, s-1. \tag{2}$$

Let $i = r$, and let $Y_r = u_1, \ldots, u_{\ell_r}$. Note that $y_r = u_{\ell_r}$. Let I_1 be the interval on the edge (x_r, u_1) at x_r, and, for $i > 1$, let I_i be the interval on the edge (u_{i-1}, u_i) at u_{i-1}. For every $i \geq 1$, we have $x_r + 1 \in I_i$, and $x_r \notin I_i$. From Lemma 1, we also have $u_1 > \ldots > u_{\ell_r} > x_r + 1$. By hypothesis, $u > x$ for every $x \in \cup_{i=r}^{s} X_i$, and $u \leq u_{\ell_r}$. Thus, since $u_i \in I_i$, we get that $u \in I_i$ for every $i = 1, \ldots, \ell_r$. Therefore, a shortest path from x_r to u goes through $u_1, \ldots, u_{\ell_r}$. As a consequence $\ell_r + d(y_r, u) \leq k_r + d(u, X_r) - 1$, and thus $d(u, y_r) \leq (\delta - 1) + (k_r - \ell_r)$.

Assume now that $d(u, y_i) \leq (\delta - 1) + \sum_{j=r}^{i}(k_j - \ell_j)$ for some i, $r \leq i < s - 1$, and let us show that $d(u, y_{i+1}) \leq (\delta - 1) + \sum_{j=r}^{i+1}(k_j - \ell_j)$. For the same reasons as for the case $i = r$, we get that a shortest path from the last node x_{i+1} of X_{i+1} to u goes through all nodes of Y_{i+1}. We get that

$$\ell_{i+1} + d(u, y_{i+1}) \leq d(x_{i+1}, u) \leq k_{i+1} + d(u, y_i) \leq (\delta - 1) + k_{i+1} + \sum_{j=r}^{i}(k_j - \ell_j),$$

and thus $d(u, y_{i+1}) \leq (\delta - 1) + \sum_{j=r}^{i+1}(k_j - \ell_j)$. This completes the proof of Equation 2.

A consequence of Equation 2 is that

$$d(u, x_s) \leq (\delta - 1) + k_s + \sum_{j=r}^{s-1}(k_j - \ell_j) = \kappa - 1.$$

So now, let I_1 be the interval on edge (x_s, v_1) at x_s, and, for $i > 1$, let I_i be the interval on edge (v_{i-1}, v_i) at v_{i-1}. If $u < v_i$ for every $i \leq \kappa - 1$, then $u \in I_i$ for $i = 1, \ldots, \kappa - 1$, and thus, $d(u, x_s) > \kappa - 1$, a contradiction. Thus $u \geq v_i$ for some $i \leq \kappa - 1$, and therefore $u > v_\kappa$ by Lemma 1. This completes the proof of Lemma 3. ■

The difference between Lemma 1 and Lemma 2 motivates the following adaptation of Lemma 3 for LIRS networks that are non necessarily strict.

Lemma 4 *Let G be a network supporting an LIRS, and let u be a node of G. Let $r \in \{1, \ldots, \varrho\}$, and assume that $d(u, X_r) \leq \delta$. Assume that there exists $s \geq r$ such that $\sum_{i=r}^{s}(\ell_i - k_i) \geq 2(s - r + 1) + \delta$, and $\sum_{i=r}^{s'}(\ell_i - k_i) < 2(s' - r + 1) + \delta$ for every s', $r \leq s' < s$. Let $\kappa = k_s + \sum_{i=r}^{s-1}(k_i - \ell_i) + 2(s - r) + \delta$, and let $Y_s = v_1, \ldots, v_{\ell_s}$. Assume that $u > x$ for every $x \in \cup_{i=r+1}^{s} X_i$, and that, for every intermediate node $w \in \cup_{i=r}^{s-1}(Y_i \setminus \{z_i\})$, $w > u$. Then $\min\{v_{\kappa+1}, v_{\kappa+2}\} \leq u$.*

Proof. Note first that, $\sum_{i=r}^{s-1}(k_i - \ell_i) > 2(r-s) - \delta$ implies $\kappa > k_s \geq 1$, and $\sum_{i=r}^{s}(k_i - \ell_i) \leq 2(r-s-1) - \delta$ implies $\kappa \leq \ell_s - 2$. We proceed is a way similar to the proof of Lemma 3. We show that, for every $i \in \{r, \ldots, s-1\}$,

$$d(u, y_i) \leq \sum_{j=r}^{i}(k_j - \ell_j) + 2(i-r) + (\delta + 1). \tag{3}$$

Let $i = r$. The shortest path from z_r to u set by the IRS goes through all nodes of Y_r. Thus $\ell_r - 1 + d(y_r, u) \leq k_r + d(u, X_r)$, and hence $d(y_r, u) \leq (k_r - \ell_r) + (\delta + 1)$. Therefore, Equation 3 holds for $i = r$. Assume Equation 3 holds for i, and let us show that it holds for $i+1$. Again, the shortest path from z_{i+1} to u set by the IRS goes through all nodes of Y_{i+1}, and thus $\ell_{i+1} - 1 + d(y_{i+1}, u) \leq k_{i+1} + 1 + d(u, y_i)$, and hence $d(y_{i+1}, u) \leq \sum_{j=r}^{i+1}(k_j - \ell_j) + (\delta + 1) + 2(i + 1 - r)$, and Equation 3 holds for $i + 1$.

A consequence of Equation 3 is that

$$d(z_s, u) \leq k_s + 1 + \sum_{i=r}^{s-1}(k_i - \ell_i) + (\delta + 1) + 2(s - r - 1) = \kappa.$$

Thus there is a node in $\{v_2, \ldots, v_{\kappa+1}\}$ such that $v_i \leq u$. Indeed, otherwise, the path set by the IRS from z_s to u would go through $v_2, \ldots, v_{\kappa+1}$, and thus would not be a shortest path. Let i be the smallest index in $\{2, \ldots, \kappa + 1\}$ such that $v_i \leq u$. If $i < \kappa + 1$, then $v_{\kappa+2} < u$ from Lemma 2. If $i = \kappa + 1$, then $v_{\kappa+1} \leq u$ (actually, $v_{\kappa+1} = u$), and the result holds. ■

Now, we know enough to start the analysis of the up/down protocol.

3 Strict Linear Interval Routing Schemes

In this section, we show that the message-complexity of the up/down protocol is at most $3n$ on a network of order n supporting a strict LIRS.

3.1 Partition: a sequence-decomposition algorithm

Assume first that the source is the node 1. We make use of the sequence W as displayed in Equation 1. Since the total number of target nodes is n, we have $\sum_{i=1}^{\varrho} |X_i| = n$. Therefore, the aim of the rest of the proof is to bound the total number $\sum_{i=0}^{\varrho} |Y_i|$ of intermediate nodes. For that purpose, we will partition the intermediate nodes of W in three types of (pairwise disjoint) subsequences. The result of this partition is called the *sequence-decomposition*. Every intermediate node appears exactly once in this decomposition. The target nodes do not appear in the decomposition. More precisely, the decomposition will be composed of an *active path*, of *dead-end paths*, and of *jumped paths*. The active path is built starting a walk from w_0 up to $w_{\tau+1}$. Along the construction of the sequence-decomposition, some parts of the active path become dead-end paths. At the end of the decomposition, the two extremities of the active path are w_0 and $w_{\tau+1}$, and the total number of nodes in the active path will be at most $O(n)$. Also, at the end of the construction, the sum of the lengths of the dead-end paths, and the sum of the lengths of the jumped paths will be both bounded by $O(n)$. Therefore, the total number of intermediate nodes will be $O(n)$, and $|W| = O(n)$. A careful analysis of the constants will actually show that $|W| \leq 3n$.

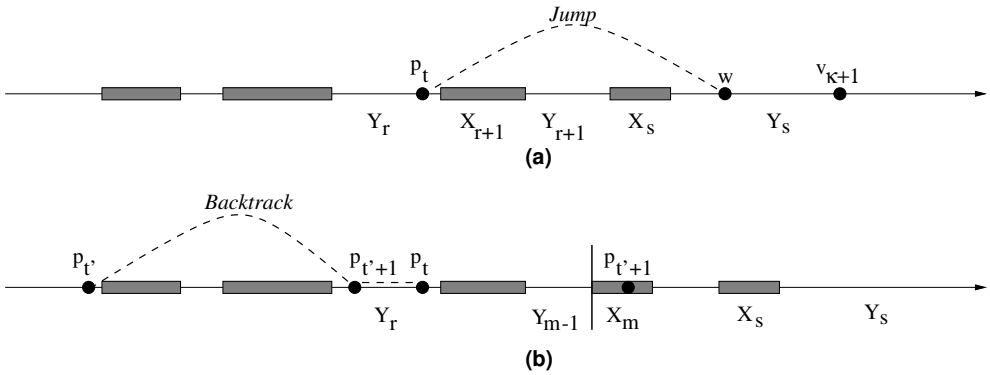

Figure 2: Construction of the sequence-decomposition

The decomposition is performed by visiting all intermediate nodes of the sequence W from w_0 to $w_{\tau+1}$, constructing in this way the active path. One may "jump" over some intermediate nodes if the length of the jump can be bounded by the number of jumped targets. The result of a jump is a jumped path. One may also backtrack along the active path for a bounded number of nodes. The result of a backtrack is a dead-end path. The decomposition requires two parameters: the *mark* m, and the *direction* d. The role of the mark is to keep in mind the progression along the sequence W. In general, the mark indicates the current position, or, more precisely, the index i of the current set Y_i. The mark is an important parameter because of the backtracks. When a backtrack occurs, the mark is set to remember the current maximum position ever reached. The direction indicates whether a backtrack recently occurred. In that case $d = -1$, and $d = +1$ otherwise.

Initially, the active path is reduced to w_0, and there is no dead-end path nor jumped path. The mark m is set to 0, and the direction is set to $+1$.

The construction of the sequence-decomposition is precisely described in Algorithm 1. The explanations of the several steps of the construction are given below.

In Case 1, the construction is currently visiting some Y_r. While the last node of this sequence of intermediate nodes is not reached, the active path is updated by adding all the forthcoming nodes of the current sequence. Informally, from Lemma 1, the size of the active path will not increase too much since the labels of the nodes are in a strictly decreasing order.

Case 2 happens in particular when the last node of the current sequence Y_r is reached (see Figure 2(a)). The definition of s is motivated by Lemma 3. If s does not exist, the construction stops. The jumped path has then a length bounded by $\sum_{i=r+1}^{\varrho} |X_i|$. In general, every jumped path b will have a length bounded by $\sum_{i \in \mathcal{I}_b} |X_i|$ for some index set $\mathcal{I}_b$, so that $\mathcal{I}_b \cap \mathcal{I}_{b'} = \emptyset$ if $b' \neq b$. If s does exist, then we make a jump in the sequence W as explained thereafter. (Note that, like in Lemma 3, $2 \leq \kappa \leq \ell_s$.)

In Case 2.1.1, we simply jump at the next intermediate node w whose label is smaller than the label of the current node. Case 2.1.2 can be seen as an extremal case of Case 2.1.1. We know from Lemma 3 that jumping at v_κ is fine since the label of that node is smaller than the label of the current node p_t. The mark is updated to contain the index of the sequence Y_σ of intermediate nodes reached after the jump.

Case 2.2 is in fact the most difficult case (see Figure 2(b)) because a backtrack occurs. This backtrack is motivated by the fact that one cannot find an intermediate node with a label smaller

that the label of p_t. Informally, we backtrack along the active path $p_0, p_1, \ldots, p_t$ until we reach a node $p_{t'}$, $t' < t$, for which Lemma 3 can be applied. Note that t' is well defined since $p_0 \notin \cup_{i=1}^{\varrho} X_i = \{1, \ldots, n\}$. We will show that the length of a dead-end path that results from a backtrack is bounded by $\sum_{i=m+1}^{\sigma} |X_i|$. We will also see that the possible jumped path set from the value of $p_{t'}$ has a length bounded by $\sum_{i=m+1}^{\sigma-1} |X_i|$. Figure 2(b) illustrates the specific case in which the backtrack goes across a former jump.

Algorithm 1 One step of the construction of the sequence-decomposition for a strict LIRS

The current active path is $P = p_0, p_1, \ldots, p_t$, with $p_0 = w_0$. Let r be such that $p_t \in Y_r$, and let $Y_r = u_1, \ldots, u_{\ell_r}$.

Case 1: $p_t = u_i, i < \ell_r$ **and** $d = +1$. Then the active path is updated to $p_0, p_1, \ldots, p_t, u_{i+1}$. There is no new dead-end path nor jumped path. The mark and the direction are not modified.

Case 2: $p_t = u_{\ell_r}$ **or** $d = -1$. Then let $\delta = d(p_t, X_{m+1})$, and $s \geq m+1$ be the smallest index such that $\sum_{i=m+1}^{s} (\ell_i - k_i) > \delta - 1$. If s does not exist, then the active path is updated to $p_0, p_1, \ldots, p_t, w_{\tau+1}$, there is no new dead-end path, all intermediate nodes between the last node of Y_m and $w_{\tau+1}$ form a new jumped path, and the construction stops. If s exists, then let $\kappa = \delta + k_s + \sum_{i=m+1}^{s-1} (k_i - \ell_i)$. Assume that $Y_s = v_1, \ldots, v_{\ell_s}$, two cases are considered:

Case 2.1: $p_t \notin \cup_{i=m+1}^{s} X_i$. Then again two cases may occur.

Case 2.1.1: There is an intermediate node w in $\{v_1, \ldots, v_{\kappa-1}\} \cup (\cup_{i=m+1}^{s-1} Y_i)$ such that $w < p_t$. Then pick the first node w of that type. The active path is updated to $p_0, p_1, \ldots, p_t, w$, and all intermediate nodes between the last node of Y_m and w form a jumped path. Assume $w \in Y_\sigma$, then the mark m is set to σ.

Case 2.1.2: For every intermediate node w in $\{v_1, \ldots, v_{\kappa-1}\} \cup (\cup_{i=m+1}^{s-1} Y_i)$, $w \geq p_t$. Then the active path is updated to $p_0, p_1, \ldots, p_t, v_\kappa$, and all intermediate nodes between the last node of Y_m and v_κ forms a jumped path. The mark m is set to s.

In both cases, there is no new dead-end path, and the direction d is set to $+1$.

Case 2.2: $p_t \in \cup_{i=m+1}^{s} X_i$. Then the direction d is set to -1. Let $t' \in \{0, \ldots, t-1\}$ be the largest index such that $p_{t'} \notin \cup_{i=m+1}^{s} X_i$. All nodes $p_{t'+1}, \ldots, p_t$ forms a dead-end path. Assume $p_{t'+1} \in X_\sigma$, then if $\sigma > m+1$, all intermediate nodes of $\cup_{i=m+1}^{\sigma-1} Y_i$ form a jumped path. The mark m is updated to $\sigma - 1$.

Now, let us analyze the construction of the sequence-decomposition.

Claim 1 *If $d = +1$, then $p_t \in Y_m$.*

Proof. This claim initially holds. Assume the claim holds before some step i, and consider the several cases of Algorithm 1. In Case 1, p_{t+1} will still be in Y_m, and d will still be $+1$. In Case 2.1, the direction is set to $+1$ and $p_{t+1} \in Y_m$ by definition of the setting of the mark m in both Cases 2.1.1, and 2.1.2. Case 2.2 sets d to -1. So the claim holds after step i too. ■

Lemma 5 *The construction of the sequence-decomposition given in Algorithm 1 produces a set of paths in which every intermediate node appears exactly once.*

Proof. After every step i, let us define m_i as the resulting mark, and

$$q_i = \begin{cases} p_t & \text{if } p_t \in Y_{m_i}; \\ \text{last node of } Y_{m_i} & \text{otherwise.} \end{cases}$$

We claim that, at every step i, all intermediate nodes before q_i appears exactly once in the sequence-decomposition. This claim initially holds. Assume it holds before step i. In Case 1, from Claim 1, $q_{i-1} = p_t \in Y_{m_{i-1}}$. The active path is upgraded by adding the next node of the sequence, $Y_{m_i} = Y_{m_{i-1}}$, and thus the claim holds after step i. In Case 2.1, m_i is set such that $p_{t+1} \in Y_{m_i}$, and thus $q_i = p_{t+1}$. Every intermediate node between the last node of $Y_{m_{i-1}}$, that is q_{i-1}, and $q_i = p_{t+1}$ are put in a jumped path, so the claim holds after step i. In Case 2.2, some part of the active path becomes a dead-end path. By the setting of m_i, all the intermediate nodes not yet assigned to any type of paths, that is all intermediate nodes in $\cup_{j=m_{i-1}+1}^{m_i} Y_j$, are put in a jumped path. Thus every intermediate node between q_{i-1} and q_i are put in a jumped path. Therefore, the claim holds after step i. This completes the proof of the claim.

We complete the proof of the lemma by noticing that, after every step, either the active path is increased, or a part of the active path is put in a dead-end path. By the setting of the mark, an intermediate-node put in a dead-end path or in a jumped path will not be considered anymore in the further steps. Therefore the construction of the sequence-decomposition of Algorithm 1 ends after a finite number of steps. ■

3.2 Message-complexity of the up/down protocol

From the sequence-decomposition algorithm, let us compute the message-complexity of the up/down protocol.

Lemma 6 *The number of intermediate nodes in the active path is at most n, excluding w_0 and $w_{\tau+1}$. More precisely, the labels of the nodes of the active path, including w_0 and $w_{\tau+1}$, form a decreasing sequence in $n+1, \ldots, 0$.*

Proof. To prove that lemma, let us go again through the several cases of a step of the decomposition. Let p_t be the last node of the current active path. In Case 1, Lemma 1 insures that $p_{t+1} < p_t$. In Case 2, if s does not exist, then the next node of the active path is $w_{\tau+1} = 0$ which is smaller than every other nodes of the current active path. So, in the remaining of this proof, we assume that s does exist. Since Case 2.2 does not add new node to the active path, we focus on case 2.1. In Case 2.1.1, the new node added to the active path is, by definition, smaller than p_t. In Case 2.1.2, $v_\kappa < p_t$ by application of Lemma 3, which completes the proof of that lemma. ■

Lemma 7 *The number of nodes in the dead-end paths is at most n.*

Proof. A dead-end path is composed of a sequence of nodes that were formerly in the active path, and through which a backtrack occurred. Every backtrack is driven by a set of target nodes $\cup_{i=m+1}^{\sigma} X_i$ where $p_{t'+1} \in X_\sigma$. Let $X_\sigma = x+1, \ldots, x+k$, and $p_{t'+1} = x+i$, $1 \leq i \leq k$. Similarly, let $p_t \in X_{m'+1} = x'+1, \ldots, x'+k'$, $p_t = x'+i'$, $1 \leq i' \leq k'$.

From Lemma 6, the number of nodes of the dead-end path corresponding to $X_{m+1}, \ldots, X_\sigma$ is bounded by $(k'-i') + \sum_{j=m'+2}^{\sigma-1} |X_j| + i$ because the active path, traversed in the reverse direction,

produced a sequence of nodes of increasing labels. Now, m is updated to $\sigma - 1$ after the backtrack, and thus $X_{m+1}, \ldots, X_{\sigma-1}$ will not be considered anymore for the counting of the number of nodes in the dead-end paths. X_σ may be considered again in another dead-end path since m' can be equal to m. However, only the $k - i$ last nodes of X_σ will be involved. As a consequence, the total number of nodes in dead-end paths is bounded by $\sum_{i=1}^{\varrho} |X_i| = n$. ■

Claim 2 *Assume that two jumps J' and J successively occurred at x (that is J' occurred, and then later a backtrack led back to x, and J occurred). Let $y' \in Y_{\sigma'} = u_1, \ldots, u_{\ell_{\sigma'}}$ and $y \in Y_\sigma$ be the respective extremities of J' and J, and assume $y' = u_i$. Let m and δ (resp. m' and δ') be the setting of the mark and the distance when J (resp. J') occurred. Then $\delta \leq (\delta' - 1) + \sum_{j=m'+1}^{\sigma'-1}(k_j - \ell_j) + (k_{\sigma'} - i)$.*

Proof. By definition $\delta = d(x, X_{m+1})$. Also, $y' \in X_{m+1}$ by the setting of m when backtracking through J' as $p_{t'+1} = y'$. Thus $\delta \leq d(x, y')$. On the other hand, by the same arguments as for Equation 2, $d(y_{m'+1}, x) + \ell_{m'+1} \leq k_{m'+1} - 1 + \delta'$, and thus $d(y_{m'+1}, x) \leq (\delta' - 1) + (k_{m'+1} - \ell_{m'+1})$. More generally, $d(y_{\sigma'-1}, x) \leq (\delta' - 1) + \sum_{j=m'+1}^{\sigma'-1}(k_j - \ell_j)$. Since $i + d(y', x) \leq k_{\sigma'} + d(y_{\sigma'-1}, x)$, we get

$$d(x, y') \leq (\delta' - 1) + \sum_{j=m'+1}^{\sigma'-1} (k_j - \ell_j) + (k_{\sigma'} - i)$$

which completes the proof of the claim. ■

Lemma 8 *The number of nodes in the jumped paths is at most n.*

Proof. As dead-end paths, the jumped paths are characterized by disjoint sets of target nodes of the form $\cup_{i=m+1}^{s} X_i$. Let us first consider the jumped paths created by application of Case 2.1 of Algorithm 1. Assume that a jump J occurred between $x \in Y_r$, and $y \in Y_\sigma$, $r < \sigma \leq s$. If $\sigma < s$, then the number of nodes of the jumped path is at most $\sum_{i=m+1}^{\sigma} \ell_i - 1 < (\delta - 1) + \sum_{i=m+1}^{\sigma} |X_i|$. If $\sigma = s$, then assume that $Y_s = v_1, \ldots, v_{\ell_s}$, $y = v_i$ with $i \leq \kappa = \delta + k_s + \sum_{i=m+1}^{s-1}(k_i - \ell_i)$. Thus, the number of nodes of the corresponding jumped path is at most $\sum_{i=m+1}^{s-1} \ell_i + \kappa - 1 \leq (\delta - 1) + \sum_{i=m+1}^{s} |X_i|$. So, in any case, the number of nodes of the jumped path is at most $(\delta - 1) + \sum_{i=m+1}^{\sigma} |X_i|$. If $\delta > 1$, then this setting corresponds to another jump J' occurred at x, say between x and $y' \in Y_{\sigma'} = u_1, \ldots, u_{\ell'}$, $y' = u_i$. Let m' and δ' be respectively the value of the mark and of the distance when J' occurred. From Claim 2, we have

$$\delta \leq (\delta' - 1) + \sum_{j=m'+1}^{\sigma'-1} (k_j - \ell_j) + (k_{\sigma'} - i).$$

On the other hand, the size of the jump at y' is $i - 1 + \sum_{j=m'+1}^{\sigma'-1} \ell_j$. It yields that the number of node for the two jumps is at most $(\delta' - 1) + \sum_{i=m'+1}^{\sigma'} |X_i| + \sum_{i=m+1}^{\sigma} |X_i|$. We can repeat for δ' what we did for δ, until we have considered all jumps that successively occurred at x. It yields that the total number of nodes of the set of jumped paths occurring at the same vertex x is at most $\sum_{i=r+1}^{\sigma} |X_i|$ where Y_σ contains the other extremity of the last jump occurred at x.

The analysis is the same for the jumped paths created in Case 2.2 by proceeding as if the jump occurred between $p_{t'}$ and the last node of $Y_{\sigma-1}$ (recall that $p_{t'+1} \in X_\sigma$).

We conclude the proof by noticing that due to the setting of the mark m, no two jumps occur above the same set of target nodes, and therefore the total number of nodes is at most $\sum_{i=1}^{\varrho} |X_i| = n$. ■

Combining Lemmas 6, 7, and 8 with Lemma 5 allows to conclude that the total number of intermediate nodes is at most $3n$. We can actually improve this upper bound to $2n$. Let x be the extremity of the active path when s cannot be defined, that is when $\sum_{i=m+1}^{s-1}(\ell_i - k_i) \leq \delta - 1$ for every s, $m+1 \leq s \leq \varrho$. Since $x \in \cup_{i=m+1}^{\varrho} X_i$, all nodes of the active path belong to $\cup_{i=m+1}^{\varrho} X_i$ by application of Lemma 6. By application of the same lemma, the total number of nodes in the active path is at most $\sum_{i=m+1}^{\varrho} |X_i|$. Since the sets $X_{m+1}, \ldots, X_{\varrho}$ were not used to bound the total number of dead-end paths, we can conclude that the sum of the number of nodes in the active path plus the number of nodes in the dead-end paths is at most n. From all what precede, the total number of nodes in the sequence-decomposition, and therefore the total number of intermediate nodes, is at most $2n$. Therefore, the total number of nodes in the sequence W is at most $3n$, and hence the message-complexity of the up/down protocol is at most $3n$.

If the source node is not the node labeled 1, then let $\nu > 1$ be the label of the source. From Lemma 1, the message complexity of the copy going upward is at most $3(n-\nu)$ whereas the message complexity of the copy going downward is at most 3ν.

To summarize, we get:

Theorem 1 *The message-complexity of the up/down broadcast protocol is at most $3n$ in a network of order n supporting a strict LIRS. As a consequence, networks supporting a strict LIRS allows broadcasting with $O(n)$ message-complexity.*

And therefore, since the complexity of the up/down protocol is equal to $\sum_{i=1}^{n-1} d(i, i+1)$, we get:

Corollary 3 *In a network supporting a strict LIRS, the average distance between two nodes labeled by two consecutive integers is at most $3 + O(1/n)$.*

Note that $K_{2,n-2}$, the complete bipartite graphs with one partition of size 2, and another partiction of size $n-2$, supports a strict LIRS for which the up/down protocol uses $2n-4$ messages.

4 Linear Interval Routing Schemes

In order to analyze the up/down protocol in networks supporting LIRS, we partition the sequence W in slightly different manner than in the previous section. This decomposition is given in Algorithm 2. As it will clearly appear to the reader, the decomposition for LIRS looks very similar to the decomposition for strict LIRS. Assume first that the source is labeled 1.

Initially, the active path is reduced to w_0, and there is neither dead-end path nor jumped path. A fourth type of path is introduced, the *auxiliary path*. This path is motivated by the fact the active path will include one node every two, according to Lemma 2. Every node in between two nodes of the active path is dropped in the auxiliary path. Initially, the auxiliary path is empty, and the mark m is set to 0.

Algorithm 2 One step of the construction of the sequence-decomposition for an LIRS

The current active path is $P = p_0, p_1, \ldots, p_t$, with $p_0 = w_0$. Let r be such that $p_t \in Y_r$, and let $Y_r = u_1, \ldots, u_{\ell_r}$.

Case 1: $p_t = u_i, i < \ell_r - 1$ **and** $d = +1$. Then the active path is updated to $p_0, p_1, \ldots, p_t, u_{i+2}$, and node u_{i+1} is put in the auxiliary path. There is neither new dead-end path nor jumped path. The mark and the direction are not modified.

Case 2: $p_t = u_{\ell_r}$ **or** $p_t = u_{\ell_r - 1}$ **or** $d = -1$. Then let $\delta = d(p_t, X_{m+1})$, and $s \geq m+1$ be the smallest index such that $\sum_{i=m+1}^{s}(\ell_i - k_i) \geq 2(s-m) + \delta$. If s does not exist, then the active path is updated to $p_0, p_1, \ldots, p_t, w_{\tau+1}$, there is no new dead-end path, all intermediate nodes between the last node of Y_m and $w_{\tau+1}$ form a new jumped path, and the construction stops. If s exists, then let $\kappa = \delta + k_s + \sum_{i=m+1}^{s-1}(k_i - \ell_i) + 2(s-m-1)$. If $p_t = y_{\ell_r - 1}$ and $d = +1$, then u_{ℓ_r} is put in the auxiliary path. Assume that $Y_s = v_1, \ldots, v_{\ell_s}$, two cases are considered:

Case 2.1: $p_t \notin \cup_{i=m+1}^{s} X_i$. Then again two cases may occur.

Case 2.1.1: There is an intermediate node w in $\{v_1, \ldots, v_\kappa\} \cup (\cup_{i=m+1}^{s-1}(Y_i \setminus \{z_i\}))$ such that $w \leq p_t$. Then pick the first node w of that type. The active path is updated to $p_0, p_1, \ldots, p_t, w$, and all intermediate nodes between the last node of Y_m and w form a jumped path. Assume $w \in Y_\sigma$, the mark m is updated to σ.

Case 2.1.2: For every intermediate node w in $\{v_1, \ldots, v_\kappa\} \cup (\cup_{i=m+1}^{s-1}(Y_i \setminus \{z_i\}))$, $w > \min\{p_t, p_{t-1}\}$. Then the active path is updated by setting $p_{t+1} = \min\{v_{\kappa+1}, v_{\kappa+2}\}$. All intermediate nodes between the last node of Y_m and p_{t+1} forms a jumped path. The mark m is set to s.

In both cases, there is no new dead-end path, and the direction d is set to $+1$.

Case 2.2: $p_t \in \cup_{i=m+1}^{s} X_i$. The direction d is set to -1. Let $t' \in \{0, \ldots, t-1\}$ be the largest index such that $p_{t'} \notin \cup_{i=m+1}^{s} X_i$. All nodes $p_{t'+1}, \ldots, p_t$ forms a dead-end path. Assume that $p_{t'+1} \in X_\sigma$, then if $\sigma > m+1$, all intermediate nodes of $\cup_{i=m+1}^{\sigma-1} Y_i$ form a jumped path. The mark m is updated to $\sigma - 1$.

Let us point out the main differences appearing at every step of the sequence-decomposition. Case 1 is roughly the same as Case 1 in Algorithm 1, that is the decomposition uses the current intermediate sequence to construct the active path. The only modification is that one node every two is dropped in the auxiliary path, and that one may stop at $u_{\ell_r - 1}$ or u_{ℓ_r}. Therefore, Case 2 considers also the case $p_t = u_{\ell_r - 1}$, which, if true, implies that u_{ℓ_r} is put in the auxiliary path. Case 2 also differs from Case 2 in Algorithm 1 by the definition of both s and κ. These settings are motivated by Lemma 4. Otherwise, the general structure of Algorithm 2 is the same as Algorithm 1.

The reader can check that the following lemma is satisfied in the LIRS sequence-decomposition.

Lemma 9 *The construction of the sequence-decomposition given in Algorithm 2 produces a set of paths in which every intermediate node appears exactly once.*

Now, recall that ϱ denotes the number of sequences X_i in W. We have:

Lemma 10 *The number of intermediate nodes in the active path is at most $n + \varrho$, excluding w_0 and $w_{\tau+1}$. More precisely, the labels of the nodes of the active path form a non increasing sequence $p_0, \ldots, p_t$ from $p_0 = n + 1$ to $p_t = 0$ such that the number of times that $p_i = p_{i-1}$ is at most ϱ.*

Proof. From lemma 2, Case 1 insures that $p_{t+1} = u_{i+2} < u_i = p_t$. In Case 2, if s does not exist, then $p_{t+1} = w_{\tau+1} = 0 < p_t$. So assume that s exists. There is no setting of the active path in Case 2.2. In case 2.1.1, a jump occurs and $p_{t+1} \leq p_t$ by definition. In case 2.1.2, a jump occurs too and $p_{t+1} \leq p_t$ from Lemma 4. The number of jumps is at most ϱ. ■

Lemma 11 *The number of nodes in dead-end paths or in the active path is at most $n + \varrho$.*

Proof. By similar arguments as in the proof of Lemma 7, and using the same notation, the number of nodes of a dead-end path corresponding to the sets $X_{m+1}, \ldots, X_\sigma$ is at most $(k' - i') + \sum_{j=m'+1}^{\sigma-1} |X_j| + i$ plus the number of jumps occurring in the portion $p_{t'+1}, \ldots, p_t$ of W. It yields a total number of nodes in dead-end paths of at most $\sum_{i=1}^{\varrho} |X_i| + \varrho = n + \varrho$.

Now, as we observed in the strict LIRS case, if $x \in \cup_{i=m+1}^{\varrho} X_i$ is the last node of the active path different from $w_{\tau+1}$, then the total number of nodes in the active path is at most $(k' - i') + \sum_{i=m+2}^{\varrho} |X_i| + j$ where j is the number of jumps of the final active path. The sets $X_{m+2}, \ldots, X_\varrho$ were not used to enumerate the nodes in the dead-end paths, and the total number of jumps cannot exceed ϱ. Therefore, the total number of nodes in the active path or in dead-end paths is at most $n + \varrho$. ■

Lemma 12 *The number of nodes in the auxiliary path is at most $n + \varrho$.*

Proof. This is a direct consequence of Lemma 11 since the number of nodes in the auxiliary path does not exceed the number of nodes in the active path or in the dead-end path. Indeed, at most one node is dropped in the auxiliary path for every node entering the active path. Some nodes formerly in the active path become member of a dead-end path. ■

Lemma 13 *The number of nodes in the jumped paths is at most $n + 3\varrho$.*

Proof. Let us first consider a jump created by application of Case 2.1. Let J be a jump corresponding to the sets $X_{m+1}, \ldots, X_s$. Assume that J occurred between $x \in Y_r$ and $y \in Y_\sigma$, $r < \sigma \leq s$. If $\sigma < s$, then the jump was over at most $\sum_{i=m+1}^{\sigma} \ell_i - 1$ intermediate nodes, with $\sum_{i=m+1}^{\sigma} \ell_i < \sum_{i=m+1}^{\sigma} k_i + 2(\sigma - m) + \delta$. If $\sigma = s$, let $Y_s = v_1, \ldots, v_{\ell_s}$, $y = v_i$. Then $i \leq \kappa + 2 = \delta + k_s + \sum_{i=m+1}^{s-1} (k_i - \ell_i) + 2(s - m)$. Therefore, in any case, the number of jumped intermediate nodes is at most $(\delta - 1) + \sum_{i=m+1}^{\sigma} |X_i| + 2(\sigma - m)$. If $\delta > 2$, then this setting corresponds to another jump J' occurred at x, say between x and $y' \in Y_{\sigma'} = u_1, \ldots, u_{\ell'}$, $y' = u_i$. Let m' and δ' be the value of the mark, and the value of the distance, respectively, when this jump occurred. We have

$$\delta \leq (\delta' - 1) + \sum_{j=m'+1}^{\sigma'-1} (k_j - \ell_j) + 2(\sigma' - m') + (k_{\sigma'} - i).$$

Indeed, $\delta = d(x, X_{m+1})$. Moreover, $y' \in X_{m+1}$ by the setting of m when backtracking through J' as $y' = p_{t'+1}$. Therefore, $\delta \leq d(x, y')$, and, by the same arguments as in the proof of Claim 2,

$$d(x, y') \leq (\delta' - 1) + \sum_{j=m'+1}^{\sigma'-1} (k_j - \ell_j) + 2(\sigma' - m') + (k_{\sigma'} - i).$$

The size of the jump J' is $i - 1 + \sum_{j=m'+1}^{\sigma'-1}(k_j - \ell_j)$, and thus the size of the two jumps is at most

$$(\delta' - 1) + \sum_{i=m+1}^{\sigma} |X_i| + 2(\sigma - m) + \sum_{i=m'+1}^{\sigma'} |X_i| + 2(\sigma' - m').$$

If $\delta' > 2$, then one can apply on δ' the same arguments. We get that the number of nodes of the set of jumped paths occurring at the same vertex $x \in Y_r$ is at most $1 + \sum_{i=r+1}^{\sigma} |X_i| + 2(\sigma - r)$ where Y_σ contains the extremity of the last jump occurred at x.

The analysis is the same for the jumped paths created in Case 2.2 by proceeding as if the jump occurred between $p_{t'}$ and the last node of $Y_{\sigma-1}$ (recall that $p_{t'+1} \in X_\sigma$).

The worst case is reached when every jump is over a single X_i, whose cost in term of nodes is $1 + |X_i| + 2 = |X_i| + 3$, for a total number of nodes at most $\sum_{i=1}^{\varrho} |X_i| + 3\varrho$. ■

The three previous lemmas, together with Lemma 9, show that $\sum_{i=1}^{\varrho-1} |Y_i| \leq 3n + 5\varrho \leq 8n$, and thus $|W| \leq 9n$. If the source node is not the node labeled 1, then let $\nu > 1$ be the label of the source. From Lemma 2, the message complexity of the copy going upward is at most $9(n - \nu)$ whereas the message complexity of the copy going downward is at most 9ν.

To summarize, we get:

Theorem 2 *The message-complexity of the up/down broadcast protocol is at most* $9n$ *in a network of order* n *supporting an LIRS. As a consequence, networks supporting an LIRS allows broadcasting with* $O(n)$ *message-complexity.*

And therefore:

Corollary 4 *In a network supporting an LIRS, the average distance between two nodes labeled by two consecutive integers is at most* $9 + O(1/n)$.

5 Interval Routing Schemes

As opposed to linear IRS, the labels of the nodes of an IRS play all the same role. Therefore, it is not surprising that results such as Corollary 1 or 2 do not hold. For instance, the nodes of the path $P_n = x_1, \ldots, x_n$ of n nodes can be labeled in an IRS by $x_1 = 1$, and $x_i = n - i + 2$ for $i \geq 2$. Let I_i be the interval on (x_i, x_{i+1}) at x_i, $i = 1, \ldots, n-1$, and let J_i be the interval on (x_{i-1}, x_i) at x_i, $i = 2, \ldots, n$. The setting $I_i = [2, n + 1 - i]$, $J_i = [n - i + 3, 1]$, for $i \geq 3$, and $J_2 = [1, 1]$ satisfies the properties of shortest path IRS. In this setting, $d(1, 2) = n - 1$. However, this IRS satisfies $\sum_{i=1}^{n-1} d(i, i + 1) = O(n)$ so the up/down protocol still has a linear message complexity. We will show that this is true for every IRS network.

We actually slightly modify the up/down protocol. If the source is labeled ν, $1 < \nu < n$, then there is only one copy of the message, going upward from ν to n, then from n to 1, and finally from 1 to $\nu - 1$, rather than two copies, one going downward from ν to 1, and the other going upward from ν to n. This protocol is denoted by up/1/up. So, as far as the up/1/up protocol is concerned, one can assume, w.l.g., that the source is labeled 1.

Again, in order to analyze the up/1/up protocol, we make use of the sequence W defined in Equation 1. For every $x \in \{0, \ldots, n-1\}$, we define

$$L_x : \{1, \ldots, n\} \mapsto \{1, \ldots, n\} \text{ such that } L_x(u) = 1 + [(n + u - x - 1) \bmod n].$$

We have $L_x(x) = n$ and $L_x(x+1) = 1$.

The reader can check that:

Lemma 14 *Let G be a network supporting an IRS. Let $u_0 = x, u_1, \ldots, u_k = x+1$ (resp. $v_0 = x, v_1, \ldots, v_k = x-1$) be the shortest path from node labeled x to node labeled $x+1$ (resp. to node labeled $x-1$) in this IRS.*

- *If the IRS is strict, then $L_x(x) = n > L_x(u_1) > L_x(u_2) > \ldots > L_x(u_{k-1}) > L_x(x+1) = 1$ (resp. $L_{x-1}(x) = 1 < L_{x-1}(v_1) < L_{x-1}(v_2) < \ldots < L_{x-1}(v_{k-1}) < L_{x-1}(x-1) = n$).*
- *Otherwise, for every i, $k > i > 1$, we have $L_x(u_i) > 1$, and $L_x(u_i) < L_x(u_j)$ for every j, $1 < j \leq i-2$ (resp., for every i, $1 < i < k$, $L_{x-1}(v_i) < n$, and $L_{x-1}(v_i) > L_{x-1}(v_j)$ for every j, $1 < j \leq i-2$).*

And that:

Lemma 15 *Let G be a network supporting an IRS, and let u be any node of G. Let $r \in \{1, \ldots, \varrho\}$, and assume that $d(u, X_r) \leq \delta$.*

- *If the IRS is strict, then assume that there exists $s \geq r$ such that $\sum_{i=r}^{s}(\ell_i - k_i) > \delta - 1$, and $\sum_{i=r}^{s'}(\ell_i - k_i) \leq \delta - 1$ for every s', $r \leq s' < s$. Let $\kappa = k_s + \sum_{i=r}^{s-1}(k_i - \ell_i) + \delta$, and let $Y_s = v_1, \ldots, v_{\ell_s}$. If, for every $i \in \{r, \ldots, s\}$, and every $x \in X_i$, $L_{x_i}(u) > L_{x_i}(x)$, and if, for every $i \in \{r, \ldots, s-1\}$ and every intermediate node $w \in Y_i$, $L_{x_i}(w) \geq L_{x_i}(u)$, then $L_{x_s}(v_\kappa) < L_{x_s}(u)$.*
- *Otherwise, assume that there exists $s \geq r$ such that $\sum_{i=r}^{s}(\ell_i - k_i) \geq 2(s-r+1) + \delta$, and $\sum_{i=r}^{s'}(\ell_i - k_i) < 2(s'-r+1) + \delta$ for every s', $r \leq s' < s$. Let $\kappa = k_s + \sum_{i=r}^{s-1}(k_i - \ell_i) + 2(s-r) + \delta$, and let $Y_s = v_1, \ldots, v_{\ell_s}$. If, for every $i \in \{r, \ldots, s\}$, and every $x \in X_i$, $L_{x_i}(u) > L_{x_i}(x)$, and if, for every $i \in \{r, \ldots, s-1\}$ and every intermediate node $w \in Y_i \setminus \{z_i\}$, $L_{x_i}(w) > L_{x_i}(u)$, then $\min\{L_{x_s}(v_{\kappa+1}), L_{x_s}(v_{\kappa+2})\} \leq L_{x_s}(u)$.*

Therefore, the sequence-decomposition of Algorithms 1, and 2 can be applied by introducing the relabeling L_x's in the appropriate places. The resulting algorithms are expressed in the Appendix. The previous lemmas of this section show that there is no major difference between the sequence-decomposition for LIRS networks and the sequence-decomposition for IRS networks. In fact, Algorithms 3, and 4 satisfy the same properties as Algorithms 1, and 2 respectively. There is only one difference: when a jump takes place in Algorithm 3, or 4, say between $p_t \in Y_r$ and $w \in Y_\sigma$,

then, by Lemma 15, $L_{x_\sigma}(w) < L_{x_\sigma}(p_t)$ in the strict IRS decomposition, and $L_{x_\sigma}(w) \leq L_{x_\sigma}(p_t)$ in the IRS decomposition. This is not enough to apply directly the arguments of Lemma 6 or 10. Nevertheless, we have:

Lemma 16 *Assume $p_i \in Y_r$, and $p_{i+1} \in Y_\sigma$, $r < \sigma$. Then $L_{x_\sigma}(p_{i+1}) < L_{x_r}(p_i)$ in the sequence-decomposition for strict IRS networks, and $L_{x_\sigma}(p_{i+1}) \leq L_{x_r}(p_i)$, where the maximum number of equality is at most ϱ, in the sequence-decomposition for IRS networks.*

Proof. According to the statement of the lemma, we are considering Case 2.1 of the sequence decomposition. Hence $p_i \notin \cup_{j=m+1}^{\sigma} X_j$. More precisely $p_i \notin \cup_{j=r+1}^{\sigma} X_j$ from the possible setting of the mark after several backtracks ending at p_i. Let us first consider the strict IRS decomposition. We have $p_{i+1} \in Y_\sigma$, and $L_{x_\sigma}(p_{i+1}) < L_{x_\sigma}(p_i)$. Let us assume that $L_{x_r}(x_\sigma) > L_{x_r}(p_i)$. Then, since from Lemma 14, $L_{x_r}(p_i) > L_{x_r}(x_r + 1)$, we get that p_i is equal to a target node in $\cup_{i=r+1}^{\sigma} X_i$, a contradiction with the hypothesis of Case 2.1. Therefore $L_{x_r}(x_\sigma) < L_{x_r}(p_i)$, and hence $L_{x_\sigma}(p_i) < L_{x_r}(p_i)$. Therefore, $L_{x_\sigma}(p_{i+1}) < L_{x_r}(p_i)$. The same type of arguments allow to show that $L_{x_\sigma}(p_{i+1}) \leq L_{x_r}(p_i)$ in the sequence-decomposition for IRS networks. ■

Therefore, if $f(i)$ denotes the index such that $p_i \in X_{f(i)}$, then the active path $p_0, p_1, \ldots, p_t, p_{t+1}$ resulting from the sequence decomposition for strict IRS networks satisfies $p_0 = n + 1$, $p_{t+1} = 0$, and $n + 1 > L_{x_{f(i)}}(p_i) > L_{x_{f(j)}}(p_j) > 0$ for any pair (i, j), $1 \leq i < j \leq t$. In IRS networks, there might be up to ϱ equalities in the sequence. As a consequence:

Theorem 3 *The message-complexity of the up/1/up broadcast protocol is respectively at most $3n$ in a network of order n supporting a strict IRS, and at most $9n$ in a network of order n supporting an IRS. As a consequence, networks supporting shortest path Interval Routing Scheme allows broadcasting with $O(n)$ message-complexity.*

And therefore:

Corollary 5

- *In a network supporting a strict IRS, the average distance between two nodes labeled by two consecutive integers is at most $3 + O(1/n)$.*
- *In a network supporting an IRS, the average distance between two nodes labeled by two consecutive integers is at most $9 + O(1/n)$.*

As a consequence, the broadcast problem can be solved with $O(kn)$ messages in any network supporting an interval routing scheme of *stretch factor* k.

A general consequence of our result is that a shortest path interval routing scheme contains ample *structural information* to avoid developing ad-hoc or network-specific solutions for basic problems that distributed systems must handle repeatedly.

Remark. The inherent serial behavior of the up/down protocol (by forwarding an increasing token and a decreasing token) imposes an $\Omega(n)$ time-complexity in any networks. We can however introduce a variant of the up/down protocol to reduce the time-complexity by using more copies of the message. In the following, w.l.g, we only consider the changes regarding the up copy (the modification for the down copy can be obtained similarly). The new protocol is obtained by modifying

the up/down protocol as follows. Each header now contains an interval $[a, b]$ of consecutive targets to reach. The minimum label a represents the current target to reach. Initially, the originator ν sets a to $\nu + 1$ and b to n, and sends the up copy toward $\nu + 1$. Any intermediate node x which receives a message with header $[a, b]$ proceeds as follows. If $x \notin [a, b]$, then x simply forwards the message toward a. If $x \in [a, b]$ then x creates two copies of the message, one which is sent toward $x+1$ with the header $[x+1, b]$, and another which is sent toward a with header $[a, x-1]$. A message of header $[x, x]$ reaching node x is removed from the network.

6 Related problems

This paper proved that n-node networks supporting shortest path interval routing schemes allow broadcasting with $O(n)$ message-complexity. Now, this result can be directly used to solve other problems, such as *leader election* or *distributed spanning tree*. Recall that, informally, the leader-election is the problem of moving the network from an initial situation where the nodes are in the same computational state, to a final situation where exactly one node is in a distinguished computational state (called *leader*) and all others are in the same state (called *defeated*). The election process may be independently started by any subset of the nodes, called awaken-nodes (any other node is said to be "asleep"). Every node x has a distinct value $I(x)$ chosen from some infinite totally ordered set and each processor is only aware of its own value. Most of the solutions commonly used solve the leader election problem using a distributed algorithm which elects the awaken-node with the highest value. In a network given with its interval routing scheme, a natural solution is to elect node labeled 1. This can be done in with $O(n)$ messages.

Theorem 4 *Networks supporting shortest-path interval routing scheme allows leader-election with $O(n)$ message-complexity.*

Proof. Every awaken node starts protocol "down" with a message (A, x) where"A" stands for awake, and x is the label of that node in the IRS. Every node executes protocol "down" on message of type (A, x), apart that an awaken target node receiving a message (A, x) removes that message from the network. When node 1 receives a message (A, x) as a target node, it awakes (if not yet awaken). Once awaken, node 1 starts the "up" protocol with message (B, i) where "B" stands for broadcast, and $i = I(1)$. Every node executes protocol "up" on message of type (B, i). When the protocol "up" completes, all nodes agree on node labeled 1 as the leader, and on the value i of $I(1)$.

The message-complexity of this algorithm is $O(n)$ because the phase "down" does not generate more than $O(n)$ messages since an awaken target node receiving a message (A, x) removes that message from the network. The "up" phase does not generate more than $O(n)$ messages thanks to Theorem 3. ■

In the distributed spanning tree problem, every node x of a network $G = (V, E)$ must chose some of its neighbors $x_1, \ldots, x_k$ so that the graph whose set of vertices is V, and whose set of edges is composed of the edges linking every node x with its chosen neighbors, form a spanning tree of G.

Theorem 5 *Networks supporting shortest-path interval routing scheme allows distributed spanning tree to be solved with $O(n)$ message-complexity.*

Proof. The protocol "up" is performed from node labeled 1 to broadcast a message "construct". Upon reception from u of the message "construct", a (non necessarily target) node v proceeds as follows:

- if it is the first time that v receives the message "construct", then (1) v selects u as its father, and (2) v sends a message "child" to u to tell u that v is one of u's children;
- v executes the instructions of protocol "up" for the message "construct".

Once the protocol "up" completes at the node labeled n, this node initiates the protocol "down" to broadcast a message "end" to propagate the information that the tree-construction is over. Once the node labeled 1 receives the message "end" as a target node, the whole process is completed. Then every node choses its neighbors in the tree as its children and its father. From Theorem 3, this protocol has $O(n)$ message-complexity: $O(n)$ messages "construct", $O(n)$ messages "end", and $n-1$ messages "child". ■

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A Sequence-decomposition for strict IRS networks

Algorithm 3 One step of the construction of the sequence-decomposition for a strict IRS

The current active path is $P = p_0, p_1, \ldots, p_t$, with $p_0 = w_0$. Let r be such that $p_t \in Y_r$, and let $Y_r = u_1, \ldots, u_{\ell_r}$.

Case 1: $p_t = u_i, i < \ell_r$ **and** $d = +1$. Then the active path is updated to $p_0, p_1, \ldots, p_t, u_{i+1}$. There is neither new dead-end path nor jumped path. The mark and the direction are not modified.

Case 2: $p_t = u_{\ell_r}$ **or** $d = -1$. Then let $\delta = d(p_t, X_{m+1})$, and $s \geq m+1$ be the smallest index such that $\sum_{i=m+1}^{s}(\ell_i - k_i) > \delta - 1$. If s does not exist, then the active path is updated to $p_0, p_1, \ldots, p_t, w_{\tau+1}$, there is no new dead-end path, all intermediate nodes between the last node of Y_m and $w_{\tau+1}$ form a new jumped path, and the construction stops. If s exists, then let $\kappa = \delta + k_s + \sum_{i=m+1}^{s-1}(k_i - \ell_i)$. Assume that $Y_s = v_1, \ldots, v_{\ell_s}$, two cases are considered:

Case 2.1: $p_t \notin \cup_{i=m+1}^{s} X_i$. Then again two cases may occur:

Case 2.1.1: There is an intermediate node w in $\{v_1, \ldots, v_{\kappa-1}\} \cup (\cup_{i=m+1}^{s-1} Y_i)$, such that $w \in Y_\sigma$ and $L_{x_\sigma}(w) < L_{x_\sigma}(p_t)$. Then pick the first node w of that type. The active path is updated to $p_0, p_1, \ldots, p_t, w$, and all intermediate nodes between the last node of Y_m and w form a jumped path. The mark m is updated to σ.

Case 2.1.2: For every intermediate node w in $\{v_1, \ldots, v_{\kappa-1}\} \cup (\cup_{i=m+1}^{s-1} Y_i)$,

$$w \in Y_\sigma \Rightarrow L_{x_\sigma}(w) \geq L_{x_\sigma}(p_t).$$

Then the active path is updated to $p_0, p_1, \ldots, p_t, v_\kappa$, and all intermediate nodes between the last node of Y_m and v_κ forms a jumped path. The mark m is set to s.

In any case, there is no new dead-end path, and the direction d is set to $+1$.

Case 2.2: $p_t \in \cup_{i=m+1}^{s} X_i$. The direction d is set to -1. Let $t' \in \{0, \ldots, t-1\}$ be the largest index such that $p_{t'} \notin \cup_{i=m+1}^{s} X_i$. All nodes $p_{t'+1}, \ldots, p_t$ forms a dead-end path. Assume $p_{t'+1} \in X_\sigma$, then if $\sigma > m+1$, all intermediate nodes of $\cup_{i=m+1}^{\sigma-1} Y_i$ form a jumped path. The mark m is updated to $\sigma - 1$.

B Sequence-decomposition for IRS networks

Algorithm 4 One step of the construction of the sequence-decomposition for an IRS

The current active path is $P = p_0, p_1, \ldots, p_t$, with $p_0 = w_0$. Let r be such that $p_t \in Y_r$, and let $Y_r = u_1, \ldots, u_{\ell_r}$.

Case 1: $p_t = u_i, i < \ell_{r-1}$ **and** $d = +1$. Then the active path is updated to $p_0, p_1, \ldots, p_t, u_{i+2}$, and node y_{i+1} is put in the auxiliary path. There is neither new dead-end path nor jumped path. The mark and the direction are not modified.

Case 2: $p_t = u_{\ell_r}$ **or** $p_t = u_{\ell_r - 1}$ **or** $d = -1$. Then let $\delta = d(p_t, X_{m+1})$, and $s \geq m+1$ be the smallest index such that $\sum_{i=m+1}^{s}(\ell_i - k_i) \geq 2(s-m) + \delta$. If s does not exist, then the active path is updated to $p_0, p_1, \ldots, p_t, w_{\tau+1}$, there is no new dead-end path, all intermediate nodes between the last node of Y_m and $w_{\tau+1}$ form a new jumped path, and the construction stops. If s exists, then let $\kappa = \delta + k_s + \sum_{i=m+1}^{s-1}(k_i - \ell_i) + 2(s-m-1)$. If $p_t = u_{\ell_r - 1}$ and $d = +1$, then y_{ℓ_r} is put in the auxiliary path. Assume that $Y_s = v_1, \ldots, v_{\ell_s}$, two cases are considered:

Case 2.1: $p_t \notin \cup_{i=m+1}^{s} X_i$. Then again two cases may occur:

Case 2.1.1: There is an intermediate node w in $\{v_1, \ldots, v_\kappa\} \cup (\cup_{i=m+1}^{s-1}(Y_i \setminus \{z_i\}))$ such that $w \in Y_\sigma$ and $L_{x_\sigma}(w) \leq L_{x_\sigma}(p_t)$. Then pick the first node w of that type. The active path is updated to $p_0, p_1, \ldots, p_t, w$, and all intermediate nodes between the last node of Y_m and w form a jumped path. The mark m is updated to σ.

Case 2.1.2: For every intermediate node w in $\{v_1, \ldots, v_\kappa\} \cup (\cup_{i=m+1}^{s-1}(Y_i \setminus \{z_i\}))$,

$$w \in Y_\sigma \Rightarrow L_{x_\sigma}(w) > L_{x_\sigma}(p_t).$$

Then the active path is updated by setting $p_{t+1} = \min\{v_{\kappa+1}, v_{\kappa+2}\}$. All intermediate nodes between the last node of Y_m and p_{t+1} forms a jumped path. The mark m is set to s.

In both cases, there is no new dead-end path, and the direction d is set to $+1$.

Case 2.2: $p_t \in \cup_{i=m+1}^{s} X_i$. Then the direction d is set to -1. Let $t' \in \{0, \ldots, t-1\}$ be the largest index such that $p_{t'} \notin \cup_{i=m+1}^{s} X_i$. All nodes $p_{t'+1}, \ldots, p_t$ forms a dead-end path. Assume that $p_{t'+1} \in X_\sigma$, then if $\sigma > m+1$, all intermediate nodes of $\cup_{i=m+1}^{\sigma-1} Y_i$ form a jumped path. The mark m is updated to $\sigma - 1$.